Robust Control of Linear Systems and Shortcut to Adiabaticity Based on Superoscillations

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With the advent of quantum technologies, control issues are becoming increasingly important. In this paper, we address control in phase space under a global constraint provided by a minimal energylike cost function and a local (in Fourier space) constraint resulting from a robustness criterion. The inverse-engineering Lagrangian formalism developed here generalizes the formalism commonly used to describe the superoscillation phenomenon. It is applied to both nondissipative and dissipative quantum mechanics, and extended to stochastic thermodynamics. Interestingly, our approach also allows one to improve sensing capabilities by appropriate control of the system.

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I. INTRODUCTION

Quantum control is nowadays a crucial and implicit step in the manipulation of a wide variety of quantum systems, including, but not limited to, quantum computation in the noisy intermediate-scale quantum era [1,2]. Many methods have been developed in this field, ranging from adiabatic control to optimal control theory [1–4] and shortcuts to adiabaticity [5]. For instance, protocols for high-fidelity fast transport or shuttling of charged ions [6–9] and neutral atoms [10,11] have been worked out, and are essential for the development of applications of quantum technologies [12–16], including quantum information processing, atom-chip manipulation, and metrology with cold atoms. The equations that encompass this field are also relevant to the transport of magnons in a spin chain [17], due to the approximate mapping between a spin chain and a single harmonically trapped particle.

Besides the question of reaching the desired target state, one cannot ignore the question of the robustness of protocols designed for that purpose against various sources of noise, including white noise, inaccurate knowledge of the experimental parameters, and model approximations for various proposals [18–25]. This concern is particularly relevant to transport of ions or neutral atoms based on moving traps. Incidentally, such systems have the same dynamical equations as for the load manipulation of mechanical cranes [26,27]. Such quantum-classical analogies allow one to use control protocols in ubiquitous linear Lagrangian systems and to compare methods [28]. Furthermore, the above strategies can be applied to linear dissipative systems. This is illustrated by the motion of harmonically trapped Brownian particles driven by a time-varying force [29,30], including the use of fluorescent nanoparticles in biology [31] and the design of Brownian heat engines [32–34].

In this paper, we address the problem of optimal control in a one-dimensional (1D) phase space for a given protocol duration, under a global constraint provided by an energy cost and a robustness criterion that can be formulated as a local constraint in Fourier space. Such a control problem encompasses transport and shuttling problems (see Refs. [5,9] and references therein) that have been addressed separately in the literature so far. The focus of the paper is on the resolution of linear systems under such constraints, in the context of an inverse-engineering Lagrangian formalism. However, solving such a classical problem also provides a solution to the corresponding quantum problem [11]. We exemplify this idea with an application of our formalism to quantum transport described by a coherent state obeying a Lindblad equation.

In the following, we first provide a short reminder of how optimal control theory is applied to linear systems. We then identify an intrinsic robustness against slight final-time uncertainties. Later, we translate mathematically the requirement of robustness against inaccurate knowledge of the strength of the moving transport potential.

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We emphasize that this is relevant in any experimental setup, since any parameter is known with a certain finite accuracy. In the following, we also explain how such inaccurate knowledge can be encapsulated in an inverse-engineering Lagrange formulation for a minimal energy cost. Finally, detailed applications are exemplified, ranging from classical and quantum mechanics to stochastic thermodynamics.

II. LAGRANGIAN FORMALISM

Consider a linear system described by the set of linear equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, where $\mathbf{x}(t) \in \mathbb{R}^n$ denotes the state vector and $\mathbf{u}(t) \in \mathbb{R}^q$ the control vector. The dynamics is assumed here to involve two time-independent matrices, $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times q}$. We assume the system to be controllable, i.e., it fulfills Kalman's controllability criterion, stating that the rank of Kalman's controllability matrix $\mathbf{K}_c = [\mathbf{B}\ \mathbf{AB}\ \mathbf{A}^2\mathbf{B}\ \cdots\ \mathbf{A}^{n-1}\mathbf{B}]$ is n; see Ref. [28]. The solution then reads $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \mathbf{I}_u(t)$, with $\mathbf{I}_u(t) = \int_0^t e^{\mathbf{A}(t-t')}\mathbf{B}\mathbf{u}(t')\,dt'$. The control problem consists in designing the time-dependent control vector to drive the system from an initial state $(t_0, \mathbf{x}_0)$ to a desired target state $(t_f, \mathbf{x}_f)$ in a time interval that lasts for t_f. The boundary condition at the final time imposes the condition $\mathbf{I}_u(t_f) = \mathbf{x}_f - e^{\mathbf{A}t_f}\mathbf{x}_0$. In most physical situations, extra requirements are needed, such as the minimization of a cost function $E = \int_0^{t_f} f[\mathbf{x}(t), \mathbf{u}(t), t]\,dt$ that depends on both the trajectory $\mathbf{x}(t)$ and the driving $\mathbf{u}$. Such a minimization problem can be solved, thanks to the Lagrangian-multiplier formalism:

$$\frac{\partial E}{\partial \mathbf{u}} - \boldsymbol{\mu}_0^T \frac{\partial \mathbf{G}_0(\mathbf{u})}{\partial \mathbf{u}} = 0, \tag{1}$$

where $\mathbf{G}_0(\mathbf{u}) = \mathbf{I}_u(t_f) - \mathbf{x}_f + e^{\mathbf{A}t_f}\mathbf{x}_0$. For the cost function $E_1 = \int_0^{t_f} \mathbf{u}^T\mathbf{u}\,dt$, the Lagrangian multiplier is given by applying the controllability Gramian matrix $\boldsymbol{\mu}_0 = \mathbf{W}(t_f)^{-1}(\mathbf{x}_f - e^{\mathbf{A}t_f}\mathbf{x}_0)$ [35]. Alternatively, the same results can be recovered using the Pontryagin formalism of optimal control theory.

III. TRANSPORT IN A MOVING TRAP

For a particle in a moving harmonic trap characterized by a time-dependent center at $x_0(t)$ and a constant angular frequency ω_0, the dynamical equation reads $\ddot{x} + \omega_0^2 x = \omega_0^2 x_0(t) \equiv u(t)$, where $u(t)$ is the control parameter that drives the system. This second-order differential equation is readily transformed into a set of two first-order differential equations with the variables x and $y = \dot{x}$. The control problem we focus on consists in designing the time-dependent control parameter $u(t)$ to transfer the system from an initial condition $(x_i = 0, y_i = 0)$ to an arbitrary target $(x_f = r\cos\varphi, y_f = r\omega_0\sin\varphi)$ in phase space in a finite amount of time t_f. For $y_f \neq 0$, such a control problem is referred to as shuttling [9,12]. The particle acquires the desired velocity at a given position after the application of an appropriate driving. For a vanishing final velocity $y_f = 0$, this problem is nothing but a transport problem, in which a particle initially at rest is displaced from $x = 0$ to $x = x_f$ and remains at rest afterwards [10,12].

Using the previous formalism for the minimization of the standard cost function $E_1 = \int_0^{t_f} u^2\,dt$, the two-dimensional Lagrangian multiplier $\boldsymbol{\mu}_0$ can be readily derived from the boundary conditions, yielding the following expression for the control parameter:

$$u_1(t) = \frac{2r\omega_0^2[\varphi_f\,\sin(\varphi+\varphi_f-\omega_0 t) - \sin(\omega_0 t+\varphi)\,\sin(\varphi_f)]}{\varphi_f^2 - \sin^2(\varphi_f)},$$

where $\varphi_f = \omega_0 t_f$. The corresponding minimal energy cost E_1 is

$$E_1 = 2r^2\omega_0^3\frac{\varphi_f + \cos(2\varphi+\varphi_f)\sin(\varphi_f)}{\varphi_f^2 - \sin^2(\varphi_f)}. \tag{2}$$

This energy cost provides a lower bound for reaching the target $(r\cos\varphi, r\omega_0\sin\varphi)$ in an amount of time t_f. In Fig. 1, we depict this energy as a function of the final time for both a transport problem (target with $\varphi = 0$, black solid line) and a shuttling process (target with $\varphi = \pi/2$, gray solid line). As intuitively expected, the shorter the driving time t_f, the larger the energy cost E_1, and E_1 is a decreasing function of $\omega_0 t_f$. Interestingly, this result reveals the existence of plateaus for which the change of t_f has a negligible impact on the energy cost, meaning that the corresponding solutions are intrinsically robust against small variations of the protocol duration t_f in a succession of windows of final time. The energy cost in Eq. (2) is the lowest possible, as it is derived without any constraints. We denote by $E_1^{(\min)}$ this lower bound.

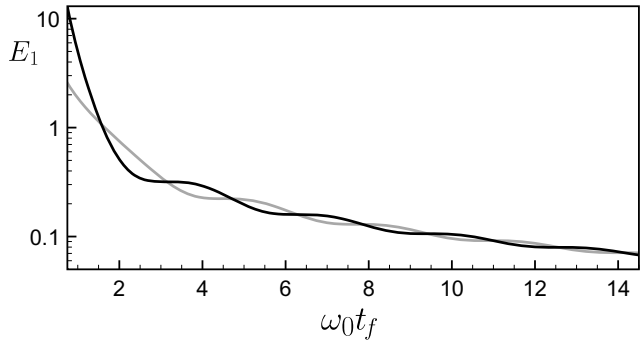

FIG. 1. Energy cost E_1 (in units of $r^2\omega_0^3$) as a function of the protocol duration t_f in units of ω_0^{-1} for transport ($\varphi = 0$, solid black line) and shuttling ($\varphi = \pi/2$, solid gray line).

Next, we add extra constraints and derive an inverse-engineering Lagrangian approach to building protocols robust against inaccurate knowledge of the trap frequency ω_0. The energy cost E_1 associated with such a requirement is quantified and compared with $E_1^{(\min)}$. As a first example, we elaborate on the transport problem, i.e., a problem with a target of $r = d$ and $\varphi = 0$ to be reached in an amount of time t_f. For an arbitrary driving $x_0(t) = u(t)/\omega_0$, the final excitation energy reads [11] (see also the detailed derivation in Appendix A)

$$\Delta E(\omega_0; t_f) = \frac{m}{2}\left|\int_0^{t_f} \ddot{x}_0(t) e^{-i\omega_0 t}\, dt\right|^2 \equiv \frac{m}{2}\left|g(\omega_0; t_f)\right|^2. \tag{3}$$

Here, ΔE provides information about the final excitation energy, while E_1 measures the energy cost. Both quantities depend on the choice of the whole trajectory $x_0(t)$ but in quite different manners; E_1 is the integral of a positive quantity (and as such is always positive), while ΔE is given by the modulus squared of a Fourier-like integral that can therefore cancel out. A vanishing value of $\Delta E(\omega_0)$ ensures transport without residual excitation, i.e., perfect transport, and, by shaping the function $\Delta E(\omega)$ about ω_0, we design a strategy to make the transport robust against the exact value of ω_0, but with the expense of an extra energy cost, estimated using E_1. Accordingly, perfect transport (without residual oscillations) requires $\Delta E(\omega_0; t_f) = 0$. Transport robust against the exact value of ω_0 requires us to smooth ΔE as a function of ω about ω_0 [11]:

$$\left.\frac{\partial^p \Delta E(\omega; t_f)}{\partial \omega^p}\right|_{\omega_0} = 0, \quad \text{with } p \geq 1. \tag{4}$$

The calculations associated with this robustness issue are made simpler by using the complex variable $z(t) = \dot{x}/\omega_0 + ix$. The equations of motion are then $\dot{z}(t) = i\omega_0 z(t) + u(t)/\omega_0$. Under the driving $u(t)$, the complex solution is $z(t) = e^{i\omega_0 t}\,(z(0) + \tilde{u}(\omega_0)/\omega_0)$, where $\tilde{u}(\omega_0) = \int_0^{t_f} u(t) e^{-i\omega_0 t}\, dt \equiv a_0$. For an arbitrary target $z(t_f) = r(\sin\varphi + i\cos\varphi)$, we have $a_0 = r\omega_0\,(\sin\varphi + i\cos\varphi)\, e^{-i\omega_0 t_f}$.

For transport, the driving function obeys the boundary conditions $u(0) = 0$, $u(t_f) = \omega_0 d$, and $\dot{u}(0) = \dot{u}(t_f) = 0$. To fulfill the robustness requirements in Eq. (4), we impose nullity of the successive derivative of $g(\omega; t_f)$ [defined in Eq. (3)] with respect to ω about ω_0. We infer the following set of extra constraints on the Fourier function $\tilde{u}(\omega)$ and its successive derivatives (see Appendix B):

$$\omega_0 \tilde{u}^{(p)}(\omega_0) + 2p\tilde{u}^{(p-1)}(\omega_0) + \frac{p(p-1)}{\omega_0}\tilde{u}^{(p-2)}(\omega_0) = \lambda, \tag{5}$$

where $\lambda = d(\varphi_f + ip)(-it_f)^{p-1} e^{-i\varphi_f}$. We find $\tilde{u}^{(m)}(\omega_0) = a_m$, with $a_1 = d(\varphi_f - i)e^{-i\varphi_f}/\omega_0$, $a_2 = d(-i\varphi_f^2 - 2\varphi_f + 2i)e^{-i\varphi_f}/\omega_0^2$, and so on.

More generally, the class of problem to be solved here requires the minimization of $E = \int_{\mathcal{D}} f(t) f^*(t)\, dt$ on a compact support $\mathcal{D}$ under a discrete number of constraints on the Fourier transform of f and its derivatives for a finite set of angular frequencies ω_j: $\tilde{f}^{(m)}(\omega_j) = \int_{\mathcal{D}} (-it)^m f(t) e^{-i\omega_j t}\, dt = \tilde{f}_j^{(m)}$. For a single constraint on the Fourier transform, such a problem has been solved in the context of superoscillations, as detailed in Refs. [36–39]. We generalize this approach here to enforce the robustness requirement. The solution relies on the Euler-Lagrange formalism,

$$\frac{\delta E}{\delta f} + \sum_j \sum_{m=0}^{p} \nu_{j,m} \frac{\delta \tilde{f}^{(m)}(\omega_j)}{\delta f} = 0, \tag{6}$$

where $\nu_{j,m}$ are Lagrange multipliers. Interestingly, this set of equations can be directly solved by a simple polynomial ansatz,

$$f(t) = \sum_j \left(\nu_{j0}^* + \sum_m (it)^m\, \nu_{j,m}^*\right) e^{i\omega_j t}. \tag{7}$$

The Lagrange multipliers $\nu_{j,m}$ are subsequently determined by the constraints. The inverse-engineering procedure amounts here to fulfilling the boundary conditions in order to set up the driving. One needs to be careful to keep $u(t)$ real, as it coincides with the position of the trap in real space. To get a real driving function, we simply exploit the linearity of the system and define the control function as $u(t) = [f(t) + f^*(t)]/2$.

In Fig. 2, we plot the excess of energy after transport in a harmonic potential as a function of the angular frequency ω for protocols optimized for a given ω_0 and for a transport duration $\omega_0 t_f = 10$. We compute the solutions under the requirements for robustness on the first (blue dot-dashed line) and second (red double-dot-dashed line) derivatives of the Fourier transform of the driving function. We clearly observe a strong improvement associated with the local flatness about ω_0 compared with the solution obtained by means of optimal control without constraints (black solid line). We also compare the solutions (see the gray lines in Fig. 2) obtained by means of the so-called Fourier method [11], which was developed to address similar requirements, as shown in Appendix C.

IV. TRANSPORT IN THE PRESENCE OF DISSIPATION

In fact, the formalism developed here is well adapted to the optimization of other classes of transport problems, since solving this classical problem amounts to solving

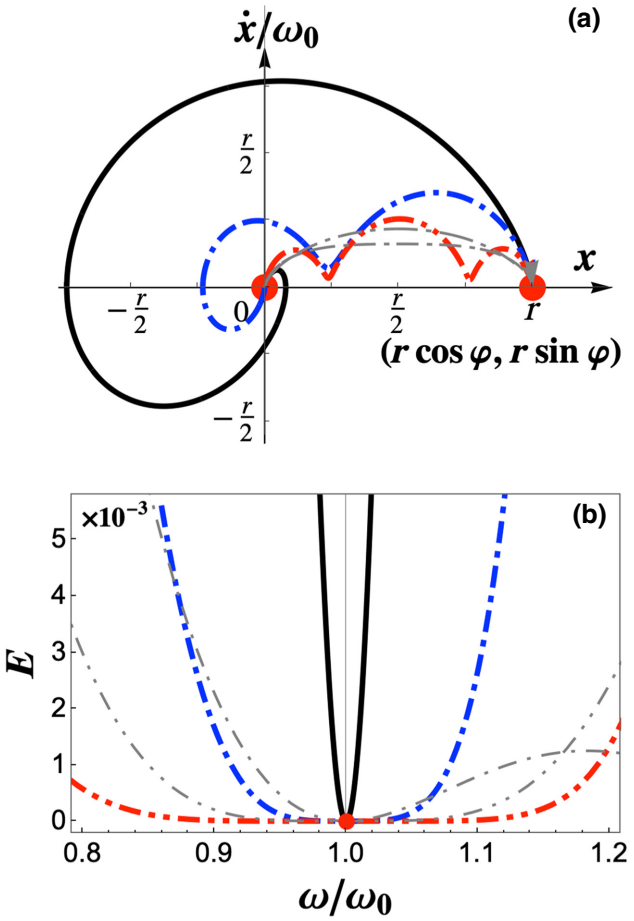

FIG. 2. In the case of transport ($\varphi = 0$), (a) trajectories in phase space, and (b) the final excitation energy (in units of $md^2\omega_0^2/2$) as a function of ω are compared between (i) the inverse-engineering Lagrange method with no extra robust-constraint protocol (black solid line), $p = 1$ robust control (blue dot-dashed line), and $p = 2$ robust control (red double-dot-dashed line), and (ii) the Fourier method for a one-frequency ($n = 2$) robust protocol (gray dot-dashed thin line) and a two-frequency ($n = 4$) robust protocol (gray double-dot-dashed thin line). Parameters: $\omega_0 t_f = 10$.

its quantum counterpart, as explained in Ref. [11]. The transport problem in the presence of dissipation requires us to solve the equation of motion for the density matrix. Regarding transport with losses, the dynamics of a coherent state $|\alpha\rangle$ is given by the following master equation:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho] + \frac{\Gamma}{2}(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a), \qquad (8)$$

where Γ is the Lindblad operator and H is the Hamiltonian, $H = p^2/2m + m\omega^2[\hat{x} - x_0(t)]^2/2$, with $x_0(t)$ being the center of the moving potential [10,12]. This problem can be solved exactly, whatever the driving may be, using a time-dependent coherent state $\rho(t) = |\alpha\rangle\langle\alpha|$ with $\alpha(t) = (i\omega/\sqrt{2}a_0)e^{-(i\omega+\Gamma/2)t}\int_0^t e^{(i\omega+\Gamma/2)t'}x_0(t')\,dt'$. This problem belongs to the same class of problems as previously studied, since it can be expressed as a linear system in the variables $x_1 = \langle\hat{x}\rangle$, $x_2 = d\langle\hat{x}\rangle/dt$: $\dot{x}_1 = x_2$ and $\dot{x}_2 = -(\omega_0^2 + \Gamma^2/4)x_1 - \Gamma x_2 + \omega_0^2 x_0(t)$.

Moreover, the transport of a mesoscopic object in stochastic thermodynamics can also be captured and optimized by our formalism. For this purpose, it turns out to be more convenient to work with the distribution function rather than with the Langevin-like stochastic differential equations. In the overdamped regime, the distribution function $\rho(x,t)$ obeys the following Fokker-Planck equation:

$$\gamma\partial_t\rho(x,t) = \partial_x[m\omega_0^2\big(x - x_0(t)\big)\rho] + \gamma D\partial_{xx}^2\rho, \qquad (9)$$

where γ is the friction coefficient and $D = k_B T/\gamma$ is the diffusion coefficient. The solution of the Fokker-Planck equation is provided by a Gaussian, $\rho(x,t) = \sqrt{\alpha/\pi}\exp[-\alpha(x - x_c(t))^2]$, with $\alpha = 2\gamma D/(m\omega_0^2)$. The time dependence of $\rho(x,t)$ is encapsulated in the variable $x_c(t)$, governed by the linear system $\dot{x}_c + (m\omega_0^2/\gamma)(x_c - x_0) = 0$, for which the previous formalism applies. These results can be readily generalized to the underdamped regime. For this purpose, we introduce the phase-space distribution $\rho(x,v,t)$, which obeys the Kramers equations [29]. A Gaussian solution can be readily worked out: $\rho(x,v,t) = \mathcal{N}\exp[-\alpha(x - x_c(t))^2 - \beta[v - v_c(t)]^2]$, with $\alpha = m\omega_0^2/2k_BT$ and $\beta = m/2k_BT\gamma$. The time variation of the parameters x_c and v_c is dictated by the following set of linear equations: $\dot{x}_c = v_c$, $\dot{v}_c = -\omega_0^2 x_c - \gamma v_c/m + \omega_0^2 x_0(t)$.

V. DISCUSSION

Besides targets in phase space located on the position axis, our protocols encompass the shuttling problem, in which the particle acquires a finite velocity. The results obtained for such shuttling protocols are summarized in Fig. 3. As a general result, the inverse-engineering Lagrangian method developed to ensure robustness provides systematically a much lower energy E_1 for a time duration t_f of the protocol larger than $10/\omega_0$, compared with the Fourier method [11]. The latter turns out to be more efficient for extremely short shuttling-protocol durations but is at the expense of a large energy cost E_1. In Fig. 3, we also provide for comparison the minimum energy obtained by optimal control theory for the minimization of the energy cost in the absence of any other constraints (black solid line). We therefore clearly identify the minimum cost of the robustness requirement.

Finally, the inverse-engineering Lagrangian formalism enables one to design a driven protocol for quantum sensing, with the objective being to measure the trap frequency or its drift as a function of time. To this end, a protocol is designed to assign different targets

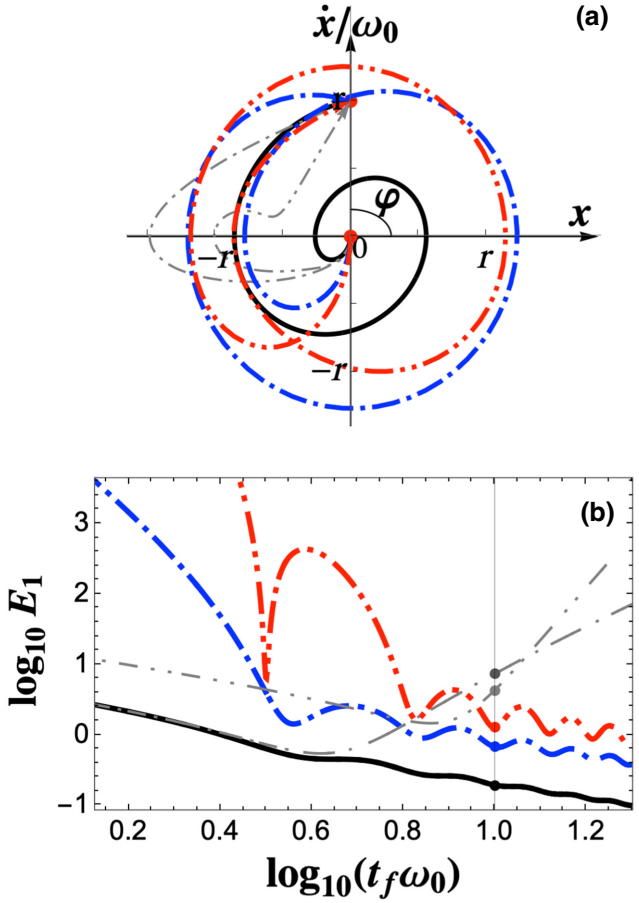

FIG. 3. Shuttling protocols (target $\varphi = \pi/2$). (a) Phase-space trajectories associated with different protocols performed over a time interval $t_f = 10/\omega_0$. (b) Energy cost of the different protocols as a function of the protocol duration. The notation is the same as in Fig. 2.

to two close but different trapping frequencies ω_1 and $\omega_2 = (1+\epsilon)\omega_1$ ($\epsilon \neq 0$). Such a strategy is reminiscent of the fingerprinting method used in NMR [40,41]. We therefore consider here the case where the same control function drives two independent set of equations, $\dot{x}_j = y_j$ and $\dot{y}_j = -\omega_j^2 y_j + u(t)$. To transfer the system from the initial state $\left(x_j(0), \dot{x}_j(0)\right) = (0,0)$ to distinguishable target states $\left(x_j(t_f), \dot{x}_j(t_f)\right) = (r_j \cos\varphi_j, r_j\omega_j \sin\varphi_j)$, the Fourier transform of the control $u(t)$ should be equal to $\tilde{u}(\omega_j) = \omega_j r_j \left(\sin\varphi_j + i\cos\varphi_j\right) e^{-i\omega_j t_f}$. Such conditions are readily fulfilled with an ansatz of the form $u(t) = \sum_j [a_j \cos(\omega_j t) - b_j \sin(\omega_j t)]$. For example, the two-frequency case refers to $j = 1,2$, and the target can be $r_1 = \nu$, $\varphi_1 = \pi/2$, $r_2 = d$, and $\varphi_2 = 0$. Such conditions are readily fulfilled with an ansatz of the form $u(t) = \sum_{j=1}^{2} [a_m \cos(\omega_m t) - b_m \sin(\omega_m t)]$. In Fig. 4, we plot in red the evolution of the trajectories for the two different ω_i towards their target, along with the final state in phase space $(x_f, \dot{x}_f)$ for different trapping angular frequencies

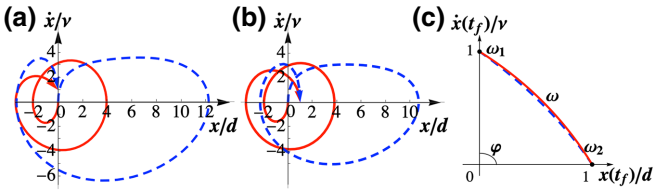

FIG. 4. Sensing the frequency between ω_1 and $\omega_2 = 1.05\omega_1$ by separating the corresponding target states into $(r_1 = \nu, \varphi_1 = \pi/2)$ and $(r_2 = d, \varphi_2 = 0)$. The comparison is between the following two protocols: the inverse-engineering Lagrange method (red solid lines) and the Fourier method with two frequencies ($n = 4$) (blue dashed lines). Parameters: $t_f = 10$ and $\omega_1 = 1$.

$\omega \in (\omega_1, \omega_2)$. This demonstrates the possibility of magnifying the sensitivity. Indeed, in the absence of driving, information about the trapping angular frequency cannot be recovered, as the system remains at rest. This sensing protocol is particularly relevant when the time duration t_f of the protocol remains smaller than the natural timescale $\pi/(2\varepsilon\omega_1)$, which would generate an angle separation in phase space equal to $\pi/4$ for a step driving (a sudden switch of the driving function to a constant value). This is the reason why in Fig. 4 we choose the parameters to fulfill the inequality $\omega_1 t_f \ll \pi/(2\varepsilon)$. The blue dashed curves are obtained using the Fourier method [11]; see Appendix C for details. The same conclusion holds concerning the sensing protocol. It is worth noticing that the Lagrange method limits considerably the extension of the trajectory required to obtain a similar sensitivity. This can be a strong asset for limiting the role of anharmonicities in such a protocol [19]. The fingerprinting sensing method discussed here can be readily generalized to a larger number of ω_i.

VI. CONCLUSION

To conclude, we develop a formalism inspired by Lagrangian minimization to tackle optimal control of linear systems, taking into account different constraints, such as energylike minimization and a robustness criterion. Our inverse-engineering Lagrangian formalism generalizes the formalism for driving quantum systems with superoscillations to enforce robustness or sensitivity requirements. This has been adopted for both nondissipative and dissipative linear systems, describing the transport of ions or neutral atoms with and without loss, and of a mesoscopic object in stochastic thermodynamics. Our results pave the way for designing optimal and robust shortcuts to adiabaticity in a class of general linear systems, with broad applications in quantum technologies.

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APPENDIX A: EXCESS OF ENERGY

Consider the motion of a particle in a moving harmonic trapping potential of angular frequency ω_0. The position $x(t)$ obeys the differential equation

$$\ddot{x} + \omega_0^2[x - x_0(t)] = 0, \tag{A1}$$

where $x_0(t)$ denotes the instantaneous position of the trap bottom. The corresponding instantaneous energy is

$$E(t) = E(0) + \frac{m}{2}\dot{x}^2 + \frac{m}{2}\omega_0^2(x - x_0)^2. \tag{A2}$$

We introduce the relative position $X = x - x_0$ of the classical particle in the frame of the moving potential. This relative position obeys the differential equation

$$\ddot{X} + \omega_0^2 X = -\ddot{x}_0. \tag{A3}$$

The solution with a boundary condition corresponding to a particle initially at rest ($x(0) = 0$ and $\dot{x}(0) = 0$) reads

$$X(t) = -\frac{1}{\omega_0}\int_0^t dt'\,\ddot{x}_0 \sin[\omega_0(t - t')]. \tag{A4}$$

We introduce the complex quantity

$$a(t) = X(t) - i\frac{\dot{X}}{\omega_0} = \frac{i}{\omega_0}e^{i\omega_0 t}\int_0^t \ddot{x}_0(t')e^{-i\omega_0 t'}\,dt'. \tag{A5}$$

The instantaneous energy then reads

$$\Delta E(t) = E(t) - E(0) = m\dot{X}\dot{x}_0 + \frac{m}{2}\dot{x}_0^2 + \frac{m\omega_0^2}{2}|a(t)|^2. \tag{A6}$$

For a transport problem, we have the boundary condition $\dot{x}_0(t_f) = 0$, yielding

$$\Delta E(t_f) = \frac{m}{2}\left|\int_0^{t_f} \ddot{x}_0(t')e^{-i\omega_0 t'}\,dt'\right|^2. \tag{A7}$$

APPENDIX B: ROBUSTNESS RECURRENCE RELATION

We provide here the derivation of Eq. (5). For this purpose, we introduce the function

$$\tilde{g}^{(p)}(\omega) = \int_0^{t_f} (-it)^p \ddot{x}_0(t)e^{-i\omega t}\,dt. \tag{B1}$$

To fulfill the criterion of Eq. (4) for robustness to order p, the cancellation of the successive derivatives of $\tilde{g}^{(k)}(\omega)$ at $\omega = \omega_0$ for $k = 1, \ldots, p$ provides a sufficient condition. Using the dimensionless notation $u(t) = \omega_0 x_0(t)$ and the boundary conditions $u(0) = 0$, $u(t_f) = \omega_0 d$, $\dot{u}(0) = \dot{u}(t_f) = 0$, a double integration by parts of the integral in Eq. (B1) yields

$$\tilde{g}^{(p)}(\omega) = \frac{(-i)^p}{\omega_0}\left(\int_0^{t_f} u(t)\ddot{h}(t)\,dt - \dot{h}(t_f)u(t_f)\right), \tag{B2}$$

with $h(t) = t^p e^{-i\omega t}$. The first two derivatives of this function read $\dot{h}(t) = t^{p-1}(p - i\omega t)e^{-i\omega t}$ and $\ddot{h}(t) = t^{p-2}[p(p - 1) - 2ip\omega t - \omega^2 t^2]e^{-i\omega t}$. We infer from these expressions the explicit form of $\tilde{g}^{(p)}(\omega)$ as a function of the derivatives of $\tilde{u}(\omega)$:

$$\begin{aligned}\tilde{g}^{(p)}(\omega) = {} & d(\omega t_f + ip)(-it_f)^{p-1}e^{-i\omega t_f} \\ & - \frac{p(p-1)}{\omega_0}\tilde{u}^{(p-2)}(\omega) \\ & - \frac{2p\omega}{\omega_0}\tilde{u}^{(p-1)}(\omega) - \frac{\omega^2}{\omega_0}\tilde{u}^{(p)}(\omega).\end{aligned} \tag{B3}$$

The cancellation to first order ($p = 1$), $\tilde{g}^{(1)}(\omega_0) = 0$, implies $\omega_0\tilde{u}^{(1)}(\omega_0) = d(\omega_0 t_f - i)e^{-i\omega_0 t_f}$. The cancellation to second order ($p = 2$), $\tilde{g}^{(2)}(\omega_0) = 0$, yields

$$\omega_0^2\tilde{u}^{(1)}(\omega_0) = d(-i\omega_0^2 t_f^2 - 2\omega_0 t_f + 2i)e^{-i\omega_0 t_f}. \tag{B4}$$

APPENDIX C: THE FOURIER METHOD

An alternative way to design the control function $u(t)$ is detailed in Refs. [11,28]. This enables one to derive the control field $\mathbf{u}(t)$ using an inverse-engineering approach. Here, we work through this method using the notation of our paper. The set of equations for the control system is linear: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}(t)$. Taking into account the constraint on $\mathbf{u}(t)$, we readily find the following constraint that must be fulfilled by $\mathbf{u}(t)$:

$$\mathbf{x}_f = e^{\mathbf{A}t_f}\int_0^{t_f} e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)\,dt. \tag{C1}$$

To design a control function $\mathbf{u}(t)$ that fulfills this equation, we introduce an auxiliary time-dependent vector $\mathbf{c}(t)$,

which is useful for expanding $\mathbf{u}(t)$ in the form

$$\mathbf{u}(t) = \sum_{k=0}^{n} c_k \mathbf{c}^{(k)}(t). \tag{C2}$$

The successive derivatives of $\mathbf{c}(t)$ satisfy the initial boundary condition

$$\mathbf{c}^{(k)}(0) = 0,\ k = 0, 1, \ldots, n-1. \tag{C3}$$

This boundary condition on the successive derivatives of $\mathbf{c}$ simplifies considerably the calculations that require integration by parts.

As explained below, the method requires us to combine Eqs. (C1) and (C2). For this purpose, we need to calculate explicitly the integrals $\mathbf{J}_k = \int_0^{t_f} e^{-\mathbf{A}t}\mathbf{B}\mathbf{c}^{(k)}(t)\,dt$. For an oscillator system, the eigenvalues of the matrix $\mathbf{A}$ are imaginary, and the corresponding integrals are nothing but Fourier integrals. This is the reason why the method is called the Fourier method. In the following, we write $\mathbf{C}(t_f) = \int_0^{t_f} e^{-\mathbf{A}t}\mathbf{B}\mathbf{c}(t)\,dt$. Using successive integration by parts, we find

$$\begin{aligned}
&\mathbf{J}_1 = e^{-\mathbf{A}t_f}\,\mathbf{B}\mathbf{c}^{(0)}(t_f) + \mathbf{A}\mathbf{C}(t_f),\\
&\mathbf{J}_2 = e^{-\mathbf{A}t_f}\left[\mathbf{B}\mathbf{c}^{(1)}(t_f) + \mathbf{A}\mathbf{B}\mathbf{c}^{(0)}(t_f)+\right] + \mathbf{A}^2\mathbf{C}(t_f),\\
&\ldots,\\
&\mathbf{J}_n = e^{-\mathbf{A}t_f}\sum_{k=0}^{n-1}\mathbf{A}^k\mathbf{B}\mathbf{c}^{(n-1-k)}(t_f) + \mathbf{A}^n\mathbf{C}(t_f).
\end{aligned}$$

The integral in Eq. (C1) with the chosen form [Eq. (C2)] for the driving involves the term $(\sum_{k=0}^{n} c_k\mathbf{A}^k)\mathbf{C}(t_f)$. According to the Cayley-Hamilton theorem, the factor in parentheses cancels out if we choose the coefficients c_k as the coefficients of the characteristic polynomial of $\mathbf{A}$. We therefore get

$$\begin{aligned}
\mathbf{x}_f &= \mathbf{B}\sum_{k=0}^{n-1} c_{k+1}\mathbf{c}^{(k)}(t_f) + \mathbf{A}\mathbf{B}\sum_{k=0}^{n-2} c_{k+2}\mathbf{c}^{(k)}(t_f)\\
&\quad + \cdots + \mathbf{A}^{n-1}\mathbf{B}c_n\mathbf{c}^{(0)}(t_f) = \sum_{k=0}^{n-1}\mathbf{A}^k\mathbf{B}\mathbf{b}_k,
\end{aligned} \tag{C4}$$

where the vectors $\mathbf{b}_k$ are defined by

$$\begin{aligned}
\mathbf{b}_0 &= \sum_{k=0}^{n-1} c_{k+1}\mathbf{c}^{(k)}(t_f),\\
\mathbf{b}_1 &= \sum_{k=0}^{n-2} c_{k+2}\mathbf{c}^{(k)}(t_f),\\
&\ldots,\\
\mathbf{b}_{n-2} &= c_n\mathbf{c}'(t_f) + \mathbf{c}_{n-1}c(t_f),\\
\mathbf{b}_{n-1} &= c_n\mathbf{c}(t_f).
\end{aligned}$$

With this set of equations, we obtain the boundary conditions at the final time for $c(t_f)$ and its derivatives. For the sake of simplicity, we choose $c_n = 1$. We have $2n$ boundary conditions that the vector $\mathbf{c}(t)$ must fulfill at the initial and final times. In the so-called Fourier method, an interpolation function is subsequently chosen to fix the function $\mathbf{c}(t)$ according to the boundary conditions, and to infer the driving vector $\mathbf{u}(t)$ through the relation in Eq. (C2). The boundary conditions can be accounted for by a polynomial interpolation of order $2n-1$: $\mathbf{c}(t) = \sum_{n}^{2n-1}\mathbf{a}_k(t/t_f)^k$. The chosen initial condition cancels out the first $n-1$ vectors $\mathbf{a}_k$.

Let us work out the simplest example, of a single ($n = 2$) 1D harmonic oscillator of angular frequency ω_0. We have

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix},\ \mathbf{B} = \mathbb{I}, \tag{C5}$$

and $\mathbf{u} = [0, u(t)]$. The objective is to drive the system from the initial phase-space point $\mathbf{x}_0 = (x(0) = 0, \dot{x}(0) = 0)$ to the final phase-space point $\mathbf{x}_f = (x(t_f) = r\cos\varphi, \dot{x}(t_f) = r\omega_0\sin\varphi)$. From the characteristic polynomial of $\mathbf{A}$, we deduce $c_2 = 1$, $c_1 = 0$, and $c_0 = \omega_0^2$. Equation (C5) reads $\mathbf{b}_0 = \mathbf{c}'(t_f)$ and $\mathbf{b}_1(t_f) = \mathbf{c}(t_f)$ here. The relation $\mathbf{x}_f = \mathbf{B}\mathbf{b}_0 + \mathbf{A}\mathbf{B}\mathbf{b}_1$ yields

$$\mathbf{c}(t_f) = \begin{pmatrix} 0 \\ r\cos\varphi \end{pmatrix} \text{ and } \mathbf{c}'(t_f) = \begin{pmatrix} 0 \\ r\omega_0\sin\varphi \end{pmatrix}. \tag{C6}$$

Choosing a polynomial interpolation function of order $2n-1 = 3$, we find an explicit expression for the auxiliary vector $\mathbf{c}$:

$$\mathbf{c}(t) = \begin{pmatrix} 0 \\ \gamma(t) \end{pmatrix}, \tag{C7}$$

with $\gamma(t) = r(3\cos\varphi - t_f\omega_0\sin\varphi)(t/t_f)^2 - r(2\cos\varphi - t_f\omega_0\sin\varphi)(t/t_f)^3$. We infer the expression for the control vector $\mathbf{u}$ using the relation

$$\mathbf{u}(t) = \ddot{\mathbf{c}}(t) + \omega_0^2\mathbf{c}(t). \tag{C8}$$

Here we emphasize that this method enables one to design a control vector $\mathbf{u}$ with extra constraints. One can accommodate them by extending the number of parameters in the polynomial interpolation.

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