Comment on "Scattering Cancellation-Based Cloaking for the Maxwell-Cattaneo Heat Waves"

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This is a comment on the assumptions, model, and calculations in the paper by Farhat *et al.* [Phys. Rev. Appl. **11**, 044089 (2019)] on cloaking of thermal waves in solids. The differences between the two thermal flux laws considered in the latter paper are also critically discussed, specifically showing that the chosen model does not correspond to the Maxwell-Cattaneo hyperbolic (wave) theory of heat transfer.

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I. INTRODUCTION

This comment discusses the recent *Physical Review Applied* publication [1], the focus of which is a proposed cloaking scheme for thermal waves in rigid solids. To begin, it is useful to briefly review the topic of hyperbolic heat transport, i.e., the theory of *heat waves* [2–4], in rigid solids.

Consider a thermally conducting, homogeneous and isotropic, rigid solid at rest. As first suggested by theory, and subsequently confirmed by experiment, at sufficiently low temperatures the transport of heat in such bodies occurs *not* via diffusion, the mechanism underlying Fourier's law for the thermal flux, but instead by the propagation of thermal waves (or *second sound*) [2]. Many constitutive relations have been proposed to describe this phenomenon [4]. Perhaps the best known is the Maxwell-Cattaneo (MC) law [5,6], which in the present context reads

$$\left(1+\tau_0\frac{\partial}{\partial t}\right)\mathbf{\Phi} = -\kappa_0\nabla T. \tag{1}$$

Unsurprisingly, the history of this relation is complex: there exists Russian-language literature describing a similar flux law prior to Cattaneo (but, of course, after Maxwell) [7,8]. Here, $T = T(\mathbf{x},t)$ and $\mathbf{\Phi} = \mathbf{\Phi}(\mathbf{x},t)$ denote the absolute temperature and the thermal flux vector, respectively, where $\mathbf{x} = (x,y,z) \in \mathbb{R}^3$. As in Ref. [1], $\tau_0(>0)$ is the thermal relaxation time for phonon processes that do not conserve phonon momentum [2] and $\kappa_0(>0)$ is the thermal conductivity of the solid under consideration.

Equation (1), which reduces to Fourier's law on setting $\tau_0 \equiv 0$, is the latter's simplest generalization that yields a hyperbolic thermal transport equation, unlike the parabolic transport equation that stems from Fourier's law. Therefore, the MC law overcomes the so-called "paradox of diffusion"—the philosophically problematic implication that thermal disturbances in continuous media propagate with infinite "speed" under Fourier's law.

The comments below on the assumptions, models, and calculations from Ref. [1] further illustrate these notions. Unless otherwise stated, the same notation as used in Ref. [1] is employed herein. First, observe that the energy-balance equation is incorrectly stated in Eq. (3) of Ref. [1]; specifically, its source term, which is denoted here by $\mathcal{S} = \mathcal{S}(\mathbf{x},t)$, is missing, and the $\partial T/\partial t$ term it contains should be multiplied by the product ϱc_p to ensure dimensional consistency. Second, a "flux diffusion" term is introduced into the MC law (1) without sufficient justification. This term turns the heat flux law into a hybrid between the MC law and model of Guyer and Krumhansl (GK) [9, Eq. (59)], which has been derived by GK from the linear Boltzmann equation. The nontrivial consequence of this manipulation is that, under the GK model, heat flow is *not* necessarily down the temperature gradient and certainly "will not permit the propagation of [heat] waves" [2, p. 46] unless the "flux diffusion" term is neglected.

Now, with the above corrections in mind and employing the "full" GK flux law, Eq. (3) of Ref. [1] becomes

$$\varrho c_p\frac{\partial T}{\partial t} = -\nabla\cdot\mathbf{\Phi} + \mathcal{S}, \tag{2a}$$

$$\underbrace{\left[1+\tau_0\frac{\partial}{\partial t} - \tau_0\sigma_0(\Delta + 2\nabla\nabla\cdot)\right]}_{=:\mathscr{K}}\mathbf{\Phi} = -\kappa_0\nabla T, \tag{2b}$$

where $\Delta := \nabla\cdot\nabla$ is the Laplacian operator. In Eq. (2), c_p (not c_v, see Ref. [10, p. 9]) and ϱ are the specific heat at constant pressure and the mass density, respectively, of the solid under consideration. From Ref. [2, Sec. IV], one

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can establish that $\sigma_0 = (1/5)\tau_N V^2$, where τ_N is the relaxation time for N processes and V carries SI units of m s^{-1}. Therefore, the SI units of σ_0 are m^2 s^{-1}; not W m^{-1} K^{-1} as reported in Ref. [1, p. 3]. The operator $\mathscr{K}$ here differs from its counterpart in Ref. [1] in that the latter is missing $\nabla\nabla\cdot$.

Remark 1.—While Eq. (1) is the $\sigma_0 \equiv 0$ special case of Eq. (2b), it is important to stress that one cannot regard the latter as being a perturbation of the former. As shown below, the MC flux law (1) predicts heat *waves*, while the GK flux law (2b) predicts heat *diffusion*.

Remark 2.—As a result of the missing ϱc_p in the energy balance, many equations in Ref. [1] are dimensionally inconsistent and, therefore, have no physical meaning. For example, consider Eq. (4) of Ref. [1]. The first and second terms on the left-hand side (lhs) have units of K s^{-1}, while the third and fourth terms on the lhs have units of W m^{-3}.

II. THE THERMAL TRANSPORT EQUATION

As in Ref. [1], regard all coefficients as constant and proceed to eliminate $\mathbf{\Phi}$ between the equations of Eq. (2), assuming sufficient smoothness of the dependent variables. The first step in this process is to employ Eq. (2a) in order to recast Eq. (2b) as

$$\underbrace{\left[1 + \tau_0\frac{\partial}{\partial t} - \tau_0\sigma_0\Delta\right]}_{=:\mathscr{H}_{\sigma_0}}\mathbf{\Phi} = -\kappa_0\nabla T + 2\tau_0\sigma_0\left[\nabla\mathcal{S} - \varrho c_p\frac{\partial(\nabla T)}{\partial t}\right]. \quad (3)$$

Next, after applying $\mathscr{H}_{\sigma_0}$ to Eq. (2a) and then using Eq. (3), one obtains a thermal transport equation

$$\frac{\partial T}{\partial t} + \tau_0\frac{\partial^2 T}{\partial t^2} = \tau_0\tilde{\sigma}_0\frac{\partial(\Delta T)}{\partial t} + \varkappa_0\Delta T + \frac{1}{\varrho c_p}\mathscr{H}_{\tilde{\sigma}_0}[\mathcal{S}], \quad (4)$$

where $\varkappa_0 := \kappa_0/(\varrho c_p)$ is the thermal *diffusivity* and, for convenience, $\tilde{\sigma}_0 := 3\sigma_0$ has been defined. In contrast to Eq. (4), the source term on the right-hand side (rhs) of Eq. (4) of Ref. [1] is not acted upon by the operator $\mathscr{H}_{\tilde{\sigma}_0}$, nor multiplied by $1/(\varrho c_p)$, and it contains σ_0 instead of $\tilde{\sigma}_0$.

Remark 3.—The $\tilde{\sigma}_0 \equiv 0$ source-free version of Eq. (4) is the multidimensional version of the *damped wave equation* [2, p. 42], which predicts that thermal signals (disturbances) propagate at a *finite* characteristic speed of $c_0 := \sqrt{\varkappa_0/\tau_0}$ (see also Refs. [11,12]). Meanwhile, the $\tilde{\sigma}_0 > 0$ source-free version of Eq. (4) is a multidimensional *Jeffreys-type equation*, which predicts an *infinite* "speed" of propagation of signals [2, p. 46].

To demonstrate this key difference between wavelike and diffusive thermal transport (see also Ref. [13]) but in a slightly simpler way, consider a related one-dimensional (1D) initial-boundary-value problem (IBVP) posed by Tanner [14] for the Jeffreys-type equation arising in the context of viscoelasticity. (This IBVP is also the one considered in Refs. [11,12] for the damped wave equation of hyperbolic heat conduction.) Recasting Tanner's problem in the present notation,

$$\frac{\partial T}{\partial t} + \tau_0\frac{\partial^2 T}{\partial t^2} = \tau_0\tilde{\sigma}_0\frac{\partial^3 T}{\partial t\partial x^2} + \varkappa_0\frac{\partial^2 T}{\partial x^2}, \quad (x,t)\in\Omega, \quad (5a)$$

$$T(0,t) = T_0 H(t), \quad \lim_{x\to\infty} T(x,t) = 0, \quad t > 0, \quad (5b)$$

$$T(x,0) = \frac{\partial T}{\partial t}(x,0) = 0, \quad x > 0, \quad (5c)$$

where $H(\cdot)$ denotes the Heaviside unit-step function, $\Omega := (0,\infty)\times(0,\infty)$ is the space-time domain of interest, and the constant $T_0(>0)$ is the amplitude of the inserted thermal signal. Following Tanner [14], one can apply the Laplace transform to Eq. (5a) and its boundary conditions (BCs) (5b). This IBVP corresponds to a *heat-pulse* experiment [15,16]. After making use of the initial conditions (5c) and then solving the resulting subsidiary equation subject to the (transformed) BCs, one obtains an algebraic expression that can be inverted back to the time domain. This exact inverse is known [14], with several other representations summarized in Ref. [17].

To illustrate this fundamental difference between the thermal transport described by the MC law and the GK-type law used in Ref. [1], the respective exact solutions of IBVP (5) are shown in Fig. 1 corresponding to $\tilde{\sigma}_0 \equiv 0$ and $\tilde{\sigma}_0 > 0$, respectively. The integral expression for the exact solution obtained by Tanner [14] is evaluated numerically using the NIntegrate subroutine of MATHEMATICA, to arbitrary precision, on a finite grid of x values [17]. Figure 1 shows that under the MC law (dashed curve), the heat pulse has only propagated slightly less than 2

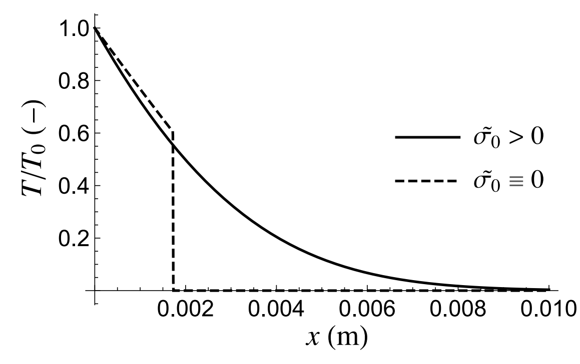

FIG. 1. The wavelike (dashed curve, $\tilde{\sigma}_0 \equiv 0$) versus diffusive (solid curve, $\tilde{\sigma}_0 = 6.4015\times10^{-6}$ m^2 s^{-1}) spread of heat from the sudden imposition of a temperature jump at $(x,t) = (0,0)$. Here, $\tau_0 = 0.991$ s and $\varkappa_0 = 2.95\times10^{-6}$ m^2 s^{-1}, as given in Ref. [15] for limestone. This plot is at $t = 1$ s, i.e., after approximately a single thermal relaxation time.

mm into the domain. Meanwhile, under the GK law (solid curve), the normalized temperature T/T_0 is nonzero *everywhere* in the domain. Since the physical parameter values given in Ref. [1] are inconsistent, to generate the plots in Fig. 1, the values for limestone are taken from Ref. [15], wherein it has been experimentally demonstrated that material microstructure can lead to GK-type heat conduction at room temperature (see also Ref. [16, Ch. 9]).

To summarize: the behavior of the $\tilde{\sigma}_0 > 0$ case is seen to be strictly diffusive (similar to the solution of the 1D thermal diffusion equation arising from Fourier's law); the signal applied at $x = 0$ is "felt" *instantly* at every point in the half-space $x > 0$. In contrast, the $\tilde{\sigma}_0 \equiv 0$ case exhibits a thermal shock (jump discontinuity), of magnitude $T_0 \exp[-t/(2\tau_0)]$, propagating (to the right) with finite speed $c_0 \approx 1.74 \times 10^{-3}$ m s^{-1} (for the chosen parameter values); see also Refs. [11, p. 545] and [2, p. 45].

III. HARMONIC DISTURBANCES

Returning to the analysis in Ref. [1], set $\mathcal{S}(\mathbf{x}, t) = 0$ and assume $T(\mathbf{x}, t) = \Theta(\mathbf{x}) \exp(-i\omega t)$, where $\omega(> 0)$ is the angular frequency of some thermal disturbance impacting the solid in question. Under these assumptions, Eq. (4) is reduced to the (source-free) Helmholtz equation

$$\Delta\Theta + \left(\frac{\tau_0\omega^2 + \omega}{\varkappa_0 - \tau_0\tilde{\sigma}_0\omega}\right)\Theta = 0. \tag{6}$$

On the other hand, in Eq. (5) of Ref. [1], "T" is reused instead of introducing a new (time-independent) function such as Θ herein, and the thermal conductivity is present in the latter equation instead of the thermal diffusivity.

Consider plane-wave propagation in a direction set by the unit vector $\hat{\mathbf{u}}$ and set $\Theta(\mathbf{x}) = \Theta_0 \exp(ik_0\hat{\mathbf{u}} \cdot \mathbf{x})$. Substitution into Eq. (6) yields

$$k_0^2(\omega) = \underbrace{\frac{\tau_0\omega^2(\varkappa_0 - \sigma_0)}{\varkappa_0^2 + \tau_0^2\tilde{\sigma}_0^2\omega^2}}_{=:\mathfrak{a}} + i\underbrace{\frac{\omega(\varkappa_0 + \tau_0^2\tilde{\sigma}_0\omega^2)}{\varkappa_0^2 + \tau_0^2\tilde{\sigma}_0^2\omega^2}}_{=:\mathfrak{b}}, \tag{7}$$

where $i = \sqrt{-1}$, $\Theta_0 > 0$, and $k_0 \in \mathbb{C}$. Enforcing $\Theta < \infty$ as $|\mathbf{x}| \to \infty$ (and, also, since $\mathfrak{b} > 0$) requires $\mathrm{Im}(k_0) \geq 0$; then, it is readily established that

$$k_0(\omega) = \sqrt{\frac{\mathfrak{a} + \sqrt{\mathfrak{a}^2 + \mathfrak{b}^2}}{2}} + i\sqrt{\frac{-\mathfrak{a} + \sqrt{\mathfrak{a}^2 + \mathfrak{b}^2}}{2}}. \tag{8}$$

The dispersion relation in Eq. (8), as well as the $\tilde{\sigma}_0 \equiv 0$ reduction to its version under the MC law, are illustrated in Fig. 2. Although the inner contours may look similar to Fig. 1(b, bottom) of Ref. [1], $\mathrm{Re}(k_0) < \mathrm{Im}(k_0)$ for the chosen set of physical parameters based on Ref. [15].

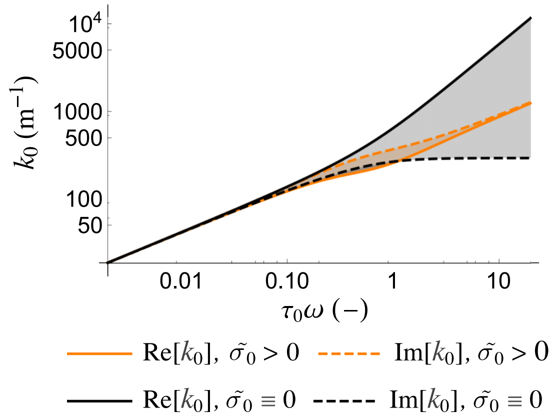

FIG. 2. The real and imaginary parts of the wave number k_0 (dark colors for the MC law and light colors for the GK law) as computed from Eq. (8) for the same parameter values as used to generate Fig. 1.

On the other hand, $\mathrm{Re}(k_0) > \mathrm{Im}(k_0)$ does hold true under the MC law ($\tilde{\sigma}_0 \equiv 0$). Furthermore, in this case of hyperbolic heat (wave) transfer, the scattering and absorption are not balanced because $\mathrm{Im}(k_0) \to (4\varkappa_0\tau_0)^{-1/2} = \text{const.}$, but $\mathrm{Re}(k_0) \sim \tau_0\omega/\sqrt{\varkappa_0\tau_0}$, as $\tau_0\omega \to \infty$.

IV. OTHER ISSUES

The following additional issues are noticed regarding the results and discussion in Ref. [1]:

(i) In Ref. [1, p. 2], it is stated that the term proportional to σ_0 is necessary to "make the discretizing process asymptotically stable." It is not clear what "asymptotically" means in this context. Furthermore, discretizing a hyperbolic heat-transport equation (without the addition of any new terms to the equation) is possible via a number of numerical methods (see, e.g., Refs. [18, 19], but note that modern schemes [20] should be used nowadays).

(ii) Below Eq. (5) of Ref. [1], it is stated that "k_0 is a complex number for all frequencies [under the GK-type flux law], which is markedly different from classical heat waves (Fourier transfer)." Note that heat waves are impossible under Fourier's law. On setting $\tau_0 \equiv 0$ in Eq. (7), it follows that $k_0(\omega) = \sqrt{i\omega/\varkappa_0} = (1 + i)\sqrt{\omega/(2\varkappa_0)} \in \mathbb{C}$. Thus, there is no discernible difference between the Fourier and non-Fourier heat-flux laws with respect to whether $k_0 \in \mathbb{C}$.

(iii) The unknown coefficients in the expansions in Eqs. (8) and (9) of Ref. [1] are found by applying a boundary condition involving "the temperature field T, as well as its flux $\kappa\nabla T$." Under the MC law, the heat flux is not (with misprints corrected) $-\kappa_0\nabla T$, as it would be under Fourier's law; rather, it is the expression obtained by solving Eq. (1) for $\mathbf{\Phi}$. In the case of harmonic time dependence, for which $\mathbf{\Phi}(\mathbf{x},t) = \mathbf{F}(\mathbf{x})\exp(-i\omega t)$, specifying the flux at the boundary of some spatial domain $\mathcal{D} \subset \mathbb{R}^3$, under the MC law, would correspond to specifying

$$\mathbf{F} = -\left(\frac{1 - i\omega\tau_0}{1 + \omega^2\tau_0^2}\right)\kappa_0\nabla\Theta \quad \text{on } \mathbf{x} \in \partial\mathcal{D}. \tag{9}$$

The corresponding expression under the GK flux law (2b) is lengthier. This error in imposing the BCs on the series expansion casts doubt on all subsequent results in Secs. III and IV of Ref. [1].

(iv) The conclusion of Ref. [1, p. 7] states that "the Fourier heat equation is not frame invariant." However, the thermal transport equation under Fourier's law is indeed frame invariant, because the material derivative $DT/Dt := \partial T/\partial t + \mathbf{v}\cdot\nabla T$ is featured on the lhs of the energy-balance equation in its derivation for heat transfer in a moving (or deforming) medium with velocity $\mathbf{v}$ (see, e.g., Ref. [21, Sec. 7.1]). Then, $DT/Dt \equiv \partial T/\partial t$ if $\mathbf{v} = \mathbf{0}$ as in Eq. (2a) above for a rigid solid at rest. Furthermore, the frame-indifferent formulation of the MC law is misattributed in Ref. [1]; it has been derived in Ref. [35], not Ref. [43], of Ref. [1].

(v) Presumably the word "photon(s)" should be replaced with "phonon(s)" everywhere in Ref. [1], given that the context is heat conduction, not electromagnetism.

V. CONCLUSION

On the basis of this commentary, it may be concluded that Farhat *et al.* [1] have not provided "the first demonstration of scattering cancellation cloaking for heat *waves* [emphasis added] obeying the Maxwell-Cattaneo transfer (*sic*) law." It would, therefore, be of interest to revisit this problem with the (unmodified) Maxwell-Cattaneo (MC) heat-flux law [i.e., the $\sigma_0 \equiv 0$ special case of Eq. (2)], which leads to the hyperbolic thermal transport equation

$$\tau_0\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} - \varkappa_0\Delta T = \frac{1}{\varrho c_p}\left(1 + \tau_0\frac{\partial}{\partial t}\right)\mathcal{S}, \tag{10}$$

also obtained by setting $\tilde{\sigma}_0 \equiv 0$ in Eq. (4) and subject to the appropriate boundary conditions under the MC law, as well as realistic values of the physical parameters. Additionally, note that if $\mathcal{S}(\mathbf{x},t) = \delta(t)$ is a point source (Dirac's δ distribution), then the rhs of Eq. (10) is $(\varrho c_p)^{-1}[\delta(t) + \tau_0\delta'(t)]$, unlike the rhs of Eq. (4) of Ref. [1].

Finally, with regard to item (iv) above, it is appropriate to mention that the reworking of classical results under the frame-indifferent generalization of the MC law has generated a scientifically and/or mathematically questionable literature [22–24]. Therefore, it is difficult to substantiate the statement "[t]he heat transfer rate thus decays for larger thermal relaxation times" [1, p. 2] on the basis of Ref. [39] of Ref. [1], because it is easily verified that the latter reference is an example of the literature discussed in Refs. [22–24].

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