Featured in Physics

Effects of Friction and Spacing on the Collaborative Behavior of Domino Toppling

David Cantor[1,*] and Kajetan Wojtacki[2,†]

[1] *Department of Civil, Geological, and Mining Engineering, Polytechnique Montréal, Québec, Canada*
[2] *Institute of Fundamental Technological Research, Polish Academy of Sciences, Warsaw, Poland*

(Received 16 December 2021; revised 29 March 2022; accepted 26 April 2022; published 10 June 2022; corrected 24 June 2022)

Inspired by the high-speed camera experiments of YouTuber Destin Sandlin (SmarterEveryDay) [D. Sandlin, Dominoes – hardcore mode (2017), [Online; accessed 15-Jul-2021].] on the toppling speed of dominoes over different surfaces, we performed discrete-element simulations of this process, varying the spacing between adjacent and evenly spaced blocks (dominoes). We also varied the block-block and block-surface friction coefficients over a wide range of values to have a complete picture of the behavior of these cooperative, dissipative mechanical systems. We found that a steady wavefront speed v exists for a specific interval of spacings between dominoes and coefficients of friction. Surprisingly, while v is more affected by the domino-domino friction, the domino-surface friction determines whether or not toppling anomalies can appear and stop the wave. Finally, our observations led us to propose a scaling law that is able to predict v based on the domino configuration and friction coefficients, and to correctly reproduce experimental tests.

DOI: 10.1103/PhysRevApplied.17.064021

I. INTRODUCTION

Dominoes can create impressive structures and acrobatics by triggering relatively simple chain reactions. It is easy to picture dominoes as a marginally stable series of objects that are put into motion by the destabilization of a single block. That single block is usually pushed, so it turns around one of its edges, gradually converting potential energy into kinetic energy. This first domino then collides against a second one, provoking a new destabilization.

The interaction between two neighboring dominoes can be described using classical mechanics principles, which dictate that the angular velocity of the second block depends on that of the first domino (i.e., single collision theory [2]). The mechanics of a series of dominoes, however, is much more complex. When evenly spaced dominoes are toppled through a consecutive destabilization of neighbors, several nonlinear physical mechanisms intervene, such as energy dissipation by collision and friction. Nevertheless, simple assumptions (e.g., prescribing nonslippage between dominoes and surface, or ignoring the friction between dominoes) have allowed mathematical models to predict the kinematics of these multibody systems (i.e., cooperative group theory [3,4]).

Both theoretical approaches mentioned above propose that, as the dominoes are toppled and lean on each other, the wavefront propagation of this motion stabilizes. A characteristic wavefront speed v seems to appear after a transient phase (usually linked to the toppling of the first 10–20 dominoes), and v becomes independent of the speed of the initial destabilization.

Other works have also explored the effect of domino spacing and block dimensions [5], mass variation of the pieces and effect of the restitution coefficient at collisions [6], and edge roundness [7], and have even developed dimensional analyses to determine a scaling law governing the wavefront speed [8].

However, despite the fact that friction between blocks and the surface are central elements in the kinematics of domino toppling, relatively little is known about their effects on the stability of the wavefront. This is mostly due to the challenges of including friction in analytical models. Since friction mobilization is a nonlinear phenomenon linked to individual contact stability, it is challenging to include its contributions in a cooperative group model. Recently, Shi *et al.* [9] used discrete-element modeling (i.e., the simulation of the dynamics of multibody systems) to study the frictional behavior between blocks and surface, and validated several of the previous conclusions drawn from theoretical approaches. Nonetheless, no systematic analysis of spacing and friction between blocks and surface has been developed until now. Neither an equation nor a law has been proposed to predict the wavefront speed based on the geometrical configuration of the dominoes and the friction between the dominoes and the surface.

Inspired by experiments developed by the science communicator Destin Sandlin, creator of the YouTube

*david.cantor@polymtl.ca
†kwojtack@ippt.pan.pl

channel "SmarterEveryDay" [1], we undertook a simulation campaign in the frame of the discrete-element modeling framework for a wide range of spacings δ_d, friction between blocks μ_{dd}, and between the blocks and the surface μ_{sd}. This work aims to provide a complete picture of these cooperative and highly nonlinear mechanical systems without neglecting the effects of friction.

II. NUMERICAL APPROACH

To simulate a series of blocks (i.e., the dominoes), we use the discrete-element method known as nonsmooth contact dynamics (NSCD) [10,11], which can model collections of rigid bodies in interaction. The NSCD method is particularly well suited for the analysis of toppling dominoes since it is capable of simulating the evolution of mechanical systems undergoing "jumps" in body velocities and contact forces due to impacts. These physical quantities, in other words, are not differentiable (i.e., they are not smooth) and a special mathematical framework has been developed to deal with these situations. For more details on the mathematical framework and implementation of the NSCD method, see Refs. [10–12]. In addition, our tests were performed using the free and open-source simulation platform LMGC90 [13,14], which provides an NSCD implementation.

Our tests consisted of 200 evenly-spaced dominoes placed on a rigid surface that were toppled as shown in Fig. 1(a). The dimensions of the blocks were $h > \ell > w$, and we set the relations $h/\ell = 3.2$ and $\ell/w = 2$, given average dimension ratios of dominoes used by manufacturing companies. We deliberately set $w = 0.015$ m. The relative spacing between dominoes is $\delta_d = s/w$, with s the distance between the faces of adjacent pieces. We varied δ_d in the range $[0.5, 5.0]$ in steps of 0.5 (i.e., ten different spacings). The friction coefficients μ_{dd} and μ_{ds} varied in the range $[0, 1.0]$ in steps of 0.1 (i.e., 11 different values). In total, we performed 1210 simulations.

While experimental high-speed camera recordings have revealed that multiple collisions occur during the interaction of two dominoes (i.e., a not perfectly inelastic collision), a recent work has shown that the coefficient of restitution between dominoes has little influence on the wavefront speed [9]. Therefore, we set the coefficients of domino-domino and domino-surface restitution to zero, meaning that there is no rebounding after collisions. Gravity g was set to 9.81 m/s^2. Finally, the first domino of the series was set in motion using an additional block that impacts the top of the first domino. We computed the wavefront speed between different dominoes from the 20th to the 200th domino and took the averaged value of ten different measures. The standard deviation of the speed for these different measures never exceeded 0.09 m/s in all the tests.

III. RESULTS AND SCALING LAW

Figure 2 gathers the values of wavefront speed v in contour plots as a function of coefficients of friction μ_{dd} and μ_{sd} for spacing (a) $\delta_d = 0.5$ to (j) $\delta_d = 5$. We note that spacings $\delta_d = 0.5$ and $\delta_d = 1.0$ [curves (a) and (b)] show complex couplings between μ_{dd}, μ_{sd}, and v. Moreover, the contour lines have discontinuities since these very narrow spacings produce anomalies in the toppling mechanism and the wavefront is sometimes not propagated. In these cases, the toppling can stop [as shown in Fig. 1(b)] due to resistant moments that stack the dominoes vertically and impede the rotation of the blocks with respect to their base.

Another anomaly occurs for large spacings, usually after $\delta_d > 3$, when the domino-surface friction is low (i.e., a slippery surface) and the domino-domino friction is high. Under such conditions, a backward sliding of blocks occurs, stopping the toppling as illustrated

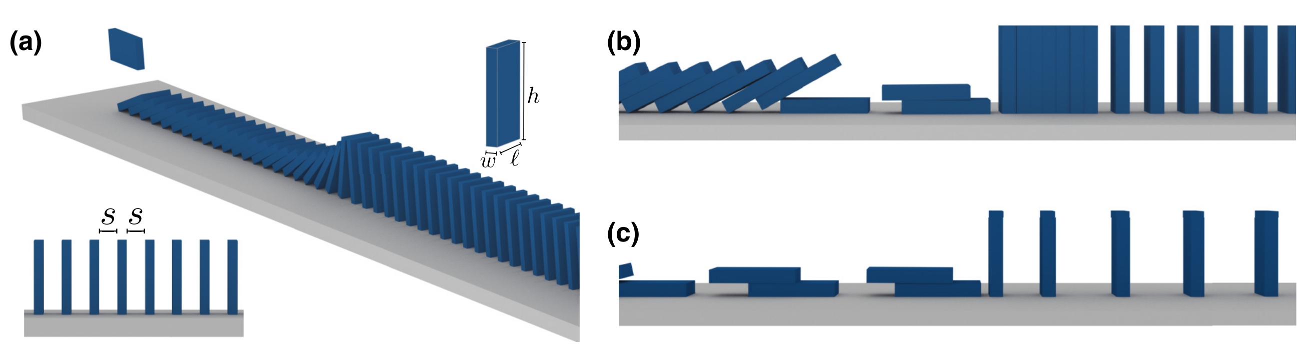

FIG. 1. (a) Screenshot for one of the tests with spacing between dominoes $\delta_d = 3$ and geometrical characteristics of the dominoes. (b) Wavefront stops under narrow spacing because of vertical stacking. (c) Wavefront stops because of backwards sliding of dominoes under low domino-surface friction (i.e., slippery surface).

in Fig. 1(c). Nevertheless, the toppling speed seems to increase when the spacing is small, the domino-domino friction is low, and the domino-surface friction is high. For instance, the wavefront speed peaks in Fig. 2(a) with $v \simeq 2.25$ m/s.

Beyond $\delta_d = 1$, and omitting the discontinuity of the contours due to the anomalies mentioned before, we observe that the wavefront speed becomes practically independent of the domino-surface friction. In turn, the distance between contour lines becomes wider when the domino-domino friction is $\mu_{dd} > 0.4$. Despite the difference of the systems, this observation is in agreement with the evolution of friction mobilization and shear strength in granular frictional materials under deformation [15,16]. For those materials, it has been shown that coefficients of friction greater than about 0.4 have a relatively low influence on the behavior of mechanical discrete systems.

As McLachlan *et al.* [17] and Sun [8] showed using dimensional analysis, the kinematics of domino toppling are linked to the scaling relation

$$v \sim \sqrt{\frac{swg}{h}}\mathcal{F}(s/h), \tag{1}$$

which includes the geometrical characteristics of the configuration of dominoes and a function $\mathcal{F}$ that seems to depend, once more, on geometrical characteristics. Since the functional form of $\mathcal{F}$ is difficult to assess, experimental tests have opted to use the relation

$$v = C\sqrt{\frac{swg}{h}}, \tag{2}$$

with C being a constant that has been found to vary widely between $\simeq 3.5$ and $\simeq 43$. It is worth noting that none of those studies considered the effects of friction between blocks or the surface.

Based on our results, we infer that the function $\mathcal{F}$ should also contain a term involving the friction coefficient between dominoes. Since the wavefront speed decreases nonlinearly with μ_{dd}, a simple way to include this observation is by setting the relation

$$v \sim \sqrt{\frac{swg}{h}}\frac{1}{(1+\mu_{dd})^{\alpha}}, \tag{3}$$

with α being a constant value. We also take into account the fact that the domino-surface friction has a negligible effect on v compared to μ_{dd}. Thus, we neglect μ_{sd} in the above relation.

Figure 3(a) shows the term on the right-hand side of relation (3) for spacings $\delta \in [1.5, 5]$ (mixing all values of μ_{sd}) versus the measured values of v. In this plot, the parameter α was set to approximately 0.4 since it linearizes the evolution of the curves for each spacing. This display of the data allows us to see that a term including the spacing between dominoes is missing in relation (3). Therefore, the simplest relation that lets us describe the scaling between

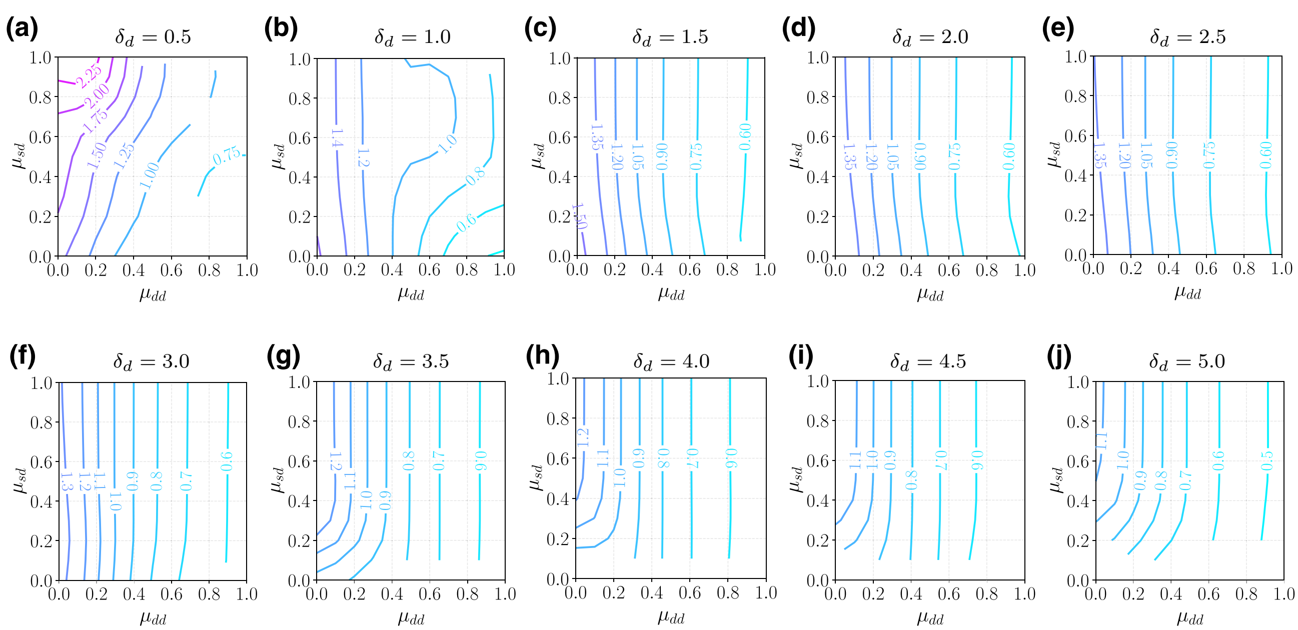

FIG. 2. Wavefront speed in m/s as a function of the coefficient of friction between dominoes μ_{dd} and between domino and surface μ_{sd}, for varying spacing between dominoes from (a) $\delta_d = 0.5$ to (j) $\delta_d = 5$ in steps of 0.5.

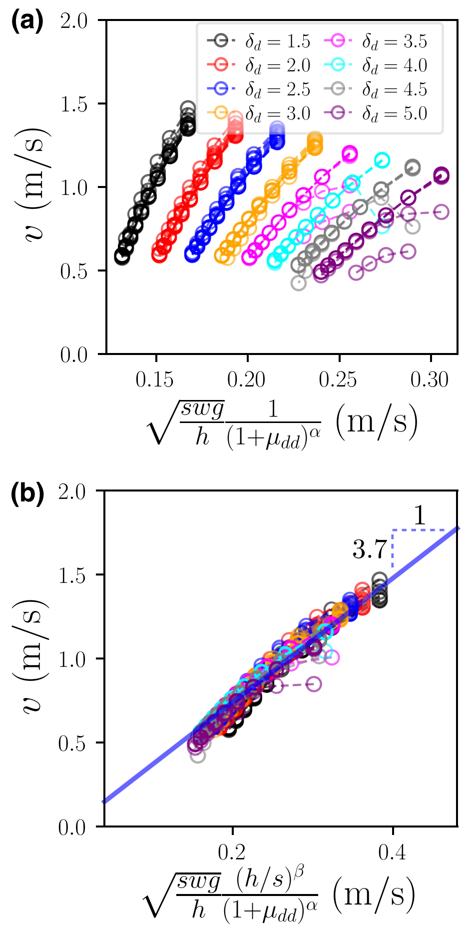

FIG. 3. (a) Relation (2) versus the measured wavefront speed v. (b) Scaling proposed in Eq. (4) versus v for a wide range of spacings and friction coefficients between dominoes.

geometry, friction, and wavefront speed can be written as

$$v = C\sqrt{\frac{swg}{h}}\frac{(h/s)^{\beta}}{(1+\mu_{dd})^{\alpha}}. \quad (4)$$

This expression includes a proportionality parameter C as well as an additional dimensionless term h/s, which uses the height and spacing between the dominoes. The exponent β is included because of the apparent nonlinearity that the factor h/s carries upon v. Based on our results, the constant $\beta \simeq 0.7$, and the proportionality parameter C remains surprisingly close to the value found experimentally by Stronge and Johnson [2], with $C \simeq 3.7$.

Figure 3(b) shows the right-hand side of Eq. (4) versus the measured values of wavefront speed v. Despite some scattered data points, this scaling produces a master curve that gathers the behavior of collaborative discrete systems over a wide range of parameters. Although we do not provide an analytical deduction of the wavefront speed in the presence of friction, this approach provides a framework to predict the wavefront speed of the toppling of solid discrete bodies.

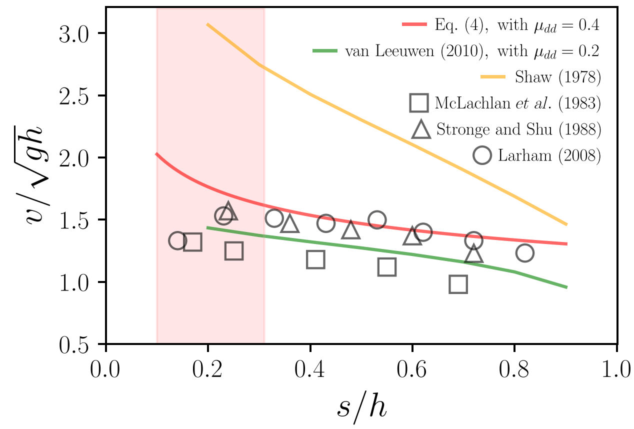

FIG. 4. Comparison between experimental data from Larham [18] (circles), McLachlan *et al.* [17] (squares), and Stronge and Shu [3] (triangles), the model proposed by Shaw [19] (orange line), the collision model by van Leeuwen [5] (green line), and the scaling law proposed in this work (red line). The shaded red zone corresponds to the spacings between dominoes that typically show anomalies.

IV. COMPARISON WITH EXPERIMENTS

Other investigations have conducted series of experiments to study the effects of spacing and dimensions of the blocks [3,17,18], revealing, for instance, that larger spacings between dominoes decrease the toppling wave speed. The results from those works are shown in Fig. 4 after a normalization by parameters to enable a direct comparison between them. In the same plot, we show the curve proposed after analytical analyses developed by Shaw [19], considered pioneering in the pursuit of an expression to define v in terms of parameters; we also add a prediction of v using an alternative iterative model proposed by van Leeuwen [5], which includes a friction coefficient between dominoes but relies on the nonslip condition between dominoes and surface.

It is clear that the early attempts to model the toppling wave velocity strongly overestimated its value compared to experiments. Alternatively, the strategy employed by van Leeuwen reproduces better predictions of the wave speed, but slightly underestimates the experimental results of Larham and McLachlan *et al.* In the same figure, we plot the proposed Eq. (4). Note that, by means of simple minimum-square optimization, we found that the coefficient of friction $\mu_{dd} \simeq 0.4$ nicely reproduces the experimental wavefront speeds by Larham and McLachlan *et al.* The shaded area in the plot corresponds to the spacings between dominoes for which we found anomalies, so the experimental results in this area are not expected to be reliable. Accordingly, our equation does not reproduce the speeds v for such range of values s/h.

It is important to note that despite the good agreement between experimental data and Eq. (4), our results can nonetheless differ from physical testing since the numerical arrangement of dominoes simulates perfectly aligned blocks with ideal edge-face contact between them. Such conditions are impossible to guarantee in experimental testing.

V. CONCLUSIONS

In summary, we undertook systematic discrete-element simulations of domino toppling for a wide range of spacings and friction coefficients between the blocks and surface. The simulations were performed using a numerical method known as contact dynamics, which is well suited to dealing with rigid discrete bodies involving impacts. Our campaign of simulations allowed us to study the evolution of wavefront speed v as a function of system parameters. Between a "well-behaved" set of spacings δ_d in the interval 1.5 to 5, and friction domino-surface $\mu_{sd} > 0.2$, we found stable wavefront speeds that decrease as the friction domino-domino increases. The wavefront speed turned out to be practically independent of the domino-surface friction. This suggests that the main toppling mechanism is rotation around the edge of the blocks and little friction mobilization occurs at the domino-surface contact. The evolution of v with μ_{dd} is, nonetheless, nonlinear and v seems to be relatively less affected by increments of μ_{dd} beyond a friction coefficient of 0.4.

Finally, based on analytical deductions on the scaling laws of domino toppling and experimental tests, we were able to enrich the relation $v \sim \sqrt{swg/h}$ by adding the effect of the domino-domino friction coefficient and additional geometrical characteristics of the domino arrangement. In doing so, we successfully identified a master curve that relates the wavefront speed to a large range of geometrical configurations and friction coefficients between dominoes. This was only possible through a massive and realistic campaign of simulations that took into account the fewest assumptions possible about the mechanics of these cooperative, dynamic systems. Despite the several efforts researchers have undertaken to study the dynamics of rigid bodies, many assumptions prevented one from understanding the different mechanisms and anomalies that can take place in the toppling of adjacent blocks. Our approach was, in turn, robust and considered the whole complexity of the mechanics of multicontact, highly nonlinear interactions between rigid bodies.

It is important to highlight that, although high-speed camera observations have shown that irregularities in domino dimensions and placement produce a rocking motion of the pieces (such motion promotes vertex-face interactions and moments out of the middle axis of the dominoes), our three-dimensional modeling could easily cope with those phenomena that involve more complex toppling kinematics. We do not intend to extend our study towards those additional toppling mechanisms, but readers are welcome to download our model and conduct their own experiments by accessing the repository [20], and running the simulations using the free and open-source software LMGC90 [14].

ACKNOWLEDGMENTS

This research was enabled in part by the support provided by Calcul Québec [21] and Compute Canada [22]. K.W. acknowledges the financial support provided by the National Science Centre (NCN) in Poland through Grant No. 2018/29/B/ST8/00729. We thank Franny McGill for her careful editing of the manuscript.

[1] D. Sandlin, Dominoes – hardcore mode (2017), [Online; accessed 15-Jul-2021].

[2] W. J. Stronge and W. Johnson, The domino effect: A wave of destabilizing collisions in a periodic array, Proc. R. Soc. London A Math. Phys. Sci. **409**, 199 (1987).

[3] W. Stronge and D. Shu, The domino effect: Successive destabilization by cooperative neighbours, Proc. R. Soc. London A Math. Phys. Sci. **418**, 155 (1988).

[4] C. J. Efthimiou and M. D. Johnson, Domino waves, SIAM Rev. **49**, 111 (2007).

[5] J. van Leeuwen, The domino effect, Am. J. Phys. **78**, 721 (2010).

[6] T. Shi, Y. Liu, N. Wang, and C. Liu, Toppling dynamics of a mass-varying domino system, Nonlinear Dyn. **98**, 2261 (2019).

[7] G. Lu, J. R. Third, and C. R. Müller, Effect of particle shape on domino wave propagation: A perspective from 3D, anisotropic discrete element simulations, Granular Matter **16**, 107 (2014).

[8] B. H. Sun, Scaling law for the propagation speed of domino toppling, AIP Adv. **10**, 1 (2020).

[9] T. Shi, Y. Liu, N. Wang, and C. Liu, Toppling dynamics of regularly spaced dominoes in an array, J. Appl. Mech. **85**, 041008 (2018).

[10] F. Dubois, V. Acary, and M. Jean, The contact dynamics method: A nonsmooth story, Comptes Rendus – Mecanique **346**, 247 (2018).

[11] M. Jean, The non-smooth contact dynamics method, Comput. Methods Appl. Mech. Eng. **177**, 235 (1999).

[12] M. Renouf, F. Dubois, and P. Alart, A parallel version of the non smooth contact dynamics algorithm applied to the simulation of granular media, J. Comput. Appl. Math. **168**, 375 (2004).

[13] F. Dubois, M. Jean, M. Renouf, R. Mozul, A. Martin, and M. Bagnéris, in *10e colloque national en calcul des structures* (2011), p. 8 p.

[14] F. Dubois and M. Jean, *et al.*, LMGC90 wiki page, https://git-xen.lmgc.univ-montp2.fr/lmgc90/lmgc90_user/wikis/home (2021), [Online: accessed 15-Jul-2021].

[15] A. Taboada, N. Estrada, and F. Radjai, Additive decomposition of shear strength in cohesive granular media

from grain-scale interactions, Phys. Rev. Lett. **97**, 098302 (2006).
[16] F. Radjai and E. Azéma, Shear strength of granular materials, Eur. J. Environ. Civil Eng. **13**, 203 (2009).
[17] B. G. McLachlan, G. Beaupre, A. B. Cox, and L. Gore, Falling dominoes (D. E. Daykin), SIAM Rev. **25**, 403 (1983).
[18] R. Larham, Validation of a Model of the Domino Effect?, 1 (2008), ArXiv:0803.2898.
[19] D. E. Shaw, Mechanics of a chain of dominoes, Am. J. Phys. **46**, 640 (1978).
[20] https://github.com/dacg/3D˙dominoes.git
[21] www.calculquebec.ca
[22] www.computecanada.ca

Correction: The spelling of the author name in Ref. [1] and in the citation in the abstract has been corrected. The year has also been corrected for the citation in the abstract.