Theoretical and simulation study of dispersive electron cooling

He Zhao,[1,2,*] Lijun Mao,[1,2] Meitang Tang,[1] Fu Ma,[1]
Xiaodong Yang,[1,2] and Jiancheng Yang[1,2]
[1]*Institute of Modern Physics, Chinese Academy of Sciences, Lanzhou 730000, China*
[2]*University of Chinese Academy of Sciences, Beijing 100049, China*

(Received 23 August 2023; accepted 22 February 2024; published 7 March 2024)

In electron cooling, the transverse cooling rate is usually smaller than the longitudinal rate, especially at high energies. By introducing dispersive cooling, it is possible to redistribute the cooling rate between longitudinal and transverse planes. Theoretically, achieving dispersive electron cooling requires an ion dispersion and a transverse gradient of longitudinal friction force. The latter depends on many factors such as the relative momentum offset, transverse displacement, e-beam density distribution, and space charge effect. Therefore, several methods can be employed to achieve dispersive electron cooling based on these factors. Based on the dc electron beam, these factors and their respective impacts on the cooling rate are discussed and analyzed. For the first time, we propose a new mechanism to achieve dispersive cooling for a uniform electron beam by placing part of the ion beam outside of the electron beam. Based on a linear friction force model, we propose a simple formula to numerically estimate the cooling rate redistribution effect of these methods. The analytical results are in good agreement with Monte Carlo calculation and numerical simulation.

DOI: 10.1103/PhysRevAccelBeams.27.033501

I. INTRODUCTION

Electron cooling has become one of the most important methods to reduce beam emittance and momentum spread, and it has been widely applied in several proton and ion storage rings [1–9]. In the future, it is desirable to extend this method to high-energy facilities, such as Electron-Ion Colliders [10,11], where an electron beam of at least tens of MeV is required. Usually, the transverse momenta of the electron beam in the particle reference frame is larger than the longitudinal momenta, resulting in weaker transverse cooling, especially at high energies. Magnetized cooling is a useful method to compensate for this. Due to the Larmor motion in the longitudinal magnetic field, the effective temperature of the electron beam is determined by its longitudinal velocity spread, thereby increasing the transverse cooling rate to the same as the longitudinal one [12]. Another method is the so-called dispersive electron cooling, which redistributes the cooling rates by introducing a dispersion function, i.e., increasing the cooling rate in one direction at the expense of the other [13].

In theory, dispersive cooling requires both ion beam dispersion and a transverse gradient of the longitudinal friction force. To simply explain that, we assume an off-momentum particle passing through the cooling section with a dispersion function D and only consider the longitudinal cooling with a linear friction force $\Delta\delta_p = -\lambda\delta_p$, the particle coordinate after cooling can be written as

$$\boldsymbol{x}_{\beta 2} = \boldsymbol{x} - D\delta_{p2} = \boldsymbol{x}_{\beta 1} + D\lambda\delta_{p1}, \qquad (1)$$

where $\boldsymbol{x}_\beta$ denotes the betatron oscillation with amplitude A_x, and $\boldsymbol{x}$ is the real coordinate which is assumed to be unchanged during passing through the cooling section. If the cooling coefficient λ is constant, the dispersion term in Eq. (1) is uncorrelated with the betatron oscillation. As a result, the center of the betatron oscillation shifts with longitudinal cooling, but the amplitude remains unchanged $A_{x2} = A_{x1}$. This means that longitudinal cooling does not contribute to transverse cooling. If the longitudinal friction force has a transverse gradient, i.e., coupled to the betatron oscillation, it is possible to redistribute the cooling rate between the two directions. For example, assuming a negative gradient $\lambda(x) = (M - |\boldsymbol{x}|)\lambda_0$ with $M > Max[\boldsymbol{x}]$, which means that particles near the center are subject to larger friction force. Then, the amplitude of the betatron oscillation turns to $A_{x2} \simeq (1 - \lambda_0|D\delta_{p1}|)A_{x1}$. It indicates the amplitude damping of the betatron motion, which exactly results from longitudinal cooling. A numerical result of these two processes based on Eq. (1) is shown in Fig. 1,

[*]hezhao@impcas.ac.cn

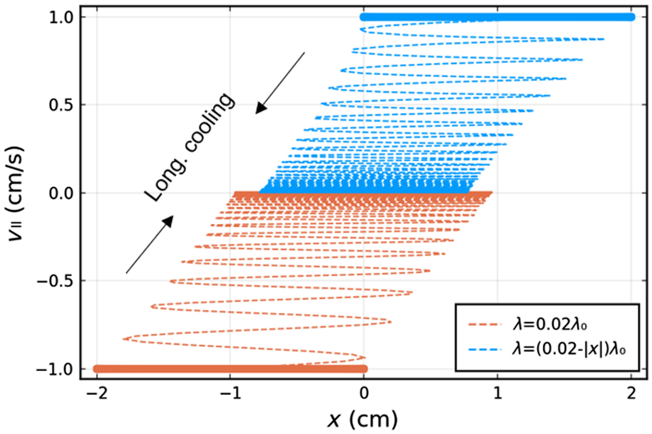

FIG. 1. Comparison of the cooling process with and without the transverse gradient of longitudinal friction force, where the x axis represents betatron oscillation under longitudinal cooling in the y axis. It demonstrates that dispersion and the transverse gradient are necessary for dispersive cooling ($x_\beta = 1$ cm, $D = 1$ m, $\delta_p = \pm 1 \times 10^{-2}$, $\lambda_0 = 1$).

where the x axis represents betatron oscillation under the longitudinal cooling with and without the transverse gradient. It clearly shows the amplitude damping process (dispersive cooling) with an appropriate longitudinal friction force setting. The mechanism is similar to the radiation decrements partitioning in electron storage rings. Particles with large transverse displacement emit more radiation power, indicating a positive damping gradient. As a result, the longitudinal damping increases at the expense of the horizontal damping, which is represented by the damping repartition number $\mathcal{D}$ [14].

We see that the transverse gradient of the longitudinal friction force plays a key role in dispersive electron cooling. Some experimental and simulation studies have demonstrated several approaches to obtain this gradient. As indicated in Refs. [15,16], one approach is to introduce a displacement between electron and ion beams and utilize the parabolic velocity profile of the e-beam that is caused by its space charge. Another method is by using a relative momentum offset (or energy offset) compared to the central momentum, a displacement, and a transverse density gradient of the e-beam [17]. Recently, it has been demonstrated that an e-beam with Gaussian transverse distribution can naturally provide this transverse gradient, thus achieving dispersive cooling. At the same time, it shows that electron dispersion is also beneficial to dispersive cooling [18]. Generally, these methods can be applied to both magnetized and nonmagnetized cooling since the magnetic field does not affect the transverse gradient. In this article, we conduct theoretical and simulation studies of these methods and show how they affect the cooling rate. For the first time, we propose a new mechanism to achieve dispersive cooling for a uniform electron beam by placing part of the ion beam outside of the electron beam. Considering a dc e-beam and linear friction force model, we finally propose a simple formula to numerically estimate the cooling rate redistribution effect of these methods, and the analytical result agrees well with Monte Carlo calculation and numerical simulation.

This paper is organized as follows: In Sec. II, the friction force is briefly described, and the space charge effect of the e-beam and its influences on the electron velocity distribution are introduced. In Sec. III, a theoretical study of dispersive cooling is presented based on a linear cooling model. Meanwhile, four different cases that can be used to achieve dispersive cooling are discussed and analyzed. In Sec. IV, a numerical simulation is carried out and compared with the analytical model. Finally, the summary and discussion are presented in Sec. V.

II. FRICTION FORCE AND E-BEAM SPACE CHARGE

When an ion moves in the accompanying electron beam with the same average velocity, it experiences the friction force resulting from the Coulomb interaction with electrons. In the particle reference frame (PRF), the nonmagnetized friction force on an ion is [19]

$$\boldsymbol{F}(\boldsymbol{r}, \boldsymbol{u_i}) = -4\pi m_e r_e^2 Z^2 c^4 \int \ln\Lambda \frac{\boldsymbol{u_i} - \boldsymbol{u_e}}{|\boldsymbol{u_i} - \boldsymbol{u_e}|^3} f_e(\boldsymbol{r}, \boldsymbol{u_e}) d\boldsymbol{u_e}, \tag{2}$$

where Z is the atomic number of the ion, m_e is the electron mass, r_e is the classical electron radius, c is the speed of light, $\ln\Lambda$ is the Coulomb logarithm, $\boldsymbol{u_i}$ and $\boldsymbol{u_e}$ are the velocities of ion and electron in PRF, respectively. For small relative velocities, a linear friction force proportional to the velocity can be used, and in some cases, an analytical formula can be obtained. For example, considering an isotropic e-beam velocity distribution (Gaussian) and only working on the leading order of the distribution, the friction force can be simplified to

$$\boldsymbol{F} \simeq K n_e(\boldsymbol{r}) \boldsymbol{u_i} \quad (|\boldsymbol{u_i}| < \sigma_{ve}), \tag{3}$$

where $K = -4\sqrt{2\pi} m_e r_e^2 Z^2 c^4 \ln\Lambda / 3\sigma_{ve}^3$, and σ_{ve} is the rms velocity spread of e-beam. Then the cooling effect can be described by $\Delta\boldsymbol{u_i} = -C n_e \boldsymbol{u_i}$, where $C = KL/\beta\gamma m_i c$, β and γ are the Lorentz factors, and L is the length of the cooling section [18]. On the other hand, magnetized cooling is more desirable for improving the cooling efficiency in conventional electron coolers, for which a semiempirical formula of the friction force is proposed by Parkhomchuk [20]:

$$\boldsymbol{F_m} = -4 n_e m_e Z^2 r_e^2 c^4 \ln\Lambda \frac{\boldsymbol{u_i}}{(u_i^2 + u_{\rm eff}^2)^{3/2}}, \tag{4}$$

where the Coulomb logarithm $\ln\Lambda$ and the effective velocity $u_{\rm eff}$ of electrons depend on the e-beam distribution and the longitudinal magnetic field. Expanding the formula to the first order, we can also estimate the cooling effect by linear friction force $\Delta \boldsymbol{u_i} = -Cn_e\boldsymbol{u_i}$, where $C = -4m_eLZ^2r_e^2c^3 \ln\Lambda/\beta\gamma m_i u_{\mathrm{eff}}^3$. For a well-cooled ion beam, most ions fall into the linear friction force region due to the small velocity spread. Therefore, the cooling process can be well and simply described using the linear friction force, as we will do in the next section. However, it is difficult to obtain an analytical solution of the cooling coefficient C for an arbitrary e-beam velocity distribution. Accurate result still needs to be calculated through numerical integration.

In addition, the space charge effect of the e-beam will cause a transverse drift velocity and a relative momentum deviation along the beam radius, especially at low energy. On the one hand, it affects the cooling process because it introduces an undesired velocity offset between electrons and ions. On the other hand, this effect is critical in dispersive electron cooling because it is one of the conditions to obtain the transverse gradient of longitudinal friction force. Before going through the study of dispersive cooling, we first introduce the space charge effect of the e-beam.

According to the e-beam density distribution, the space charge field $\boldsymbol{E}_{\mathrm{sc}}$ can be easily calculated. Also, a longitudinal magnetic field $\boldsymbol{B}_g$ is usually applied in an electron cooler, which is essential for the e-beam adiabatic expansion [8], e-beam focusing [9], magnetized cooling, etc. As a consequence, an azimuthal drift velocity v_{drift} will be generated by these two fields, i.e., $\boldsymbol{E}\times\boldsymbol{B}$ drift. Moreover, due to the space charge potential depression, electrons inside the beam have a radial-dependant longitudinal velocity Δv_s after being accelerated by the electric field [21]. Here, we assume a round dc electron beam, these effects in PRF can be described as follows:

$$\begin{aligned}\boldsymbol{E}_{\mathrm{sc}}(r) &= \frac{\int_0^r x\rho(x)dx}{\varepsilon_0 r}\hat{\boldsymbol{r}}\\ v_{\mathrm{drift}}(r) &= \frac{\boldsymbol{E}_{\mathrm{sc}}\times\boldsymbol{B}_g}{\boldsymbol{B}_g^2} = \frac{\int_0^r x\rho(x)dx}{\varepsilon_0 r B_g}\hat{\boldsymbol{r}}\times\hat{\boldsymbol{s}}\\ \delta_e(r) &= \gamma^2\frac{\Delta v_s(r)}{v_0} = \frac{1}{\gamma\beta^2E_0}\int_0^r E_{\mathrm{sc}}(r)dr, \end{aligned} \tag{5}$$

where $\rho(r)$ is the electron beam distribution in the transverse direction, ϵ_0 is the vacuum permittivity, E_0 is the rest energy of the electron, v_{drift} is the radial drift velocity, δ_e is the momentum deviation in the longitudinal direction, and v_0 is the reference particle velocity. The e-beam space charge can also introduce a tune spread and may drive resonances in the ion beam. In this paper, we do not include this effect because it is negligible as long as the e-beam current is not too large, which is generally true for a typical cooling process [22,23].

Based on Eq. (5), velocity distributions of the e-beam with various transverse profiles can be calculated. In this paper, we only discuss the dc e-beam with a Gaussian and uniform profile, for which Eq. (5) can be solved analytically [24]. Considering a Gaussian e-beam with $\rho(r) = I_e e^{-r^2/2\sigma_r^2}/2\pi\gamma\beta c\sigma_r^2$, we have the two velocity distributions inside the e-beam

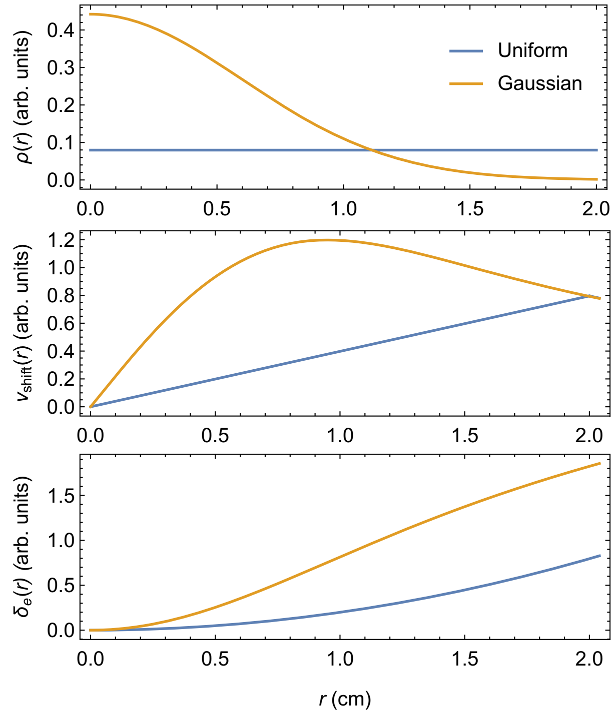

FIG. 2. Comparison of the space charge induced velocity distribution of two e-beam profiles under the same current: $r_0 = 0.0$ cm and $\sigma_r = 0.6$ cm for Gaussian beam, and $R_e = 2$ cm for uniform beam.

$$\begin{aligned}v_{\mathrm{drift}}(r) &= \frac{I_e}{2\pi\epsilon_0\gamma\beta c B_g r}\left(1-e^{-\frac{r^2}{2\sigma_r^2}}\right)\\ \delta_e(r) &= \frac{I_e}{4\pi\epsilon_0E_0\gamma^2\beta^3c}\left[\gamma_e + \ln\left(\frac{r^2}{2\sigma_r^2}\right) + E_1\left(\frac{r^2}{2\sigma_r^2}\right)\right], \end{aligned}\tag{6}$$

and for a uniform e-beam with $\rho(r) = I_e/\pi\gamma\beta c R_e^2$, we have

$$\begin{aligned}v_{\mathrm{drift}}(r) &= \frac{I_e r}{2\pi\epsilon_0\gamma\beta c B_g R_e^2}\\ \delta_e(r) &= \frac{I_e r^2}{4\pi\epsilon_0E_0\gamma^2\beta^3 c R_e^2}, \end{aligned}\tag{7}$$

where I_e is the current in the laboratory frame, R_e is the beam radius, $\gamma_e \approx 0.57721566$ is the Euler's constant, and $E_1(z) = \int_z^\infty e^{-t}/t dt$ is the exponential integral. A comparison of the two e-beams under at same current is shown in Fig. 2. In the next section, we will discuss dispersive cooling of Gaussian and uniform e-beams, respectively. For simplicity, we assume $v_{\rm drift} = L_{\rm sc} r$ and $\delta_e = K_{\rm sc} r^2$ for both e-beams, where the coefficients $L_{\rm sc}$ and $K_{\rm sc}$ are obtained by numerical fitting according to Eqs. (6) and (7).

As discussed in Ref. [15,16], the transverse gradient of longitudinal friction force can be achieved by a radial displacement of the e-beam, which takes advantage of the velocity distribution that is determined by $\delta_e(r)$. However, this displacement will directly introduce a coherent tune shift. Additionally, it will consequently produce an average velocity offset both in transverse and longitudinal directions, which may lead to a beam circular attractor in phase space and even the anticooling effect [25]. In this paper, our focus remains exclusively on dispersive cooling, other effects arising from beam displacement will not be discussed.

III. DISPERSIVE ELECTRON COOLING

In this section, we discuss several methods to achieve dispersive electron cooling and develop a simple analytical tool to estimate the cooling rate redistribution effect. To begin with, we assume a linear friction force $\Delta \boldsymbol{u} = -C n_e \boldsymbol{u}$ both in transverse and longitudinal directions, where n_e is the electron beam density, and C is the cooling coefficient that depends on the velocity distribution of the electron beam. Consider a beam displacement x_o, a relative momentum offset δ_o, and horizontal dispersion D of ions in the cooling section, the momentum change of a single particle after cooling can be described by $\Delta\delta \simeq -C_p n_e(\delta - \delta_e - \delta_o)$, where $\delta_e = K_{\rm sc}(x^2 + y_\beta^2)$ is the electron momentum deviation due to space charge and $x = x_\beta + x_o + D\delta$. Then we have

$$\Delta\delta^2 \simeq -2C_p n_e \delta^2 + 2C_p n_e \delta\delta_o + 2C_p n_e K_{\rm sc}\delta(x^2 + y_\beta^2). \quad (8)$$

For transverse, we only discuss the horizontal direction. Assume $\alpha = 0$ and ignore the betatron evolution in the cooling section, the single particle emittance is $\epsilon_x = (x - x_o - D\delta)^2/2\beta_x + \beta_x x'^2/2$, and the cooling effect can be written by

$$\Delta\epsilon_x \simeq -D x_\beta \Delta\delta/\beta_x + \beta_x x' \Delta x', \quad (9)$$

where $\Delta x' = -C_x n_e(x' - x'_e)$, and $x'_e = -L_{\rm sc} y_\beta \hat{\boldsymbol{x}}$ is the drift velocity caused by the space charge and magnetic fields. Note that we think the e-beam space charge is not very strong, so the drift velocity does not affect the assumption of the linear friction force.

Expanding Eqs. (8) and (9) and ignoring high-order and noncorrelated terms, the longitudinal and horizontal cooling effects of the ion beam can be described as

$$\begin{aligned}
\langle\Delta\delta^2\rangle &= -2C_p\langle n_e\delta^2\rangle + 2C_p\delta_o\langle n_e\delta\rangle \\
&\quad + 2C_p K_{\rm sc} x_o (x_o\langle n_e\delta\rangle + 2\langle n_e x_\beta\delta\rangle + 2D\langle n_e\delta^2\rangle) \\
\langle\Delta\epsilon\rangle &= -C_x\epsilon_0\langle n_e\rangle + \frac{C_p D}{\beta_x}\langle n_e x_\beta\delta\rangle - \frac{C_p D\delta_o}{\beta_x}\langle n_e x_\beta\rangle \\
&\quad - \frac{C_p D K_{\rm sc} x_o}{\beta_x}(x_o\langle n_e x_\beta\rangle + 2\langle n_e x_\beta^2\rangle + 2D\langle n_e x_\beta\delta\rangle),
\end{aligned} \quad (10)$$

where $\langle\rangle$ denotes averaging over the ion beam phase space and $n_e(x, y, s)$ is the local density of e-beam at the location of an ion particle. Since the e-beam density n_e is uncorrelated with x', so here we have $\beta_x\langle n_e x'^2\rangle = \epsilon_0\langle n_e\rangle$, where ϵ_0 is the rms emittance. The equation shows that momentum offset, beam displacement, space charge effect as well as the e-beam density are the main factors affecting the cooling rate and that the coupling between horizontal position, longitudinal momentum, and e-beam density caused by dispersion is responsible for the rate redistribution. The term related to drift velocity is eliminated as it is not relevant to longitudinal cooling. Additionally, the average drift velocity can be compensated by adjusting the angle of the e-beam with respect to the trajectory. Based on the ion and electron beam distributions, Eq. (10) can be calculated analytically according to the law of the unconscious statistician (LOTUS) [26]. As shown in Ref. [18], the cooling rate redistribution for the e-beam with a Gaussian profile is studied. In this section, we will discuss the effect of rate redistribution for both Gaussian and uniform electron beams, while including all the mentioned factors.

We define the cooling rates $\lambda_p = \langle\Delta\delta^2\rangle/\delta_p^2$ and $\lambda_x = \langle\Delta\epsilon\rangle/\epsilon_0$, where δ_p and ϵ_0 are the rms momentum spread and emittance, respectively. And the gain factor is the ratio of the cooling rate with and without dispersion and other factors $k = \lambda/\lambda_0$. Using the same method in Ref. [18], several cases that can realize dispersive electron cooling are studied based on the e-beams with transverse Gaussian and uniform distributions. It is worth noting that we briefly give the derivation details in the Appendix, and for each case, a Monte Carlo calculation based on Eq. (10) is performed and compared with the analytical formula in the next.

A. Case 1: Gaussian e-beam with momentum offset δ_o and beam displacement x_o

As Derbenev introduced in Ref. [13,17], dispersive electron cooling can be achieved by a longitudinal velocity offset, a beam displacement as well as a transverse gradient of electron density. For this case, Eq. (10) can be written as

$$\begin{aligned}
\langle\Delta\delta^2\rangle &= -2C_p\langle n_e\delta^2\rangle + 2C_p\delta_o\langle n_e\delta\rangle \\
\langle\Delta\epsilon\rangle &= -C_x\epsilon_0\langle n_e\rangle + \frac{C_p D}{\beta_x}\langle n_e x_\beta\delta\rangle - \frac{C_p D\delta_o}{\beta_x}\langle n_e x_\beta\rangle.
\end{aligned} \quad (11)$$

Here, we use a Gaussian e-beam to produce the transverse density gradient

$$n_e(x,y,s) = \frac{N_{e0} e^{-\frac{(x_\beta+x_o+D_i\delta)^2}{2\sigma_{ex}^2}-\frac{y_\beta{}^2}{2\sigma_{ey}^2}-\frac{s^2}{2\sigma_{es}^2}}}{(2\pi)^{3/2}\sigma_{ex}\sigma_{ey}\sigma_{es}}. \quad (12)$$

Furthermore, we assume that the ion beam also has a Gaussian distribution in the transverse direction. Then, Eq. (11) can be calculated and the final cooling rates and the gain factors are

$$\begin{aligned}
a &= \sqrt{\sigma_{ex}^2+\sigma_{ix}^2}\\
b &= \sqrt{\sigma_{ex}^2+\sigma_{ix}^2+D^2\delta_p^2}\\
\lambda_p &= -\frac{2e_0C_{YI}}{\sqrt{2\pi}}e^{-\frac{x_o^2}{2b^2}}\left[C_p\frac{a^2}{b^3}+C_p\frac{D^2\delta_p^2x_o^2}{b^5}+C_p\frac{D\delta_o x_o}{b^3}\right]\\
\lambda_x &= -\frac{e_0C_{YI}}{\sqrt{2\pi}}e^{-\frac{x_o^2}{2b^2}}\left[\frac{C_x}{b}-C_p\frac{D^2\delta_p^2}{b^5}(x_o^2-b^2)-C_p\frac{D\delta_o x_o}{b^3}\right]\\
k_p &= e^{-\frac{x_o^2}{2b^2}}\left[\frac{a^3}{b^3}+\frac{a}{b^5}D^2\delta_p^2x_o^2+\frac{a}{b^3}D\delta_o x_o\right]\\
k_x &= e^{-\frac{x_o^2}{2b^2}}\left[\frac{a}{b}+\frac{C_pa}{C_xb^5}D^2\delta_p^2(b^2-x_o^2)-\frac{C_pa}{C_xb^3}D\delta_o x_o\right], \quad (13)
\end{aligned}$$

where C_{YI} is a constant that depends on beam distribution (see Appendix).

Using arbitrary parameters with $\sigma_{ex}=1$ cm, $\sigma_{ix}=1$ cm, $\delta_p=1\times10^{-2}$, and $C_p/C_x=2$, the dependence of the gain factors on the dispersion function under different conditions is calculated and shown in Fig. 3. The Monte Carlo results show a good agreement with the analytical formula. We see that using dispersion alone can realize dispersive cooling and a factor of 1.4 is achieved for the horizontal cooling rate. This effect is mainly due to the Gaussian e-beam distribution, which naturally provides the transverse gradient of the longitudinal force. The details of the explanation can be found in Ref. [18]. When beam displacement is applied, it shows that the horizontal gain factor drops off and the maximum value of k_x decreases to 1.1. This is due to the fact that the displacement reduces the average friction force on the ion beam, which is density dependent, thereby weakening the dispersive cooling effect as shown by the second term in the bracket of Eq. (13). Meanwhile, the third term in the bracket depends on both momentum offset and beam displacement. The increase and decrease of the horizontal cooling rate can be adjusted by the product of the two values. This conclusion agrees with Refs. [13,17]. As shown in Fig 3, the maximum value of k_x can reach 1.75 by using $x_o=1$ cm and $\delta_o=-1.5\times10^{-2}$, even though the beam displacement introduces a certain degradation of the horizontal cooling rate.

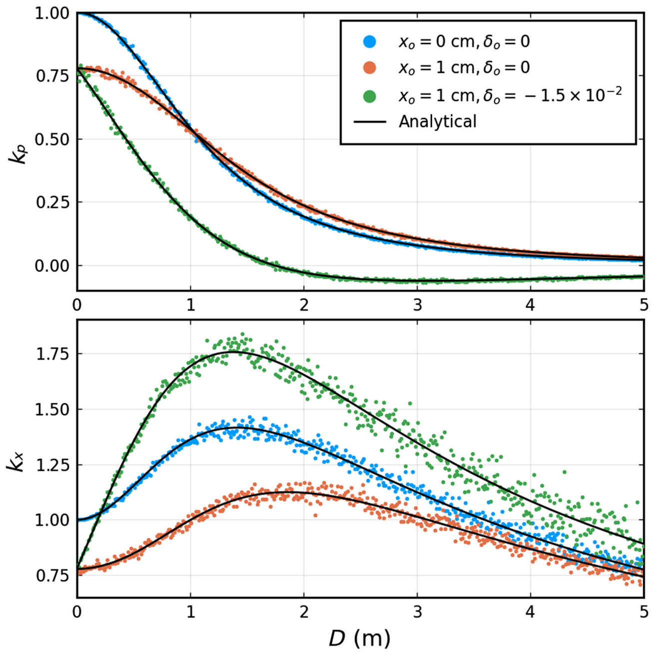

FIG. 3. Monte Carlo and analytical results of the gain factor dependence on dispersion for case 1 with $\sigma_{ex}=1$ cm, $\sigma_{ix}=1$ cm, $\delta_p=1\times10^{-2}$, $C_p/C_x=2$.

However, an energy mismatch between electron and ion beams may result in a circular attractor in the longitudinal phase space of the ion bunch. If the relative shift exceeds a critical value, beam heating instead of cooling may occur [25]. Therefore, the method of using a momentum offset to realize dispersive cooling needs to be carefully calculated and evaluated in practical applications.

B. Case 2: Gaussian e-beam with space charge K_{sc} and beam displacement x_o

Another method to realize dispersive cooling relies on the space charge effect of the e-beam, which in combination with a beam displacement can generate the transverse gradient of the longitudinal force [15,16]. As discussed in Sec. II, a radial-dependant velocity deviation in the longitudinal direction will be produced due to the space charge effect. Here we assume a parabolic velocity profile, i.e., $\delta_e(r)=K_{sc}r^2$, then Eq. (10) can be written as

$$\begin{aligned}
\langle\Delta\delta^2\rangle &= -2C_p\langle n_e\delta^2\rangle\\
&\quad + 2C_pK_{sc}x_o(x_o\langle n_e\delta\rangle+2\langle n_ex_\beta\delta\rangle+2D\langle n_e\delta^2\rangle)\\
\langle\Delta\epsilon\rangle &= -C_x\epsilon_0\langle n_e\rangle+\frac{C_pD}{\beta_x}\langle n_ex_\beta\delta\rangle\\
&\quad -\frac{C_pDK_{sc}x_o}{\beta_x}(x_o\langle n_ex_\beta\rangle+2\langle n_ex_\beta^2\rangle+2D\langle n_ex_\beta\delta\rangle). \quad (14)
\end{aligned}$$

The final result for the cooling rate and gain factor is

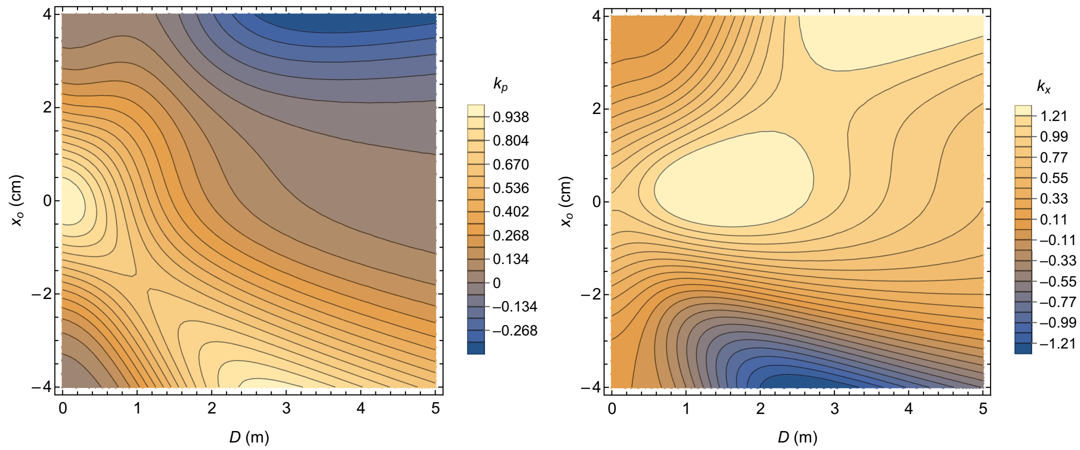

FIG. 4. Dependence of gain factors on ion dispersion and beam displacement considering Gaussian electron and ion beams with $\sigma_{ex} = 1$ cm, $\sigma_{ix} = 1$ cm, $\delta_p = 1 \times 10^{-2}$, $C_p/C_x = 2$, $K_{\rm sc} = 20$ m^{-2}.

$$
\begin{aligned}
a &= \sqrt{\sigma_{ex}^2 + \sigma_{ix}^2} \\
b &= \sqrt{\sigma_{ex}^2 + \sigma_{ix}^2 + D^2\delta_p^2} \\
c &= \sqrt{\sigma_{ex}^2 - \sigma_{ix}^2 - D^2\delta_p^2} \\
\lambda_p &= -\frac{2e_0 C_{YI}}{\sqrt{2\pi}} e^{-\frac{x_o^2}{2b^2}} \left[C_p \frac{a^2}{b^3} + C_p \frac{D^2\delta_p^2 x_o^2}{b^5} \right. \\
&\quad \left. - C_p \frac{D K_{\rm sc} x_o}{b^5} (2\sigma_{ex}^2 b^2 - x_o^2 c^2) \right] \\
\lambda_x &= -\frac{e_0 C_{YI}}{\sqrt{2\pi}} e^{-\frac{x_o^2}{2b^2}} \left[\frac{C_x}{b} + C_p \frac{D^2\delta_p^2}{b^5} (b^2 - x_o^2) \right. \\
&\quad \left. + C_p \frac{D K_{\rm sc} x_o}{C_x b^5} (2\sigma_{ex}^2 b^2 - x_o^2 c^2) \right] \\
k_p &= e^{-\frac{x_o^2}{2b^2}} \left[\frac{a^3}{b^3} + \frac{a}{b^5} D^2\delta_p^2 x_o^2 - \frac{a}{b^5} D K_{\rm sc} x_o (2\sigma_{ex}^2 b^2 - x_o^2 c^2) \right] \\
k_x &= e^{-\frac{x_o^2}{2b^2}} \left[\frac{a}{b} + \frac{C_p a}{C_x b^5} D^2\delta_p^2 (b^2 - x_o^2) \right. \\
&\quad \left. + \frac{C_p a}{C_x b^5} D K_{\rm sc} x_o (2\sigma_{ex}^2 b^2 - x_o^2 c^2) \right].
\end{aligned} \tag{15}
$$

It shows that the first and second terms in the bracket of the cooling rate formula come from the Gaussian e-beam distribution, which has already been discussed above. The third term is of interest to us, and it is directly determined by the e-beam space charge and beam displacement. We see that the sign of this term also depends on the electron and ion beam parameters. Considering arbitrary parameters, the dependence of the gain factor on ion dispersion and beam displacement is shown in Fig. 4, and a comparison between the Monte Carlo calculation and analytical formula is shown in Fig. 5. It is clear that beam displacement and the space charge effect contribute to dispersive electron cooling. As discussed in Refs. [15,16], an outward displacement of the e-beam is required for the increase of the horizontal cooling rate, which is consistent with our result. The equation also shows that a larger $K_{\rm sc}$ can improve the rate redistribution effect. However, a strong space charge field is not desirable for cooling since the transverse drift velocity and the longitudinal velocity deviation have a significant influence on the cooling process. Therefore, when studying dispersive electron cooling, the value of $K_{\rm sc}$ or the e-beam density should be carefully determined according to the beam energy and cooling requirements.

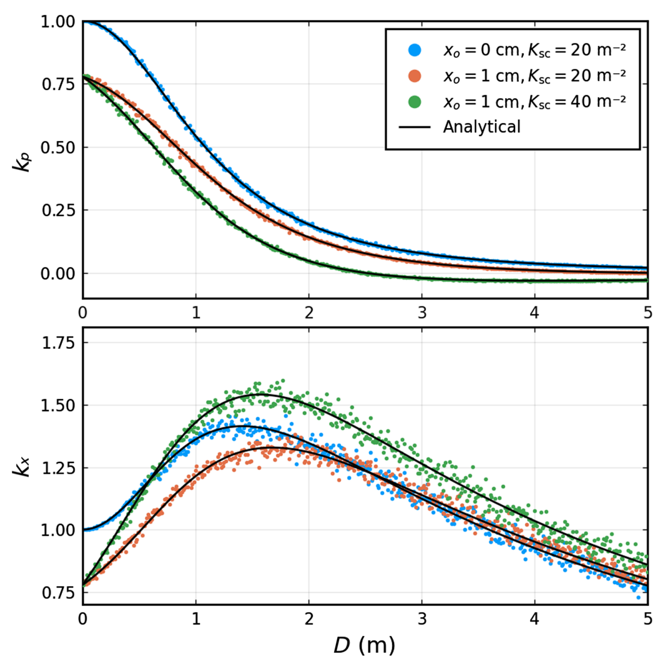

FIG. 5. Monte Carlo and analytical results of the gain factor dependence on dispersion for case 2 with $\sigma_{ex} = 1$ cm, $\sigma_{ix} = 1$ cm, $\delta_p = 1 \times 10^{-2}$, $C_p/C_x = 2$.

C. Case 3: Uniform e-beam with infinite radius R_e, space charge K_{sc}, and beam displacement x_o

In the above, we investigated two methods that can be used to realize dispersive cooling for a Gaussian e-beam. In addition to the momentum offset and space charge effect, we see that the e-beam itself also contributes to the rate redistribution effect since the Gaussian density distribution naturally provides a transverse gradient of the longitudinal force. In fact, a uniform or hollow e-beam is much preferable for most electron coolers to avoid beam losses due to recombination, overcooling, and instabilities [27,28]. Therefore, it is necessary to study the rate redistribution effect for these two e-beams. The analytical process of the hollow e-beam is too complicated, and it can be further studied by numerical simulation. Here we only discuss the uniform e-beam.

We first assume a uniform e-beam with an infinite radius that is $n_e = \rho_0$. Then there is no correlation between the three variables n_e, x_β, and δ, and Eq. (10) becomes

$$\begin{aligned}\langle\Delta\delta^2\rangle &= -2C_p n_e\langle\delta^2\rangle + 4C_p D K_{\mathrm{sc}} n_e x_o\langle\delta^2\rangle\\ \langle\Delta\epsilon\rangle &= -C_x\epsilon_0 n_e - \frac{2C_p D K_{\mathrm{sc}} n_e x_o}{\beta_x}\langle x_\beta^2\rangle.\end{aligned} \tag{16}$$

Since there is no density gradient, the momentum offset does not affect the dispersion cooling as discussed in case 1, and the density distribution will not provide the transverse gradient of the friction force. So, using the space charge effect of the e-beam is the only useful approach. In this case, the gain factors can be easily calculated

$$\begin{aligned}\lambda_p &= -2\rho_0(C_p - 2C_p D K_{\mathrm{sc}} x_o)\\ \lambda_x &= -\rho_0(C_x + 2C_p D K_{\mathrm{sc}} x_o)\\ k_p &= 1 - 2DK_{\mathrm{sc}} x_o\\ k_x &= 1 + 2DK_{\mathrm{sc}} x_o C_p/C_x.\end{aligned} \tag{17}$$

It shows the same conclusion that dispersive electron cooling can be achieved by the velocity deviation caused by the e-beam space charge, combined with a beam displacement. This result is consistent with Eq. (5) in Ref. [16], which was benchmarked by the experimental measurement.

D. Case 4: Uniform e-beam with finite radius R_e, space charge K_{sc}, and beam displacement x_o

For a uniform e-beam with a finite radius R_e, the density distribution is

$$n_e(r) = \begin{cases}\rho_0, & r \le R_e\\ 0, & r > R_e.\end{cases} \tag{18}$$

For this distribution, we think that there is a density gradient which is created by the density difference between the inside and outside of the e-beam. Due to the betatron motion, particles with large amplitude will cross the boundary of the e-beam back and forth. After averaging over several turns, the particle will be subject to a transverse gradient of the longitudinal friction force. As a result, a uniform e-beam itself can also be applied to achieve dispersive cooling as well as a Gaussian e-beam.

For simplicity, we only consider the space charge effect $\delta_e(r) = K_{\mathrm{sc}} r^2$ and beam displacement. So, the longitudinal and horizontal cooling effects can be described by Eq. (14). Based on the uniform density distribution [Eq. (18)], the analytical result of the gain factors is calculated below

$$\begin{aligned}
m &= Erf\left[\frac{R_e}{\sqrt{2\sigma_{ix}^2}}\right]\\
\sigma &= \sqrt{\sigma_{ix}^2 + D^2\delta_p^2}\\
a &= Erf\left[\frac{R_e + x_o}{\sqrt{2}\sigma}\right] + Erf\left[\frac{R_e - x_o}{\sqrt{2}\sigma}\right]\\
b &= \frac{e^{-\frac{(R_e - x_o)^2}{2\sigma^2}}(R_e - x_o) + e^{-\frac{(R_e + x_o)^2}{2\sigma^2}}(R_e + x_o)}{\sigma^3}\\
c &= \frac{e^{-\frac{(R_e + x_o)^2}{2\sigma^2}} - e^{-\frac{(R_e - x_o)^2}{2\sigma^2}}}{\sigma}\\
\lambda_p &= -2e_0\rho_0\left[C_p\frac{a}{2} - C_p\frac{D^2\delta_p^2 b}{\sqrt{2\pi}}\right.\\
&\qquad \left. + C_p\frac{DK_{\mathrm{sc}}x_o}{\sqrt{2\pi}}(2\sigma^2 b - \sqrt{2\pi}a - x_o c)\right]\\
\lambda_x &= -e_0\rho_0\left[C_x\frac{a}{2} + C_p\frac{D^2\delta_p^2 b}{\sqrt{2\pi}}\right.\\
&\qquad \left. - C_p\frac{DK_{\mathrm{sc}}x_o}{\sqrt{2\pi}}(2\sigma^2 b - \sqrt{2\pi}a - x_o c)\right]\\
k_p &= \frac{a}{2m} - \frac{D^2\delta_p^2 b}{\sqrt{2\pi}m} + \frac{DK_{\mathrm{sc}}x_o}{\sqrt{2\pi}m}(2\sigma^2 b - \sqrt{2\pi}a - x_o c)\\
k_x &= \frac{a}{2m} + \frac{C_p}{C_x}\left[\frac{D^2\delta_p^2 b}{\sqrt{2\pi}m} - \frac{DK_{\mathrm{sc}}x_o}{\sqrt{2\pi}m}(2\sigma^2 b - \sqrt{2\pi}a - x_o c)\right]
\end{aligned} \tag{19}$$

It shows that the first and second terms of the cooling rate are due to the e-beam distribution, and the third term comes from the space charge effect. As an example, the dependence of the gain factors on the ion dispersion and beam displacement is shown in Fig. 6, which proves that proper dispersion and beam displacement can achieve dispersive cooling. A comparison of different settings is shown in Fig. 7, and the Monte Carlo results agree well with the analytical formula. We see that the rate redistribution effect strongly depends on the e-beam radius

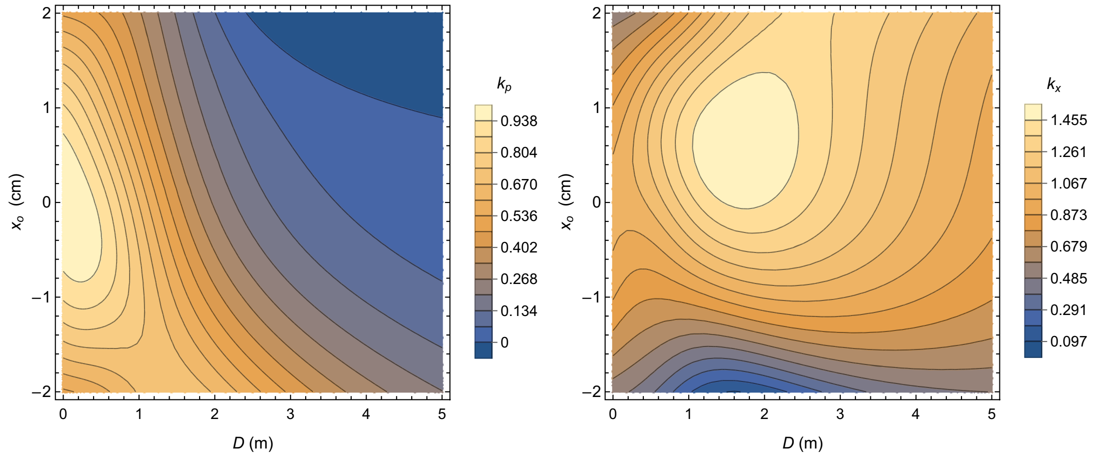

FIG. 6. Dependence of gain factors on ion dispersion and beam displacement considering uniform electron beam with $R_e = 2$ cm, $\sigma_{ix} = 1$ cm, $\delta_p = 1 \times 10^{-2}$, $C_p/C_x = 2$, $K_{\mathrm{sc}} = 20\,\mathrm{m}^{-2}$.

since it directly determines how many particles can see the density gradient. If the e-beam radius is smaller than the ion beam, it is easy to realize dispersive cooling with small dispersion. Otherwise, dispersive cooling is less likely to occur unless the dispersion is large enough. However, the difference in the radius of the two beams should be reasonable to ensure the overall cooling performance and avoid any aberrations in the ion beam phase space distribution.

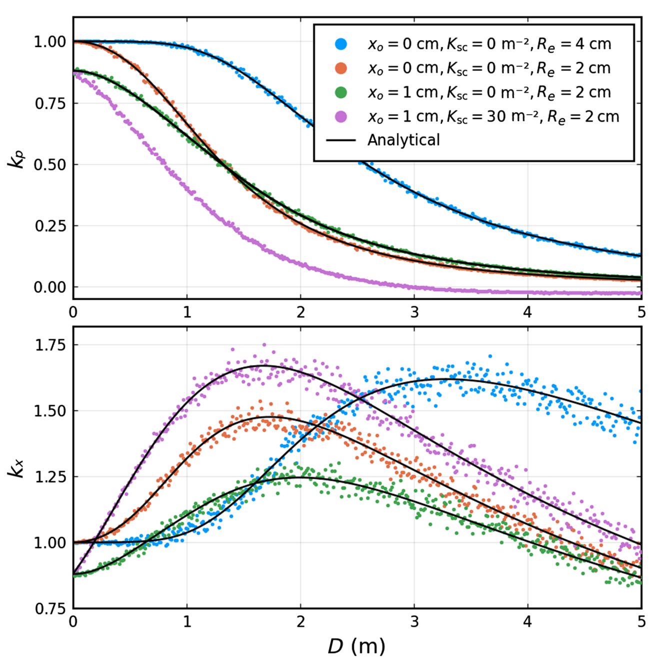

FIG. 7. Monte Carlo and analytical results of the gain factor dependence on dispersion for case 4 with $\sigma_{ix} = 1$ cm, $\delta_p = 1 \times 10^{-2}$, $C_p/C_x = 2$.

IV. NUMERICAL SIMULATION

To verify the above conclusions, a numerical simulation is carried out and compared with the analytical result. We use the multiparticle tracking code TRACKIT and mainly consider the magnetized cooling, e-beam space charge, and ion dispersion [29]. Other effects such as intrabeam scattering (IBS) and ion space charge are not included in our simulation. Since the code has already been well applied in several cooling experiments [30–32], we directly use it for the dispersive cooling simulation. The ion and electron beam parameters in the simulation are listed in Table I, which are mainly based on the HIRFL-CSRe facility and its electron cooler EC300 [6].

In order to effectively include the space charge effect of the e-beam, we choose the ion beam Fe^{26+} at a relatively low energy of 35 MeV/u. The energy and current of the e-beam are 19.2 keV and 30–50 mA, respectively. Figure 8 shows the calculated results of the longitudinal velocity deviation for Gaussian and uniform electron beams, as well as the dependence of the magnetized friction force on the electron-ion relative velocity (Parkhomchuk model [20]). According to the density and velocity distribution of the ion beam, we know that most particles are located in the linear friction force region. Moreover, because the radius of the ion beam with Gaussian distribution is less than 1.0 cm, the electron velocity deviation has little effect on the relative velocity distribution between electrons and ions. As a result, the analytical model with the assumption of linear friction force is suitable for benchmarking with the simulation.

Figures 9 and 10 show the simulation results of the dispersion dependence of the horizontal cooling gain factor k_x for Gaussian and uniform e-beam, respectively. In the simulation, the cooling rate is estimated based on a short cooling process, during which ion emittance and

TABLE I. Beam parameters in the simulation.

	$^{56}Fe^{26+}$	Electron	
Circumference (m)		128.8	
Length of cooler (m)		3.4	
Transverse displacement	Gaussian	Gaussian	Uniform
Longitudinal displacement	Coasting	dc	dc
Energy (MeV/u)	35.0	0.0192	0.0192
Beam current (mA)	0.5	50.0	30.0
Beam radius (cm)	1.0/0.5	2.0	2.0
rms ϵ_x/ϵ_y (μm)	0.5/0.1	...	...
rms δ_p	1.0×10^{-4}	...	...
β_x/β_y @cooler (m)	25/25	...	...
Longitudinal temperature (eV)	0.7	5.0×10^{-5}	5.0×10^{-5}
Transverse temperature (eV)	1.4/0.3	0.5	0.5
B field in cooler (Gs)	...	500	500
B field parallelity	...	2.0×10^{-4}	2.0×10^{-4}
Longitudinal rate λ_p (s^{-1})	...	63.5	16.4
Transverse rate λ_x/λ_y (s^{-1})	...	25.3/33.9	6.8/9.0

distribution change small enough that the cooling rate is not affected by these parameters. According to the simulation result without the dispersion and other factors, we get the original cooling rate $\lambda_x/\lambda_y/\lambda_p = 25.3/33.9/63.5\ s^{-1}$ for Gaussian e-beam, $\lambda_x/\lambda_y/\lambda_p$=10.2/13.1/24.4$s^{-1}$ (R_e=1.5cm) and $\lambda_x/\lambda_y/\lambda_p = 6.8/9.0/16.4\ s^{-1}$ ($R_e = 2.0$ cm) for uniform e-beam. Then, we estimate the value of C_p/C_x is about 1.2. Moreover, the value of K_{sc} is based on the fitting result of the velocity deviation using the power function, which gives $K_{sc} \sim 0.3\,m^{-2}$ for Gaussian e-beam with $R_e(3\sigma) = 2.0$ cm, and $K_{sc} \sim 0.2\ m^{-2}$ ($R_e = 1.5$ cm), ~0.15 m^{-2} ($R_e = 2.0$ cm) for uniform e-beam. Using these parameters, the analytical result is calculated and compared with the simulation, as shown in Figs. 9 and 10. Typical simulation results for the cooling rates are listed in Table II. It shows that the analytical model agrees well with the simulation results. However, there is a discrepancy between the simulation and analytical model, especially for large dispersion. One reason is the nonlinear part of the friction force, which is related to both the dispersion function and the e-beam space charge effect. And this is not considered in the analytical model. Another reason is that the calculated cooling rate depends on the beam distribution, so the value of C_p/C_x is actually dispersion dependent. We equate dispersion to the horizontal beam size in the simulation and estimate that the value of C_p/C_x with dispersion is in the range of 1.18–1.31, which we think is an acceptable error for the analytical model.

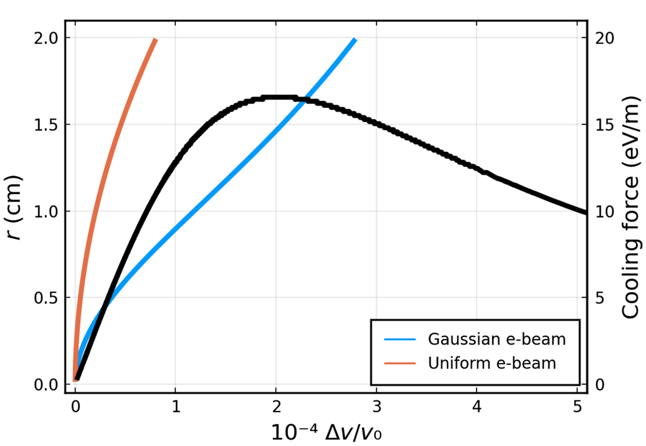

FIG. 8. Calculated velocity deviation in the longitudinal direction for Gaussian ($R_e(3\sigma) = 2.0$ cm) and uniform ($R_e = 2.0$ cm) e-beam, as well as the dependence of the magnetized friction force on the relative velocity.

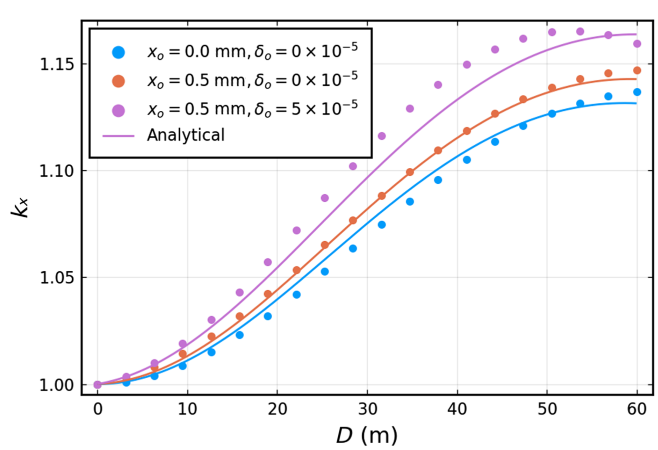

FIG. 9. Comparison between the simulation (dots) and analytical model (solid line) of the horizontal gain factor for Gaussian e-beam with $R_e(3\sigma) = 2.0$ cm, $C_p/C_x = 1.2$ and $K_{sc} = 0.3\,m^{-2}$.

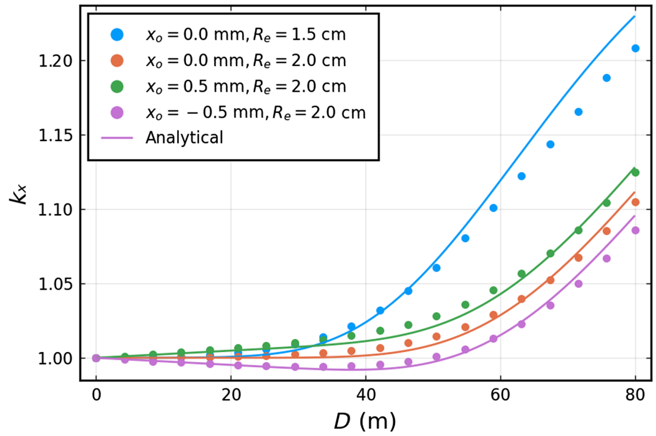

FIG. 10. Comparison between the simulation (dots) and analytical model (solid line) of the horizontal gain factor for uniform e-beam with $C_p/C_x = 1.2$ and $K_{sc} = 0.15$–$0.2\ \mathrm{m}^{-2}$.

V. SUMMARY AND DISCUSSION

In electron cooling, transverse cooling is usually weaker than the longitudinal direction. For this reason, dispersive electron cooling is an effective scheme to redistribute the cooling rate, especially for future high-energy coolers. In this paper, we investigated several approaches that can be applied to achieve dispersive electron cooling. It is demonstrated that beam relative momentum offset, transverse displacement, density distribution, and space charge effect of e-beam all contribute to the rate redistribution in dispersive cooling. For the first time, we propose a new mechanism for dispersive electron cooling in the case of uniform e-beam by placing part of the ion beam outside of the electron beam. Based on a linear friction force model, we present an analytical formula for numerically estimating the cooling rate redistribution effect. Moreover, a Monte Carlo calculation and numerical simulation are carried out, and all results show good agreement with the analytical model.

TABLE II. Typical cooling rates in simulation.

Gaussian e-beam	
$D = 50$ m	$\lambda_x/\lambda_y/\lambda_p\,(s^{-1})$
$x_o = 0.0$ mm, $\delta_o = 0 \times 10^{-5}$	28.5/29.4/38.5
$x_o = 0.5$ mm, $\delta_o = 0 \times 10^{-5}$	28.8/29.4/37.8
$x_o = 0.5$ mm, $\delta_o = 5 \times 10^{-5}$	29.6/28.4/32.2

Uniform e-beam	
$D = 80$ m	$\lambda_x/\lambda_y/\lambda_p\,(s^{-1})$
$x_o = 0.0$ mm, $R_e = 1.5$ cm	12.3/13.3/17.6
$x_o = 0.0$ mm, $R_e = 2.0$ cm	7.6/9.2/14.9
$x_o = 0.5$ mm, $R_e = 2.0$ cm	7.7/9.3/14.8
$x_o = -0.5$ mm, $R_e = 2.0$ cm	7.4/9.2/15.0

However, we only discuss the cooling rate redistribution effect between horizontal and longitudinal directions. To increase the vertical cooling rate, the same method of applying vertical dispersion can be used or simply by introducing betatron coupling in the ion machine.

As previously discussed, the beam relative momentum offset and displacement may affect the cooling performance and may induce some undesired effects such as circular attractor or even beam heating. So, these two approaches should be carefully calculated and evaluated in practice. Moreover, since the strong dependence of the space charge effect on beam energy, the method using the velocity deviation is only suitable for low-energy beam cooling, such as conventional electron cooling with electron energies below a few MeV. For high-energy beam cooling, such as EIC, where e-beam energy would reach tens or hundreds of MeV, the method employing beam density is much preferable. However, these factors in dispersive electron cooling have not been well explored through experiments. The influences of these factors on the cooling rate need to be further investigated. Additionally, the effects induced by the dispersion function, such as ion beam dynamics and IBS, require comprehensive exploration. It is also important to note that the dispersion function in accelerators is quite limited. Therefore, considering all the above points, dispersive cooling needs further extensive research in the future.

ACKNOWLEDGMENTS

This work is supported by the National Natural Science Foundation of China (No. 12275323, No. 12275325, and No. 12205346), the Hundred Talents Project of the Chinese Academy of Sciences, and the National Key R&D Program of China (No. 2019YFA0405400).

APPENDIX: DERIVATION OF THE DISPERSIVE COOLING RATE

In this section, we present the derivation of the cooling rate for Eq. (10). It is clear to see that the dispersion function couples the transverse motion and longitudinal momentum and affects the electron density encountered by the ions. Therefore, the calculation of the cooling rate requires averaging over the entire ion phase space based on the electron and ion beam distributions. For example $\langle n_e \delta x_\beta \rangle = \int n_e \delta x_\beta P_i d\mathrm{X} d\mathrm{Y} d\mathrm{S}$, where P_i is the ion probability density function (PDF). Here, we discuss Gaussian and uniform e-beams, respectively, and always assume a Gaussian ion beam in phase space,

$$P_i = \frac{e^{-\frac{x_\beta^2}{2\sigma_{ix}^2} - \frac{y_\beta^2}{2\sigma_{iy}^2} - \frac{s^2}{2\sigma_{is}^2} - \frac{x'^2}{2\sigma_{ix'}^2} - \frac{y'^2}{2\sigma_{iy'}^2} - \frac{\delta^2}{2\delta_{ip}^2}}}{(2\pi)^3 \sigma_{ix}\sigma_{iy}\sigma_{is}\sigma_{ix'}\sigma_{iy'}\delta_{ip}}, \tag{A1}$$

where $(\sigma_{ix}, \sigma_{iy}, \sigma_{is}, \sigma_{ix'}, \sigma_{iy'}, \delta_{ip})$ are the standard deviation of the beam distribution.

1. Gaussian e-beam

First, we assume a Gaussian e-beam and the local number density of at the position $(x, y, s) = (x_\beta + x_o + D_i\delta, y_\beta, s)$ is given by

$$n_e(x,y,s) = \frac{N_{e0} e^{-\frac{(x_\beta + x_o + D_i\delta)^2}{2\sigma_{ex}^2} - \frac{y_\beta{}^2}{2\sigma_{ey}^2} - \frac{s^2}{2\sigma_{es}^2}}}{(2\pi)^{3/2}\sigma_{ex}\sigma_{ey}\sigma_{es}}. \quad \text{(A2)}$$

Then we have

$$\langle n_e \rangle = \int n_e P_i d\mathrm{X} d\mathrm{Y} d\mathrm{S} = C_{YI} \int \frac{e^{-\frac{(x_\beta + x_o + D_i\delta)^2}{2\sigma_{ex}^2} - \frac{x_\beta{}^2}{2\sigma_{ix}^2} - \frac{\delta^2}{2\delta_{ip}^2}}}{(2\pi)^{3/2}\sigma_{ex}\sigma_{ix}\delta_{ip}} dx_\beta d\delta, \quad \text{(A3)}$$

where

$$C_{YI} = \frac{N_{e0}}{2\pi\sqrt{(\sigma_{ey}^2 + \sigma_{iy}^2)(\sigma_{es}^2 + \sigma_{is}^2)}}. \quad \text{(A4)}$$

For a dc e-beam with the linear number density ρ_L, we have $C_{YI} = \rho_L/\sqrt{2\pi(\sigma_{ey}^2 + \sigma_{iy}^2)}$. Rewrite the exponential function in Eq. (A3) into the general form:

$$e^{-\frac{(x_\beta + x_o + D_i\delta)^2}{2\sigma_{ex}^2} - \frac{x_\beta{}^2}{2\sigma_{ix}^2} - \frac{\delta^2}{2\delta_{ip}^2}} = e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_\beta - M)^2}{2\sigma_1^2} - \frac{2\rho(x_\beta - M)(\delta - N)}{\sigma_1\sigma_2} + \frac{(\delta - N)^2}{2\sigma_2^2}\right] + K}, \quad \text{(A5)}$$

where

$$\sigma_1 = \sqrt{\frac{\sigma_{ix}^2(\sigma_{ex}^2 + D^2\delta_p^2)}{\sigma_{ix}^2 + \sigma_{ex}^2 + D^2\delta_{ip}^2}}$$
$$\sigma_2 = \sqrt{\frac{\delta_{ip}^2(\sigma_{ix}^2 + \sigma_{ex}^2)}{\sigma_{ix}^2 + \sigma_{ex}^2 + D^2\delta_{ip}^2}}$$
$$\rho = -\frac{D\delta_{ip}\sigma_{ix}}{\sqrt{(\sigma_{ix}^2 + \sigma_{ex}^2)(\sigma_{ex}^2 + D^2\delta_{ip}^2)}}$$
$$M = -\frac{x_o\sigma_{ix}^2}{\sigma_{ix}^2 + \sigma_{ex}^2 + D^2\delta_{ip}^2}$$
$$N = -\frac{x_o D\delta_{ip}^2}{\sigma_{ix}^2 + \sigma_{ex}^2 + D^2\delta_{ip}^2}$$
$$K = -\frac{x_o^2}{2(\sigma_{ix}^2 + \sigma_{ex}^2 + D^2\delta_{ip}^2)}. \quad \text{(A6)}$$

Based on this general form and after some trivial derivation, we finally get

$$n_0 = \frac{e^K C_{YI}}{(2\pi)^{3/2}\sigma_{ix}\sigma_{ex}\delta_p}, \quad a = \frac{1}{2(1-\rho^2)\sigma_1^2}$$
$$b = \frac{1}{2(1-\rho^2)\sigma_2^2}, \quad c = -\frac{\rho}{2(1-\rho^2)\sigma_1\sigma_2}$$
$$\langle n_e \rangle = e_0 n_0 \frac{\pi}{\sqrt{ab - c^2}}$$
$$\langle n_e x_\beta \rangle = e_0 n_0 M \frac{\pi}{\sqrt{ab - c^2}}$$
$$\langle n_e \delta \rangle = e_0 n_0 N \frac{\pi}{\sqrt{ab - c^2}}$$
$$\langle n_e x_\beta^2 \rangle = e_0 n_0 \left[\frac{\pi b(ab - c^2)^{-3/2}}{2} + M^2 \frac{\pi}{\sqrt{ab - c^2}}\right]$$
$$\langle n_e \delta^2 \rangle = e_0 n_0 \left[\frac{\pi a(ab - c^2)^{-3/2}}{2} + N^2 \frac{\pi}{\sqrt{ab - c^2}}\right]$$
$$\langle n_e x_\beta \delta \rangle = e_0 n_0 \left[-\frac{\pi c(ab - c^2)^{-3/2}}{2} + MN \frac{\pi}{\sqrt{ab - c^2}}\right]. \quad \text{(A7)}$$

Substituting the above into Eq. (10), the dispersive cooling rates can be calculated.

2. Uniform e-beam

We only consider a dc e-beam with a uniform profile and radius R_e, the average on the phase space turns into a surface integral on the e-beam profile

$$\int n_e P_i d\mathrm{X} d\mathrm{Y} d\mathrm{S} = \iint_S \rho_0 \frac{e^{-\frac{x_\beta^2}{2\sigma_{ix}^2} - \frac{y_\beta^2}{2\sigma_{iy}^2} - \frac{\delta^2}{2\delta_{ip}^2}}}{(2\pi)^{3/2}\sigma_{ix}\sigma_{iy}\delta_{ip}} dx_\beta dy_\beta d\delta, \quad \text{(A8)}$$

where ρ_0 is the e-beam density and $S\colon (x_\beta + D\delta + x_o)^2 + y_\beta^2 \le R_e^2$. Since the dispersion only couples the horizontal amplitude and the longitudinal momentum, we ignore the vertical betatron motion, then Eq. (A8) can be simplified to

$$\langle n_e \rangle = \iint_{S_1} \rho_0 \frac{e^{-\frac{x_\beta^2}{2\sigma_{ix}^2} - \frac{\delta^2}{2\delta_{ip}^2}}}{(2\pi)\sigma_{ix}\delta_{ip}} dx_\beta d\delta, \quad \text{(A9)}$$

where $S_1\colon (x_\beta + D\delta + x_o)^2 \le R_e^2$. This integral is much easier to solve. After some derivation, we finally get

$$
\begin{aligned}
\sigma &= \sqrt{\sigma_{ix}^2 + D^2\delta_p^2} \\
a &= Erf\left[\frac{R_e + x_o}{\sqrt{2}\sigma}\right] + Erf\left[\frac{R_e - x_o}{\sqrt{2}\sigma}\right] \\
b &= \frac{e^{-\frac{(R_e-x_o)^2}{2\sigma^2}}(R_e - x_o) + e^{-\frac{(R_e+x_o)^2}{2\sigma^2}}(R_e + x_o)}{\sigma^3} \\
c &= \frac{e^{-\frac{(R_e+x_o)^2}{2\sigma^2}} - e^{-\frac{(R_e-x_o)^2}{2\sigma^2}}}{\sigma} \\
\langle n_e \rangle &= e_0\rho_0 a/2 \\
\langle n_e x_\beta \rangle &= e_0\rho_0\sigma_{ix}^2 c/\sqrt{2\pi} \\
\langle n_e \delta \rangle &= e_0\rho_0 D\delta_p^2 c/\sqrt{2\pi} \\
\langle n_e x_\beta^2 \rangle &= e_0\rho_0\sigma_{ix}^2 a/2 - e_0\rho_0\sigma_{ix}^4 b/\sqrt{2\pi} \\
\langle n_e \delta^2 \rangle &= e_0\rho_0\delta_p^2 a/2 - e_0\rho_0 D^2\delta_p^4 b/\sqrt{2\pi} \\
\langle n_e x_\beta \delta \rangle &= -e_0\rho_0 D\delta_p^2\sigma_{ix}^2 b/\sqrt{2\pi}. \qquad \text{(A10)}
\end{aligned}
$$

Substituting the above into Eq. (10), the dispersive cooling rates can be calculated.

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