

Hyperbolic-space spectral characteristics for a network of mechanical linkages

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We investigate the dynamic properties of elastic lattices defined by tessellations of a curved hyperbolic space. The lattices are obtained by our projecting nodes of a regular hyperbolic tessellation onto a flat disk and then connecting those sites with simple linkages. Numerical and experimental investigations illustrate how their vibrational spectral properties are characterized by a high density of modes that are localized at the domain boundaries. Such properties govern the propagation of waves induced by broadband inputs. This suggests the potential for applications seeking the protection of bulk media from boundary-incident perturbations. We uncover the boundary-dominated nature of an exemplary hyperbolic lattice through the evaluation and analysis of its integrated density of states and vibrational spectrum. The dynamics of the lattice are also contextualized by our comparing them with those of continuous disks characterized by Euclidean and hyperbolic property distributions, which confirms that the lattice retains the boundary-dominated spectrum observed in the hyperbolic plane. We then numerically investigate the response of the lattice to transient pulses incident on the boundary and find that edge-confined wave propagation occurs. The modal and transient pulse-propagation behavior of the lattice is experimentally validated in a milled aluminum sample. By leveraging hyperbolic geometry, our mechanical lattice ushers in a novel class of mechanical metamaterials with boundary-dominated wave phenomena reminiscent of topologically protected systems suitable for applications in advanced wave control.

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I. INTRODUCTION

Waves can be controlled and steered by modulation of the material properties of a wave-bearing medium in space or time. However, it is often difficult in practice to exert the amount of control necessary on the material properties to achieve the desired wave properties. A passive approach to wave control is to curve the underlying geometric space hosting the waves [1–7]. In this case, the material can remain a simple homogeneous material, while the wave propagation is controlled by the curvature of the medium. Curved materials are ubiquitous in the natural world, where they are known to induce collective cellular wave phenomena in biological tissues [8,9], scatter or focus waves on surfaces [10–12], and allow topologically protected waves in acoustic systems as well as Earth's atmosphere and oceans [13,14].

A curved space of note is the hyperbolic plane, which is characterized by a constant negative curvature over its entire domain. This curious space challenges our geometric intuition by rejecting Euclid's parallel postulate, resulting in peculiar mathematical implications reported as early as the 18th century [15]. Studies in the 19th century developed the canonical differential geometry of hyperbolic space [16–19], while more-recent investigations have

developed its geometric group theory [20–26] and classical lattice dynamics [27–35]. In the last few years, hyperbolic lattices have been studied as platforms for informing the connection between quantum mechanics and general relativity through simulations of quantum lattices embedded in hyperbolic space [36–39]. In another vein of recent research, the theory of topological insulators, previously considered only in flat, Euclidean space, has been investigated in hyperbolic space, introducing new platforms for topologically protected systems [40–48]. These active areas of research have been supported by advances in the mathematical descriptions of waves in hyperbolic space, including hyperbolic crystallography and band structure [44,49–53]. Because of the inherent difficulties in creating practical structures with constant negative curvature, work has been done to mimic hyperbolic space in flat domains. Of note is recent work with circuit networks that has shown how dynamics in a hyperbolic space can be realized in tabletop experiments [37,38]. For electrical circuits, where the lengths of coupling wires are orders smaller than a wavelength, the resonators, placed at lattice sites, are effectively connected with equivalent couplings regardless of individual-wire lengths. Circuit networks have since been leveraged to experimentally explore topological phases in hyperbolic space [46,47]. Recently, the exploration of topological wave phenomena in hyperbolic systems has been extended to electromagnetic waves, with

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a notable study in a nonreciprocal scattering network hosting microwave frequencies [48] reporting robust chiral topological edge modes of Chern and anomalous types.

Previous work in mechanics includes the theoretical study conducted in Ref. [54], which highlights properties of hyperbolic tessellations with length-dependent interactions. In the present work, we expand these investigations to specifically focus on the analysis of spectral properties and the existence of high densities of states that are confined to the boundaries. This is a property that appears tightly connected to the generation of the tessellation, which produces approximants of hyperbolic distributions of material properties. Our experimentally validated numerical simulations confirm the existence of a high density of boundary-dominated modes, and suggest that the lattice considered is an illustration of hyperbolic-space dynamics in a mechanical system.

Following this introduction, we begin by reviewing hyperbolic geometry and the Poincaré disk model of hyperbolic space used to design the proposed mechanical lattice. To illustrate our lattice's ability to simulate hyperbolic space, we also review the dynamics governing harmonic vibrations in continuous Euclidean and hyperbolic spaces. With these models established, we show, by examining the integrated density of states (IDS) of all three systems, that the lattice exhibits a boundary-dominated spectrum similar to that of the hyperbolic plane. Next, the time-dependent response to boundary-incident pulses is numerically explored in connection with the previously investigated spectral properties. We find that edge-confined wave propagation occurs in the lattice, which we experimentally corroborate through laser-Doppler-vibrometry measurements of a milled aluminum sample. Although these boundary states are reminiscent of topological edge modes, we do not attempt to explore the potential connection with their associated reciprocal-space topology, which we reserve for future studies. The findings presented demonstrate the promise of elastic hyperbolic lattices as a feasible class of metamaterials for advanced wave control with applications including vibration mitigation, vibration-based sensing, and waveguiding. The proposed design presents a simple platform with potential for advanced studies in hyperbolic wave phenomena, such as novel topological states and the realization of curved spaces with novel physical properties and waveguiding capabilities.

II. HYPERBOLIC GEOMETRY ON A FLAT SURFACE

The hyperbolic plane is a two-dimensional surface characterized by constant negative Gaussian curvature, in contrast to the surface of a sphere, which has everywhere a constant positive Gaussian curvature. Hilbert's theorem states that the hyperbolic plane cannot be isometrically

mapped to the three-dimensional Euclidean space [55,56], meaning that there are no complete surfaces with constant negative curvature in Euclidean space that faithfully represent hyperbolic space. Thus, all models of hyperbolic space in subspaces of $\mathbb{R}^3$ necessarily distort lengths. One common representation of the hyperbolic plane is the positive sheet of a two-sheet hyperboloid that extends infinitely in a three-dimensional Euclidean space. This is the hyperboloid model of the hyperbolic plane, a portion of which is shown in orange in Fig. 1. This three-dimensional model provides a visually intuitive curvature but is an unrealistic specimen for tabletop experiments since it requires significantly more surface area than other models describing equal regions of interest in hyperbolic space. A more-practical alternative is to project the hyperboloid to a compact, lower-dimensional domain, where it is easier to manufacture and study. Much like how cartographers project Earth's positively curved surface to a flat map, the curved surface of the hyperboloid can be projected onto a flat unit disk with all projection lines meeting at the point $(0, 0, -1)$ as shown in Fig. 1(a). This flat model of

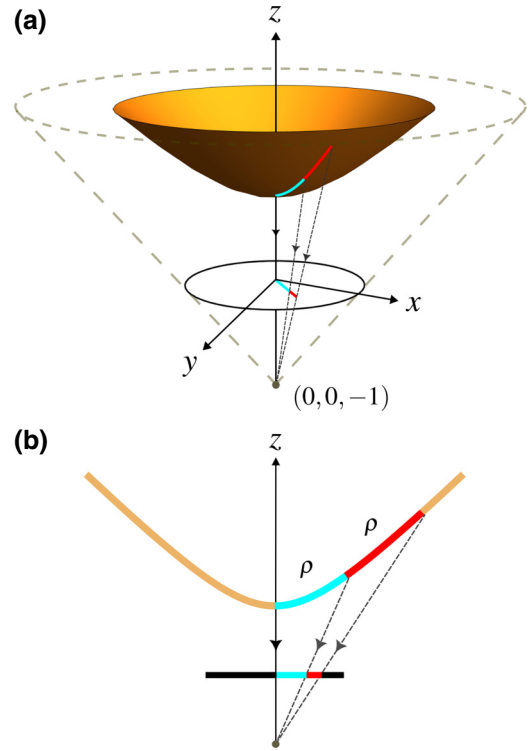


FIG. 1. The Poincaré disk model of hyperbolic space as a projection of the hyperboloid model of hyperbolic space. (a) The Poincaré disk is obtained by one projecting the hyperboloid (in orange) onto a disk of radius 1 through the point $(0, 0, -1)$. Cyan and red line segments show two geodesics in both models, which are of equal length ρ in hyperbolic space. (b) Cross section of the diagram in (a) illustrating the projection of hyperbolic geodesics from the hyperboloid to the disk, where compression of distances along the radius is seen.

the hyperbolic plane, known as the Poincaré disk model [57], is a conformal mapping that locally preserves hyperbolic angles and thus the shape of hyperbolic polygons in a tessellation.

Each model of the hyperbolic plane is paired with a metric that defines distances in the hyperbolic plane as a function of distances in the model. We illustrate these metrics by drawing two arcs, in cyan and red, in Fig. 1. These arcs, which are of equal length ρ in hyperbolic space, are seen as arcs of unequal length on the hyperboloid and on the disk. In the former they appear to stretch away from the origin, whereas in the latter they appear to compress away from the origin. For the Poincaré disk, this compressive effect is mathematically reconciled by one equipping the disk with the non-Euclidean Poincaré metric [16,58]

$$d\rho^2 = -\frac{4}{K} \frac{dx^2 + dy^2}{(1 - r^2)^2}, \quad (1)$$

where the hyperbolic line element $d\rho$ is related to Euclidean distances through the Gaussian curvature K and the radial coordinate $r = \sqrt{x^2 + y^2}$. K quantifies the curvature of the hyperbolic space and is commonly assigned the value -4 or -1 [38,52].

III. WAVE DYNAMICS ON THE POINCARÉ DISK

A direct result of the Poincaré metric in Eq. (1) is not only the distortion of distances on the disk but also the distortion of wave speeds. In this section, we illustrate how wave dynamics are distorted in the Poincaré disk by first considering the classical wave equation for an isotropic Euclidean space, given by

$$\Delta_E u = \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2}, \quad (2)$$

where Δ_E denotes the Laplacian in Euclidean space, u is the field variable, and c_0 is the characteristic wave speed.

In the Poincaré disk, we see that the spatial derivatives of the wave equation must be modified to account for Eq. (1). The Laplacian generalizes to the Laplace-Beltrami operator [28,59,60]. Acting on a smooth function ϕ , this operator is given by

$$\Delta\phi = \frac{1}{\sqrt{|g|}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(\sqrt{|g|} g^{ij} \frac{\partial \phi}{\partial x_j} \right), \quad (3)$$

where $|g|$ is the determinant of the metric tensor g_{ij} relative to coordinates x_1 and x_2 of a two-dimensional space and g^{ij} is the inverse of g_{ij} , following Ricci-calculus tensor notation. In Euclidean space, the metric tensor is given by the identity matrix, $g_{ij} = \delta_{ij}$, and Eq. (3) gives the Laplacian $\Delta_E = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$, where (x, y) defines an ordered pair of Cartesian coordinates on the reference

plane. The metric tensor for the Poincaré disk with curvature $K = -1$ is $g_{ij} = [4/(1 - x^2 - y^2)^2] \delta_{ij}$, leading to a Laplace-Beltrami operator, which on comparison with Eq. (2), can be expressed as

$$\Delta_H = \frac{(1 - x^2 - y^2)^2}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = \frac{(1 - r^2)^2}{4} \Delta_E, \quad (4)$$

where the subscript H signifies action on functions defined on the hyperbolic plane. At the origin of the Poincaré disk, $r = 0$, Δ_H reduces to $\frac{1}{4} \Delta_E$, which suggests that near this point the dynamics of waves on the Poincaré disk are governed by the expression $\frac{1}{4} \Delta_E u \approx (1/c_0^2)(\partial^2 u/\partial t^2)$. This corresponds to a Euclidean surface whose wave speed is reduced by a factor of 2. As we move away from the origin in a straight path, and with reference to Fig. 1(b), equidistant steps in the hyperbolic plane are increasingly compressed on the Poincaré disk, due to its radially dependent metric. As we approach the outer rim at $r = 1$, we approach an infinite compression of hyperbolic space and the step size becomes vanishingly small. This geometric feature appears in the wave equation through the radially dependent spatial operator, which becomes the zero operator in the limit $r \rightarrow 1$. Equation (4) suggests that Poincaré-disk wave dynamics could be achieved in a flat circular system where the wave speed is a function of r . This can be seen by one considering the wave equation on the Poincaré disk,

$$\Delta_H u = \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2}, \quad (5)$$

which on substitution of Eq. (4) gives

$$\frac{(1 - r^2)^2}{4} \Delta_E u = \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2}. \quad (6)$$

This is formally equivalent to

$$\Delta_E u = \frac{1}{c(r)^2} \frac{\partial^2 u}{\partial t^2}, \quad (7)$$

and corresponds to a system whose wave speed is quadratic in r , i.e., $c(r) = [(1 - r^2)/2]c_0$.

IV. POINCARÉ-DISK LATTICES

The properties of a system governed by an equation similar to Eq. (5) may be investigated by one considering a hyperbolic tessellation in the Poincaré disk as illustrated in Fig. 2(a), which shows a regular triangular tessellation on the hyperboloid and its resulting projection on the disk. The connectivity pattern for an elastic hyperbolic lattice

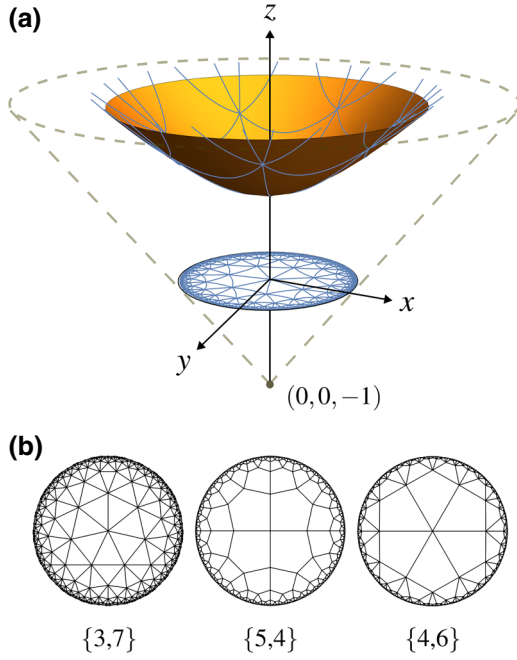


FIG. 2. Tessellations of the Poincaré disk model of hyperbolic space. (a) A hyperboloid sheet (orange) tessellated by regular triangles with vertex coordination number 7 projected onto the flat unit disk (blue), illustrating a $\{3, 7\}$ tessellation of the Poincaré disk model of hyperbolic space. (b) Example hyperbolic lattices with sites defined by vertices of tessellations of the Poincaré disk connected by straight Euclidean line segments. The tessellations are denoted by their Schläfli symbols, where the first number defines the number of sides of each regular polygon and the second number defines the number of polygons shared by each vertex. The $\{3, 7\}$ lattice of (a) is shown in (b) in a top-down view.

can be created by one connecting the vertices of a regular hyperbolic tessellation via straight Euclidean ligaments, as shown in Fig. 2(b), which provides three example lattices. Each lattice is distinguished by its Schläfli symbols $\{p, q\}$, where p is the number of sides of the tessellating polygon and q is the vertex coordination number. A hyperbolic polygon, or tile, hosts an interior-angle sum less than $(p - 2)\pi$ due to its distortion in curved space, giving the governing constraint $(p - 2)(q - 2) > 4$ [61,62]. This rule is satisfied by an infinite ensemble of $\{p, q\}$ sets, granting a limitless number of possible tessellations. In contrast, the Euclidean plane requires that tessellations satisfy the equality $(p - 2)(q - 2) = 4$, a constraint met by only three possibilities: $\{3, 6\}$, $\{4, 4\}$, and $\{6, 3\}$. We form the hyperbolic tessellations by first nucleating a seed tile at the origin and then repeatedly reflecting that tile about its sides until the desired tile count is reached. Each group of tile reflections surrounding the previous group marks a new generation (sometimes called an “epoch”), where an infinite number of generations is required to tile the full Poincaré disk [54]. The vertex count of hyperbolic tessellations grows exponentially as a function of the generation

[34,40,46,49,53,63]. As a result, all hyperbolic tessellations produce highly connected boundaries independent of p and q , and we find that different tilings produce similar spectral properties. The following investigations focus on the $\{3, 7\}$ tessellation, which is selected, without loss of generality, as a representative of properties also found in other tessellations (see note I in Supplemental Material [64]).

Figure 2 illustrates how even though the edges of regular triangles in hyperbolic space are all equal, in the hyperboloid and disk they are distorted. The challenge, then, is how to account for these distortions so that tessellations built in Euclidean space can still exhibit hyperbolic characteristics. One approach is to build a system where these lengths are unimportant. In this scenario, continuous dynamics are recovered because the graph Laplacian of the system (which depends only on the lattice connectivity pattern, described as a graph) reduces to the continuous Laplace-Beltrami operator in the long-wavelength limit [37,38]. An alternative approach is to increase the effective distances between lattice vertices by one stretching the paths connecting these sites. For example, when these paths are made to no longer follow the system’s geodesics, they can be carefully arranged so that they are of equal lengths while maintaining the same vertex connectivity pattern [39]. This observation is particularly relevant for this study, which considers the mechanical implementation of these lattices, wherein interactions are dependent on the Euclidean distance between connected sites. These distances the tessellations from the continuous-hyperbolic-medium description of Eq. (5). However, the results presented below show how some interesting properties of the system, and specifically the existence of boundary-dominated frequency ranges, are preserved.

V. NUMERICAL INVESTIGATION OF SPECTRAL PROPERTIES

We investigate the ability of the $\{3, 7\}$ lattice to localize vibrations near its boundary by evaluating its spectral properties and comparing them with those of continuous hyperbolic and Euclidean disks. The hyperbolic and Euclidean disks are both modeled via finite-element discretizations of the variational formulations of Eqs. (2) and (5). With use of the weak-form module in the COMSOL MULTIPHYSICS environment, these statements are approximated by Galerkin forms whereby the problem is defined over a discrete space of admissible trial and weighting functions. This allows us to assemble a finite-element matrix formulation over a discrete mesh to approximate the eigenvalue problem. To ensure proper convergence, we partition each disk into 2880 quadrilateral elements via an unstructured mesh with a maximum element size of $1/25$ the disk radius, which is at $r = 0.918$ in the Poincaré and Euclidean disks in order to match the maximal radial

coordinate of the hyperbolic lattice studied. We use Hermite cubic shape functions for the element-displacement interpolation. The eigenvalue problem is solved for the first 600 modes.

The $\{3, 7\}$ lattice is modeled as a network of one-dimensional Timoshenko beams undergoing strictly-out-of-plane vibrations. We discretize each beam into a finite element mesh with a maximum element length of $1/10$ the shortest lattice beam length, which is expected to ensure the accurate capture of the first 600 modes. The resulting mesh comprises 3024 one-dimensional elements, which again use Hermite cubic shape functions for the element-displacement interpolation. The beams are made of 6061 aluminum (Young's modulus 70 GPa, density 2700 kg/m³, Poisson's ratio 0.33) with cross-section width 3.81 mm (0.15 in.) and height 4.83 mm (0.19 in.). Constrained by the small features that arise near the rim of Poincaré-disk tessellations, we study a three-generation version of the lattice, holding 85 lattice vertices that fill the disk to radial coordinate $r \leq 0.918$. This is scaled to a physical radius $R = 150$ mm. These dimensions reflect the experimental geometry described in later sections [see Figs. 4(a), 4(b) and 5]. The summarized design provides a proper balance between the computational cost required for analysis and the ability to retain the key properties under investigation. All considered systems have free boundary conditions.

In our analyzing the spectral properties, the eigenvalues are characterized in terms of an integrated density of states, while the eigenstates are classified on the basis of a localization index that quantifies the degree of spatial localization near the disk boundaries. We express this IDS [65–68] as a function of modal index, i , by

$$\text{IDS}(i) = \frac{\#\{n \mid f_n \leq f_i\}}{N}, \quad (8)$$

where $\#$ is a counting operator returning the cardinality n of the set of eigenfrequencies available below the i th eigenfrequency f_i , while N denotes the total number of eigenfrequencies considered. Degenerate eigenfrequencies are repeated according to their multiplicity. The IDS naturally lies in the range $[0, 1]$ as it is normalized by N . We here consider the first $N = 600$ states of each system, which captures the IDS within a frequency range corresponding to wavelengths of dynamic deformation on the order of the smallest lattice beam length. Our choice of truncating the lattice to the third generation implicitly limits the occurrence of such wavelengths and therefore informs a suitable upper bound to N .

We classify each state by estimating the extent to which it is localized toward the domain center or outer boundary. This classification is based on the computation of a *localization index*, adapted from the index first applied to

discrete hyperbolic lattices [54] and defined as

$$\mathcal{L}_i = \frac{\iint \left(\frac{r}{R}\right)^2 u_i^2(\mathbf{r}) dA}{\iint u_i^2(\mathbf{r}) dA}, \quad (9)$$

where r is the magnitude of the position vector $\mathbf{r} = r(\cos \theta \hat{i} + \sin \theta \hat{j}) = r\hat{r}$ with origin at the center of the domain, $u_i(\mathbf{r})$ is the scalar displacement field of the i th eigenstate being classified, and R is the radius of either the disk or the lattice. The localization index is the mean squared (normalized) radial position of the squared displacement field. Simply put, it is a weighted average of the squared radial position $\left(\frac{r}{R}\right)^2$, where the weight $u_i(\mathbf{r})^2$ emphasizes locations along the radius with significant displacements. The square root of $\mathcal{L}_i$ provides an effective nondimensional radius between 0 and 1, representative of the radial position where displacement values converge in an average sense. A value close to 1 indicates the state is predominantly localized near the boundary (i.e., $r \rightarrow R$), while a value near 0 indicates localization near the origin ($r \rightarrow 0$).

Equation (9) is a high-level description of $\mathcal{L}_i$ acting on continuous domains. As we discretize our domains into finite elements, we compute the localization index numerically as

$$\mathcal{L}_i = \frac{\sum_{e=1}^{n_{\text{el}}} \int_{\Omega^e} \left(\frac{r}{R}\right)^2 \langle N^e(\mathbf{r}) \mid |d^e\rangle^2 d\Omega}{\sum_{e=1}^{n_{\text{el}}} \int_{\Omega^e} \langle N^e(\mathbf{r}) \mid |d^e\rangle^2 d\Omega}, \quad (10)$$

where our displacement field is now expressed as the inner product $u_i(\mathbf{r}) = \langle N^e(\mathbf{r}) \mid |d^e\rangle = \sum_{a=1}^{n_{\text{en}}} N_a^e(\mathbf{r}) d_a^e$, where $\langle N^e(\mathbf{r}) \mid$ is the vector of Hermite cubic shape functions and $|d^e\rangle$ is the vector of nodal displacements, each of length n_{en} , the number of element nodes. This is the element interpolation of the displacement at position vector $\mathbf{r}$ within the element domain Ω^e based on the computed nodal displacements. The integration is done numerically in the material (Lagrangian) frame via seven-point Gaussian quadrature, giving exact solutions for our polynomial integrands, which are at most of order 12. We integrate over each individual element domain Ω^e so as to integrate over the entire discretized domain, given by $\Omega^h = \mathbf{A}_{e=1}^{n_{\text{el}}} \Omega^e$, with assembly operator $\mathbf{A}$ [69] acting over the number of elements n_{el} . We then sum these integrals in both the numerator and the denominator over n_{el} (2880 for the disks and 3024 for the lattice), taking the ratio as the localization index.

Through $\mathcal{L}_i$, we classify each mode as either interior dominated or boundary dominated by setting a decision boundary, or threshold, $\mathcal{L}_i = 0.5$. This value is chosen because it corresponds to a displacement field averaged to a normalized radial position of $(r/R)_i = \sqrt{0.5} \approx 0.707$, which is the radius that exactly separates equal areas between the interior and boundary regions of the disks

or lattice. Thus, $\mathcal{L}_i > \mathcal{L}_t$ indicates high modal activity in the boundary region, while $\mathcal{L}_i < \mathcal{L}_t$ indicates high modal activity in the interior region. Values near $\mathcal{L}_t$ describe states with global behavior, which are ultimately classified relative to $\mathcal{L}_t$ for simplicity.

After applying $\mathcal{L}_t$ to separate the spectra into subsets of boundary and interior states, we evaluate their respective, separate integrated densities of states. In Fig. 3, we plot the boundary and interior IDS of each system, coloring the curves corresponding to boundary states red and those corresponding to interior states blue. The horizontal axis marks the total states plotted among both red and blue curves up to a modal index such that the sum of both corresponding IDS values gives the proportion of the 600 modes considered. Evidently, by the 600th mode, the sum of the bulk IDS and the boundary IDS is 1. The curves for the Euclidean and hyperbolic continuous disks governed by Eqs. (2) and (5), respectively, are shown in Figs. 3(a) and 3(b). The Euclidean disk [Fig. 3(a)] shows an overall higher number of interior modes than boundary modes, as indicated by the blue curve ultimately exceeding the red curve, a trend retained in Euclidean lattices (see note II in Supplemental Material [64]). The opposite happens for the hyperbolic disk, which has a consistently and significantly higher boundary IDS than interior IDS [Fig. 3(b)]. This is due to two features, highlighted in other studies of Poincaré-disk wave dynamics [38,54]. The first is that the mode shapes on the Poincaré disk are similar to those of the Euclidean disk, but with more-prominent localization toward the outer rim. The second is that more boundary modes populate lower frequencies in the hyperbolic Poincaré disk than in the Euclidean disk. This can be understood by one considering that hyperbolic distances are most distorted near the disk's boundary, where they are longer than their Euclidean counterparts. Therefore, the wavelengths of angular, boundary-localized waves in the Poincaré disk are longer than those more centrally located. This is directly deduced from Eq. (1) in geodesic polar coordinates with curvature $K = -1$: $d\rho^2 = dr^2 + \sinh^2(r)d\theta^2$ (see page 133 in Ref. [70]). We consider the path of a boundary-localized wave along an origin-centered circle parametrically described by $r(t) = R$ and $\theta(t) = t$. The hyperbolic circumference C_H of this circle is given by the arc length in one revolution, computed from the metric by $C_H = \int_0^{2\pi} \sqrt{(dr(t)/dt)^2 + \sinh^2(r(t))(d\theta(t)/dt)^2} dt$. We directly find that the hyperbolic circumference is given by $C_H = 2\pi \sinh(R)$, which grows superlinearly in radius by $dC_H/dR = 2\pi \cosh(R)$. On the other hand, the circumference of a Euclidean circle grows constantly by $(d/dR)2\pi R = 2\pi$. Therefore, boundary modes in the Poincaré disk with angular oscillations at radial point $R > 0$ have longer circumferential wavelengths than Euclidean boundary modes at the same R with the same number of angular zeros. Thus, boundary modes in the Poincaré disk

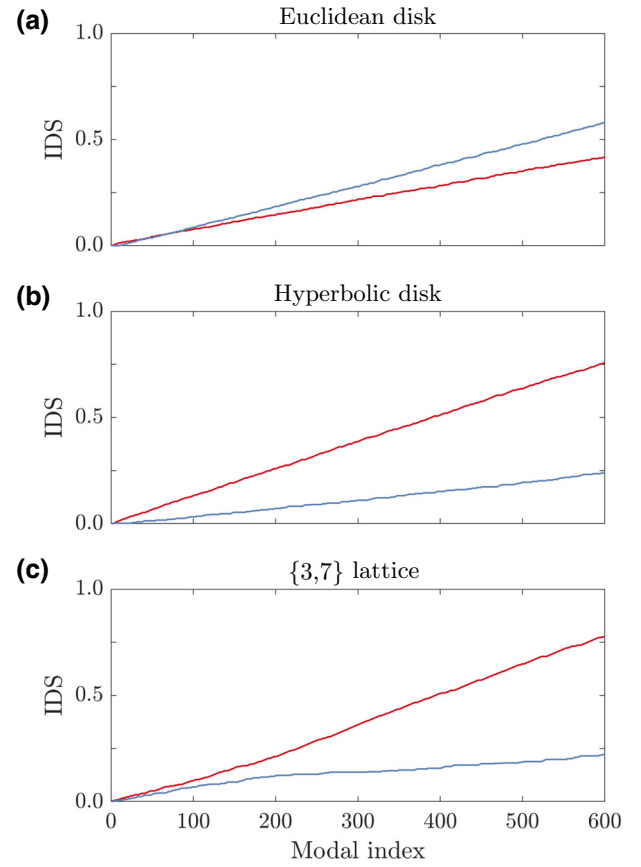


FIG. 3. IDS versus modal index for (a) a Euclidean disk, (b) a hyperbolic disk, and (c) the $\{3, 7\}$ lattice. The blue IDS curves correspond to interior-dominated states, while the red curves correspond to boundary-dominated states. The Euclidean spectrum begins with a similar number of interior (blue) and boundary (red) states and becomes dominated by interior modes for increasing mode index, while the hyperbolic disk shows a dominance of boundary modes. The $\{3, 7\}$ lattice holds an approximately equal proportion of interior and boundary modes until around the 80th mode, when the boundary-mode count begins to dominate and ultimately gives a proportion similar to that for the hyperbolic disk.

have lower frequencies and hence lower modal indices than similar ones in the Euclidean disk.

By conducting a similar analysis on the $\{3, 7\}$ lattice, we can ascertain to what extent such a lattice captures the spectral signature of the hyperbolic plane. The lattice's IDS shown in Fig. 3(c) reveals a spectral dominance of boundary states in the $\{3, 7\}$ lattice similar to that of the hyperbolic disk. While the IDS of interior and boundary modes is approximately balanced at the lower modal indices, a tendency for modes to localize along the boundary is observed to begin around the 80th mode. Below this region, at the start of the spectrum where wavelengths are long relative to lattice beam lengths, the lattice behaves more like the Euclidean domain, with a similar proportion of interior and boundary modes. In contrast, as the

characteristic wavelengths decrease, we observe the surfacing of features similar to those of the hyperbolic disk, whereby a larger proportion of boundary modes in the lattice exists. This demonstrates the ability of the lattice considered to mimic the boundary-dominated signature of the hyperbolic disk above a certain threshold frequency. At the highest mode considered, the lattice's spectrum is populated by 78% boundary modes, which is close to the hyperbolic disk's 76% boundary modes, while the Euclidean disk has only about 42% boundary modes.

VI. NUMERICAL DEMONSTRATION OF EDGE-CONFINED WAVE PROPAGATION

Next we investigate the lattice response to transient vibratory inputs to understand how it is influenced by its boundary-dominated spectrum. We do so

via time-domain simulations of the lattice response when it is subjected to broadband pulse excitations applied to its boundary. These simulations are done with the COMSOL MUULTIPHYSICS finite-element solver with the same geometry, parameters, boundary conditions, and mesh previously used for the eigenfrequency study. The excitation is a Hanning window–modulated sine wave including six cycles of the signal's center frequency. The duration of each simulation is 12 times the duration of the excitation, which is equal to the number of cycles times the period of the excitation center frequency. Figures 4(a) and 4(b) show the location of the input excitation and the color plots of the root-mean-square (rms) lattice displacement averaged over the simulation time for broadband inputs centered at 2.2 kHz [Fig. 4(a)] and 23.6 kHz [Fig. 4(b)]. The rms displacement fields are normalized by the maximum rms value

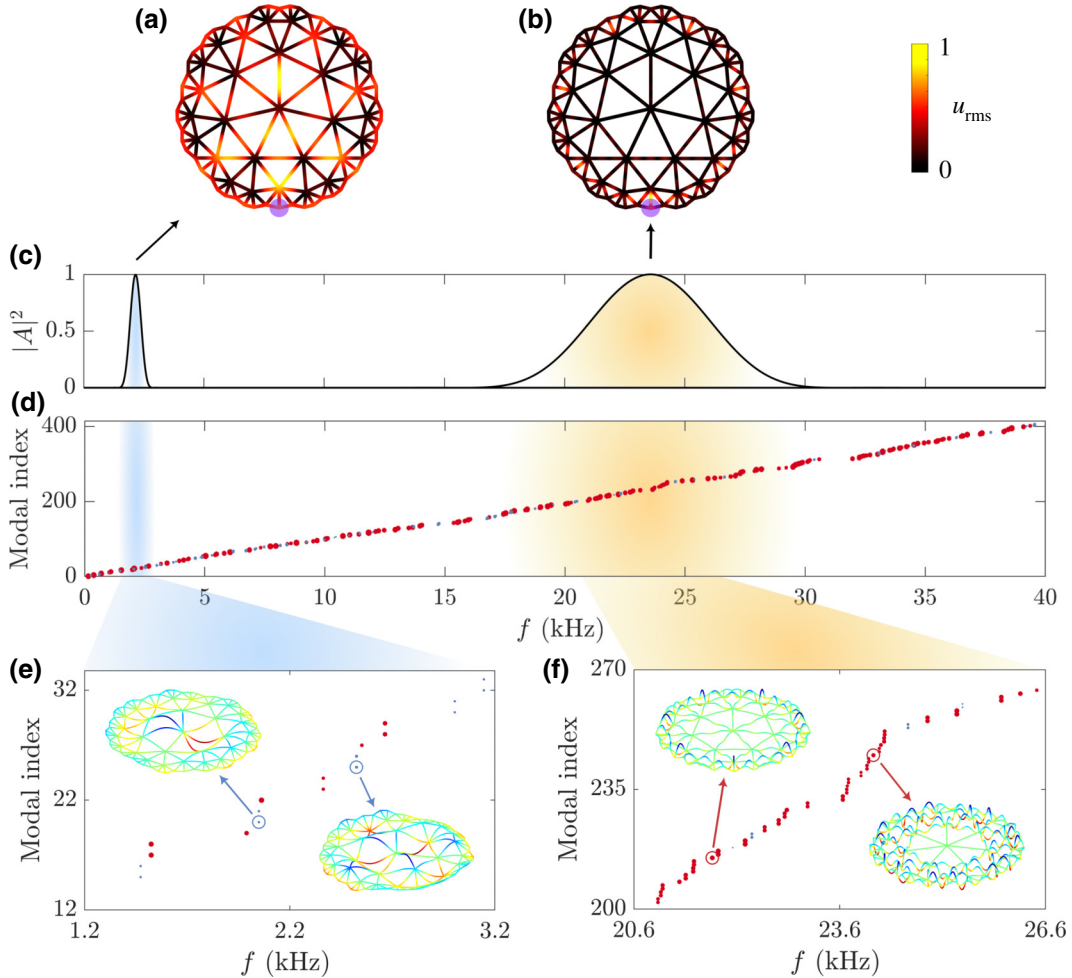


FIG. 4. (a),(b) Time-averaged squared displacement (rms) fields of a $\{3, 7\}$ lattice excited at its boundary (transparent purple dot) by transient inputs centered at (a) 2.2 kHz and (b) 23.6 kHz. (c) Frequency content of the 2.2- and 23.6-kHz pulses, each with equivalent fractional bandwidths. (d) Spectrum of the $\{3, 7\}$ lattice in the 0–40-kHz range: eigenvalues are plotted with red and blue dots on the basis of the associated localization index, which also determines the size of the dot. (e) Enlarged portion of the spectrum in the frequency range of the 2.2-kHz pulse highlighting two interior-dominated modes. (f) Enlarged portion of the spectrum in the frequency range of the 23.6-kHz pulse highlighting two boundary-dominated modes.

according to the reported color bar, which shows a variation in intensity from black (0) to yellow (1). In Fig. 4(a), the response amplitude is globally distributed, with the highest displacements near the excitation point and in the center. In Fig. 4(b), the response amplitude is significantly larger at the boundary, with virtually no motion in the bulk, confirming the boundary-governed spectral properties of the lattice at high frequencies.

This observation is supported by the spectral content of the inputs, plotted in Fig. 4(c) as normalized power spectral densities. The bandwidths of the two frequency inputs considered are highlighted by shaded blue and gold areas that are projected onto Fig. 4(d), the lattice's spectrum in the 0–40-kHz range. Each mode in the spectrum is represented as a dot, color coded again in red (boundary mode) and blue (interior mode). The size of the dot is also scaled by the localization index $\mathcal{L}_i$, with very large and very small dots indicating high degrees of localization. The highlighted bands aid in visualizing which modes are excited by the applied input and therefore contribute to the lattice's transient response. Figures 4(e) and 4(f) provide magnified portions of the spectrum within the frequency contents of each pulse. We see that for the 2.2-kHz pulse, there are roughly equal numbers of boundary and interior modes, whereas the 23.6-kHz pulse excites significantly more boundary modes. The boundary-mode:interior-mode ratio in Fig. 4(e) is 10:9, compared with 56:7 in Fig. 4(f). Example mode shapes are inset in Figs. 4(e) and 4(f), which show two interior modes and two boundary modes, respectively.

VII. EXPERIMENTAL CORROBORATION OF SPECTRUM AND EDGE-CONFINED WAVES

Transient-vibration measurements are conducted on a $\{3, 7\}$ elastic lattice to corroborate the spectral properties predicted by the theoretical investigation presented above. The experimental specimen is machined from a single 6061 aluminum plate by our milling away triangles. The resulting lattice, pictured in Fig. 5, comprises beams of width 3.81 mm and thickness 4.83 mm, as in the numerical model. The overall lattice radius R is 150 mm. In these tests, the lattice is suspended vertically by thin nylon threads bound to optical posts mounted on an optical table. The lattice faces a laser Doppler vibrometer that takes the vibration measurements, and it is excited by a piezoelectric disc bonded at the boundary site corresponding to the excitation point numerically considered and depicted in Fig. 4(a).

First, the modal content of the lattice is evaluated via a frequency-domain experiment. Here, the affixed piezoelectric transducer excites the lattice with pseudorandom noise containing frequencies from 0 to 40 kHz, which is the range numerically studied in the previous section. These measurements highlight the resonance frequencies

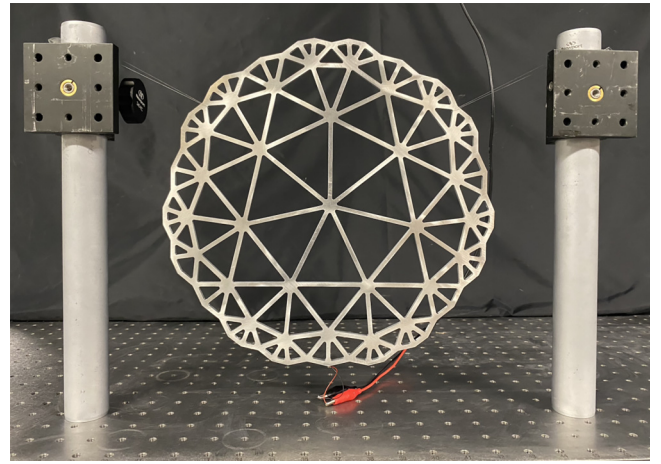


FIG. 5. Measurement setup for the experimental $\{3, 7\}$ lattice. The lattice, machined out of an aluminum plate, is suspended via nylon strings bound to damped optical posts affixed to an optical table in order to achieve free boundary conditions. A function generator is used to excite the lattice via a piezoelectric transducer at its bottom-most point, where there are red and black alligator clips attached to the leads of this transducer. A scanning laser Doppler vibrometer measures the response of the lattice, and it is connected to a data-acquisition unit, which processes and stores the results.

of the lattice and allow the estimation of dynamic deflection shapes related to the eigenstates of the system. They are used to compute correlation with numerical mode shapes and to corroborate the corresponding localization indices. Second, transient-vibration measurements are conducted to verify the edge-dominated transient response of the lattice when it is excited by broadband signals in the boundary-dominated high-frequency regime compared with the bulk, interior-dominated low frequencies.

In the frequency-domain study, the lattice's response is obtained by our averaging the responses at each scan point, producing the averaged frequency response shown in Fig. 6(b). The plot is limited to the 7–13-kHz range, which contains the portion of the spectrum where the IDS begins to become boundary dominated; thus, it contains the lower-frequency boundary modes that can be captured in experiments with less noise. Figure 6(a) maps the measured resonant peaks to those measured experimentally, providing a visual comparison with the numerically calculated spectrum. Each of these numerical-experimental mode pairs is labeled with the letters A–F for tracking purposes. Table I compares the numerical eigenfrequencies and localization indices with the experimental resonance peak frequencies and localization indices for pairs A–F, showing an overall good agreement. Some of the discrepancies can be attributed to the fact that the finite size of the computer numerical control router bit produces some rounding of the triangle vertices in the milled sample, which are not included in the numerical description of

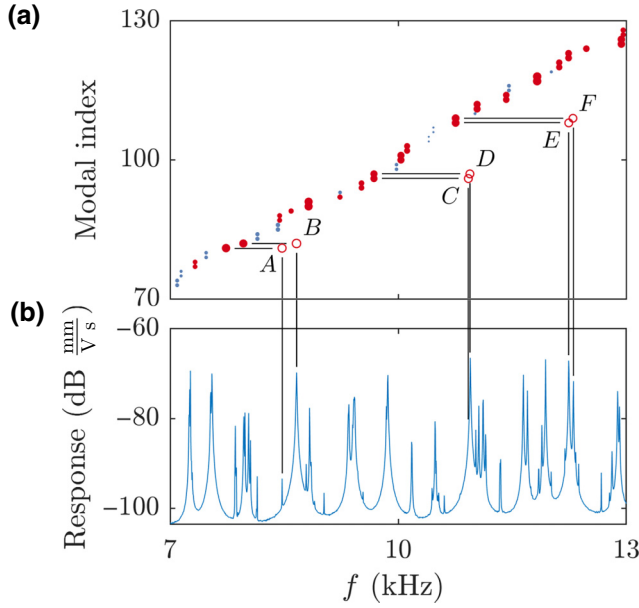


FIG. 6. (a) Numerical spectrum of the $\{3,7\}$ lattice in the 7–13-kHz range. Modes are ordered by frequency and sized according to their localization index. Solid red dots indicate boundary modes, while solid blue dots indicate interior modes. Selected experimental boundary modes are plotted in hollow red circles. Labels A – F indicate the six pairs of modes being compared. (b) Experimental frequency-response function in the 7–13-kHz range.

the lattice. This is expected to lead to local stiffening effects at the vertices relative to the idealized numerical model, which may systematically augment experimental frequencies compared with numerical ones.

The experimental response provides the necessary information to construct the operational deflection shapes of the lattice based on measured velocities at scanned points for a given frequency peak. These operational deflection shapes are plotted along with numerically obtained mode shapes in Fig. 7. By organizing individual experimental and numerical shapes as vector entries in matrices, $\{\phi_e\}$

TABLE I. Natural frequencies and localization indices of six modes (A – F) in numerical simulations and experimental measurements.

Mode	Frequency (kHz)		Localization index	
	Simulation	Expt.	Simulation	Expt.
A	7.73	8.47	0.87	0.83
B	7.96	8.66	0.85	0.87
C	9.68	10.92	0.75	0.81
D	9.68	10.94	0.75	0.81
E	10.75	12.24	0.82	0.80
F	10.75	12.29	0.82	0.79

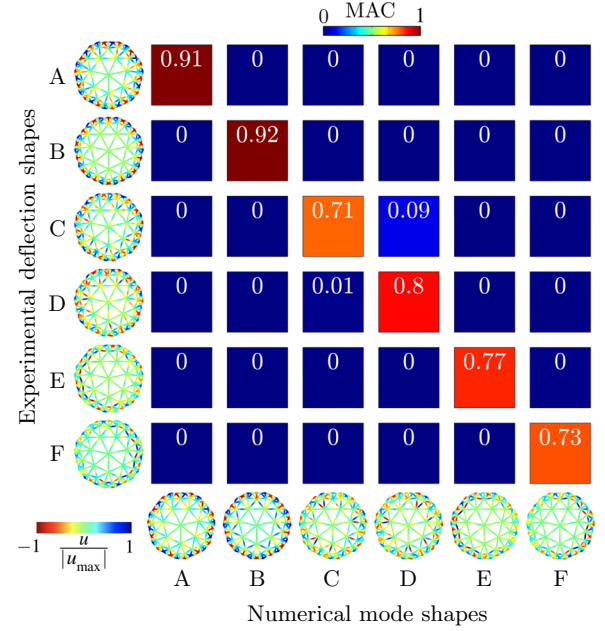


FIG. 7. Modal-assurance-criterion matrix comparing measured deflection shapes with numerically calculated mode shapes. Six experimentally obtained operational deflection shapes of the manufactured 6061-aluminum $\{3,7\}$ elastic hyperbolic lattice are shown on the left. Six numerical mode shapes of the modeled $\{3,7\}$ elastic hyperbolic lattice are shown at the bottom.

and $\{\phi_n\}$, respectively, we can further quantify their consistency via the modal assurance criterion (MAC), which provides a matrix with entries (r, q) given by [71]

$$\text{MAC}(r, q) = \frac{|\{\phi_e\}_r \{\phi_n\}_q|^2}{(\{\phi_e\}_r^T \{\phi_e\}_r)(\{\phi_n\}_q^T \{\phi_n\}_q)}, \quad (11)$$

where $\{\phi_e\}_r$ and $\{\phi_n\}_q$ are the individual experimental and numerical mode shapes being compared for the given (r, q) entry. The MAC is effectively a normalized inner product that is evaluated to 1 if the measured deflection shape is fully consistent with the numerically calculated mode shape and to 0 if they are linearly independent, although a true orthogonality check warrants the inclusion of the mass or stiffness matrix. This evaluation is particularly useful in comparing intricate mode shapes that are challenging to examine with the naked eye, while at the same time providing quantifiable validation of the numerical model. To compute the MAC, the numerical and experimental shapes must lie in the same vector space. Therefore, the numerical mesh is interpolated linearly at the experimental scan points, giving $\{\phi_e\}$ and $\{\phi_n\}$ equal ranks.

To confirm the time-dependent simulations, we suspend and excite the lattice by the same nylon threads and piezoelectric transducer as before. The transducer, affixed at the marked purple dot in Fig. 8(a), excites the lattice with

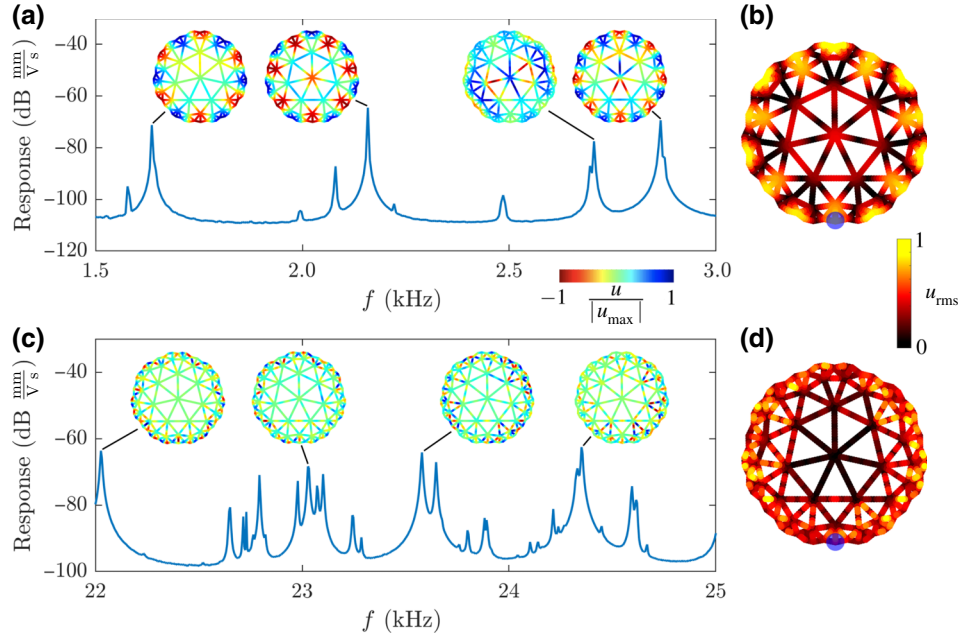


FIG. 8. Experimental rms time-averaged displacement field of the $\{3,7\}$ lattice in response to (b) a 2.2-kHz pulse in the low-frequency regime and (d) a 23.6-kHz pulse in the high-frequency regime, both excited through a piezoelectric transducer affixed to a boundary vertex (marked with purple dots). Averages are taken over the course of 12 times the duration of each excitation. Corresponding experimental frequency responses within the frequency content of each pulse are shown in (a),(c). The four experimental deflection shapes with the highest response peak are inset in each response spectrum.

the 2.2- and 23.6-kHz pulses from the numerical simulations. A scanning laser Doppler vibrometer measures out-of-plane displacements across the entire lattice. The rms displacement fields averaged over the course of 12 times the duration of each excitation are plotted in Figs. 8(b) and 8(d), where they are normalized to a maximum of 1. Accompanying these averages in Figs. 8(a) and 8(c) are the experimental frequency responses in the neighborhood of each pulse's center frequency. Inset in each figure are the four experimental deflection shapes with the highest response amplitude. We see that for the 2.2-kHz pulse, the four-highest peaks correspond to one boundary-dominated mode and three global modes. For the 23.6-kHz pulse, we see that the four-highest peaks correspond to four boundary modes. Therefore, the anticipated global rms from Fig. 8(b) and the boundary-dominated rms in Fig. 8(d) are observed. Compared with the numerical study, we see more boundary activity in experiments overall for both the low-frequency and the high-frequency rms displacements. We anticipate this is due to the systematic upward shift of experimental frequencies, and thus the inclusion of additional higher-frequency modes in the excitation-frequency contents. Nonetheless, the high-frequency pulse excites virtually zero activity in the lattice center, which encapsulates the entire first generation of beams. This feature could find application in protecting bulk media from vibrations incident on the boundary. Such edge states are reminiscent of those in topologically protected systems.

VIII. CONCLUSIONS

In this paper we report numerical and experimental observations of the boundary-dominated nature of vibrations in a two-dimensional elastic hyperbolic lattice. We first numerically compute the spectrum of the hyperbolic lattice compared with that of Euclidean and hyperbolic continuous disks, and we show that the straight-edged hyperbolic lattice retains some of the boundary-dominated features of hyperbolic media, in particular at high mode numbers. This spectrum is experimentally confirmed via frequency-domain experiments using a laser Doppler vibrometer. We then numerically investigate the response of the lattice to boundary-incident pulses, which reveals its capability to confine incident vibrations to its edge for excitations with high-enough-frequency content. Examples are provided for pulses centered at 2.2 and 23.6 kHz, which highlight the boundary-dominated behavior at high frequency, where vibrations tend to excite the smaller intricacies of the lattice located at the boundary. Experimental measurements are then conducted to confirm the numerical observations and specifically to verify the boundary-dominated nature of the vibratory response at high frequency. By our leveraging the boundary-dominated spectrum of the hyperbolic plane, the hyperbolic lattice considered acts as a host for edge-confined states reminiscent of topologically protected systems, some of which have recently been uncovered in hyperbolic space. This work presents a feasible way to

study hyperbolic lattices where couplings between sites are not all equal, allowing for a natural and simple experimental platform to study elastic waves in hyperbolic lattices. Therefore, it may allow the realization of as-yet largely unexplored hyperbolic metastructures that are deployed in various geometrical domains as well as applications in architected materials with broadband vibration-confinement capabilities.

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The authors declare that they have no competing interests.

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- [1] J. Zhu, Y. Liu, Z. Liang, T. Chen, and J. Li, Elastic waves in curved space: Mimicking a wormhole, *Phys. Rev. Lett.* **121**, 234301 (2018).
 - [2] D. Lee, Y. Hao, J. Park, I. S. Kang, S.-H. Kim, J. Li, and J. Rho, Singular lenses for flexural waves on elastic thin curved plates, *Phys. Rev. Appl.* **15**, 034039 (2021).
 - [3] M. Mazzotti, M. Gupta, C. Santangelo, and M. Ruzzene, Elastic wave manipulation in geodesic lenses, *ArXiv*: 2203.08266.
 - [4] V. H. Schultheiss, S. Batz, A. Szameit, F. Dreisow, S. Nolte, A. Tünnermann, S. Longhi, and U. Peschel, Optics in curved space, *Phys. Rev. Lett.* **105**, 143901 (2010).
 - [5] R. Bekenstein, Y. Kabessa, Y. Sharabi, O. Tal, N. Engheta, G. Eisenstein, A. J. Agranat, and M. Segev, Control of light by curved space in nanophotonic structures, *Nat. Photonics* **11**, 664 (2017).
 - [6] J. G. Ratcliffe, S. Axler, and K. Ribet, *Foundations of Hyperbolic Manifolds* (Springer, New York, 1994), Vol. 149.
 - [7] S. Helgason, Geometric analysis on symmetric spaces, *Math. Surv. Monogr.* **39**, 468 (1994).
 - [8] T. Brandstätter, D. B. Brückner, Y. L. Han, R. Alert, M. Guo, and C. P. Broedersz, Curvature induces active velocity waves in rotating spherical tissues, *Nat. Commun.* **14**, 1643 (2023).
 - [9] B. Schamberger, R. Ziege, K. Anselme, M. Ben Amar, M. Bykowski, A. P. Castro, A. Cipitria, R. A. Coles, R. Dimova, M. Eder, *et al.*, Curvature in biological systems: Its quantification, emergence, and implications across the scales, *Adv. Mater.* **35**, 2206110 (2023).
 - [10] R. Mitchell-Thomas, O. Quevedo-Teruel, T. McManus, S. Horsley, and Y. Hao, Lenses on curved surfaces, *Opt. Lett.* **39**, 3551 (2014).
 - [11] B. Rulf, Rayleigh waves on curved surfaces, *J. Acoust. Soc. Am.* **45**, 493 (1969).
 - [12] L. La Spada, C. Spooner, S. Haq, and Y. Hao, Curvilinear metasurfaces for surface wave manipulation, *Sci. Rep.* **9**, 3107 (2019).
 - [13] S. Shankar, M. J. Bowick, and M. C. Marchetti, Topological sound and flocking on curved surfaces, *Phys. Rev. X* **7**, 031039 (2017).
 - [14] P. Delplace, J. Marston, and A. Venaille, Topological origin of equatorial waves, *Science* **358**, 1075 (2017).
 - [15] G. Saccheri, *Euclides Vindicatus*, edited by G. B. Halsted (Open Court Publishing Company, Chicago, 1920).
 - [16] E. Beltrami, Saggio di interpretazione della geometria non-euclidea, *Gior. Mat.* **6**, 248 (1868).
 - [17] W. Killing, Ueber zwei Raumformen mit constanter positiver Krümmung, *J. Reine Angew. Math.* **86**, 72 (1878).
 - [18] J. Bolyai, Appendix, Scientiam Spatii absolute veram exhibens: A veritate aut falsitate Axiomatis XI. Euclidei (a priori haud unquam decidenda) independentem; adjecta ad casum falsitatis, quadratura circuli geometrica. Auctore Johanne Bolyai de eadem, Geometrarum in Exercitu Caesareo Regio Austriaco Castrensium Capitaneo. (Coll. Ref., 1832).
 - [19] F. Klein, Ueber liniengeometrie und metrische Geometrie, *Math. Ann.* **5**, 257 (1872).
 - [20] C. Series, The infinite word problem and limit sets in Fuchsian groups, *Ergod. Theory Dyn. Syst.* **1**, 337 (1981).
 - [21] A. Beardon, *The Geometry of Discrete Groups*. Springer, New York, Zbl **528**, 30001 (1983).
 - [22] W. Woess, Context-free languages and random walks on groups, *Discrete Math.* **67**, 81 (1987).
 - [23] W. J. Floyd and S. P. Plotnick, Growth functions on Fuchsian groups and the Euler characteristic, *Invent. Math.* **88**, 1 (1987).
 - [24] K. Corlette, Archimedean superrigidity and hyperbolic geometry, *Ann. Math.* **135**, 165 (1992).
 - [25] A. Lubotzky, Free quotients and the first Betti number of some hyperbolic manifolds, *Trans. Groups* **1**, 71 (1996).
 - [26] S. Katok, *Fuchsian Groups* (University of Chicago Press, Chicago, 1992).
 - [27] A. Comtet, On the Landau levels on the hyperbolic plane, *Ann. Phys. (N. Y.)* **173**, 185 (1987).
 - [28] N. Balazs and A. Voros, Chaos on the pseudosphere, *Phys. Rep.* **143**, 109 (1986).
 - [29] R. Rietman, B. Nienhuis, and J. Oitmaa, The Ising model on hyperlattices, *J. Phys. A: Math. Gen.* **25**, 6577 (1992).
 - [30] C. Monthus and C. Texier, Random walk on the Bethe lattice and hyperbolic Brownian motion, *J. Phys. A: Math. Gen.* **29**, 2399 (1996).
 - [31] R. Aurich and J. Marklof, Trace formulae for three-dimensional hyperbolic lattices and application to a strongly chaotic tetrahedral billiard, *Physica D* **92**, 101 (1996).
 - [32] L. A. Bunimovich, Y. G. Sinai, and N. I. Chernov, Statistical properties of two-dimensional hyperbolic billiards, *Russ. Math. Surv.* **46**, 47 (1991).
 - [33] K. Ueda, R. Krcmar, A. Gendiar, and T. Nishino, Corner transfer matrix renormalization group method applied to the Ising model on the hyperbolic plane, *J. Phys. Soc. Jpn.* **76**, 084004 (2007).
 - [34] S. K. Baek, P. Minnhagen, and B. J. Kim, Percolation on hyperbolic lattices, *Phys. Rev. E* **79**, 011124 (2009).
 - [35] S. K. Baek, S. Do Yi, and B. J. Kim, Diffusion on a heptagonal lattice, *Phys. Rev. E* **77**, 022104 (2008).
 - [36] P. Bienias, I. Boettcher, R. Belyansky, A. J. Kollár, and A. V. Gorshkov, Circuit quantum electrodynamics in hyperbolic space: From photon bound states to frustrated spin models, *Phys. Rev. Lett.* **128**, 013601 (2022).

- [37] I. Boettcher, P. Bienias, R. Belyansky, A. J. Kollár, and A. V. Gorshkov, Quantum simulation of hyperbolic space with circuit quantum electrodynamics: From graphs to geometry, *Phys. Rev. A* **102**, 032208 (2020).
- [38] P. M. Lenggenhager, A. Stegmaier, L. K. Upreti, T. Hofmann, T. Helbig, A. Vollhardt, M. Greiter, C. H. Lee, S. Imhof, H. Brand, *et al.*, Simulating hyperbolic space on a circuit board, *Nat. Commun.* **13**, 1 (2022).
- [39] A. J. Kollár, M. Fitzpatrick, and A. A. Houck, Hyperbolic lattices in circuit quantum electrodynamics, *Nature* **571**, 45 (2019).
- [40] N. Cheng, F. Serafin, J. McInerney, Z. Rocklin, K. Sun, and X. Mao, Band theory and boundary modes of high-dimensional representations of infinite hyperbolic lattices, *Phys. Rev. Lett.* **129**, 088002 (2022).
- [41] Z.-R. Liu, C.-B. Hua, T. Peng, and B. Zhou, Chern insulator in a hyperbolic lattice, *Phys. Rev. B* **105**, 245301 (2022).
- [42] D. M. Urwyler, P. M. Lenggenhager, I. Boettcher, and R. Thomale, Hyperbolic topological band insulators, *Phys. Rev. Lett.* **129**, 246402 (2022).
- [43] A. Chen, H. Brand, T. Helbig, T. Hofmann, S. Imhof, A. Fritzsche, T. Kießling, A. Stegmaier, L. K. Upreti, T. Neupert, *et al.*, Hyperbolic matter in electrical circuits with tunable complex phases, *Nat. Commun.* **14**, 622 (2023).
- [44] F. R. Lux and E. Prodan, Converging periodic boundary conditions and detection of topological gaps on regular hyperbolic tessellations, *Phys. Rev. Lett.* **131**, 176603 (2023).
- [45] S. Yu, X. Piao, and N. Park, Topological hyperbolic lattices, *Phys. Rev. Lett.* **125**, 053901 (2020).
- [46] W. Zhang, H. Yuan, N. Sun, H. Sun, and X. Zhang, Observation of novel topological states in hyperbolic lattices, *Nat. Commun.* **13**, 1 (2022).
- [47] W. Zhang, F. Di, X. Zheng, H. Sun, and X. Zhang, Hyperbolic band topology with non-trivial second Chern numbers, *Nat. Commun.* **14**, 1083 (2023).
- [48] R. Fleury, Q. Chen, Z. Zhang, H. Qin, A. Bossart, Y. Yang, and H. Chen, Anomalous and Chern topological waves in hyperbolic networks (2023), <https://doi.org/10.21203/rs.3.rs-3286219/v1>.
- [49] I. Boettcher, A. V. Gorshkov, A. J. Kollár, J. Maciejko, S. Rayan, and R. Thomale, Crystallography of hyperbolic lattices, *Phys. Rev. B* **105**, 125118 (2022).
- [50] J. Maciejko and S. Rayan, Automorphic Bloch theorems for hyperbolic lattices, *Proc. Natl. Acad. Sci.* **119**, e2116869119 (2022).
- [51] J. Maciejko and S. Rayan, Hyperbolic band theory, *Sci. Adv.* **7**, eabe9170 (2021).
- [52] A. Attar and I. Boettcher, Selberg trace formula in hyperbolic band theory, *Phys. Rev. E* **106**, 034114 (2022).
- [53] A. Stegmaier, L. K. Upreti, R. Thomale, and I. Boettcher, Universality of Hofstadter butterflies on hyperbolic lattices, *Phys. Rev. Lett.* **128**, 166402 (2022).
- [54] M. Ruzzene, E. Prodan, and C. Prodan, Dynamics of elastic hyperbolic lattices, *Extreme Mech. Lett.* **49**, 101491 (2021).
- [55] D. R. Hilbert, Über die theorie der algebraischen formen, *Math. Ann.* **36**, 473 (1890).
- [56] M. P. Do Carmo, *Differential Geometry of Curves and Surfaces: Revised and Updated* (Courier Dover Publications, New York, 2016), 2nd ed., p. 446.
- [57] H. Poincaré, Mémoire sur les Fonctions Fuchsiennes, *Acta Math.* **1**, 193 (1882).
- [58] H. Poincaré, Mémoire sur les courbes définies par une équation différentielle, *J. Math. Pure Appl.* **8**, 251 (1882).
- [59] J. Jost, *Riemannian Geometry and Geometric Analysis* (Springer-Verlag, Berlin, 2011), p. 115.
- [60] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces* (Academic Press, New York, 1979).
- [61] A. Beardon, Hyperbolic polygons and Fuchsian groups, *J. London Math. Soc.* **2**, 247 (1979).
- [62] M. J. Greenberg, *Euclidean and Non-Euclidean Geometries: Development and History* (Macmillan, New York, 1993).
- [63] H. Gu and R. M. Ziff, Crossing on hyperbolic lattices, *Phys. Rev. E* **85**, 051141 (2012).
- [64] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevApplied.22.014048> for the IDS of other hyperbolic tessellations as well as Euclidean tessellations.
- [65] J. Bellisard, in *Geometric and Topological Methods for Quantum Field Theory* (World Sci. Publishing, River Edge, NJ, 2003), p. 86.
- [66] R. Carmona and J. Lacroix, *Spectral Theory of Random Schrödinger Operators* (Birkhäuser Boston, 1990), p. xv.
- [67] Y. Liu, L. F. Santos, and E. Prodan, Topological gaps in quasiperiodic spin chains: A numerical and k -theoretic analysis, *Phys. Rev. B* **105**, 035115 (2022).
- [68] R. K. Pal, M. I. Rosa, and M. Ruzzene, Topological bands and localized vibration modes in quasiperiodic beams, *New J. Phys.* **21**, 093017 (2019).
- [69] T. J. Hughes, *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis* (Dover Publications Mineola, 2000), p. 44.
- [70] M. Berger, *A Panoramic View of Riemannian Geometry* (Springer, Berlin, 2003).
- [71] M. Pastor, M. Binda, and T. Harčarik, Modal assurance criterion, *Proc. Eng.* **48**, 543 (2012).