

Experimentally Accessible Lower Bounds for Genuine Multipartite Entanglement and Coherence Measures

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Experimentally quantifying entanglement and coherence are extremely important for quantum resource theory. However, because the quantum state tomography requires exponentially growing measurements with the number of qubits, it is hard to quantify entanglement and coherence based on the full information of the experimentally realized multipartite states. Fortunately, other methods have been found to directly measure the fidelity of experimental states without quantum state tomography. Here we present a fidelity-based method to derive experimentally accessible lower bounds for measures of genuine multipartite entanglement and coherence. On the one hand, the method works for genuine multipartite entanglement measures including the convex-roof extended negativity, the concurrence, the G-concurrence, and the geometric measure for genuine multipartite entanglement. On the other hand, the method also delivers observable lower bounds for the convex roof of the l_1 -norm of coherence, the geometric measure of coherence, and the coherence of formation. Furthermore, all the lower bounds are based on the fidelity between the chosen pure state and the target state, and we obtain the lower bounds of several real experimental states as examples of our results.

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I. INTRODUCTION

Quantum entanglement is one of the most important concept in quantum mechanics. In recent years, entanglement has been considered as a valuable resource for quantum information processing and has attracted much attention. For quantifying entanglement, several entanglement measures have been proposed for bipartite systems [1–5], such as the negativity or extensions thereof [6–10], the concurrence [11–16], the G-concurrence [17–20], and the geometric measure of entanglement [20–22].

Compared with bipartite systems, the situation for multipartite systems is much more complex. Considering an N -partite system, there exist different classes of entanglement [23–25]. A multipartite quantum state that is not a convex combination of biseparable states with respect to any bipartition contains genuinely multipartite entanglement (GME) [26–28]. It is worth noting that the GME denotes the strongest entanglement type in multipartite quantum states, which is considered as a rich resource for quantum information processing. Consequently, many criteria have been introduced for multipartite entanglement detection [29–31].

In fact, analytical results concerning the computation of entanglement measures have been obtained only for special measures and for two-qubit states or some special kinds of higher-dimensional mixed states [10,13,21,32–35]. It has been proved that computing faithful entanglement measures is NP-hard for a general state [36,37]. For general higher-dimensional states and multipartite states, lower or upper bounds are usually presented to quantify entanglement [38–49].

Coherence is another crucial quantum mechanical phenomenon and was characterized in quantum optics. Although quantum entanglement can only occur in bipartite or multipartite systems, quantum coherence is usually defined for a single system [50–53]. Recently, it has been recognized that coherence, just like entanglement, can be treated as a physical resource, and many coherence measures have been proposed [50–59]. Thus, one may ask whether methods to obtain lower bounds for entanglement measures can also be used for coherence measures.

Experimentally quantifying entanglement and coherence are also extremely important. However, the quantum state tomography requires exponentially growing measurements with the number of qubits. To quantify GME in a large-qubit system such as 8–12 qubits, one needs a method to directly measure its GME, because it is difficult to perform quantum state tomography.

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The purpose of this work is two-fold. On the one hand, we present a method to obtain lower bounds for GME measures of multipartite quantum states, such as the convex-roof extended negativity (CREN) of GME, the concurrence of GME, the G-concurrence of GME, and the geometric measure of GME. On the other hand, we show that our method is also useful to obtain lower bounds for coherence measures, such as the convex roof of the l_1 -norm of coherence, the geometric measure of coherence, and the coherence of formation. Our lower bounds are experimentally accessible without quantum state tomography, by determining the fidelity between the chosen pure state and the target state. Moreover, we present several examples including real experimental states.

II. FIDELITY-BASED LOWER BOUNDS FOR GME MEASURES

For the quantification of multipartite entanglement, there exists a simple way to generalize an arbitrary bipartite entanglement measure to a GME measure. Considering a bipartite entanglement measure E , one can define the GME measure as

$$E_{\text{GME}}(|\psi\rangle) = \min_{\alpha} E_{\alpha}(|\psi\rangle), \quad (1)$$

for an N -partite pure state $|\psi\rangle$, where α represents all possible bipartitions $\alpha|\bar{\alpha}$ of $\{1, 2, \dots, N\}$ see Fig. 1, and $E_{\alpha}(|\psi\rangle)$ is the bipartite entanglement measure E under the bipartition $\alpha|\bar{\alpha}$. The GME measure E_{GME} can be generalized to mixed states ρ via a convex roof construction, that is,

$$E_{\text{GME}}(\rho) = \inf_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E_{\text{GME}}(|\psi_i\rangle), \quad (2)$$

where the minimization runs over all possible decompositions $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. Clearly, evaluating this minimization is not straightforward.

Before embarking on our main results, let us recall some facts about an arbitrary N -partite pure entangled state $|\phi\rangle$, which will be used in Theorems 1–4. As we mentioned previously, α denotes any possible bipartition $\alpha|\bar{\alpha}$ of $\{1, 2, \dots, N\}$, the arbitrarily chosen $|\phi\rangle$ has its Schmidt decomposition $|\phi\rangle = V_{\alpha} \otimes V_{\bar{\alpha}} \sum_{i=1}^{m_{\alpha}} (s_i^{(\alpha)})^{1/2} |ii\rangle$ under the bipartition $\alpha|\bar{\alpha}$, where $\{(s_i^{(\alpha)})^{1/2}\}$ are its Schmidt coefficients in decreasing order and m_{α} is the total number of nonvanishing Schmidt coefficients. We can define $s'_1 := \max_{\alpha} \{s_1^{(\alpha)}\}$ and $m' := \max_{\alpha} \{m_{\alpha}\}$, which can be easily calculated once $|\phi\rangle$ has been chosen.

For a bipartite pure state $|\psi\rangle$, the CREN is defined by the negativity $\mathcal{N}(|\psi\rangle) = \|\psi\rangle\langle\psi|^{T_B}\| - 1$ [8–10]. For an N -partite pure state $|\psi\rangle$, the CREN of GME can be defined as $\mathcal{N}_{\text{GME}}(|\psi\rangle) = \min_{\alpha} \mathcal{N}_{\alpha}(|\psi\rangle)$, and it can be generalized

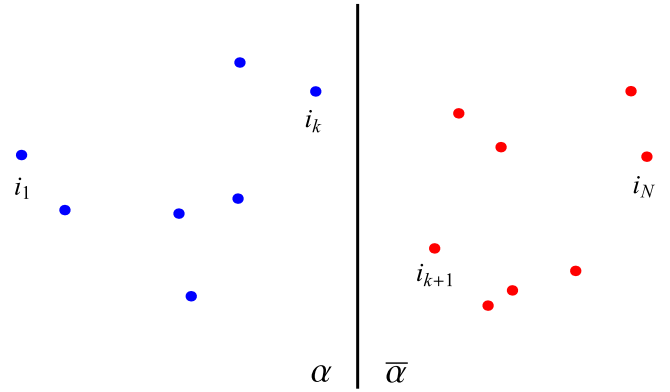


FIG. 1. Illustration for any possible bipartition. The index α denotes any possible bipartition $\alpha|\bar{\alpha}$ of $\{1, 2, \dots, N\}$. For instance, suppose that $i_1, i_2, \dots, i_N$ is an arbitrary order of $1, 2, \dots, N$. The subset α contains $i_1, \dots, i_k$ and $\bar{\alpha} = \{i_{k+1}, \dots, i_N\}$ with $1 \leq k \leq N-1$, because α and $\bar{\alpha}$ are two nonempty subsets.

to mixed states by the convex roof. We can formulate the following result.

Theorem 1. For any N -partite state ρ , its CREN of GME satisfies

$$\mathcal{N}_{\text{GME}}(\rho) \geq S - 1, \quad (3)$$

where $S = \max\{\langle\phi|\rho|\phi\rangle/s'_1, 1\}$. Here, $|\phi\rangle$ is an arbitrary pure state.

The proof is presented in the Appendix A. The interpretation of the lower bound in Theorem 1 is the following: if ρ has a high overlap with some highly entangled state $|\phi\rangle$, this can be used to estimate the entanglement. In experiments one only needs the fidelity with respect to $|\phi\rangle$.

The concurrence was introduced for two-qubit states in Refs. [11–13]. In bipartite higher-dimensional systems, the concurrence is defined by $C(|\psi\rangle) = \sqrt{2(1 - \text{Tr}\rho_A^2)}$ for pure states, and by the convex roof for mixed states [14–16]. Similar to the CREN of GME, one can obtain a lower bound for the concurrence of GME in multipartite systems.

Theorem 2. For any N -partite state ρ , a lower bound on the concurrence of GME is given by

$$C_{\text{GME}}(\rho) \geq \sqrt{\frac{2}{m'(m'-1)}}(S - 1), \quad (4)$$

where $S = \max\{\langle\phi|\rho|\phi\rangle/s'_1, 1\}$.

The proof is similar to Theorem 1, details are given in the Appendix A.

For an $m \otimes n$ ($m \leq n$) pure state $|\psi\rangle$, the G-concurrence is defined by $G(|\psi\rangle) = m(\det \varrho_A)^{1/m}$ [18–20], and by the convex roof for mixed states [17]. An interesting feature of the G-concurrence is that it indicates entanglement of maximal Schmidt rank, that is, it vanishes for pure states where ϱ_A is not of maximal rank. A lower bound for G-concurrence of GME can also be provided as follows.

Theorem 3. *For any N -partite state ϱ , its G-concurrence of GME satisfies*

$$G_{\text{GME}}(\varrho) \geq 1 - m' + S, \quad (5)$$

where $S = \max\{\langle\phi|\varrho|\phi\rangle/s'_1, 1\}$.

The proof is similar to Theorem 1, details are given in the Appendix A.

For an arbitrary bipartite pure state $|\psi\rangle = U_A \otimes U_B \sum_i \sqrt{\mu_i} |ii\rangle$ with $\sqrt{\mu_i}$ being its Schmidt coefficients, the geometric measure of entanglement is defined by $\mathcal{G}(|\psi\rangle) = 1 - \max_i\{\mu_i\}$ [20,21]. Similarly, the geometric measure of entanglement is extended to mixed states by the convex roof. The last entanglement measure for which we provide the lower bound is the geometric measure of GME.

Theorem 4. *For an arbitrary N -partite state ϱ , its geometric measure of GME $\mathcal{G}(\varrho)$ satisfies*

$$\mathcal{G}_{\text{GME}}(\varrho) \geq 1 - \gamma(S), \quad (6)$$

where $\gamma(S) = [\sqrt{S} + \sqrt{(m' - 1)(m' - S)}]^2 / m'^2$ with $S = \max\{\langle\phi|\varrho|\phi\rangle/s'_1, 1\}$.

The proof is similar to Theorem 1, details are given in the Appendix A.

From Theorems 1–4, all lower bounds provide a simple and convenient way to evaluate GME for finite-dimensional multipartite states. Instead of finding the minimal pure-state decomposition, we only need to seek the maximal Schmidt coefficient of the chosen state $|\phi\rangle$ among all possible bipartitions, and obtain the fidelity between $|\phi\rangle$ and the target state ϱ . It is remarkable that the choice of the state $|\phi\rangle$ is significant for determining the bounds. Note that all the bounds are related to the entanglement witness [60] $W = s'_1 1 - |\phi\rangle\langle\phi|$, and this witness detects only states that have a negative partial transpose (NPT) for any bipartition [61]. On the other hand, if this witness does not detect the states (and $S \leq 1$), then there is a biseparable state compatible with the measured fidelity of $|\phi\rangle$ and one cannot conclude that the state contained GME. In this sense, our bounds are optimal. It would be interesting to study whether our bounds also deliver the best possible value for $S > 0$ in the sense of Ref. [47].

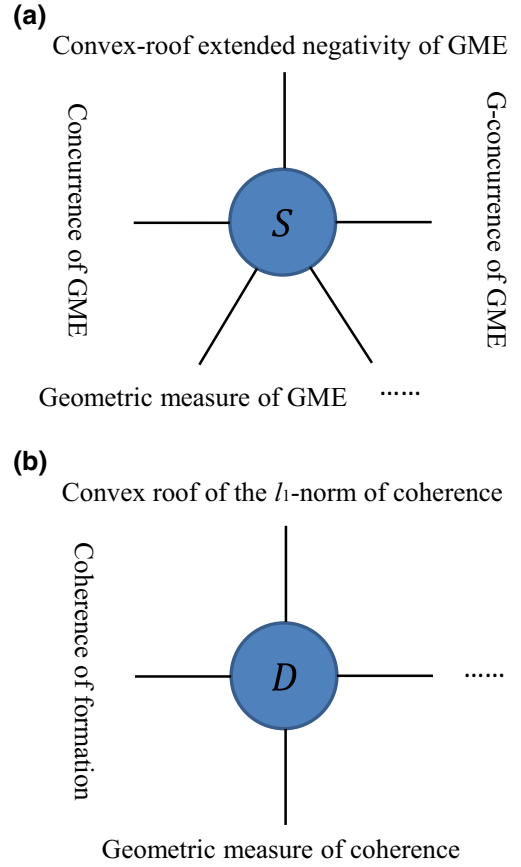


FIG. 2. Estimating GME and coherence measures via S and D . (a) All lower bounds of GME measures in Theorems 1–4 are related to S . In principle, one can obtain lower bounds for all convex-roof constructed GME measures (2) from S . (b) Convex-roof constructed coherence measures in Theorems 5–7 are related to D . In fact, D is analogous to S , which is a simpler case in a single system.

III. FIDELITY-BASED LOWER BOUNDS FOR COHERENCE MEASURES

Before focusing on coherence measures, let us first compare the lower bounds of GME measures and coherence measures. As shown in Fig. 2, one can find that all the lower bounds of GME measures in Theorems 1–4 are related to S , which can be viewed as renormalized fidelity between the chosen pure state $|\phi\rangle$ and the target state ϱ . In the following, we will also find coherence measures in Theorems 5–7 are related to an analog renormalized fidelity D .

For an m -dimensional pure state, a simple and reliable measure of coherence is the l_1 -norm of coherence [50]. It is defined as $C_{l_1}(\varrho) = \sum_{i \neq j} |\langle i|\varrho|j\rangle|$, that is, the summation of all nondiagonal elements of density matrix. For mixed states, the convex roof can also be used in the l_1 -norm measure [54], just like entanglement measures. The convex roof of the l_1 -norm is defined as

$\tilde{C}_{l_1}(\rho) = \inf_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C_{l_1}(|\psi_i\rangle)$ for all possible ensemble realizations of pure states $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. Our method can also be used for estimating the coherence as follows.

Theorem 5. *For an arbitrary state ρ in an m -dimensional system the convex roof of the l_1 -norm of coherence satisfies*

$$\tilde{C}_{l_1}(\rho) \geq D - 1, \quad (7)$$

where $D = \max\{\langle\phi|\rho|\phi\rangle/|d_{\max}|^2, 1\}$, and $|\phi\rangle = \sum_{i=1}^m d_i|i\rangle$ is an arbitrary m -dimensional pure state with $|d_{\max}| = \max_i\{|d_i|\}$.

The proof is presented in the Appendix B.

The idea behind the geometric measure of coherence comes from the geometric measure of entanglement. For a pure state $|\psi\rangle$, we have its geometric measure of coherence $C_g(|\psi\rangle) = 1 - \max_i |\langle i|\psi\rangle|^2$. For a general mixed state ρ , the convex roof construction is used [55].

Theorem 6. *For an arbitrary state ρ in an m -dimensional system, the geometric measure of coherence satisfies*

$$C_g(\rho) \geq 1 - \gamma(D), \quad (8)$$

where $\gamma(D) = [\sqrt{D} + \sqrt{(m-1)(m-D)}]^2/m^2$, $D = \max\{\langle\phi|\rho|\phi\rangle/|d_{\max}|^2, 1\}$, and $|\phi\rangle = \sum_{i=1}^m d_i|i\rangle$ is an arbitrary m -dimensional pure state with $|d_{\max}| = \max_i\{|d_i|\}$.

The proof is similar to Theorem 5, details are given in the Appendix B.

For a pure state $|\psi\rangle$, its coherence of formation is defined as $C_f(|\psi\rangle) = S[\Delta(|\psi\rangle\langle\psi|)]$, where S is the von Neumann entropy and $\Delta(\rho) = \sum_i |i\rangle\langle i|\rho|i\rangle\langle i|$. For a general mixed state ρ , the convex roof is used [56,57]. Moreover, it is proved that for any state ρ the coherence cost is given by the coherence of formation, that is, $C_c(\rho) = C_f(\rho)$ [56]. Thus, the lower bound for coherence of formation is also the lower bound for coherence cost.

Theorem 7. *For an arbitrary state ρ in an m -dimensional system, the coherence of formation satisfies*

$$C_f(\rho) \geq \mathcal{R}(D), \quad (9)$$

where

$$\mathcal{R}(D) = \begin{cases} H_2[\gamma(D)] + [1 - \gamma(D)] \log_2(m-1), & \text{when } D \in \left[1, \frac{4(m-1)}{m}\right]; \\ (D-m) \frac{\log_2(m-1)}{m-2} + \log_2 m, & \text{when } D \in \left[\frac{4(m-1)}{m}, m\right]. \end{cases} \quad (10)$$

$H_2(x) = -x \log_2 x - (1-x) \log_2(1-x)$, $\gamma(D) = [\sqrt{D} + \sqrt{(m-1)(m-D)}]^2/m^2$, $D = \max\{\langle\phi|\rho|\phi\rangle/|d_{\max}|^2, 1\}$, and $|\phi\rangle = \sum_{i=1}^m d_i|i\rangle$ is an arbitrary m -dimensional pure state with $|d_{\max}| = \max_i\{|d_i|\}$.

The proof is similar to Theorem 5, details are given in the Appendix B.

Note that it is not surprising that the form of the l_1 -norm bound is similar to the CREN bound. In Ref. [58], Zhu and colleagues suggested that for pure states the negativity of $\sum_j \sqrt{\lambda_j} |jj\rangle$ is equal to the l_1 -norm of coherence for $\sum_j \sqrt{\lambda_j} |j\rangle$, and the two measures share the same convex roof. The lower bounds for geometric measure of coherence is also similar to the lower bounds for geometric measure of entanglement. In Ref. [56], Winter and Yang pointed out that for a general state $\rho = \sum_{ij} \rho_{ij} |i\rangle\langle j|$, the coherence of formation is equal to the entanglement of formation of the corresponding maximally correlated state $\rho_{mc} = \sum_{ij} \rho_{ij} |ii\rangle\langle jj|$. Thus, the observable lower bounds are naturally extended from entanglement to coherence.

Similar to the entanglement witness, the concept of coherence witness has been proposed in Ref. [59]. From Theorems 5–7, all the observable lower bounds of coherence measures are related to the same coherence witness, that is, $W = |d_{\max}|^2 1 - |\phi\rangle\langle\phi|$, where $|\phi\rangle = \sum_{i=1}^m d_i|i\rangle$ is an arbitrary m -dimensional pure state with $|d_{\max}| = \max_i\{|d_i|\}$ and $\{|i\rangle\}$ is the reference basis. For all the incoherent states $\delta = \sum_i p_i |i\rangle\langle i|$, we have $\text{Tr}(W\delta) = |d_{\max}|^2 - \sum_i p_i |d_i|^2 \geq 0$. If there exists a state ρ such that $\text{Tr}(W\rho) < 0$, the state ρ must be a coherent state. Finally, concerning other coherence measures than those stated previously, a proposal has recently been made to estimate the relative entropy of coherence [62].

IV. EXAMPLES

We provide two examples to contrast our results with the results from genuine multiparticle negativity (GMN) in Ref. [63].

The GMN measurement is built on the analysis of bipartite decompositions of the multipartite state ρ . If we can find an entanglement witness W that is decomposable for all possible bipartite decompositions $\alpha|\bar{\alpha}$ and it can detect the state $\text{Tr}(\rho W) < 0$, the state ρ is a GME state. Here $W = P_\alpha + Q_\alpha^{T_\alpha}$ with positive operators P_α and Q_α . The GMN is defined as $\tilde{N}_g(\rho) = -\min \text{Tr}(\rho W)$, where $0 \leq P_\alpha \leq 1$ and $0 \leq Q_\alpha \leq 1$. Based on the GMN, Ref. [63] proposed the renormalized GMN, where the operator P is no longer bounded by 1. Interestingly, the renormalized GMN is equal to a mixed-state convex roof of bipartite negativity:

$$N_g(\rho) = \inf_{\{P_\alpha, Q_\alpha\}} \sum_\alpha P_\alpha N_\alpha(Q_\alpha), \quad (11)$$

where the summation runs over all possible decompo-

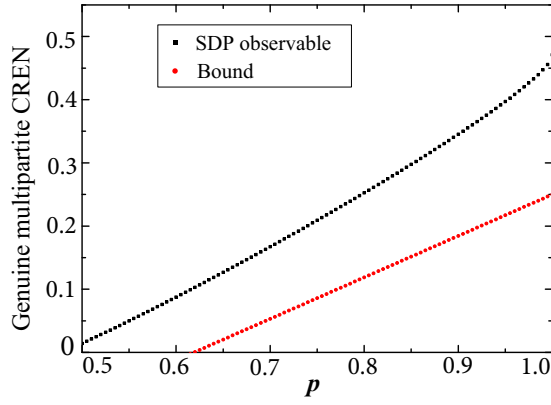


FIG. 3. Comparison of SDP results with the GME observable lower bound. The results of SDP and the GME observable lower bound are shown here. The black square dots denote the renormalized GMN of ρ_W by using SDP, where $\rho_W = p|W\rangle\langle W| + (1-p)1/8$. The red dots denote the GME observable lower bound of CREN for ρ_W based on Eq. (3), where we have used $|\phi\rangle = |W\rangle$ with $s'_1 = 2/3$. The mixed coefficient p is plotted on the horizontal axis whereas the entanglement measures are plotted on the vertical axis.

sitions $\alpha|\bar{\alpha}$ of the system and the minimization is performed over all mixed-state decompositions of the state $\rho = \sum_{\alpha} p_{\alpha} \rho_{\alpha}$. Different from the pure convex roof, such as the CREN, this definition is built on the mixed-state decomposition. The mixed-state decompositions also include the pure-state decompositions. Therefore the bound described by the mixed convex roof is a lower bound of the pure-state convex roof.

Example 1. Considering a n -qubit Greenberger-Horne-Zeilinger (GHZ) diagonal state ρ , its renormalized GMN $N_g(\rho)$ satisfies $N_g(\rho) = \max_i \{2F_i - 1, 0\}$ (we use a different prefactor compared with Ref. [63]), where $F_i = \langle \psi_i | \rho | \psi_i \rangle$ denotes the fidelity with the GHZ basis state $|\psi_i\rangle$.

Our bound on the CREN of GME can also give the same result. We adopt the GHZ basis state with the maximal fidelity as the observable state $|\phi\rangle$ then s_1 can only be the value $1/2$. Therefore, Eq. (3) reads as $\mathcal{N}(\rho) \geq \max\{2\langle \psi_i | \rho | \psi_i \rangle - 1, 0\}$. This lower bound is just equal to the analytical result in Ref. [63].

Example 2. Consider a 3-qubit W state with white noise $\rho_W = p|W\rangle\langle W| + (1-p)1/8$, where $|W\rangle = 1/\sqrt{3}(|001\rangle + |010\rangle + |100\rangle)$. We measure its entanglement with the renormalized GMN and CREN of GME. A semidefinite program (SDP) proposed in Ref. [63] is used here to calculate the renormalized GMN. The result is shown in Fig. 3. The GME lower bound is less than the SDP result for each p . The advantage of the GME lower bound is that we only need to measure the expectation value $\langle \phi | \rho | \phi \rangle$ rather than detect ρ itself.

The GMN in Ref. [63] needs quantum state tomography. In the following, we provide two examples using real experimental data in which quantum state tomography is usually impractical.

Example 3. We adopt three experimental results of multipartite systems [64–66] to calculate the GME lower bound. All the three experiments are experimental demonstrations of multipartite quantum entanglement. The produced states in these experiments achieve 6–10 photons and it is very hard to perform quantum tomography. In these experiments, coincidence counts measured in the $|H\rangle/|V\rangle$ basis and the fidelity with N -GHZ state is used to describe the produced states. In these cases, it is impossible to obtain exact results of most entanglement measures. Our method provides a reliable way to solve the question. From Table I, one can see that the genuine entanglement exists in the experimentally realized states of Refs. [64–66], quantified by most entanglement measures except the G-concurrence of GME. We can also estimate the coherence of the produced states with the lower bound.

Example 4. We consider the experiment of a genuine 12-qubit entanglement in a superconducting processor in Ref. [67]. Linear cluster states have been prepared from 4 to 12 qubits by implementing a set of controlled-phase gates on the superconducting quantum processor. As the quantum state tomography requires exponentially growing measurements with the number of qubits, the lower bounds of the state fidelities are measured with probability distributions, which are 0.9176(28), 0.9196(28), 0.8870(27), 0.8827(27), 0.8536(27), 0.7988(27), 0.7136(26), 0.5720(25), and 0.5544(25), from 4 to 12 qubits [67]. It is worth noting that the right-hand sides of Eqs. (3), (4), (5), and (6) are

TABLE I. Results of entanglement and coherence measures on N -photon entanglement states. The superscripts a, b, and c denote different data sources of Refs. [64–66], respectively. The fidelity is the result of overlap $\langle \phi | \rho | \phi \rangle$ with $|\phi\rangle = 1/\sqrt{2}(|H\rangle^{\otimes N} + |V\rangle^{\otimes N})$.

$\rho(N)$	6 ^a	8 ^a	8 ^b	8 ^c	10 ^a
Fidelity	0.710(16)	0.644(22)	0.610(26)	0.59(2)	0.573(23)
$\mathcal{N}_{\text{GME}}(\rho)$	0.420	0.288	0.220	0.18	0.146
$C_{\text{GME}}(\rho)$	0.0794	0.0263	0.0201	0.016	0.0066
$G_{\text{GME}}(\rho)$	0	0	0	0	0
$\mathcal{G}_{\text{GME}}(\rho)$	0.00540	0.00122	0.00073	0.00049	0.00016
$\tilde{C}_1(\rho)$	0.420	0.288	0.220	0.18	0.146
$C_g(\rho)$	5.85×10^{-4}	7.14×10^{-5}	4.29×10^{-5}	2.9×10^{-5}	4.86×10^{-6}
$C_f(\rho)$	0.0106	0.00165	0.00102	7.1×10^{-4}	1.41×10^{-4}

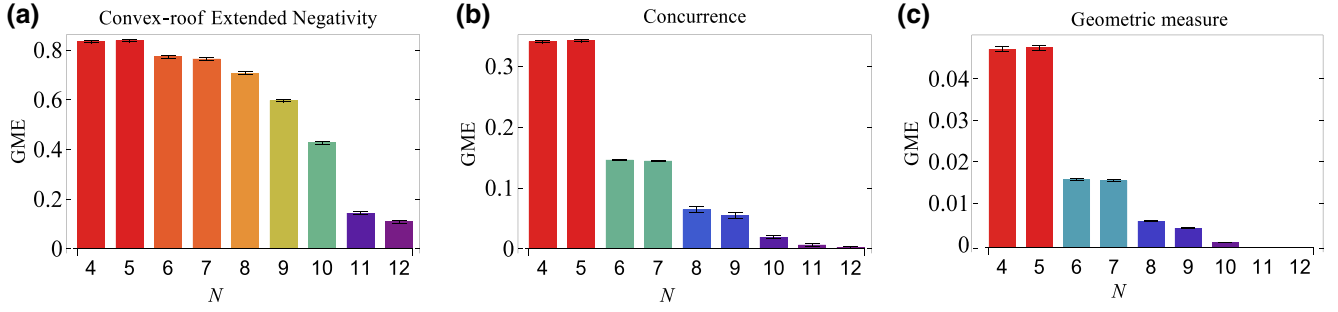


FIG. 4. Lower bounds of GME measures for linear cluster states. (a) The lower bound of the CREN of GME versus the qubit number from 4 to 12. (b) The lower bound of concurrence of GME versus the qubit number from 4 to 12. (c) The lower bound of the geometric measure of GME versus the qubit number from 4 to 12.

monotonically increasing functions of S . Thus, if we use a lower bound of fidelity in S , one can still obtain a valid lower bound of GME measure. In Fig. 4, the lower bounds of GME measures including CREN of GME, concurrence of GME and geometric measure of GME have been shown for the experimentally realized linear cluster states from 4 to 12 qubits.

V. DISCUSSION AND CONCLUSIONS

The observable lower bounds for the entanglement of bipartite states have been discussed in Ref. [68], and we have extended it to lower bounds for GME and coherence. Unlike the constant m and s_1 for bipartite states, m_α and $s_1^{(\alpha)}$ often change with different bipartite decompositions. For calculating GME lower bounds, the key point is to find the maximal m_α and $s_1^{(\alpha)}$ among all bipartite decompositions $\alpha|\bar{\alpha}$. Although the fidelity and Schmidt coefficients are related to the entanglement witness and the geometric measure of entanglement, our lower bounds of GME are substantially different from previous entanglement lower bounds.

The form of the lower bounds for coherence is very similar to the bounds for entanglement. This indicates a strong relation between entanglement and coherence. In certain cases, we can measure coherence just as measure entanglement. We find this relation in the l_1 -norm of coherence, the geometric measure of coherence, and the coherence of formation. It is an open question whether other coherence measures can apply for such an observable lower bound.

In conclusion, we have proposed a method to obtain lower bounds for some genuine entanglement and coherence measures defined by the convex roof construction. The entanglement measures include the CREN, the concurrence, the G-concurrence, and the geometric measure of entanglement, whereas coherence measures include the convex roof of l_1 -norm of coherence, the geometric measure of coherence, and the coherence of formation. The

lower bounds estimate these measures for arbitrary finite-dimensional multipartite states. Moreover, these lower bounds can be easily obtained from the fidelity between a chosen pure state $|\phi\rangle$ and the target state. For future work, it would be very interesting to extend our results to other figures of merit in quantum information, such as distillability rates, or the usefulness of a state for tasks such as teleportation or metrology.

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APPENDIX A: PROOFS OF THEOREMS 1–4

Theorem 1. For any N -partite state ρ , its CREN of GME satisfies

$$\mathcal{N}_{\text{GME}}(\rho) \geq S - 1, \quad (\text{A1})$$

where $S = \max\{\langle\phi|\rho|\phi\rangle/s_1', 1\}$. Here, $|\phi\rangle$ is an arbitrary pure state.

Proof: Assume that $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$ is the optimal decomposition for ρ to achieve the infimum of $\mathcal{N}_{\text{GME}}(\rho) = \inf_{\{p_i, |\psi_i\rangle\}} \sum_i p_i \mathcal{N}_{\text{GME}}(|\psi_i\rangle)$, and then $\mathcal{N}_{\text{GME}}(\rho) = \sum_j p_j \mathcal{N}_{\text{GME}}(|\psi_j\rangle)$. For all possible bipartitions $\alpha|\bar{\alpha}$, we can calculate the CREN of each $|\psi_j\rangle$: $\mathcal{N}_\alpha(|\psi_j\rangle) = \|\langle\psi_j|^{T_\alpha} - 1\| = \left[\sum_{i=1}^{m_\alpha} (\mu_i^{(j,\alpha)})^{1/2} \right]^2 - 1$, where $(\mu_i^{(j,\alpha)})^{1/2}$ are Schmidt coefficients of $|\psi_j\rangle$ in decreasing order under the bipartition $\alpha|\bar{\alpha}$, and $\|\cdot\|$ is the trace norm. Then we

have

$$\begin{aligned}\mathcal{N}_{\text{GME}}(|\psi_j\rangle) &= \min_{\alpha} \mathcal{N}_{\alpha}(|\psi_j\rangle) \geq \min_{\alpha} \frac{\langle\phi|\psi_j\rangle\langle\psi_j|\phi\rangle}{s_1^{(\alpha)}} - 1 \\ &\geq \frac{\langle\phi|\psi_j\rangle\langle\psi_j|\phi\rangle}{s'_1} - 1,\end{aligned}\quad (\text{A2})$$

where we have used the inequality

$$\left(\sum_{i=1}^{m_{\alpha}} \sqrt{\mu_i^{(j,\alpha)}}\right)^2 \geq \frac{\langle\phi|\psi_j\rangle\langle\psi_j|\phi\rangle}{s_1^{(\alpha)}}, \quad (\text{A3})$$

which can be proved according to Refs. [68,69]. Then we can find the GME lower bound

$$\mathcal{N}_{\text{GME}}(\varrho) = \sum_j p_j \mathcal{N}_{\text{GME}}(|\psi_j\rangle) \geq S - 1. \quad (\text{A4})$$

Thus, Theorem 1 is proved. ■

Theorem 2. For any N -partite state ϱ , its GME lower bound of concurrence satisfies

$$C_{\text{GME}}(\varrho) \geq \sqrt{\frac{2}{m'(m'-1)}}(S-1), \quad (\text{A5})$$

where $S = \max\{\langle\phi|\varrho|\phi\rangle/s'_1, 1\}$.

Proof: Similarly, suppose that $\varrho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$ is the optimal decomposition for ϱ to achieve the infimum of $C_{\text{GME}}(\varrho) = \inf_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C_{\text{GME}}(|\psi_i\rangle)$, and then $C_{\text{GME}}(\varrho) = \sum_j p_j C_{\text{GME}}(|\psi_j\rangle)$. For the bipartite decomposition $\alpha|\bar{\alpha}$, we can calculate the concurrence $C_{\alpha}(|\psi_j\rangle) = \sqrt{2\left[1 - \sum_{i=1}^{m_{\alpha}} (\mu_i^{(j,\alpha)})^2\right]}$ with $(\mu_i^{(j,\alpha)})^{1/2}$ being Schmidt coefficients of $|\psi_j\rangle$ in decreasing order under the bipartition $\alpha|\bar{\alpha}$. Thus,

$$\begin{aligned}C_{\text{GME}}(|\psi_j\rangle) &= \min_{\alpha} C_{\alpha}(|\psi_j\rangle) \\ &\geq \min_{\alpha} \sqrt{\frac{2}{m_{\alpha}(m_{\alpha}-1)}} \left(\frac{\langle\phi|\psi_j\rangle\langle\psi_j|\phi\rangle}{s_1^{(\alpha)}} - 1 \right) \\ &\geq \sqrt{\frac{2}{m'(m'-1)}} \left(\frac{\langle\phi|\psi_j\rangle\langle\psi_j|\phi\rangle}{s'_1} - 1 \right),\end{aligned}\quad (\text{A6})$$

where the first inequality is proved in Ref. [68]. Therefore,

$$C_{\text{GME}}(\varrho) = \sum_j p_j C_{\text{GME}}(|\psi_j\rangle) \geq \sqrt{\frac{2}{m'(m'-1)}}(S-1). \quad (\text{A7})$$

Thus, Theorem 2 has been proved. ■

Theorem 3. For any N -partite state ϱ , its G -concurrence of GME satisfies

$$G_{\text{GME}}(\varrho) \geq 1 - m' + S, \quad (\text{A8})$$

where $S = \max\{\langle\phi|\varrho|\phi\rangle/s'_1, 1\}$.

Proof: Similarly, suppose that $\varrho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$ is the optimal decomposition for ϱ to achieve the infimum of $G_{\text{GME}}(\varrho) = \inf_{\{p_i, |\psi_i\rangle\}} \sum_i p_i G_{\text{GME}}(|\psi_i\rangle)$ and, thus, $G_{\text{GME}}(\varrho) = \sum_j p_j G_{\text{GME}}(|\psi_j\rangle)$. For the bipartite decomposition $\alpha|\bar{\alpha}$, one obtains $G_{\alpha}(|\psi_j\rangle) = m_{\alpha} \left(\prod_{i=1}^{m_{\alpha}} \mu_i^{(j,\alpha)} \right)^{1/m_{\alpha}}$. Similar to the proofs of Theorem 1 and 2, we can find the inequality following the proof of bipartite systems [68]:

$$\begin{aligned}G_{\text{GME}}(|\psi_j\rangle) &= \min_{\alpha} G_{\alpha}(|\psi_j\rangle) \\ &\geq \min_{\alpha} \left(1 - m_{\alpha} + \frac{\langle\phi|\psi_j\rangle\langle\psi_j|\phi\rangle}{s_1^{(\alpha)}} \right) \\ &\geq 1 - m' + \frac{\langle\phi|\psi_j\rangle\langle\psi_j|\phi\rangle}{s'_1}.\end{aligned}\quad (\text{A9})$$

Therefore, we obtain the lower bound,

$$G_{\text{GME}}(\varrho) = \sum_j p_j G_{\text{GME}}(|\psi_j\rangle) \geq 1 - m' + S, \quad (\text{A10})$$

where $m' = \max_{\alpha} \{m_{\alpha}\}$ and $s'_1 = \max_{\alpha} \{s_1^{(\alpha)}\}$.

Before proving Theorem 4, let us prove a lemma first. ■

Lemma 1. The function $\gamma(S) = [\sqrt{S/m'} + \sqrt{(m'-1)(1-S/m')}]^2/m'$ with $S = \max\{\langle\phi|\varrho|\phi\rangle/s'_1, 1\}$ is a monotone increasing function of s'_1 and m' .

Proof: First, we prove that γ is a monotone increasing function of s'_1 . For simplicity, let $S = \max\{\langle\phi|\varrho|\phi\rangle/s'_1, 1\}$. We can find that

$$\begin{aligned}\frac{\partial \gamma}{\partial S} &= \left[\sqrt{\left(1 - \frac{S}{m'}\right)(m'-1)} + \sqrt{\frac{S}{m'}} \right] \\ &\quad \left(\frac{1 - m'}{\sqrt{\left(1 - \frac{S}{m'}\right)(m'-1)}} + \sqrt{\frac{m'}{S}} \right) / m'^2.\end{aligned}\quad (\text{A11})$$

For determining the monotone of $\gamma(S)$, we only need to discuss $(1 - m')/\sqrt{(1 - S/m')(m'-1)} + \sqrt{m'/S}$. After simplifying the function, the condition of $\partial \gamma / \partial S < 0$ is $S > 1$. Therefore, γ is a monotone decreasing function of S , that is, a monotone increasing function of s'_1 .

Then we prove that γ is a monotone increasing function of m' . Similarly, we calculate the partial derivative of γ with respect to m'

$$\frac{\partial \gamma}{\partial m'} = \left[\sqrt{\left(1 - \frac{S}{m'}\right)(m' - 1)} + \sqrt{\frac{S}{m'}} \right] \left(\frac{\frac{S}{m'}(m' - 2) + 1}{\sqrt{\left(1 - \frac{S}{m'}\right)(m' - 1)}} - 2\sqrt{\frac{S}{m'}} \right) / m'^2. \quad (\text{A12})$$

As the above proof, we only consider $\left\{ \left[\left(\frac{S}{m'} \right) (m' - 2) + 1 \right] / \sqrt{\left(1 - \frac{S}{m'}\right)(m' - 1)} \right\} - 2\sqrt{S/m'}$. After simplifying, the condition of $\partial \gamma / \partial m' > 0$ is $(S - 1)^2 > 0$. Therefore, γ is a monotone increasing function of m' . ■

Theorem 4. For an arbitrary N -partite state ϱ , its geometric measure of GME $\mathcal{G}(\varrho)$ satisfies

$$\mathcal{G}_{\text{GME}}(\varrho) \geq 1 - \gamma(S), \quad (\text{A13})$$

where $\gamma(S) = [\sqrt{S} + \sqrt{(m' - 1)(m' - S)}]^2 / m'^2$ with $S = \max\{\langle \phi | \varrho | \phi \rangle / s'_1, 1\}$.

Proof: Similarly, suppose that $\varrho = \sum_j p_j |\psi_j\rangle \langle \psi_j|$ is the optimal decomposition for ϱ to achieve the infimum of $\mathcal{G}_{\text{GME}}(\varrho) = \inf_{\{p_i, |\psi_i\rangle\}} \sum_i p_i \mathcal{G}_{\text{GME}}(|\psi_i\rangle)$. Thus, $\mathcal{G}_{\text{GME}}(\varrho) = \sum_j p_j \mathcal{G}_{\text{GME}}(|\psi_j\rangle)$. For the bipartition $\alpha | \bar{\alpha}$, one obtains $\mathcal{G}_\alpha(|\psi_j\rangle) = 1 - \mu_{\max}^{(j, \alpha)}$ where $\mu_{\max}^{(j, \alpha)} = \max_i \{\mu_i^{(j, \alpha)}\}$. Therefore,

$$\begin{aligned} \mathcal{G}_{\text{GME}}(|\psi_j\rangle) &= \min_{\alpha} \mathcal{G}_{\alpha}(|\psi_j\rangle) \\ &\geq 1 - \min_{\alpha} \frac{\left[\sqrt{S_{\alpha}^{(j)}} + \sqrt{(m_{\alpha} - 1)(m_{\alpha} - S_{\alpha}^{(j)})} \right]^2}{m_{\alpha}^2} \\ &\geq 1 - \frac{\left[\sqrt{S^{(j)}} + \sqrt{(m' - 1)(m' - S^{(j)})} \right]^2}{m'^2}, \end{aligned} \quad (\text{A14})$$

where the first inequality is proved in Ref. [68], $S_{\alpha}^{(j)} = \max\{\langle \phi | \psi_j \rangle \langle \psi_j | \phi \rangle / s'_1, 1\}$ and $S^{(j)} = \max\{\langle \phi | \psi_j \rangle \langle \psi_j | \phi \rangle / s'_1, 1\}$. In Lemma 1, we have proved that $\gamma(S)$ is a monotone increasing function of m' and s'_1 . Therefore, the last inequality of Eq. (A14) holds. In Ref. [68], Zhang and colleagues proved that $1 - \gamma(S)$ is a convex function, so

$1 - \sum_j p_j \gamma(S^{(j)}) \geq 1 - \gamma(\sum_j p_j S^{(j)})$. Thus,

$$\begin{aligned} \mathcal{G}_{\text{GME}}(\varrho) &= \sum_j p_j \mathcal{G}_{\text{GME}}(|\psi_j\rangle) \geq 1 - \sum_j p_j \gamma(S^{(j)}) \\ &\geq 1 - \gamma\left(\sum_j p_j S^{(j)}\right) = 1 - \gamma(S). \end{aligned} \quad (\text{A15})$$

■

APPENDIX B: PROOFS OF THEOREMS 5–7

Theorem 5. For an arbitrary state ϱ in an m -dimensional system, the convex roof of the l_1 -norm of coherence satisfies

$$\tilde{C}_{l_1}(\varrho) \geq D - 1, \quad (\text{B1})$$

where $D = \max\{\langle \phi | \varrho | \phi \rangle / |d_{\max}|^2, 1\}$, and $|\phi\rangle = \sum_{i=1}^m d_i |i\rangle$ is an arbitrary m -dimensional pure state with $|d_{\max}| = \max_i \{|d_i|\}$.

Proof: Similarly, one can assume that $\varrho = \sum_j p_j |\psi_j\rangle \langle \psi_j|$ is the optimal decomposition for ϱ to achieve the infimum of $\tilde{C}_{l_1}(\varrho) = \inf_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C_{l_1}(|\psi_i\rangle)$, where $|\psi_j\rangle = \sum_k a_k^{(j)} |k\rangle$. Therefore, $\tilde{C}_{l_1}(\varrho) = \sum_j p_j C_{l_1}(|\psi_j\rangle)$. Each $C_{l_1}(|\psi_j\rangle)$ can be calculated as follows

$$\begin{aligned} C_{l_1}(|\psi_j\rangle) &= \sum_{l \neq k} |a_l^{(j)}| |a_k^{(j)}| = \left(\sum_k |a_k^{(j)}| \right)^2 - 1 \\ &\geq \frac{\left(\sum_k |d_k| |a_k^{(j)}| \right)^2}{|d_{\max}|^2} - 1 \geq \frac{\langle \phi | \psi_j \rangle \langle \psi_j | \phi \rangle}{|d_{\max}|^2} - 1. \end{aligned} \quad (\text{B2})$$

Therefore, one has

$$\tilde{C}_{l_1}(\varrho) \geq \frac{\sum_j p_j \langle \phi | \psi_j \rangle \langle \psi_j | \phi \rangle}{|d_{\max}|^2} - 1 = D - 1. \quad (\text{B3})$$

Therefore, we have proved Theorem 5. ■

Theorem 6. For an arbitrary state ϱ in an m -dimensional system, its geometric measure of coherence satisfies

$$C_g(\varrho) \geq 1 - \gamma(D), \quad (\text{B4})$$

where $\gamma(D) = [\sqrt{D} + \sqrt{(m - 1)(m - D)}]^2 / m^2$, $D = \max\{\langle \phi | \varrho | \phi \rangle / |d_{\max}|^2, 1\}$, and $|\phi\rangle = \sum_{i=1}^m d_i |i\rangle$ is an arbitrary m -dimensional pure state with $|d_{\max}| = \max_i \{|d_i|\}$.

Proof: Similarly, suppose that $\varrho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$ is the optimal decomposition for ϱ to achieve the infimum of $\inf_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C_g(|\psi_i\rangle)$. Thus, $C_g(\varrho) = \sum_j p_j C_g(|\psi_j\rangle)$. Similar to the proof of lower bound for geometric measure of entanglement in Ref. [68], each $C_g(|\psi_j\rangle)$ can be calculated as $C_g(|\psi_j\rangle) \geq 1 - \gamma(D^{(j)})$, where $D^{(j)} = \max\{|\langle\phi|\psi_j\rangle\langle\psi_j|\phi\rangle|/|d_{\max}|^2, 1\}$. As proved in Ref. [68], $1 - \gamma(D)$ is a convex function. Thus,

$$\begin{aligned} C_g(\varrho) &= \sum_j p_j C_g(|\psi_j\rangle) \geq 1 - \sum_j p_j \gamma(D^{(j)}) \\ &\geq 1 - \gamma\left(\sum_j p_j D^{(j)}\right) = 1 - \gamma(D). \end{aligned} \quad (\text{B5})$$

■

Theorem 7. For an arbitrary state ϱ in an m -dimensional system, its coherence of formation satisfies

$$C_f(\varrho) \geq \mathcal{R}(D), \quad (\text{B6})$$

where

$$\mathcal{R}(D) = \begin{cases} H_2[\gamma(D)] + [1 - \gamma(D)] \log_2(m-1), & \text{when } D \in \left[1, \frac{4(m-1)}{m}\right]; \\ (D-m) \frac{\log_2(m-1)}{m-2} + \log_2 m, & \text{when } D \in \left[\frac{4(m-1)}{m}, m\right]. \end{cases} \quad (\text{B7})$$

Here $H_2(x) = -x \log_2 x - (1-x) \log_2(1-x)$, $\gamma(D) = [\sqrt{D} + \sqrt{(m-1)(m-D)}]^2/m^2$, $D = \max\{|\langle\phi|\varrho|\phi\rangle|/|d_{\max}|^2, 1\}$, and $|\phi\rangle = \sum_{i=1}^m d_i |i\rangle$ is an arbitrary m -dimensional pure state with $|d_{\max}| = \max_i\{|d_i|\}$.

Proof: Similarly, suppose that $\varrho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$ is the optimal decomposition for ϱ to achieve the infimum of $\inf_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C_f(|\psi_i\rangle)$. Thus, $C_f(\varrho) = \sum_j p_j C_f(|\psi_j\rangle)$. Similar to the proof of the lower bound for entanglement of formation in Ref. [68], each $C_f(|\psi_j\rangle)$ can be calculated as $C_f(|\psi_j\rangle) \geq \mathcal{R}(D^{(j)})$, where $D^{(j)} = \max\{|\langle\phi|\psi_j\rangle\langle\psi_j|\phi\rangle|/|d_{\max}|^2, 1\}$. As proved in Ref. [68], $\mathcal{R}(D)$ is a convex function. Thus,

$$\begin{aligned} C_f(\varrho) &= \sum_j p_j C_f(|\psi_j\rangle) \geq \sum_j p_j \mathcal{R}(D^{(j)}) \\ &\geq \mathcal{R}\left(\sum_j p_j D^{(j)}\right) = \mathcal{R}(D), \end{aligned} \quad (\text{B8})$$

where $D = \max\{|\langle\phi|\varrho|\phi\rangle|/|d_{\max}|^2, 1\}$. ■

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