3D theory of microscopic instabilities driven by space-charge forces

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Microscopic, or short-wavelength, instabilities are known for a drastic reduction of the beam quality and strong amplification of the noise in a beam. Space charge and coherent synchrotron radiation are known to be the leading causes of such instabilities. In this paper, we present a rigorous 3D theory of such instabilities driven by the space-charge forces. We define the condition when our theory is applicable to an arbitrary accelerator system with 3D coupling. Finally, we derive a linear integral equation describing such instability and identify conditions when it can be reduced to an ordinary second-order differential equation.

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I. INTRODUCTION

High-quality lepton and hadron beams play important role in various applications of accelerators ranging from colliders to x-ray free-electron lasers (FELs) [1–14]. These beams undergo through processes of generation, acceleration, transport, and compression, during which instabilities could cause significant degradation of the beam's quality. On the other hand, some of these instabilities can be tamed and used for the generation of coherent radiation [15–19] or hadron-beam cooling [20–23].

There are several 1D theories of microbunching instabilities [24–32], but none of them fully account for the coupling between instabilities in transverse and longitudinal degrees of freedom. In this paper, we attempt to develop a general 3D theory of microscopic instabilities driven by space-charge (SC) forces. There are compelling theoretical and experimental reasons why coupling between various degrees of motion should be included in the analysis of SC-driven instabilities. In fact, we observed a variety of coupled SC-driven instabilities in our superconducting accelerator and its beamlines [33]. Two measured beam profiles illustrating such coupling in SC-driven instabilities are shown in Fig. 1: Figure 1(a) is an example of strong coupling between radial and axial modes, while Fig. 1(b) exemplifies coupling between longitudinal and transverse (vertical) modes. Both instabilities occur in low-energy high-charge (1.25 MeV kinetic energy, 1.5 nC) electron bunches propagating along a straight line and compressed to a peak current of 50 A. In this beam, the space-charge forces dominate other forces exciting microscopic instabilities. Both instabilities shown in Fig. 1 are excited intentionally by modifying transverse focusing in the beamline. Figure 1(a) shows the transverse beam profile at a monitor located 3 m downstream of the SRF (superconducting radiofrequency) gun and shows how the ninth-order axial mode is coupled with the second-order radial mode. Figure 1(b) shows how a THz-scale longitudinal instability generates a density modulation in the vertical direction.

In this paper, we are considering an accelerator with the most general beam transport, described by a symplectic 6×6 transport map, which takes into account all macroscopic effects including SC forces. By linearizing the transport maps (in a vicinity of a chosen phase-space trajectory), we reduce the self-consistent Vlasov equation

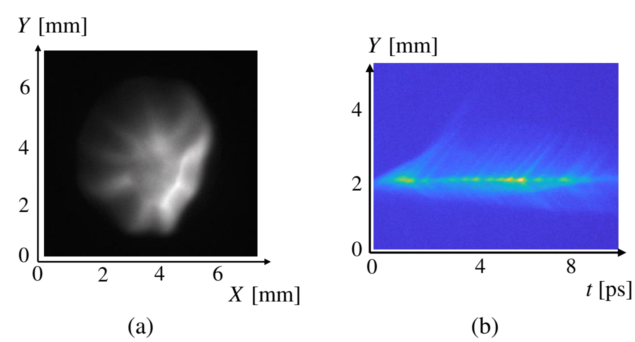

FIG. 1. Samples of measured electron beam distributions in the CeC (coherent electron cooling) accelerator [33] illustrate some aspects of 3D coupling in SC-driven instabilities: (a) Coupling between radial and axial modes in SC-driven instability; (b) Featherlike coupling between vertical and longitudinal degrees of freedom in SC-driven instability.

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to a linear integral equation describing the evolution of 3D Fourier harmonics of the beam density distribution.

For the derivations presented in this paper, we pursue the classical plasma physics methods that are specifically modified for the modern accelerator lingo:

(i) We consider an accelerator without any limitations on its components, acceleration, deceleration, compression, focusing, coupling, or its 3D beam trajectory.

(ii) We use the length along the reference trajectory, s, as an independent variable. Particle motion is described as an evolution of a full set of six canonical variables driven by the Hamiltonian, which includes macroscopic SC forces.

(iii) We assume that the effects of microscopic instability can be treated as a perturbation.

(iv) We consider that the beam transport map is evaluated as a function of s for the unperturbed Hamiltonian including all macroscopic effects, and it is known from beam dynamics simulations.

(v) We use Canonical transformation to the initial condition to remove macroscopic components and arrive at the linearized Vlasov equation.

(vi) We identify a range when and where our microscopic approach is applicable and derive an equation for perturbation Hamiltonian.

(vii) We use local linearization of the transport map with a symplectic 6×6 matrix in Alex Dragt's notation [34]. The use of this notation allows us to clearly identify the roles of the 3×3 matrix blocks in the evolution of the beam and the perturbation parameters.

(viii) We apply Fourier transform and arrive at the explicit form of a linear integral equation describing the evolution of the microscopic perturbations.

(ix) Finally, we identify the conditions when the linear integral equation can be reduced to an ordinary second-order differential equation for the electron beam density perturbation.

It was very tempting to expand our approach in order to include coherent synchrotron radiation (CSR) effects. But such inclusion requires nonlocal interactions, the consideration of which is outside of the scope of this paper. The CSR inclusion would, at least, double the length of this paper and make it convoluted.

To keep the main portion of the text compact, we appended the detailed discussions and derivations in five Appendixes.

II. THEORY

Let us consider a charged particle beam with a reference particle moving along a curved trajectory $\vec{r}_o(s)$. Its motion can be described using a standard Frenet-Serret coordinate system with three orthogonal unit vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ and the length along the reference trajectory $s = \int |d\vec{r}_o|$ (azimuth) serving as an independent variable [34–36]:

$$\hat{e}_3(s) = \frac{d\vec{r}_o}{ds}; \qquad \hat{e}_1(s) = -\frac{\vec{r}_o''}{|\vec{r}_o''|}; \qquad \hat{e}_2(s) = [\hat{e}_1(s) \times \hat{e}_3(s)];$$
$$\hat{e}_3' = -K_o(s) \cdot \hat{e}_1(s); \qquad \hat{e}_1' = K_o(s) \cdot \hat{e}_3(s) - \kappa_o(s) \cdot \hat{e}_2(s); \qquad \hat{e}_2' = \kappa_o(s) \cdot \hat{e}_1(s);$$
$$\vec{r} = \vec{r}_o(s) + q_1 \cdot \hat{e}_1(s) + q_2 \cdot \hat{e}_2(s), \tag{1}$$

where $f' \equiv \frac{df}{ds}$, $f'' \equiv \frac{d^2 f}{ds^2} \ldots$, $K_o(s) = \frac{1}{\rho_o(s)}$ is the curvature of the trajectory, and $\kappa_o(s)$ is its torsion (see Fig. 2).

Any vector $\vec{A}$ can be expanded at azimuth s using this coordinate system as[1]

$$\vec{A} = A_1 \cdot \hat{e}_1 + A_2 \cdot \hat{e}_2 + A_3 \cdot \hat{e}_3; \qquad A_i = \hat{e}_i \cdot \vec{A}.$$

Further in this paper, we will use the traditional notations, where c is the speed of light, e, m, and $\vec{v}$ are the particle's charge, mass, and velocity, respectively, $\vec{\beta} = \vec{\mathrm{v}}/c$; $\gamma = (1 - \vec{\beta}^2)^{-\frac{1}{2}}$ are the relativistic factors, $\vec{p} = \gamma m \vec{\mathrm{v}}$ and $\mathbf{E} = \gamma mc^2$ are the mechanical momentum and energy of a particle, and φ, $\vec{A}$ are scalar and vector potentials of electromagnetic (EM) field. We will also use subscript "o" to indicate values obtained by the variables at the reference trajectory $\vec{r} = \vec{r}_o(s)$ for the reference particle with momentum $\vec{p}_o(s) = m \cdot \gamma_o(s) \cdot \vec{\mathrm{v}}_o(s)$ and energy $\mathbf{E}_o(s) = \gamma_o(s) \cdot mc^2$ reaching azimuth s at time

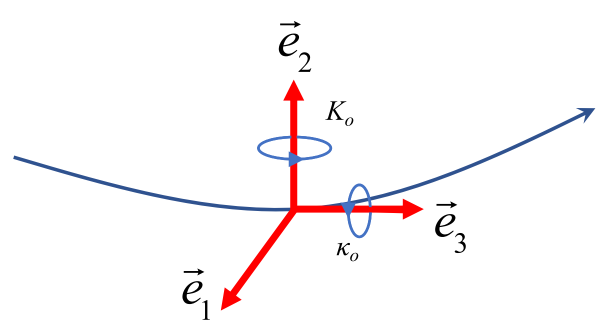

FIG. 2. Accelerator coordinate system including curvature $K_o(s)$ and torsion $\kappa_o(s)$.

[1]We intentionally do not use contravariant $\vec{A} = A^1 \cdot \hat{e}_1 + A^2 \cdot \hat{e}_2 + A^3 \cdot \hat{e}_3$ and covariant indices $\vec{A} = A_1 \cdot \hat{e}^1 + A_2 \cdot \hat{e}^2 + A_3 \cdot \hat{e}^3$, commonly used in curvilinear coordinate systems, to avoid confusion between squares (cubes) of the value and the second (third) contravariant component of a vector. We are also avoiding the use of zeroth components of four-vectors, such as $\boldsymbol{A}_o, x_o$ to avoid confusion either with the initial condition or values at the reference orbit. This important distinction is not needed and will not be used in the rest of the paper.

$$t_o(s) = \int_0^s \frac{d\zeta}{|\vec{\mathrm{v}}_o(\zeta)|}. \tag{2}$$

In classical Hamiltonian mechanics, the time plays the role of an independent variable and components of particles position, $\vec{r}$, and canonical momenta, $\vec{P}$, are known as canonical pairs, (q_i, P_i). The full set of Canonical pairs and system Hamiltonian fully describe particle's evolution [37,38]

$$H \equiv \mathbf{E} + e\varphi = \sqrt{m^2c^4 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 c^2} + e\varphi; \qquad \vec{P} = \vec{p} + \frac{e}{c}\vec{A};$$
$$\frac{dq_i}{dt} = \frac{\partial H}{\partial P_i}; \qquad \frac{dP_i}{dt} = -\frac{\partial H}{\partial q_i}, \tag{3}$$

Using s as an independent variable retains two canonical pairs, $(q_1,P_1),(q_2,P_2)$, and generates a new third canonical pair: $(-ct, H/c)$. The arrival time of a particle, t, and H become s-dependent variables with the accelerator Hamiltonian of [34–36]:

$$h_1 = -(1 + K_o \cdot q_1)\sqrt{\frac{(H - e\varphi)^2}{c^2} - m^2c^2 - \left(P_1 - \frac{e}{c}A_1\right)^2 - \left(P_2 - \frac{e}{c}A_2\right)^2} - \frac{e}{c}(1 + K_o \cdot q_1)A_3 + \kappa_o(q_1 \cdot P_2 - q_2 \cdot P_1). \tag{4}$$

Canonical transformation with the generation function [37]

$$\Phi = q_1 \cdot P_1 + q_2 \cdot P_2 + q_3 \cdot \frac{H - \mathbf{E}_o(s) - e\varphi(\vec{r}_o(s), t)}{c} + H \cdot (t_o(s) - t)$$

reduces the third canonical pair to (q_3, P_3):

$$q_3 = c \cdot (t_o(s) - t); \qquad P_3 = \frac{\mathbf{E} - \mathbf{E}_o(s)}{c} + e\frac{\varphi(\vec{r}, t) - \varphi(\vec{r}_o(s), t)}{c}, \tag{5}$$

with zero values for reference particle, $q_3|_o = 0$, $P_3|_o = 0$, and it reduces the Hamiltonian (4) to

$$h(q, P) = h_1 - \frac{c}{\mathrm{v}_o(s)} \cdot P_3 + q_3 \cdot \frac{d}{ds}\left(\mathbf{E}_o(s) + e\frac{\varphi(\vec{r}_o(s), t)}{c}\right). \tag{6}$$

For compactness, we define a set of notations for coordinates, q, and corresponding Canonical momenta, P, as well as a phase-space vector ξ as

$$q^T = [q_1, q_2, q_3], \qquad P^T = [P_1, P_2, P_3], \qquad \xi^T = [q^T, P^T], \tag{7}$$

where index ^{T} indicates the transposition of matrices, including transferring a column into a row and vice versa. We will call sets $[q_1, q_2, q_3]$ and $[P_1, P_2, P_3]$ phase-space coordinates and momenta, correspondingly. Using the definition from Eq. (7), one can write the 3D equations of motion in a compact symplectic form [34][2]:

$$\frac{d\xi_i}{ds} = S_{ij} \cdot \frac{\partial h}{\partial \xi_j} \Leftrightarrow \frac{d\xi}{ds} = \mathbf{S} \cdot \frac{\partial h}{\partial \xi};$$
$$\mathbf{S} \equiv [S_{ij}] = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{3\times3} \\ -\mathbf{I}_{3\times3} & \mathbf{0} \end{bmatrix}; \qquad \mathbf{I}_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \qquad \mathbf{S}^2 = -\mathbf{I}_{6\times6}; \tag{8}$$

where $i, j = (1, 2, 3)$, $\mathbf{0}$ is a 3×3 zero matrix (see Appendix A for further discussion). The number of components can be proportionally reduced for the 2D and 1D cases.

The motion of particles is determined by the initial conditions[3]

$$\underline{q} \equiv q(s = 0), \qquad \underline{P} \equiv P(s = 0), \qquad \underline{\xi} \equiv \xi(s = 0),$$

[2]Further, in the paper, we will use Einstein's convention of summation by repeated indices, e.g., $a_i \cdot b_i \equiv \sum_i a_i \cdot b_i$; $a_i \cdot b_{ki} \cdot c_{nk} \equiv \sum_i \sum_k a_i \cdot b_{ki} \cdot c_{nk}$.

[3]We will continue using underscore $\underline{f}$ for initial values at $s = 0$: $\underline{f} \equiv f(...., s = 0)$.

with solved equations of motion

$$q = q(\underline{q}, \underline{P}, s), \qquad P = P(\underline{q}, \underline{P}, s), \quad \xi = \xi(\underline{\xi}, s), \quad (9)$$

representing the Canonical transformation from $(\underline{q}, \underline{P})$ to (q, P) [37]. The inverse transformation

$$\underline{q} = \underline{q}(q, P, s), \quad \underline{P} = \underline{P}(q, P, s), \quad \underline{\xi} = \underline{\xi}(\xi, s), \quad (10)$$

not only exists but also is a Canonical transformation from (q, P) to $(\underline{q}, \underline{P})$. The transformation (10) results in zero Hamiltonian for the set of canonical variables $(\underline{q}, \underline{P})$[4]:

$$\underline{h}(\underline{q}, \underline{P}, s) = 0,$$

which is a traditional way of solving evolution for the background distribution function upon which instability can develop. This method is called “the variation of initial values” in analytical mechanics [39] or “the method of trajectories” in plasma physics [40]. By assuming that the solutions for the self-consistent trajectories in Eqs. (8)–(10) are known, it allows us to remove the dynamic terms and reduce the Vlasov equations to the ones comprising of the perturbation terms only. Hence, we assume that the solution for an unperturbed distribution function f_o is known and satisfies the self-consistent Vlasov equation[5] [41]:

$$\frac{\partial f_o(\xi, s)}{\partial s} + S_{ik} \cdot \frac{\partial f_o(\xi, s)}{\partial \xi_i} \cdot \frac{\partial h_o(\xi, s)}{\partial \xi_k} = 0;$$
$$\underline{f}_o(\underline{\xi}) \equiv \underline{f}_o(\underline{\xi}, s = 0) \Rightarrow f_o(\xi, s) = \underline{f}_o[\underline{\xi}(\xi, s)]. \quad (11)$$

Let’s now consider an infinitesimally small perturbation of the distribution function, $\tilde{f}$,

$$f(\xi, s) = f_o(\xi, s) + \tilde{f}(\xi, s); \qquad |\tilde{f}(\xi, s)| \ll |f_o(\xi, s)|, \quad (12)$$

e.g., $|\tilde{f}(\xi, s)| = O(\varepsilon)|f_o(\xi, s)|$; $\varepsilon \ll 1$, and the corresponding weak perturbation in the Hamiltonian:

$$h(\xi, s) = h_o(\xi, s) + \tilde{h}(\xi, s); \qquad \tilde{h}(\xi, s) = O(\varepsilon). \quad (13)$$

By applying the Canonical transformation (10), we reduce the Hamiltonian (13) to the perturbation term

$$\underline{h}(\underline{\xi}, s) = \underline{\tilde{h}}(\underline{\xi}, s) \equiv \tilde{h}(\xi(\underline{\xi}, s), s), \quad (14)$$

With the Vlasov equations for the corresponding variation of the initial distribution function $\underline{\tilde{f}}$:

$$\underline{f}(\underline{\xi}, s) = \underline{f}_o(\underline{\xi}) + \underline{\tilde{f}}(\underline{\xi}, s); \quad \tilde{f}(\xi, s) \equiv \underline{\tilde{f}}(\underline{\xi}(\xi, s), s);$$
$$\frac{\partial \underline{\tilde{f}}}{\partial s} + S_{ik} \cdot \frac{\partial \underline{f}_o}{\partial \underline{\xi}_i} \cdot \frac{\partial \underline{\tilde{h}}}{\partial \underline{\xi}_k} + S_{ik} \cdot \frac{\partial \underline{\tilde{f}}}{\partial \underline{\xi}_i} \cdot \frac{\partial \underline{\tilde{h}}}{\partial \underline{\xi}_k} = 0. \quad (15)$$

Next standard step is the linearization of the Vlasov equation by recognizing that the third term in second line in Eq. (15) is on the order of $O(\varepsilon^2)$:

$$\frac{\partial \underline{\tilde{f}}}{\partial s} + S_{ik} \cdot \frac{\partial \underline{f}_o}{\partial \underline{\xi}_i} \cdot \frac{\partial \underline{\tilde{h}}}{\partial \underline{\xi}_k} = -S_{ik} \cdot \frac{\partial \underline{\tilde{f}}}{\partial \underline{\xi}_i} \cdot \frac{\partial \underline{\tilde{h}}}{\partial \underline{\xi}_k} = O(\varepsilon^2) \to 0;$$
$$\frac{\partial \underline{\tilde{f}}}{\partial s} + \frac{\partial \underline{f}_o}{\partial \underline{q}_i} \cdot \frac{\partial \underline{\tilde{h}}}{\partial \underline{P}_i} - \frac{\partial \underline{f}_o}{\partial \underline{P}_i} \cdot \frac{\partial \underline{\tilde{h}}}{\partial \underline{q}_i} \cong 0. \quad (16)$$

It is known that a generic 3D evolution of a finite-size charged beam is analytically intractable. Rare exceptions, such as nonphysical but self-consistent Kapchinsky-Vladimirsky (KV) distribution [42], only attest to the case. Several further assumptions are needed to analytically derive a solvable equation.[6]

One typical simplification used in the theory of beam instabilities is an assumption of homogenous background density. While this approach is not applicable to all collective effects in a beam with finite sizes, it has a limited applicability for analyzing the evolution of perturbations with periods significantly smaller than the typical scales of the beam’s uniformity.

It is intuitively understandable that the scales of the beam uniformity a_i

$$\left| \frac{\partial f_o}{\partial q_i} \right| \propto \frac{f_o}{a_i} \quad (17)$$

define the scale of the perturbations when the homogenous background density can be used as a good approximation. Appendix B has detailed studies of these requirements. It can be summarized as follows: the k vector, $k^T = [k_1, k_2, k_3]$ of Fourier component with the exponential factor $e^{ik^T q} = e^{i\vec{k}\cdot\vec{q}}$ must satisfy the following conditions[7]:

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[4]This transformation can leave, $\underline{h}(\underline{q}, \underline{P}, s) = f(s)$, which can be easily removed by a Canonical transformation $F = \underline{q}_i \underline{P}_i - \int^s f(z) dz$.

[5]Self-consistent distribution function, which we use as the known background, would include all macroscopic collective effects such as SC and wakefields induced by the bunch. The self-consistent Hamiltonian would have functional dependence on the initial beam distribution $\underline{f}_o(\underline{\xi})$, e.g., $h = h_o[q, P, s, \underline{f}_o(\underline{\xi})]$. This fact does not change the validity and applicability of the Vlasov equation (11).

[6]Typically, the combination of Vlasov and Maxwell equations is not directly solvable because it contains partial derivatives.

[7]Where it is convenient, we will use objects such as $\vec{x} = \sum_{i=1}^{3} \hat{e}_i x_i$ for $x^T = [x_1, x_2, x_3]$, with product defined as $\vec{x} \cdot \vec{y} = \sum_{i=1}^{3} x_i y_i \equiv x^T \cdot y \equiv y^T \cdot x$. It is important to note that these vectors are not real 3D vectors. We will also use compact notation for convolution of objects and matrices: $\overleftrightarrow{\mathbf{A}} \cdot \vec{y} = \sum_{i=1}^{3} \hat{e}_i \sum_{j=1}^{3} A_{ij} \cdot y_j$; $\vec{x} \cdot \overleftrightarrow{\mathbf{A}} \cdot \vec{y} = \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij} \cdot x_i \cdot y_j$.

$$a_{1,2}\cdot\sqrt{(\beta_o\gamma_o)^2\cdot(k_1{}^2+k_2{}^2)+k_3{}^2}\gg\beta_o\gamma_o;$$
$$a_3\cdot\sqrt{(\beta_o\gamma_o)^2\cdot(k_1{}^2+k_2{}^2)+k_3{}^2}\gg 1. \quad (18)$$

Since we are considering a generic accelerator, which can include beam's focusing and bending of its trajectory, acceleration, compression, or decompression, we shall also assume that changes in the beam and the accelerator parameters at the scale of the density modulation are negligible:

$$|\vec{\nabla} g|\ll|\vec{k}|\cdot|g|, \quad (19)$$

where g is any generic parameter of the accelerator, including but not limited to the beam's energy, velocity, sizes, the accelerator EM fields, the curvature, and the torsion of the reference beam trajectory.

Nonlinearity of the transfer map $\xi=\mathrm{M}{:}X$ could cause distortions resulting in coupling between the Fourier harmonics of the density perturbation. Hence, we are considering linearization of the self-consistent symplectic map $\xi=\mathrm{M}(s){:}\underline{\xi}$ with small deviations $\Delta\xi(s)$ nearby a selected phase-space trajectory $\xi_o(s)=\mathrm{M}(s){:}\underline{\xi}_o$[8]:

$$\underline{\xi}=\underline{\xi}_o+\Delta\underline{\xi},$$
$$\mathbf{M}_{\underline{\xi}_o}(s)=\frac{\partial\xi}{\partial\underline{\xi}}\bigg|_{\underline{\xi}=\underline{\xi}_o}\equiv\frac{\partial}{\partial\underline{\xi}}(\mathrm{M}(s){:}\underline{\xi})\bigg|_{\underline{\xi}=\underline{\xi}_o};$$
$$\xi(s)=\xi_o(s)+\Delta\xi(s)=\mathrm{M}(s){:}\underline{\xi}_o+\mathbf{M}_{\underline{\xi}_o}(s)\cdot\Delta\underline{\xi}+O(|\Delta\underline{\xi}|^2),$$

with linear map (matrix) implicitly depending on the starting point of the phase-space trajectory, $\underline{\xi}_o$. Rewriting this map expansion as

$$\begin{bmatrix}q(s)\\P(s)\end{bmatrix}=\begin{bmatrix}\mathrm{M}_q(s){:}\underline{\xi}_o\\\mathrm{M}_P(s){:}\underline{\xi}_o\end{bmatrix}+\begin{bmatrix}\mathbf{A}&\mathbf{B}\\\mathbf{C}&\mathbf{D}\end{bmatrix}\cdot\begin{bmatrix}\Delta\underline{q}\\\Delta\underline{P}\end{bmatrix}+O(|\Delta\underline{q}|^2,|\Delta\underline{P}|^2),$$

[8]Linearization of the transfer map—in a relevant phase-space volume Ω—is critical for analytical studies beyond generally unsolvable linear Vlasov equation (16). As indicated on this page, we identify a phase-space volume $\Omega(s)=\mathrm{M}(s){:}\underline{\Omega}$ around a selected phase-space trajectory $\xi_o(s)=\mathrm{M}(s){:}\underline{\xi}_o$, determined by the initial condition $\underline{\xi}_o=\xi(s=0)$, where the transfer map can be linearized to allow further analytical investigation of the microscopic 3D instabilities. Unless someone is very lucky to encounter a completely linear system, the study of an SC instability with wave vector $\underline{k}$ involves an evaluation of a self-consistent transfer map, identification of points of interest in the phase $\underline{\xi}_o$, and a phase-space volumes $\underline{\Omega}(\underline{\xi}_o,\underline{k})$, where linear expansion of the map (20) satisfies conditions (19).

we can define an area of the phase space, Ω, where nonlinear distortion and coupling between the Fourier harmonics can be neglected

$$\delta q=\mathrm{M}_q(s){:}(\underline{\xi}_o+\Delta\underline{\xi})-\mathrm{M}_q(s){:}\underline{\xi}_o-\mathbf{A}\cdot\Delta\underline{q}-\mathbf{B}\cdot\Delta\underline{P};$$
$$|\vec{k}\cdot\delta\vec{q}|\ll 1;\qquad\{\Delta\underline{q},\Delta\underline{P}\}\in\Omega. \quad (20)$$

The transfer map should be evaluated self-consistently, including macroscopic collective effects. Further, in the paper, we will drop Δ in front of ξ and will use 6×6 symplectic transport matrix [34–36]

$$\xi=\mathbf{M}(s)\cdot\underline{\xi};\quad\underline{\xi}=\mathbf{M}^{-1}(s)\cdot\xi;\quad\mathbf{M}(0)=\mathbf{I}_{6\times6};$$
$$\mathbf{M}^T\cdot\mathbf{S}\cdot\mathbf{M}=\mathbf{M}\cdot\mathbf{S}\cdot\mathbf{M}^T=\mathbf{S}\Rightarrow\det\mathbf{M}=1;$$
$$\mathbf{M}^{-1}=-\mathbf{S}\cdot\mathbf{M}^T\cdot\mathbf{S}, \quad (21)$$

in the vicinity Ω of the initial condition $\underline{\xi}_o$. It is convenient to identify four 3×3 block matrices in the transport and inverse matrices:

$$\begin{bmatrix}q\\P\end{bmatrix}=\mathbf{M}(s)\cdot\begin{bmatrix}\underline{q}\\\underline{P}\end{bmatrix};\qquad\begin{bmatrix}\underline{q}\\\underline{P}\end{bmatrix}=\mathbf{M}^{-1}(s)\cdot\begin{bmatrix}q\\P\end{bmatrix};$$
$$\mathbf{M}=\begin{bmatrix}\mathbf{A}&\mathbf{B}\\\mathbf{C}&\mathbf{D}\end{bmatrix};\qquad\mathbf{M}^{-1}=-\mathbf{S}\cdot\mathbf{M}^T\cdot\mathbf{S}=\begin{bmatrix}\mathbf{D}^T&-\mathbf{B}^T\\-\mathbf{C}^T&\mathbf{A}^T\end{bmatrix}, \quad (22)$$

providing explicit connections between the local and initial coordinates and momenta:

$$q=\mathbf{A}\cdot\underline{q}+\mathbf{B}\cdot\underline{P};\qquad P=\mathbf{C}\cdot\underline{q}+\mathbf{D}\cdot\underline{P};$$
$$\underline{q}=\mathbf{D}^T\cdot q-\mathbf{B}^T\cdot P;\qquad\underline{P}=-\mathbf{C}^T\cdot q+\mathbf{A}^T\cdot P. \quad (23)$$

It is worth noticing that in this notation [34], three degrees of motion are decoupled when all four 3×3 matrices, **A**, **B**, **C**, and **D** are diagonal (see Appendix A for more details).

Matrix **A** plays a special role in this instability since its determinant represents the degree of the three-dimensional bunch compression:

$$\rho(q,s)=e\int_{-\infty}^{\infty}dP^3\cdot f(q,P)$$
$$=\frac{e}{\det\mathbf{A}}\int_{-\infty}^{\infty}d\underline{P}^3\cdot\underline{f}(\mathbf{A}^{-1}(q-\mathbf{B}\cdot\underline{P}),\underline{P}). \quad (24)$$

where we used one of Eq. (23) to connect local beam densities (at azimuth s) with their initial values at $s=0$:

$$\begin{aligned} &\underline{P} = -\mathbf{C}^T \cdot q + \mathbf{A}^T \cdot P \Rightarrow P = (\mathbf{A}^T)^{-1} \cdot (\underline{P} + \mathbf{C}^T q) \Rightarrow dP^3|_{q=\text{const}} = \frac{1}{\det \mathbf{A}} \cdot d\underline{P}^3|_{q=\text{const}}; \\ &q = \mathbf{A} \cdot \underline{q} + \mathbf{B} \cdot \underline{P} = \text{const} \Rightarrow \underline{q} = \mathbf{A}^{-1}(q - \mathbf{B} \cdot \underline{P}); \\ &\quad f[q, (\mathbf{A}^T)^{-1}(\underline{P} + \mathbf{C}^T \cdot q)] = \underline{f}(\mathbf{A}^{-1}(q - \mathbf{B} \cdot \underline{P}), \underline{P}). \end{aligned} \tag{25}$$

One of the important consequences of using the assumption of a homogeneous background density, described by the background distribution function of $\underline{f}_o(\underline{P})$, results in the requirement of $\det \mathbf{A} > 0$. Because of the assumption of a homogeneous background density, the beam is effectively infinite, and its density would become infinitely large, e.g., unphysical, when $\det \mathbf{A} = 0$. While it is already indicated by $\det \mathbf{A}$ in the denominator in Eq. (24) for the particle density $\rho(q, s) = \frac{1}{\det \mathbf{A}} \int_{-\infty}^{-\infty} d\underline{P}^3 \cdot \underline{f}_o(P)$, this is most evident in a 1D case

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}; \qquad \mathbf{M}^{-1} = -\mathbf{S} \cdot \mathbf{M}^T \cdot \mathbf{S} = \begin{bmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{bmatrix},$$

where m_{11} plays a role of the $\det \mathbf{A}$ and the change in the line density can be easily expressed as

$$\rho(q, s) = \int_{-\infty}^{\infty} \underline{f}_o \cdot (-m_{21} q + m_{11} P) \cdot dP = \frac{1}{m_{11}} \int_{-\infty}^{\infty} \underline{f}_o(\underline{P}) \cdot d\underline{P} = \frac{n_o}{m_{11}}. \tag{26}$$

We discuss the consequences and solutions for handling cases of $\det \mathbf{A} \to 0$ in Sec. V of this paper.

As shown in Appendixes C and D, density perturbation will generate additional potentials of the EM field resulting in the following perturbation of the accelerator Hamiltonian [see Eq. (D5)]:

$$\tilde{h} = \frac{4\pi e^2}{c} \cdot \int \frac{\tilde{\rho}_{\vec{k}} \cdot e^{i\vec{k}\cdot\vec{q}} \cdot dk^3}{\gamma_o{}^2 \cdot \beta_o{}^2 \cdot \vec{k}_\perp^2 + k_3{}^2}; \qquad \delta\left(\frac{d\vec{P}}{ds}\right) = -\frac{\partial \tilde{h}}{\partial \vec{q}} = -\frac{4\pi e^2}{c} \cdot \int \frac{i\vec{k} \cdot \tilde{\rho}_{\vec{k}} \cdot e^{i\vec{k}\cdot\vec{q}} \cdot dk^3}{\gamma_o{}^2 \cdot \beta_o{}^2 \cdot \vec{k}_\perp^2 + k_3{}^2}, \tag{27}$$

where $\vec{k} \cdot \vec{q} = k^T \cdot q = \sum_{i=1}^3 k_i \cdot q_i$; $dk^3 \equiv \prod_{i=1}^3 dk_i$; $\vec{k}_\perp^2 = k_1^2 + k_2^2$. Here, and further in the text, we use $\delta(...)$ as a change of the value of the expression inside the brackets caused by the forces induced by the beam density perturbation, which we consider to be infinitesimally small.

We can easily connect $\tilde{\rho}_{\vec{k}}$ at a location s with the Fourier harmonic of $\tilde{f}$ and $\tilde{\underline{f}}$. Considering the conservation of the phase-space volume $dq^3 \cdot dP^3 = \det \mathbf{M} \cdot d\underline{q}^3 \cdot d\underline{P}^3 = d\underline{q}^3 \cdot d\underline{P}^3$ and conservation of the phase-space density $\underline{\tilde{f}}(\underline{\xi}, s) \equiv \tilde{f}(\xi(\underline{\xi}, s), s)$, we get

$$\begin{aligned} \tilde{\rho}_{\vec{k}} \equiv \tilde{\rho}(s, \vec{k}) &= \frac{1}{(2\pi)^3} \int\int \tilde{f}(q, P, s) \cdot e^{-i\vec{k}\cdot\vec{q}} \cdot dq^3 \cdot dP^3 \\ &= \frac{1}{(2\pi)^3} \int\int \underline{\tilde{f}}(\underline{q}, \underline{P}, s) e^{-i\vec{k}\cdot\vec{q}(\underline{\xi},t)} \cdot d\underline{q}^3 \cdot d\underline{P}^3 \\ &= \frac{1}{(2\pi)^3} \int\int \underline{\tilde{f}}(\underline{q}, \underline{P}, s) \cdot e^{-i\vec{k}\cdot(\overleftrightarrow{\mathbf{A}}\cdot\vec{\underline{q}} + \overleftrightarrow{\mathbf{B}}\cdot\vec{\underline{P}})} \cdot d\underline{q}^3 \cdot d\underline{P}^3, \end{aligned} \tag{28}$$

where we used $\vec{q} = \overleftrightarrow{\mathbf{A}} \cdot \vec{\underline{q}} + \overleftrightarrow{\mathbf{B}} \cdot \vec{\underline{P}}$ as an object equivalent to $q = \mathbf{A} \cdot \underline{q} + \mathbf{B} \cdot \underline{P}$ in Eq. (23).

It is a natural place to discuss the evolution of wave numbers. As can be seen from Eq. (28), $\vec{k}$-vector of the density modulation at azimuth s is connected to that at $s = 0$:

$$\begin{aligned} &\underline{k} = k(s = 0); \\ &\underline{k}^T = k^T(s) \cdot \mathbf{A}(s) \Rightarrow k^T(s) = \underline{k}^T \cdot \mathbf{A}^{-1}(s). \end{aligned} \tag{29}$$

or in vector form[9]:

$$\vec{k}(s) = \vec{\underline{k}} \cdot \overleftrightarrow{\mathbf{A}}^{-1}(s); \qquad \vec{k}(s) \cdot \vec{q} = \vec{\underline{k}} \cdot (\vec{\underline{q}} + \overleftrightarrow{\mathbf{A}^{-1}\mathbf{B}} \cdot \vec{\underline{P}}).$$

It means that the matrix $\mathbf{A}$, the spatial components of the transport matrix, also defines the evolution of the k vector with an initial value of $\vec{\underline{k}} = \vec{k}(s = 0)$:

$$\begin{aligned} &k^T(s) = [k_1(s), k_2(s), k_3(s)]; \\ &\underline{k} = \mathbf{A}^T k(s) \Leftrightarrow k(s) = (\mathbf{A}^T)^{-1} \underline{k}. \end{aligned} \tag{30}$$

[9]For compactness, in places where it cannot cause confusion, we omit the explicit indication of s dependence, for example using $\overleftrightarrow{\mathbf{A}^{-1} \cdot \mathbf{B}}$ instead of $\overleftrightarrow{\mathbf{A}(s)^{-1} \cdot \mathbf{B}(s)}$.

We can assume, without a loss of generality, that the initial background distribution is an arbitrary integrable function of the momenta[10]:

$$\underline{f}_o \Rightarrow n_o \cdot \underline{f}_o(P); \qquad \int_{-\infty}^{\infty} \underline{f}_o(\underline{P}) \cdot d\underline{P}^3 = 1; \qquad n_o = \frac{j_o}{ec}, \tag{31}$$

where j_o is the initial beam current density. It is important to note that in contrast with the velocity-dependent spatial density of the beam, $n_l = j_o/e\mathrm{v}_o$, the $n_o = \beta_o \cdot n_l$ has a well-defined finite value.

Using relations in Eq. (25) and taking into account that $\delta(\frac{dq_j}{ds}) = \frac{\partial \tilde{h}(\xi,s)}{\partial P_j} = 0$, we can rewrite Vlasov equation (16) as follows:

$$\underline{P} = \mathbf{A}^T \cdot P - \mathbf{C}^T \cdot q; \qquad d\underline{P}_i = A_{ji} \cdot dP_j - C_{ji} \cdot dq_j;$$
$$\frac{\partial \tilde{f}}{\partial s} = -n_o \cdot \frac{\partial \underline{f}_o}{\partial \underline{P}_i} \cdot \frac{\partial \underline{P}_i}{\partial P_j} \cdot \delta\left(\frac{dP_j}{ds}\right) - n_o \cdot \frac{\partial \underline{f}_o}{\partial \underline{P}_i} \cdot \frac{\partial \underline{P}_i}{\partial q_j} \cdot \delta\left(\frac{dq_j}{ds}\right) = n_o \cdot \frac{\partial \underline{f}_o}{\partial \underline{P}_i} \cdot A_{ji} \cdot \frac{\partial \tilde{h}}{\partial q_j},$$

and introducing the perturbation Hamiltonian (27) to arrive at the self-consistent Vlasov equations:

$$\frac{\partial \tilde{f}}{\partial s} = n_o \cdot \frac{\partial \underline{f}_o}{\partial \underline{P}_i} \cdot A_{ji}(s) \cdot \mathbf{F}_j(q,s);$$
$$\mathbf{F}_j(q,s) = \frac{\partial \tilde{h}}{\partial q_j} = \frac{4\pi e^2}{c} \int \frac{ik_j \cdot \tilde{\rho}_{\vec{k}} \cdot e^{i\vec{k}\cdot\vec{q}} \cdot dk^3}{\gamma_o{}^2 \cdot \beta_o \cdot{}^2 \vec{k}_\perp^2 + k_3{}^2}. \tag{32}$$

Applying Fourier transform $\underline{f}_{\vec{\underline{k}}}(\underline{P},s) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \underline{f}(\underline{q},\underline{P},s) \cdot e^{-i\vec{\underline{k}}\cdot\vec{\underline{q}}} \cdot d\underline{q}^3$ to this equation, we get

$$\frac{\partial \tilde{f}_{\vec{\underline{k}}}}{\partial s} = \frac{n_o}{(2\pi)^3} \cdot \frac{\partial \underline{f}_o}{\partial \underline{P}_i} \cdot A_{ji}(s) \cdot \int d\underline{q}^3 \cdot e^{-i\vec{\underline{k}}\cdot\vec{\underline{q}}} \cdot \mathbf{F}_j(q,s). \tag{33}$$

The latter must be evaluated at $\vec{\underline{P}} = \mathrm{const}$ using the established relations between the k vectors (30):

$$\mathbf{F}_{\vec{\underline{k}}} = \frac{1}{(2\pi)^3} \cdot \int e^{-i\vec{\underline{k}}\cdot\vec{\underline{q}}} \cdot \mathbf{F}(q,s) \cdot d\underline{q}^3 \bigg|_{\underline{P}=\mathrm{const}} = \frac{4\pi e^2}{c} \cdot \int \frac{i\vec{k} \cdot \tilde{\rho}_{\vec{k}} \cdot dk^3}{\gamma_o{}^2 \cdot \beta_o{}^2 \cdot \vec{k}_\perp^2 + k_3{}^2} \cdot \frac{1}{(2\pi)^3} \int e^{i\vec{k}\cdot\vec{q}} e^{-i\vec{\underline{k}}\cdot\vec{\underline{q}}} \cdot d\underline{q}^3;$$
$$\frac{1}{(2\pi)^3} \cdot \int e^{i\vec{k}\cdot\vec{q}} e^{-i\vec{\underline{k}}\cdot\vec{\underline{q}}} \cdot d\underline{q}^3 = \frac{e^{i\vec{k}\cdot\overleftrightarrow{\mathbf{B}}\cdot\vec{\underline{P}}}}{(2\pi)^3} \int e^{i(\vec{k}\cdot\overleftrightarrow{\mathbf{A}} - \vec{\underline{k}})\cdot\vec{\underline{q}}} \cdot d\underline{q}^3 = e^{i\vec{k}\cdot\overleftrightarrow{\mathbf{B}}\cdot\vec{\underline{P}}} \delta(\vec{k} \cdot \overleftrightarrow{\mathbf{A}} - \vec{\underline{k}}) = \frac{e^{i\vec{\underline{k}}\cdot\overleftrightarrow{\mathbf{A}^{-1}\cdot\mathbf{B}}\cdot\vec{\underline{P}}}}{\det \mathbf{A}} \delta(\vec{k} - \vec{\underline{k}} \cdot \overleftrightarrow{\mathbf{A}}^{-1}),$$

resulting in

$$\frac{\partial \tilde{\underline{f}}_{\vec{\underline{k}}}(\underline{P},s)}{\partial s} = \frac{4\pi n_o e^2}{c} \cdot \frac{\tilde{\rho}(s,\vec{k}(s))}{\gamma_o(s)^2 \cdot \beta_o(s)^2 \cdot \vec{k}_\perp(s)^2 + k_3(s)^2} \frac{e^{i\vec{\underline{k}}\cdot\overleftrightarrow{\mathbf{A}^{-1}(s)\cdot\mathbf{B}(s)}\cdot\vec{\underline{P}}}}{\det \mathbf{A}(s)} \left(i\underline{k}_i \cdot \frac{\partial \underline{f}_o}{\partial \underline{P}_i}\right), \tag{34}$$

where we took into account that $k_j(s) A_{ji}(s) = \underline{k}_i$.

[10]Requirements for the local uniformity of the beam density (18) and (20), local phase-space distribution function must have a form of $f(P + \mathbf{V} \cdot q)$. Initial linear correlations between P and q can be incorporated into the transport matrix (22):

$$\mathbf{M}(s=0) = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{V} & \mathbf{I} \end{bmatrix}.$$

This equation can be easily integrated:

$$\underline{\tilde{f}}_{\vec{k}}(\underline{P},s)=\underline{\tilde{f}}_{\vec{k}}(\underline{P},0)+\frac{4\pi i n_o e^2}{c}\left(\underline{k}_i\frac{\partial \underline{f}_o}{\partial \underline{P}_i}\right)\int_o^s \frac{e^{i\vec{\underline{k}}\cdot\overleftrightarrow{\mathbf{A}^{-1}(\zeta)\mathbf{B}(\zeta)}\cdot\vec{\underline{P}}}}{\det \mathbf{A}(\zeta)}\frac{\tilde{\rho}_{\vec{k}}(\zeta)d\zeta}{\gamma_o(\zeta)^2\beta_o(\zeta)^2\vec{k}_\perp^2(\zeta)+k_3{}^2(\zeta)}, \tag{35}$$

where $\underline{\tilde{f}}_{\vec{k}}(\underline{P},0)\equiv\tilde{f}_{\vec{k}(0)}(P,s=0)$ is a Fourier harmonic of the initial perturbation. Rewriting (28) as

$$\tilde{\rho}[s,\vec{k}(s)]=\int\underline{\tilde{f}}_{\vec{k}}(\underline{P},s)\cdot e^{-i\vec{\underline{k}}\cdot\overleftrightarrow{\mathbf{A}(s)^{-1}\mathbf{B}(s)}\cdot\vec{\underline{P}}}d\underline{P}^3, \tag{36}$$

turns Eq. (35) into a directly solvable integral equation:

$$\tilde{\rho}[s,\vec{k}(s)]=\tilde{\rho}_{o\underline{k}}(s)+\frac{4\pi i\cdot e^2\cdot n_o}{c}\cdot\int_o^s\frac{\tilde{\rho}[\zeta,\vec{k}(\zeta)]\cdot d\zeta}{\det\mathbf{A}(\zeta)}\int\frac{e^{i[\vec{k}(\zeta)\cdot\overleftrightarrow{\mathbf{B}}(\zeta)-\vec{k}(s)\cdot\overleftrightarrow{\mathbf{B}}(s)]\cdot\vec{\underline{P}}}}{\gamma_o(\zeta)^2\cdot\beta_o(\zeta)^2\cdot\vec{k}_\perp^2(\zeta)+k_3{}^2(\zeta)}\cdot\underline{k}_i\cdot\frac{\partial\underline{f}_o}{\partial\underline{P}_i}\cdot d\underline{P}^3;$$
$$\tilde{\rho}_{o\underline{k}}(s)=\int e^{-i\vec{k}(s)\cdot\overleftrightarrow{\mathbf{B}}(s)\cdot\vec{\underline{P}}}\cdot\underline{\tilde{f}}_{\vec{k}}(\underline{P},0)\cdot d\underline{P}^3. \tag{37}$$

While this equation can already be used for evaluation of the instability, it can be further simplified by eliminating convolution $\sum_{i=1}^3\underline{k}_i\cdot\frac{\partial\underline{f}_o}{\partial\underline{P}_i}$. Integrating by parts

$$\int\frac{\partial\underline{f}_o}{\partial\underline{P}_i}\cdot\phi\cdot d\underline{P}_i=\underline{f}_o\cdot\phi|_{P_i=-\infty}^{P_i=\infty}-\int\underline{f}_o\cdot\frac{\partial\phi}{\partial\underline{P}_i}\cdot d\underline{P}_i \tag{38}$$

and $\underline{f}_o(\underline{P}_i=\pm\infty)=0$, we get

$$\sum_{i=1}^3\underline{k}_i\cdot\frac{\partial}{\partial\underline{P}_i}e^{i\vec{\underline{k}}\cdot[\overleftrightarrow{\boldsymbol{U}}(\zeta)-\overleftrightarrow{\boldsymbol{U}}(s)]\cdot\vec{\underline{P}}}=-i\cdot[u(s)-u(\zeta)]\cdot e^{i\vec{\underline{k}}\cdot[\overleftrightarrow{\boldsymbol{U}}(\zeta)-\overleftrightarrow{\boldsymbol{U}}(s)]\cdot\vec{\underline{P}}};$$
$$u(\zeta)=\vec{k}\cdot\overleftrightarrow{\boldsymbol{B}}(\zeta)\cdot\vec{\underline{k}}\equiv\sum_{i,j}\boldsymbol{B}_{ij}(\zeta)\cdot k_i\cdot\underline{k}_j=\sum_{i,j}[\boldsymbol{A}(\zeta)^{-1}\boldsymbol{B}(\zeta)]_{ij}\cdot\underline{k}_i\cdot\underline{k}_j \tag{39}$$

Combining Eqs. (37) and (38) brings us to the final form of the integral equation for the 3D SC instability:

$$\tilde{\rho}[s,\vec{k}(s)]=-\int_o^s\tilde{\rho}[\zeta,\vec{k}(\zeta)]\cdot K(\zeta)\cdot[u(s)-u(\zeta)]\cdot L_d(s,\zeta)\cdot d\zeta+\tilde{\rho}_{o\underline{k}}(s);$$
$$K(\zeta)=\frac{4\pi\cdot n_o\cdot e^2}{c\cdot\det\boldsymbol{A}(\zeta)\cdot\upsilon(\zeta)};\qquad L_d(\vec{\underline{k}},s,\zeta)=\int e^{i[\vec{k}(\zeta)\cdot\overleftrightarrow{\boldsymbol{B}}(\zeta)-\vec{k}(s)\cdot\overleftrightarrow{\boldsymbol{B}}(s)]\cdot\vec{\underline{P}}}\cdot\underline{f}_o(\underline{P})\cdot d\underline{P}^3;$$
$$u(\zeta)=\vec{k}(\zeta)\cdot\overleftrightarrow{\boldsymbol{B}}(\zeta)\cdot\vec{\underline{k}}\equiv\vec{\underline{k}}(\zeta)\cdot\overleftrightarrow{\boldsymbol{U}}(\zeta)\cdot\vec{\underline{k}};\qquad\upsilon(s)=\gamma_o(s)^2\cdot\beta_o(s)^2\cdot\vec{k}_\perp(s)^2+k_3(s)^2, \tag{40}$$

which can be solved numerically for any accelerator. Here we defined $\mathbf{U}=\mathbf{A}^{-1}\cdot\mathbf{B}$.

It is important to note that in the kernel of the integral equation (40), there is only one term, the Landau damping, L_d, which depends on both, the value and direction of the k vector. The u/υ is defined by the geometry (e.g., the direction of the initial k vector) and the components of the accelerator transport matrix in a form of matrix $\mathbf{U}=\mathbf{A}^{-1}\cdot\mathbf{B}$ and s-dependent denominator:

$$\frac{u(s)-u(\zeta)}{\upsilon(\zeta)} = \frac{\underline{\vec{\vartheta}} \cdot [\overleftrightarrow{\mathbf{U}}(\zeta) - \overleftrightarrow{\mathbf{U}}(s)] \cdot \underline{\vec{\vartheta}}}{\gamma_o(s)^2 \cdot \beta_o(s)^2 \cdot \vec{\vartheta}_\perp(s)^2 + \vec{\vartheta}_3(s)^2};$$
$$\vec{\vartheta}(s) = \frac{\vec{k}(s)}{|\vec{k}|}; \qquad \underline{\vec{\vartheta}} = \vec{\vartheta}(0) = \frac{\underline{\vec{k}}}{|\underline{\vec{k}}|}.$$

We show in Eq. (A3) of Appendix A that $\mathbf{A} \cdot \mathbf{B}^T = \mathbf{B} \cdot \mathbf{A}^T$, which also means that $\mathbf{U} = \mathbf{A}^{-1} \cdot \mathbf{B}$ is a symmetric matrix.

The most nontrivial construction is actually $\upsilon(\zeta)$, which is the result of the asymmetry between the longitudinal and transverse degrees of freedom introduced by the Lorentz transformation:

$$\upsilon(s) = \vec{k} \cdot \overleftrightarrow{\mathbf{G}}(s) \cdot \vec{k};$$
$$\overleftrightarrow{\mathbf{G}} = \mathbf{A}^{-1} \cdot \begin{bmatrix} \gamma_o{}^2\beta_o{}^2 & 0 & 0 \\ 0 & \gamma_o{}^2\beta_o{}^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot (\mathbf{A}^{-1})^T. \tag{41}$$

Furthermore, the convolution $u(s) = \underline{\vec{k}} \cdot \overleftrightarrow{\mathbf{U}}(\zeta) \cdot \underline{\vec{k}}$ has important nontrivial properties that it is a nonnegative monotonic function of s with positive derivative [see Eq. (A11) in Appendix A]:

$$u(s) \geq 0; \qquad u'(s) > 0. \tag{42}$$

Generally speaking, for a beam with an arbitrary momentum spread, Eq. (40) cannot be either evaluated analytically or reduced in complexity. But the physical nature of various terms can be identified by considering specific cases. For example, the integral over the momenta, known as Landau damping, can be easily evaluated for Gaussian distribution:

$$\underline{f}_o(P) = \prod_{i=1}^{3} \frac{1}{\sqrt{2\pi} \cdot \sigma_i} \cdot \exp\left(-\frac{\underline{P}_i^2}{2 \cdot \sigma_i{}^2}\right) \tag{43}$$

generating exponential term

$$L_d = \int e^{i[\vec{\eta}(\zeta)-\vec{\eta}(s)]\cdot\vec{P}} \cdot F_o(P) \cdot dP^3$$
$$= \prod_{i=1}^{3} \exp\left(-\frac{\sigma_i{}^2 \cdot [\eta_i(\zeta)-\eta_i(s)]^2}{2}\right);$$
$$\vec{\eta}(\zeta) = \vec{k}(\zeta) \cdot \overleftrightarrow{\boldsymbol{B}}(\zeta) \tag{44}$$

corresponding to the decay of the modulation during the interval (ξ, s).

To conclude this section, we would like to summarize that Eq. (40) is the most general equation that describes the evolution of the high-frequency modulation in beams driven by space-charge effects. It can describe all space-charge-driven instabilities from one-dimensional to three-dimensional. For example, it is easy to show that longitudinal microwave instability can be also described by this equation under a number of simplified assumptions. Specifically, the conventional theory of longitudinal microwave instability assumes that transverse and longitudinal motion are decoupled, and the energy modulation accumulated from SC forces is transferred into the density by R_{56} of the system, which in convention used in this paper equal to $\mathbf{M}_{36}$ (or $\mathbf{B}_{33}$).

Furthermore, coupling between transverse and longitudinal modulations, which occur in any bending magnetic system, is completely neglected in the conventional theory of longitudinal microwave instability. With uncoupled transverse and longitudinal degrees of freedom, Eq. (40) for longitudinal microbunching instability in compressing or decompressing electron bunch will take the following form:

$$\tilde{\rho}(s,k(s)) = -\frac{4\pi \cdot n_o \cdot e^2}{c} \int_0^s \frac{\tilde{\rho}(\zeta,k(\zeta))}{A_{11}(\zeta)A_{22}(\zeta)} \cdot \left(\frac{A_{33}(\zeta)}{A_{33}(s)} \cdot B_{33}(s) - B_{33}(\zeta)\right) \cdot L_d(s,\zeta) \cdot d\zeta + \tilde{\rho}_{o\underline{k}}(s);$$
$$L_d(s,\zeta) = \int e^{i(k(\zeta)\cdot B_{33}(\zeta) - k(s)\cdot B_{33}(s))\cdot \underline{P}} \underline{f}_o(\underline{P}) d\underline{P}; \qquad \tilde{\rho}_{o\underline{k}}(s) = \int e^{-ik(s)\cdot B_{33}(s)\cdot \underline{P}} \cdot \underline{\tilde{f}}_{\underline{k}}(\underline{P},0) \cdot d\underline{P},$$

where we considered that the wavelength of the modulation evolves with the longitudinal compression parameter A_{33}: $k(s) = \underline{k}/A_{33}$. Using the definitions and the set of parameters traditional for the description in the conventional theory of longitudinal microwave instability:

$$R_{11} = A_{11}; \qquad R_{33} = A_{22}; \qquad R_{55} = A_{33}; \qquad R_{56} = B_{33}; \qquad r_c = \frac{e^2}{mc^2};$$
$$\underline{P} = \delta \cdot E_o/c; \qquad E_o = \gamma_o mc^2; \qquad \underline{\tilde{f}}_o(\delta) = \frac{1}{\sqrt{2\pi}\sigma_z} \exp\left(-\frac{\delta^2}{2}\right).$$

one can get, for a noncompressing beam ($R_{55} = 1$) and $k = \text{const}$, a simple equation

$$\tilde{\rho}_k(s) = -\frac{4\pi \cdot n_o \cdot r_c}{\gamma_o} \int_0^s \frac{\tilde{\rho}_k(\zeta)}{R_{11}(\zeta)R_{33}(\zeta)} \cdot [R_{56}(s) - R_{56}(\zeta)] \cdot e^{-\frac{k^2\cdot[R_{56}(\zeta)-R_{56}(s)]\cdot\sigma_\delta{}^2}{2}} \cdot d\zeta + \tilde{\rho}_{ok}(s),$$

Representing the traditional 1D model for microbunching instability.

Since our theory includes an arbitrary local, linear coupling, Eq. (40) is a universal equation for the description of instabilities driven by the SC.

III. REDUCTION TO AN ORDINARY DIFFERENTIAL EQUATION

In this section, we review some specific cases when we can reduce the linear integral equation (40) to a second-order ordinary differential equation (ODE).

Let's consider cases when the Landau damping term allows the separation of variables s and ζ[11]:

$$L_d(s,\zeta) = \Lambda(\zeta)\cdot\Lambda^{-1}(s); \qquad \Lambda(s) = e^{-\phi(s)}; \tag{45}$$

[12] and the integral equation (40) becomes

$$\tilde{q}(s) = -\int_a^s \tilde{q}(\zeta)\cdot K(\zeta)\cdot\Big[u(s)-u(\zeta)\Big]\cdot d\zeta + \tilde{q}_o(s); \tag{46}$$

for a scaled density modulation $\tilde{q}(s) = e^{\phi(s)}\cdot\tilde{\rho}[s,\vec{k}(s)]$. A combination of the first and the second derivatives of Eq. (46) transfers it into a second-order ODE:

$$\begin{aligned}&\tilde{q}'' - \alpha'\cdot\tilde{q}' + K\cdot u'\cdot\tilde{q} = \tilde{q}_o'' - \alpha'\cdot\tilde{q}_o';\\ &\tilde{q}_o(s) = e^{\phi(s)}\cdot\tilde{\rho}_{\vec{k}0}(s); \quad \alpha = \ln\frac{u'}{u_o'}; \quad u_o' = u'(0),\end{aligned} \tag{47}$$

where we used the fact that $u' > 0$. This equation can be also reduced to an inhomogeneous Hill's equation

$$\begin{aligned}&\hat{q}'' + \hat{K}(s)\cdot\hat{q} = \varsigma(s);\\ &\hat{K}(s) = K(s)\cdot u'(s) - \frac{\alpha'(s)^2}{4} + \frac{\alpha''(s)}{2};\\ &\hat{q} = e^{-\frac{\alpha(s)}{2}}\tilde{q} \equiv \sqrt{\frac{u_o'}{u'(s)}}\cdot e^{\phi(s)}\cdot\tilde{\rho}(s,\vec{k}s);\\ &\varsigma(s) = e^{-\frac{\alpha(s)}{2}}\cdot[\tilde{q}_o''(s) - \tilde{q}_o'(s)\cdot\alpha'(s)].\end{aligned} \tag{48}$$

It is known [43] that the solution of a homogeneous Hill's equation is represented by a symplectic matrix:

$$\begin{bmatrix}\hat{q}(s)\\ \hat{q}'(s)\end{bmatrix} = \mathbf{R}(s)\cdot\begin{bmatrix}\hat{q}(0)\\ \hat{q}'(0)\end{bmatrix}; \qquad \mathbf{R} = \begin{bmatrix}r_{11} & r_{12}\\ r_{21} & r_{22}\end{bmatrix};$$
$$\mathbf{R}' = \begin{bmatrix}0 & 1\\ -\hat{K}(s) & 0\end{bmatrix}\cdot\mathbf{R}; \qquad \det\mathbf{R} = 1, \tag{49}$$

which also defines general solutions of the inhomogeneous equation:

$$\begin{aligned}\hat{q}(s) &= r_{11}(s)\cdot\hat{q}(0) + r_{12}(s)\cdot\hat{q}'(0)\\ &\quad + \int_0^s [r_{11}(\zeta)\cdot r_{12}(s) - r_{11}(s)\cdot r_{12}(\zeta)]\cdot\varsigma(\zeta)\cdot d\zeta.\end{aligned} \tag{50}$$

Hence, the solution of the homogenous Hill's equation $\hat{q}'' + \hat{K}(s)\cdot\hat{q} = 0$ can be used for the investigation of this instability when the separation (45) is possible.

A cold beam, which is very popular in the studies of instabilities, has momenta distribution,

$$\underline{f}_0(P) = \delta(P_1)\cdot\delta(P_2)\cdot\delta(P_3)$$

that definitely satisfies this requirement with $\phi = 1$. A more general and more interesting case is a beam with $\kappa - 1$, also known as Lorentzian, momentum distribution in all three directions:

$$F_0(P) = F_{\kappa-1}(P) = \frac{1}{\pi^3}\prod_{i=1}^{3}\frac{\sigma_i}{\sigma_i{}^2 + P_i{}^2}, \tag{51}$$

allowing to integrate over the momenta:

$$\begin{aligned}L_d(\zeta,s) &= \int e^{i[\vec{\eta}(\zeta)-\vec{\eta}(s)]\cdot\vec{P}}\cdot F_o(P)\cdot dP^3\\ &= e^{-\sum_{i=1}^{3}|\sigma_i|\cdot|\eta_i(s)-\eta_i(\zeta)|}.\end{aligned} \tag{52}$$

If condition $|\eta_i(s)| \geq |\eta_i(\zeta)|$; $s \geq \zeta$ is satisfied for all three components $i = 1,\ 2,\ 3$, we can use $|\eta_i(s) - \eta_i(\zeta)| = |\eta_i(s)| - |\eta_i(\zeta)|$ and the separated variables

$$L_d(s,\zeta) = e^{\phi(\zeta)}\cdot e^{-\phi(s)}; \quad \phi(\zeta) = \sum_{i=1}^{3}|\sigma_i\cdot\eta_i(\zeta)|; \tag{53}$$

resulting in a second-order ODE (50) for $\tilde{q}_{\vec{k}}(s) = \tilde{\rho}[s,\vec{k}(s)]\cdot\exp[\phi(s)]$, and in Hill's equation (48) for $\hat{q}(s) = \tilde{\rho}[s,\vec{k}(s)]\cdot\sqrt{\frac{u_o'}{u'(s)}}\cdot\exp[\phi(s)]$.

This is a good place to discuss the driving term, $\varsigma(s)$, on the right-hand side (rhs) of the Hill's equation

[11]Unfortunately, as can be seen from Eq. (44), such separation is impossible for Gaussian momenta distribution.

[12]It is easy to show that the separation $L_d(s,\zeta) = \Lambda_1(\zeta)\Lambda_2(s)$, with $\Lambda_2(s) \neq 0$ is also a sufficient condition. But we did not find cases, when such generalization is needed.

$$\varsigma(s) = e^{-\frac{\alpha}{2}} \cdot (\tilde{q}_o'' - \tilde{q}_o' \cdot \alpha');$$
$$\tilde{q}_o(s) = e^{\phi(s)} \cdot \tilde{\rho}_{\underline{\vec{k}}}(s) = e^{\phi(s)} \cdot \int e^{-i\vec{\eta}(s)\cdot\vec{\underline{P}}} \cdot \tilde{\underline{f}}_{\underline{\vec{k}}}(\underline{P}) \cdot d\underline{P}^3. \tag{54}$$

Generally speaking, for an arbitrary initial perturbation $\tilde{\underline{f}}(\underline{q}, \underline{P})$, both $\tilde{q}''$ and $\tilde{q}_o'$ are not equal to zero and Hill's equation remains inhomogeneous. One case is an exception: when the initial perturbation is a product of the density perturbation and a $\kappa - 1$ momenta distribution:

$$\tilde{\underline{f}}(\underline{q}, \underline{P}) = \tilde{\underline{\rho}}_o(\underline{q}) \cdot f_{\kappa 1}(\underline{P}) \to \int e^{-i\vec{\eta}(s)\cdot\vec{\underline{P}}} f_{\kappa 1}(\underline{P})$$
$$= \tilde{\underline{\rho}}_o(\underline{q}) \cdot e^{-\phi(s)};$$
$$\tilde{q}_o(s) = \tilde{\rho}_{\underline{\vec{k}}} = \int \tilde{\underline{\rho}}_o(\underline{q}) e^{-i\underline{\vec{k}}\cdot(\underline{q})} d\underline{q}^3 = \text{const}, \tag{55}$$

all derivatives of $\tilde{q}_o$ are equal to zero, and the Hill's equation becomes homogenous:

$$\tilde{\underline{f}}(P, Q) = \tilde{\underline{\rho}}_o(\underline{q}) \cdot f_{\kappa 1}(\underline{P}) \to \hat{q}'' + \hat{K}(s) \cdot \hat{q} = 0. \tag{56}$$

While the conditions $|\eta_i(s)| \geq |\eta_i(\zeta)|$; $s \geq \zeta$ are frequently satisfied, they also can be violated in the case of an arbitrary coupling. In fact, it is possible to construct a matrix $\mathbf{U}$ such that one component of the vector $\vec{\eta} = \underline{\vec{k}} \cdot \overleftrightarrow{\mathbf{U}}$ turns from a nonzero value at ζ to zero at $s > \zeta$. Emittance exchange lattices can serve as an example [44]. If even one of $|\eta_i(s)| \geq \eta_i(\zeta)$ conditions is violated, the separation becomes impossible and the use of the ODE is invalid. Nevertheless, the linear integral equation (40) is always solvable [43,45].

IV. SPECIAL CASES

We show in Appendix A [see Eq. (A14)], that in the case of an uncoupled motion, the matrix $\mathbf{U}$ is diagonal with monotonically growing diagonal terms:

$$\mathbf{U}(s) = [\delta_{ij}] \cdot \mu_i(s); \qquad \mu_i(0) = 0;$$
$$\mu_i(s) > \mu_i(\zeta) \quad \forall\ \zeta < s$$

where $[\delta_{ij}]$ is a unit matrix. It means that

$$|\eta_i(s)| = |k_{io}| \cdot \mu_i(s) \tag{57}$$

are also monotonic functions satisfying conditions $|\eta_i(s)| \geq |\eta_i(\zeta)|$; $s \geq \zeta$. Hence, we proved that in an arbitrary accelerator with decoupled motion, one can use a second-order ODE (47) or Hill's equation (48) for the beam with $\kappa - 1$ momentum (energy) distributions. This also includes linear accelerators using solenoids—the equations of motion can be easily decoupled by using torsion $\kappa_o(s) = -\frac{eB_s}{2p_o c}$, see Eq. (1) and Ref. [46].

For the beam with a constant density and constant energy propagating in a drift space, all matrices and all components in the equations can be easily evaluated:

$$\boldsymbol{A} = \boldsymbol{D} = \boldsymbol{I}; \qquad \boldsymbol{C} = 0; \qquad \boldsymbol{U}(s) = \boldsymbol{B}(s) = \frac{1}{\gamma_o \cdot \beta_o \cdot mc} \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s/(\gamma_o{}^2 \cdot \beta_o{}^2) \end{bmatrix};$$
$$\vec{k} = \text{const}; \qquad u = \frac{s}{\gamma_o{}^3 \cdot \beta_o{}^3 \cdot mc} (\gamma_o{}^2 \cdot \beta_o{}^2 \cdot \vec{k}_\perp^2 + k_3{}^2);$$
$$u'K = \frac{4\pi \cdot n_o \cdot e^2}{\gamma_o{}^3 \cdot \beta_o{}^3 \cdot mc} = \text{const}; \qquad u'' = 0. \tag{58}$$

For cold plasma oscillations, it results in $\vec{k}$-independent equation:

$$\frac{d^2\tilde{\rho}_{\vec{k}}}{ds^2} + k_p^2 \cdot \tilde{\rho}_{\vec{k}} = 0; \qquad k_p^2 = \frac{4\pi \cdot n_o \cdot e^2}{\gamma_o{}^3 \cdot \beta_o{}^3 \cdot mc}, \tag{59}$$

which, after applying the inverse Fourier transformation, becomes a carbon copy of the known plasma oscillations equation but in the laboratory frame:

$$\frac{d^2\tilde{\rho}(\vec{q})}{ds^2} + k_p^2 \cdot \tilde{\rho}(\vec{q}) = 0; \qquad \tilde{\rho}(\vec{q}) = \tilde{\rho}_o(\vec{q}) e^{i(k_p \cdot s - \omega_{pl} \cdot t)}; \qquad \omega_{pl} = c \cdot \beta_o \cdot k_p. \tag{60}$$

For beams with $\kappa - 1$ momentum distribution (51) propagating in a drift space with a constant density and constant energy, $\vec{k}$ dependence occurs via the Landau damping term and $q''_{\underline{\vec{k}}}$ in the driving term:

$$\tilde{q}'' + k_p^2(s)\cdot\tilde{q} = \tilde{q}_o''; \qquad \tilde{q} = e^{\phi(s)}\cdot\tilde{\rho}_{\vec{\underline{k}}}; \qquad \tilde{q}_o = e^{\phi(s)}\cdot\tilde{\rho}_{\underline{k}};$$
$$\phi(s) = \frac{s}{\gamma_o\cdot\beta_o\cdot mc}\cdot(\sigma_1\cdot|k_1| + \sigma_2\cdot|k_2| + (\gamma_o\beta_o)^{-2}\cdot\sigma_3\cdot|k_3|);$$
$$\tilde{\rho}_{\vec{\underline{k}}} = \int \exp\left(i\frac{s}{\gamma_o\cdot\beta_o\cdot mc}\cdot(k_1\cdot\sigma_1\cdot\underline{P}_1 + \sigma_2\cdot k_2\cdot\underline{P}_2 + (\gamma_o\beta_o)^{-2}\cdot\sigma_3\cdot k_3\cdot\underline{P}_3)\right)\cdot\underline{\tilde{f}}_{\vec{\underline{k}}}(\underline{P})\cdot d\underline{P}^3. \tag{61}$$

One important consequence of Eq. (61) is that for a beam propagating in a straight section, the Landau damping decrement for the transverse modulation is boosted by a factor ${\gamma_o}^2{\beta_o}^2$, which is typically $\gg 1$. In other words, for $|k_{1,2}|\cdot\sigma_{1,2}\propto|k_3|\cdot\sigma_3$, the Landau damping term is significantly larger than for $k_{1,2}\neq 0$. This is one of the reasons why the longitudinal instability plays a special role and is of a special interest for accelerators with ${\gamma_o}^2 \gg 1$.

Let's consider 1D longitudinal instability in a beam propagating along a straight trajectory, e.g., when the longitudinal and transverse motion is decoupled:

$$\mathbf{A}(s) = \begin{bmatrix}\mathbf{A}_\perp(s) & 0\\ 0 & a_\parallel(s)\end{bmatrix}; \qquad \mathbf{B}(s) = \begin{bmatrix}\mathbf{B}_\perp(s) & 0\\ 0 & b_\parallel(s)\end{bmatrix}; \qquad \vec{k}(s) = \hat{e}_3\cdot k(s) = \hat{e}_3\cdot\frac{k_o}{a_\parallel(s)}. \tag{62}$$

The evolution of this instability can be described either by an integral equation

$$\tilde{\rho}[s,k(s)] = -\frac{4\pi\cdot n_o\cdot e^2}{c}\cdot\int_o^s\tilde{\rho}[\zeta,k(\zeta)]\cdot K_\parallel(\zeta)\cdot[u(s)-u(\zeta)]\cdot d\zeta\int e^{-ik_o\cdot(u(s)-u(\zeta))\cdot\underline{P}}\cdot\underline{f}_\parallel(\underline{P})\cdot d\underline{P} + \tilde{\rho}_{\underline{k}}(s);$$
$$K_\parallel(\zeta) = \frac{4\pi\cdot n_o\cdot e^2}{c}\cdot\frac{a_\parallel(\zeta)}{\det\boldsymbol{A}_\perp(\zeta)}; \qquad u(\zeta) = \frac{b_\parallel(\zeta)}{a_\parallel(\zeta)}; \qquad \tilde{\rho}_{ok}(s) = \int e^{-ik_o u(s)\underline{P}}\cdot\underline{\tilde{f}}_{k'\parallel}(\underline{P})\cdot d\underline{P}; \tag{63}$$

or for $\kappa - 1$ longitudinal momentum distribution by the differential equation:

$$\tilde{q}'' - \xi(s)\cdot\tilde{q}' + k_p^2(s)\cdot\tilde{q} = \tilde{q}_o'' - \xi(s)\cdot\tilde{q}_o';$$
$$k_p^2(s) = \frac{4\pi}{(\gamma_o\beta_o)^3}\cdot\frac{n_o\cdot r_c}{a_\parallel\cdot\det\mathbf{A}_\perp}; \qquad \xi(s) = \frac{d}{ds}(\ln{a_\parallel}^2\cdot(\gamma_o\beta_o)^3); \qquad \tilde{q}(s) = \tilde{\rho}\left(s,\frac{k_o}{a_\parallel(s)}\right)e^{\frac{k_o\cdot b_\parallel(s)\sigma_3}{a_\parallel(s)}}, \tag{64}$$

where $k_p(s)$ is s-dependent plasma "frequency" (in s domain), $k/a_\parallel(s)$ is scaled wavenumber of the perturbation, and $\xi(s)$ represents an additional term, which, depending on its sign, either damps or amplifies modulation. The corresponding Hill's equation has the same driving term but slightly different s-dependent "frequency":

$$\hat{q}'' + k_p'^2\cdot\hat{q} = a_\parallel\cdot(\gamma_o\beta_o)^{3/2}\cdot\left(\tilde{q}_o'' - \tilde{q}_o'\frac{u''}{u'}\right); \qquad k_p'^2 = k_p^2 - \frac{\xi'^2}{4} + \frac{\xi''}{2}. \tag{65}$$

For a beam with constant energy and no compression ($a_\parallel = 1$), Eqs. (64) and (65) become identical. They describe the instability driven by transverse focusing:

$$a_\parallel = 1; \qquad \gamma_o\cdot\beta_o = \text{const};$$
$$b_\parallel = \frac{s}{(\gamma_o\cdot\beta_o)^3\cdot mc}\to\tilde{q}'' + k_p^2(s)\cdot\tilde{q} = \tilde{q}_o'';$$
$$k_p^2(s) = \frac{4\pi}{(\gamma_o\cdot\beta_o)^3}\cdot\frac{n_o\cdot r_c}{\det\boldsymbol{A}_\perp(s)}; \qquad \rho_k(s) = \tilde{q}(s)e^{-\frac{k\cdot s}{(\gamma_o\cdot\beta_o)^2}\cdot\frac{\sigma_{\gamma o}}{\gamma_o\cdot\beta_o}}. \tag{66}$$

V. DISCUSSIONS

To our knowledge, the analytical solution for the evolution of the finite-size beams that are strongly affected by the SC forces has never been found before. In this paper, we presented the most general theoretical description of 3D microscopic (short wavelength) instabilities driven by the SC forces. Our approach of solving this problem is based on the local linearization of the nonlinear transfer map, which includes the macroscopic SC forces. This approach allows arriving at a linear Vlasov equation for the microscopic perturbations at a scale much smaller than the other important scales of the problem (beam sizes, changes in the beam trajectory, scales of nonlinearity in the transfer map, etc.—see Eqs. (18)–(20).

As was suggested in the previous section, matrix **A** plays a special role in the evolution of the microscopic perturbation in beams with strong SC. First and foremost, approximation of the homogenous background results in the beam density, $n(s)$, to be inversely proportional to the determinant of the matrix **A**: $n(s) = n_o \det \mathbf{A}^{-1}$. It is also easy to show that this is no longer a problem for a beam with finite sizes and finite emittances where the beam density remains finite when $\det \mathbf{A} = 0$.[13] It means, that in the final expression for instability, we can use the calculated finite local density $n(s)$:

$$\tilde{\rho}[s, \vec{k}(s)] = -\int_o^s \frac{4\pi \cdot n(\zeta) \cdot e^2}{c \cdot v(\zeta)} \cdot \tilde{\rho}[\zeta, \vec{k}(\zeta)] \cdot [u(s) - u(\zeta)] \cdot L_d(s, \zeta) \cdot d\zeta + \tilde{\rho}_{\vec{k}_o}(s). \quad (67)$$

The second complication related to $\det \mathbf{A} \to 0$ arises with k vector transformation (29):

$$k^T(\zeta) = \underline{k}^T \cdot \mathbf{A}^{-1}(\zeta),$$

which involves an inversion of the matrix **A**. In contrast with the beam density, which remains finite for a finite beam size, the module of k vector is generally not limited: $|k|_{\det \mathbf{A} \to 0} \to \infty$.

This challenge can be resolved by noticing that the kernel in the integral remains finite:

$$\left|\frac{u(s) - u(\zeta)}{v(\zeta)}\right| \propto \frac{|k(s)|}{|k(\zeta)|^2} + \frac{|k(\zeta)|}{|k(\zeta)|^2} \propto \frac{\det \mathbf{A}(\zeta)^2}{\det \mathbf{A}(s)} + \det \mathbf{A}(\zeta).$$

For any finite momenta spreads and $|\mathbf{B}| \neq 0$, the Landau damping term vanishes when $|k| \to \infty$:

$$L_d = \int e^{i[\vec{k}(\zeta)\cdot\overleftrightarrow{\boldsymbol{B}}(\zeta) - \vec{k}(s)\cdot\overleftrightarrow{\boldsymbol{B}}(s)]\cdot\vec{P}} \cdot \underline{f}_o(P) \cdot dP^3 \underset{|\vec{k}(\zeta)\cdot\overleftrightarrow{\boldsymbol{B}}(\zeta)| + |\vec{k}(s)\cdot\overleftrightarrow{\boldsymbol{B}}(s)| \to \infty}{\Rightarrow} 0,$$

which resolves the issues in the evaluation of the critical compression points for an arbitrary initial perturbation in density momenta. With the known—to be exact, properly evaluated in the specific portion of the phase space—transport matrices, one can numerically solve equations (40) or (67)—see example in Appendix E.

An additional complication of using the method presented in this paper can arise when simultaneous compliance with conditions (18) and (20) is impossible. In other words, an accelerator lattice with strong filamentation of phase space caused by nonlinearity may not allow to define a phase-space area where linearization of the map (20) and conditions for homogenous background density (18) are compatible. For example, a multimillion turn map with strong amplitude-dependent tune shifts could epitomize such case. Still, any coupling and nonlinearities in the transfer map that can be linearized on a microscopic scale are included in our theory.

It is desirable to include CSR in the evolution of the microscopic density evolution, especially for the high energy accelerators with strong bends. Unfortunately, CSR perturbations are not local, and the application of the procedure used in this paper could be erroneous.

VI. CONCLUSIONS

In this paper, we derived a linear integral equation uniquely describing the evolution of the 3D microscopic density modulation driven by SC forces, including instabilities. Our theory includes the coupling between the density modulation in various degrees of freedom, for example, occurring in a bend, in an SQ quadrupole, or in a transverse deflecting cavity. It also includes effects of compression in all three directions, rotation, energy chirp, and acceleration (deceleration) of the beams as well as Landau damping. Most likely it will be the most useful for investigating beam stability in low-energy linear accelerators with SC-dominated beams.

We also derived the conditions when the linear integral equation can be reduced to an ordinary second-order

[13] For a Gaussian distribution, $m_{11} = 0$ simply means a rotation by 90° in the phase space and the momentum spread determines the density:

$$\underline{f}_o = \frac{1}{2\pi\sigma_q\sigma_P}\exp\left(-\frac{\underline{q}^2}{2\sigma_q^2} - \frac{\underline{P}^2}{2\sigma_P^2}\right) = \frac{1}{2\pi\sigma_q\sigma_P} \times \exp\left(-\frac{(m_{22}q - m_{12}P)^2}{2\sigma_Q^2} - \frac{(-m_{21}q + m_{11}P)^2}{2\sigma_P^2}\right);$$

$$\rho(q, s) = \int_{-\infty}^{\infty} \underline{f}_o d\underline{P} = \frac{1}{\sqrt{2\pi({m_{11}}^2\sigma_q^2 + {m_{12}}^2\sigma_P^2)}} \times \exp\left(-\frac{q^2}{2({m_{11}}^2\sigma_q^2 + {m_{12}}^2\sigma_P^2)}\right).$$

differential equation and demonstrated applications of our method for a set of special cases.

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APPENDIX A: SYSTEM HAMILTONIAN AND EQUATIONS OF MOTION

Traditionally in the accelerator physics literature, the phase-space vector consists of Canonical pairs of coordinates and momenta $[\ldots(q_i, P^i)\ldots] \equiv [\ldots(x_{2i-1}, x_{2i})\ldots]$. In this paper, following Dragt [34], we use an equivalent, but the different structure of the *phase-space* vector, which clearly separates coordinates and momenta and simplifies the form of the matrix of a symplectic generator, $\mathbf{S}$;

$$\xi^T = [\xi_1, \xi_2, .., \xi_{2n}] \equiv [q^T, P^T]; \quad q^T = [q_1, .., q_n]; \quad P^T = [P_1, \ldots, P_n]; \left\{ \begin{array}{l} \frac{dq_i}{ds} = \frac{\partial H}{\partial P_i} \\ \frac{dP_i}{ds} = -\frac{\partial H}{\partial q_i} \end{array} \right\} \Leftrightarrow \frac{d\xi}{ds} = \boldsymbol{S} \cdot \frac{\partial H}{\partial \xi} \Leftrightarrow \frac{d\xi_i}{ds} = S_{ik} \frac{\partial H}{\partial \xi_k};$$

$$\boldsymbol{S} = [S_{ik}] = \begin{bmatrix} \mathbf{0} & \boldsymbol{I}_{n\times n} \\ -\boldsymbol{I}_{n\times n} & \mathbf{0} \end{bmatrix}; \quad \boldsymbol{I}_{n\times n} = \begin{bmatrix} 1 & .... & 0 \\ ... & ... & ... \\ 0 & ... & 1 \end{bmatrix}. \tag{A1}$$

The use of these notations is especially convenient for linear maps in the form of $2n \times 2n$ symplectic transport matrices:

$$\begin{gathered} \xi(s_2) \equiv \begin{bmatrix} q(s_2) \\ P(s_2) \end{bmatrix} = \mathbf{M}(s_1|s_2) \cdot \xi(s_1) \equiv \mathbf{M}(s_1|s_2) \cdot \begin{bmatrix} q(s_1) \\ P(s_1) \end{bmatrix}; \\ \mathbf{M}^T \cdot \mathbf{S} \cdot \mathbf{M} = \mathbf{M} \cdot \mathbf{S} \cdot \mathbf{M}^T = \mathbf{S}; \qquad \mathbf{M}^{-1} = -\mathbf{S} \cdot \mathbf{M}^T \cdot \mathbf{S}; \\ \mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}; \qquad \mathbf{M}^{-1} = \begin{bmatrix} \mathbf{D}^T & -\mathbf{B}^T \\ -\mathbf{C}^T & \mathbf{A}^T \end{bmatrix}; \\ q(s_2) = \mathbf{A}q(s_1) + \mathbf{B} \cdot P(s_1); \qquad P(s_2) = \mathbf{C}q(s_1) + \mathbf{D} \cdot P(s_1); \\ q(s_1) = \mathbf{D}^T q(s_2) - \mathbf{B}^T \cdot P(s_2); \qquad P(s_1) = -\mathbf{C}^T q(s_2) + \mathbf{A}^T \cdot P(s_2); \end{gathered} \tag{A2}$$

providing explicit connection between the coordinates and the momenta with their initial values and vice versa. It also provides important properties of the block matrices which can be very useful for the evaluation of complex expressions. Specifically, symplecticity of the transport matrix requires that four $n \times n$ matrices $\mathbf{A} \cdot \mathbf{B}^T$, $\mathbf{D} \cdot \mathbf{C}^T$, $\mathbf{A}^T \cdot \mathbf{C}$, $\mathbf{D}^T \cdot \mathbf{B}$ will be symmetric

$$(\mathbf{A} \cdot \mathbf{B}^T)^T = \mathbf{A} \cdot \mathbf{B}^T; \qquad (\mathbf{D} \cdot \mathbf{C}^T)^T = \mathbf{D} \cdot \mathbf{C}^T, \qquad (\mathbf{A}^T \cdot \mathbf{C})^T = \mathbf{A}^T \cdot \mathbf{C}, \qquad (\mathbf{D}^T \cdot \mathbf{B})^T = \mathbf{D}^T \cdot \mathbf{B} \tag{A3}$$

and that

$$\mathbf{A} \cdot \mathbf{D}^T - \mathbf{B} \cdot \mathbf{C}^T = \mathbf{I}; \qquad \mathbf{A}^T \cdot \mathbf{D} - \mathbf{C}^T \cdot \mathbf{B} = \mathbf{I}. \tag{A4}$$

In the case of uncoupled motion, all four matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ become diagonal automatically satisfying conditions (A3) and turning (A4) into simpler conditions for diagonal components of the block matrices:

$$A_{ii} \cdot D_{ii} - B_{ii} \cdot C_{ii} = 1; \qquad i = 1, .., n \tag{A5}$$

equivalent to the unity of determinants for individual 2×2 matrices in notations (A2):

$$M_i = \begin{bmatrix} A_{ii} & B_{ii} \\ C_{ii} & D_{ii} \end{bmatrix}; \qquad \det M_i = 1. \tag{A6}$$

Matrix $\mathbf{A}^{-1} \cdot \mathbf{B}$ plays a critical role in the instability integral equation (40). Its properties can be studied for a Hamiltonian system describing a generic linear system:

$$H = \frac{1}{2}\xi^T \cdot \mathbf{H}(s) \cdot \xi \equiv \frac{1}{2}\sum_{i,k=1}^{2n} h_{ik} \cdot \xi \cdot_i \xi_k \equiv \frac{1}{2}\left(\sum_{i,k=1}^{n} h_{q\_ik} \cdot q_i \cdot q_k + \sum_{i,k=1}^{n} h_{p\_ik} \cdot P_i \cdot P_k + \sum_{i,k=1}^{n} h_{m\_ik} \cdot q_i \cdot P_k\right)$$
$$\mathbf{H} = [h_{ik}] = \begin{bmatrix} \mathbf{H}_q & \mathbf{H}_m^T \\ \mathbf{H}_m & \mathbf{H}_p \end{bmatrix}; \qquad \mathbf{H}^T = \mathbf{H}. \tag{A7}$$

Using equations of motion, the derivative of the matrix M becomes

$$\mathbf{M}' \equiv \frac{d\mathbf{M}}{ds} = \mathbf{S} \cdot \mathbf{H} \cdot \mathbf{M}; \qquad \mathbf{A}' = \mathbf{H}_m \cdot \mathbf{A} + \mathbf{H}_p \cdot \mathbf{C}; \qquad \mathbf{B}' = \mathbf{H}_m \cdot \mathbf{B} + \mathbf{H}_p \cdot \mathbf{D}. \tag{A8}$$

Taking into account that $(\mathbf{A}^{-1})' = -\mathbf{A}^{-1} \cdot \mathbf{A}' \cdot \mathbf{A}^{-1}$, we get

$$(\mathbf{A}^{-1} \cdot \mathbf{B})' = \mathbf{A}^{-1} \cdot \mathbf{B}' - \mathbf{A}^{-1} \cdot \mathbf{A}' \cdot \mathbf{A}^{-1} \cdot \mathbf{B} = \mathbf{A}^{-1} \cdot \mathbf{H}_p \cdot (\mathbf{D} - \mathbf{C} \cdot \mathbf{A}^{-1} \cdot \mathbf{B}), \tag{A9}$$

and using symplecticity conditions (A4) $\mathbf{A}^{-1}\mathbf{B} = \mathbf{B}^T(\mathbf{A}^T)^{-1}$ and $\mathbf{D}\mathbf{A}^T - \mathbf{C}\mathbf{B}^T = \mathbf{I}$, one can show that it can be turned into

$$\mathbf{D} - \mathbf{C} \cdot \mathbf{A}^{-1} \cdot \mathbf{B} = (\mathbf{D}\mathbf{A}^T - \mathbf{C} \cdot \mathbf{B}^T)(\mathbf{A}^{-1})^T = (\mathbf{A}^{-1})^T;$$
$$(\mathbf{A}^{-1} \cdot \mathbf{B})' = \mathbf{A}^{-1} \cdot \mathbf{H}_p \cdot (\mathbf{A}^{-1})^T. \tag{A10}$$

It is possible to show for an arbitrary accelerator [34–36] that $\mathbf{H}_p$ is a diagonal matrix with positive diagonal terms:

$$\mathbf{H}_p = \frac{1}{\gamma_o \beta_o mc}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (\gamma_o\beta_o)^{-2} \end{bmatrix} \tag{A11}$$

This allows us to prove that for an arbitrary accelerator with an invertible matrix $\mathbf{A}$, the convolution $u(s)$ in Eq. (40) is a non-negative monotonically growing function with $u'(s) > 0$.

$$u(\zeta) = \vec{k}(\zeta) \cdot \overleftrightarrow{\boldsymbol{B}}(\zeta) \cdot \underline{\vec{k}} \equiv \vec{k}(\zeta) \cdot \overleftrightarrow{\boldsymbol{U}}(\zeta) \cdot \underline{\vec{k}}$$
$$u(s) = \underline{k}^T \cdot [\boldsymbol{A}(s)^{-1} \cdot \boldsymbol{B}(s)] \cdot \underline{k} \equiv \underline{\vec{k}} \cdot \overleftrightarrow{(\mathbf{A}^{-1} \cdot \boldsymbol{B})} \cdot \underline{\vec{k}};$$
$$\boldsymbol{B}(0) = 0 \to u(0) = 0; \qquad \vec{k}(s) = \boldsymbol{A}^T(s)^{-1}\underline{\vec{k}};$$
$$u'(s) = \underline{k}^T \cdot (\mathbf{A}^{-1} \cdot \boldsymbol{B})' \cdot \underline{k} = k^T(s) \cdot \boldsymbol{H}_p(s) \cdot k(s) = \sum_{i=1}^{3} \boldsymbol{H}_{ii}(s) \cdot k_i{}^2(s) > 0.$$
$$u(s) = \int_0^s \left(\sum_{i=1}^{3} \boldsymbol{H}_{ii}(s) \cdot k_i{}^2(s)\right) \cdot d\zeta \geq 0. \tag{A12}$$

In other words, we proved that the convolution of any constant vector with matrix $\mathbf{A}^{-1} \cdot \mathbf{B}$ is a non-negative monotonically growing function.

In the case of uncoupled motion, all 3×3 matrices are diagonal and

$$\frac{d}{ds}(\mathbf{A}^{-1} \cdot \mathbf{B}) = \begin{bmatrix} \alpha_1(s) & 0 & 0 \\ 0 & \alpha_2(s) & 0 \\ 0 & 0 & \alpha_3(s) \end{bmatrix} = \frac{1}{\gamma_o \cdot \beta_o \cdot mc}\begin{bmatrix} a_{11}{}^{-2} & 0 & 0 \\ 0 & a_{22}{}^{-2} & 0 \\ 0 & 0 & (a_{33} \cdot \gamma_o \cdot \beta_o)^{-2} \end{bmatrix};$$
$$\alpha_i(s) > 0;$$
$$\mu_i(s) = \int_0^s \alpha_i(\zeta) \cdot d\zeta \geq 0; \qquad \mathbf{A}^{-1} \cdot \mathbf{B} = \int_0^s (\mathbf{A}^{-1} \cdot \mathbf{B})' d\zeta = \begin{bmatrix} \mu_1(s) & 0 & 0 \\ 0 & \mu_2(s) & 0 \\ 0 & 0 & \mu_3(s) \end{bmatrix} = \delta_{ij}\mu_i. \tag{A13}$$

i.e., diagonal terms of matrix $\mathbf{A}^{-1}\mathbf{B}$ are monotonically growing positive functions:

$$\forall\ s_1 > s_2; \qquad \mu_i(s_1) > \mu_i(s_2). \tag{A14}$$

APPENDIX B: CONDITIONS FOR APPLICABILITY OF THE SHORT PERIOD (MICROSCOPIC) PERTURBATIONS

These conditions are also known as assumptions of homogeneous infinite plasma. Fourier or Laplace transformations are frequently used to solve the linearized Vlasov equation. The main problem arising from the use of an inhomogeneous distribution (or finite size of the beam) is that it results in coupling between the Fourier harmonics of the perturbation and those of the background, e.g., applying a Fourier transformation to Eq. (11)

$$\begin{aligned}&\int dQ^3\cdot e^{-i\vec{k}\cdot\vec{Q}\cdot}\left(\frac{\partial\tilde{\underline{f}}}{\partial s}+\frac{\partial \underline{f}_o}{\partial\vec{\underline{q}}}\cdot\frac{\partial\tilde{\underline{h}}}{\partial\vec{\underline{P}}}-\frac{\partial\underline{f}}{\partial\vec{\underline{P}}}\cdot\frac{\partial\tilde{\underline{h}}}{\partial\vec{\underline{q}}}\right)\\&=\frac{\partial\tilde{\underline{f}}_{\vec{k}}}{\partial t}+i\int d\mathbf{k}\,.^3\left\{\underline{f}_{o\vec{\mathbf{k}}}\cdot\left(\vec{\mathbf{k}}\cdot\frac{\partial\tilde{\underline{h}}_{\vec{k}-\vec{\mathbf{k}}}}{\partial\vec{\underline{P}}}\right)\right.\\&\quad\left.-\tilde{\underline{h}}_{\vec{k}-\vec{\mathbf{k}}}\cdot\left((\vec{k}-\vec{\mathbf{k}})\cdot\frac{\partial\underline{f}_{o\vec{\mathbf{k}}}}{\partial\vec{\underline{P}}}\right)\right\},\end{aligned}$$

does not result in the separation of the Fourier harmonics. In this sense, this equation is as complicated as the original Vlasov equation. The conditions for the separation of the Fourier harmonics are easiest to derive in the comoving frame of reference. In this Appendix, we will use indices "*cm*" and "*lab*" to distinguish between the comoving and laboratory frames, correspondingly.

In the vicinity of azimuth s_o, the particle's trajectory in the laboratory frame can be described using the Cartesian coordinate system with three fixed orthogonal unit vectors $\hat{e}_{1,2,3}(s_o)$ (see Sec. II):

$$\vec{r}_{\rm lab}=\vec{r}_o(s_o)+\hat{e}_1(s_o)\cdot x+\hat{e}_2(s_o)\cdot y+\hat{e}_3(s_o)\cdot z.$$

Let's consider an instantaneous comoving frame that propagates in the vicinity of azimuth s_o with velocity

$$\vec{\mathrm{v}}_o=\mathrm{v}_o(s_o)\cdot\hat{e}_3(s_o),$$

along the local z axis. The next step is to establish relations between the parameters in the laboratory and the comoving frame. It is known that Lorentz transformation does not affect transverse coordinates x, y, but boosts longitudinal coordinates by the relativistic factor $\gamma_o=(1-\frac{\mathrm{v}_o{}^2}{c^2})^{-1/2}$:

$$z_{cm}=\gamma_o z\Rightarrow\vec{r}_{cm}=\hat{x}\cdot x+\hat{y}\cdot y+\gamma_o\hat{z}\cdot z;$$

and also transforms the 4-vectors $\hat{k}=(\omega/c,\vec{k})$ as [37]:

$$\begin{aligned}&\vec{k}_{cm}\equiv\vec{\boldsymbol{k}}=\hat{z}\cdot\boldsymbol{k}_z+\vec{\boldsymbol{k}}_\perp;\qquad\omega_{cm}=0;\\&\vec{k}_{\rm lab}\equiv\vec{k}=\hat{z}\cdot k_z+\vec{k}_\perp;\\&\vec{k}_\perp=\vec{\boldsymbol{k}}_\perp;\qquad k_z=\gamma_o\cdot\left(\boldsymbol{k}_z+\beta_o\cdot\frac{\omega_{cm}}{c}\right)=\gamma_o\boldsymbol{k}_z;\\&\omega_{\rm lab}=\gamma_o\cdot(\omega_{cm}+v_o\cdot\boldsymbol{k}_z)=v_o\cdot k_z,\end{aligned}\tag{B1}$$

with known relation between exponents in Fourier transforms:

$$e^{i\vec{\mathbf{k}}\cdot\vec{r}_{cm}}=e^{i\{\vec{k}\cdot\vec{r}_{\rm lab}-\mathrm{v}_o\cdot k_z\cdot[t-t_o(s_o)]\}}=e^{i(\vec{k}\cdot\vec{r}_{\rm lab}+\frac{k_3\cdot q_3}{\beta_o(s_o)})}$$

providing us with an important connection with the k vector defined in Eq. (18) of the main text:

$$\begin{aligned}&k_1=k_x=\mathbf{k}_x;\qquad k_2=k_y=\mathbf{k}_y;\\&k_3=\beta_o\cdot k_z=\gamma_o\cdot\beta_o\cdot\mathbf{k}_z;\\&a_1=a_x;\qquad a_2=a_y;\qquad a_3=a_z/\beta_o.\end{aligned}\tag{B2}$$

Let's consider a beam with typical scales of inhomogeneity, $a_{x,y,z}$, which are not necessarily of the same order of magnitude:

$$\left|\frac{\partial f_o}{\partial x}\right|\propto\frac{f_o}{a_x};\qquad\left|\frac{\partial f_o}{\partial y}\right|\propto\frac{f_o}{a_y};\qquad\left|\frac{\partial f_o}{\partial z}\right|\propto\frac{f_o}{a_z}$$

defined in the laboratory frame. Mathematically, we can define inhomogeneity parameters in the phase-space section Ω as

$$a_{x,y,z}=\min\left(\left|\frac{f_o(X)}{\partial_{x,y,z}f_o(X)}\right|\right);\qquad X\in\Omega.$$

As indicated above, transverse scales will remain the same in the comoving frame, but the longitudinal scale will be boosted by a factor γ_o.

For simplicity, we will consider that the particle motion in the comoving frame is nonrelativistic, and we can neglect the effects of the magnetic field, e.g., assume $\vec{B}_{cm}=0$. In this case, Maxwell's equations are reduced to two equations for the electric field:

$$\mathrm{div}\vec{E}=4\pi\cdot\rho;\qquad\mathrm{curl}\vec{E}=0.\tag{B3}$$

Further in this Appendix, we will use the comoving frame and will drop the index "*cm*." The natural condition for neglecting the beam's edges, transitions, and reflection effects is that there must be a significant number of oscillations in each direction at the typical scales of the inhomogeneity, e.g.,

$$\mathbf{k}_{x,y}\cdot a_{x,y}=k_{x,y}\cdot a_{x,y}\gg 2\pi;\qquad\gamma\mathbf{k}_z\cdot a_z=k_z\cdot a_z\gg 2\pi.\tag{B4}$$

But as we find out in this Appendix, not all of these "natural" conditions are necessary. For example, it is intuitively understandable that for a one-dimensional perturbation with $\mathbf{k}_z \neq 0$, $\mathbf{k}_{x,y} = 0$, two requirements in (B4), $\mathbf{k}_{x,y} \cdot a_{x,y} \gg 2\pi$ are not necessary. As we will also show, the condition (B4) in ith direction is not necessary if $|\mathbf{k}_i| \ll |\vec{\mathbf{k}}|$.

Second, and much more convoluted, condition is that the Fourier harmonic of the induced electric field (and therefore of the perturbation in the Hamiltonian) are linear functions of the harmonic of the charge density perturbation,

$$\rho_{\vec{\mathbf{k}}} = e \cdot \int_{-\infty}^{\infty} d\vec{r} \cdot e^{-i\vec{\mathbf{k}}\cdot\vec{r}} \cdot \int_{-\infty}^{\infty} \tilde{f}_{cm}(\vec{r},\vec{v},t) \cdot d\vec{\mathrm{v}}. \tag{B5}$$

where $\tilde{f}_{cm}$ is a perturbation of the distribution function in the comoving frame. In an infinite charged plasma, a periodic density perturbation results in a periodic electric field aligned with the $\vec{\mathbf{k}}$-vector:

$$\vec{E} = \vec{E}_{\vec{\mathbf{k}}} \cdot e^{i\vec{\mathbf{k}}\cdot\vec{r}}; \qquad \vec{E}_{\vec{\mathbf{k}}} = \vec{E}_{\vec{\mathbf{k}}_\parallel} + \vec{E}_{\vec{\mathbf{k}}_\perp};$$
$$\mathrm{curl}\vec{E} = 0 \to \vec{\mathbf{k}} \times \vec{E}_{\vec{\mathbf{k}}} = 0 \to \vec{E}_{\vec{\mathbf{k}}\perp} = 0 \to \vec{E}_{\vec{\mathbf{k}}\parallel} = \frac{\vec{\mathbf{k}}}{|\vec{\mathbf{k}}|} E_{\vec{\mathbf{k}}};$$
$$\mathrm{div}\vec{E} = 4\pi \cdot \rho_{\vec{\mathbf{k}}} \cdot e^{i\vec{\mathbf{k}}\cdot\vec{r}} \to i\vec{\mathbf{k}} \cdot \vec{E}_{\vec{\mathbf{k}}} = i|\vec{\mathbf{k}}| \cdot E_{\vec{\mathbf{k}}} = 4\pi \cdot \rho_{\vec{\mathbf{k}}}, \tag{B6}$$

resulting in

$$\vec{E}_{\vec{\mathbf{k}}} = -4\pi i \cdot \rho_{\vec{\mathbf{k}}} \cdot \frac{\vec{\mathbf{k}}}{\vec{\mathbf{k}}^2}. \tag{B7}$$

For a nonuniform density, the electric field will deviate from the intuitive extensions of (B7) by some $\delta\vec{E}$:

$$\vec{E} = 4\pi \cdot \rho_{\vec{\mathbf{k}}}(\vec{r}) \cdot \frac{\vec{\mathbf{k}}}{i\vec{\mathbf{k}}^2} e^{i\vec{k}\cdot\vec{r}} + \delta\vec{E}. \tag{B8}$$

We can neglect $\delta\vec{E}$ when compared with the main rhs term in (B7) when $|\vec{\mathbf{k}}| \cdot |\delta\vec{E}| \ll 4\pi \cdot |\rho_{\vec{\mathbf{k}}}|$. We get the following using (B3):

$$\begin{aligned}
\mathrm{div}\vec{E} &= 4\pi \cdot \rho_{\vec{k}}(\vec{r}) e^{i\vec{k}\cdot\vec{r}} + 4\pi \cdot \frac{[\vec{k} \cdot \vec{\nabla}\rho_{\vec{k}}(\vec{r})]}{\vec{k}^2} e^{i\vec{k}\cdot\vec{r}} + \mathrm{div}\delta\vec{E} \\
&= 4\pi \cdot \rho_{\vec{k}}(\vec{r}) \cdot e^{i\vec{k}\cdot\vec{r}}; \\
\mathrm{curl}\vec{E} &= 4\pi \cdot \frac{[\vec{k} \times \vec{\nabla}\rho_{\vec{k}}(\vec{r})]}{\vec{k}^2} e^{i\cdot\vec{k}\cdot\vec{r}} + \mathrm{curl}\delta\vec{E} = 0; \\
\mathrm{div}\delta\vec{E} &= -4\pi \frac{[\vec{k} \cdot \vec{\nabla}\rho_{\vec{k}}(\vec{r})]}{\vec{k}^2} e^{i\vec{k}\cdot\vec{r}} \sim |\vec{k}| \cdot |\delta\vec{E}| \\
\mathrm{curl}\delta\vec{E} &= -4\pi \frac{(\vec{k} \times \vec{\nabla}\rho_{\vec{k}}(\vec{r}))}{\vec{k}^2} e^{i\vec{k}\cdot\vec{r}} \sim |\vec{k}| \cdot |\delta\vec{E}|
\end{aligned} \tag{B9}$$

While the error estimation resulting from $\mathrm{div}\vec{E} = 4\pi \cdot \rho_{\vec{\mathbf{k}}}$ improves on the intuitive requirement (B4):

$$\begin{aligned}
&\frac{\partial\delta E_x}{\partial x} + \frac{\partial\delta E_y}{\partial y} + \frac{\partial\delta E_z}{\partial z} = -4\pi \cdot \frac{e^{i\vec{\mathbf{k}}\cdot\vec{r}}}{\vec{\mathbf{k}}^2} \cdot \left(\mathbf{k}_x \cdot \frac{\partial\rho_{\vec{k}}}{\partial x} + \mathbf{k}_y \cdot \frac{\partial\rho_{\vec{k}}}{\partial y} + \mathbf{k}_z \cdot \frac{\partial\rho_{\vec{k}}}{\partial y}\right); \qquad \left|\frac{\partial\rho_{\vec{\mathbf{k}}}}{\partial x_i}\right| \sim \frac{|\rho_{\vec{\mathbf{k}}}|}{a_i}; \\
&\left|-ie^{i\vec{\mathbf{k}}\cdot\vec{r}} \cdot \left(\frac{\partial\delta E_x}{\partial x} + \frac{\partial\delta E_y}{\partial y} + \frac{\partial\delta E_z}{\partial z}\right)\right| \sim (|\mathbf{k}_x \cdot \delta E_x| + |\mathbf{k}_y \cdot \delta E_y| + |\mathbf{k}_z \cdot \delta E_y|) \sim 4\pi \cdot \frac{|\rho_{\vec{\mathbf{k}}}|}{k^2} \cdot \left(\frac{|\mathbf{k}_x|}{a_x} + \frac{|\mathbf{k}_y|}{a_y} + \frac{|\mathbf{k}_z|}{\gamma a_z}\right) \\
&\left|\frac{\mathrm{div}\delta\vec{E}}{\mathrm{div}\vec{E}}\right| \sim \frac{|\frac{\mathbf{k}_x}{a_x}| + |\frac{\mathbf{k}_y}{a_y}| + |\frac{\mathbf{k}_z}{\gamma a_z}|}{\mathbf{k}^2} \ll 1
\end{aligned} \tag{B10}$$

the error estimations resulting from $\mathrm{curl}\vec{E} = 0$

$$\begin{aligned}
\mathrm{curl}\delta\vec{E} &= -4\pi \frac{(\vec{k} \times \vec{\nabla}\rho_{\vec{k}}(\vec{r}))}{\vec{k}^2} e^{i\vec{k}\cdot\vec{r}} = \frac{4\pi}{\vec{k}^2} \cdot e^{i\vec{k}\cdot\vec{r}} \\
&\quad \times \left\{\hat{x} \cdot \left(\mathbf{k}_y \cdot \frac{\partial\rho_{\vec{k}}}{\partial z} - \mathbf{k}_z \cdot \frac{\partial\rho_{\vec{k}}}{\partial y}\right) + \hat{y} \cdot \left(\mathbf{k}_z \cdot \frac{\partial\rho_{\vec{k}}}{\partial x} - \mathbf{k}_x \cdot \frac{\partial\rho_{\vec{k}}}{\partial z}\right) + \hat{z} \cdot \left(\mathbf{k}_x \cdot \frac{\partial\rho_{\vec{k}}}{\partial y} - \mathbf{k}_y \cdot \frac{\partial\rho_{\vec{k}}}{\partial x}\right)\right\} \\
&\frac{\partial\delta E_z}{\partial y} - \frac{\partial\delta E_y}{\partial z} = \frac{4\pi}{\vec{k}^2} e^{i\vec{k}\cdot\vec{r}} \cdot \left(\mathbf{k}_y \cdot \frac{\partial\rho_{\vec{k}}}{\partial z} - \mathbf{k}_z \cdot \frac{\partial\rho_{\vec{k}}}{\partial y}\right); \qquad \frac{\partial\delta E_x}{\partial z} - \frac{\partial\delta E_z}{\partial x} = \frac{4\pi}{\vec{k}^2} \cdot e^{i\vec{k}\cdot\vec{r}} \left(\mathbf{k}_z \cdot \frac{\partial\rho_{\vec{k}}}{\partial x} - \mathbf{k}_x \cdot \frac{\partial\rho_{\vec{k}}}{\partial z}\right); \\
&\frac{\partial\delta E_y}{\partial x} - \frac{\partial\delta E_x}{\partial y} = \frac{4\pi}{\vec{k}^2} \cdot e^{i\vec{k}\cdot\vec{r}} \cdot \left(\mathbf{k}_x \cdot \frac{\partial\rho_{\vec{k}}}{\partial y} - \mathbf{k}_y \cdot \frac{\partial\rho_{\vec{k}}}{\partial x}\right)
\end{aligned} \tag{B11}$$

is much more important because it links all three dimensions:

$$\left|\frac{\partial \delta E_z}{\partial y}\right| + \left|\frac{\partial \delta E_y}{\partial z}\right| \sim |\mathbf{k}_y \cdot \delta E_z| + |\mathbf{k}_z \cdot \delta E_y| \sim \frac{4\pi}{\vec{\mathbf{k}}^2} \cdot |\rho_{\vec{\kappa}}| \cdot \left(\frac{|\mathbf{k}_y|}{\gamma \cdot a_z} + \frac{|\mathbf{k}_z|}{a_y}\right);$$
$$\left|\frac{\partial \delta E_x}{\partial z}\right| + \left|\frac{\partial \delta E_z}{\partial x}\right| \sim |\mathbf{k}_z \cdot \delta E_x| + |\mathbf{k}_x \cdot \delta E_z| \sim \frac{4\pi}{\vec{\mathbf{k}}^2} \cdot |\rho_{\vec{\kappa}}| \cdot \left(\frac{|\mathbf{k}_x|}{\gamma \cdot a_z} + \frac{|\mathbf{k}_z|}{a_x}\right);$$
$$\left|\frac{\partial \delta E_x}{\partial y}\right| + \left|\frac{\partial \delta E_y}{\partial x}\right| \sim |\mathbf{k}_y \cdot \delta E_x| + |\mathbf{k}_x \cdot \delta E_y| \sim \frac{4\pi}{\vec{\mathbf{k}}^2} \cdot |\rho_{\vec{\kappa}}| \cdot \left(\frac{|\mathbf{k}_x|}{a_y} + \frac{|\mathbf{k}_y|}{a_x}\right). \tag{B12}$$

This allows us to estimate errors for each component of the electric field:

$$|\vec{E}| \cong 4\pi \cdot \frac{|\rho_{\vec{\kappa}}|}{|\vec{\mathbf{k}}|}; \qquad |\delta E_x| \sim |\vec{E}| \cdot \left(\frac{1}{|\vec{\mathbf{k}}| \cdot a_x} + \frac{1}{|\vec{\mathbf{k}}| \cdot a_y} \cdot \frac{|\mathbf{k}_y|}{|\mathbf{k}_x|} + \frac{1}{\gamma \cdot |\vec{\mathbf{k}}| \cdot a_z} \cdot \frac{|\mathbf{k}_z|}{|\mathbf{k}_x|}\right);$$
$$|\delta E_y| \sim |\vec{E}| \cdot \left(\frac{1}{|\vec{\mathbf{k}}| \cdot a_x} \cdot \frac{|\mathbf{k}_x|}{|\mathbf{k}_y|} + \frac{1}{|\vec{\mathbf{k}}| \cdot a_y} + \frac{1}{\gamma \cdot |\vec{\mathbf{k}}| \cdot a_z} \cdot \frac{|\mathbf{k}_z|}{|\mathbf{k}_y|}\right);$$
$$|\delta E_z| \sim |\vec{E}| \cdot \left(\frac{1}{|\vec{\mathbf{k}}| \cdot a_x} \cdot \frac{|\mathbf{k}_x|}{|\mathbf{k}_z|} + \frac{1}{|\vec{\mathbf{k}}| \cdot a_y} \cdot \frac{|\mathbf{k}_y|}{|\mathbf{k}_z|} + \frac{1}{\gamma \cdot |\vec{\mathbf{k}}| \cdot a_z}\right); \tag{B13}$$

and

$$|\delta E_y| \sim |\vec{E}| \cdot \left(\frac{1}{\gamma \cdot |\vec{\mathbf{k}}| \cdot a_z} \cdot \frac{|\mathbf{k}_y|}{|\mathbf{k}_z|} + \frac{1}{|\vec{\mathbf{k}}| \cdot a_y}\right); \qquad |\delta E_y| \sim |\vec{E}| \cdot \left(\frac{1}{|\vec{\mathbf{k}}| \cdot a_y} + \frac{1}{|\vec{\mathbf{k}}| \cdot a_x} \cdot \frac{|\mathbf{k}_y|}{|\mathbf{k}_x|}\right);$$
$$|\delta E_z| \sim |\vec{E}| \cdot \left(\frac{1}{\gamma \cdot |\vec{\mathbf{k}}| \cdot a_z} + \frac{1}{|\vec{\mathbf{k}}| \cdot a_y} \cdot \frac{|\mathbf{k}_z|}{|\mathbf{k}_y|}\right); \qquad |\delta E_z| \sim |\vec{E}| \cdot \left(\frac{1}{\gamma \cdot |\vec{\mathbf{k}}| \cdot a_z} + \frac{1}{|\vec{\mathbf{k}}| \cdot a_x} \frac{|\mathbf{k}_z|}{|\mathbf{k}_x|}\right);$$
$$|\delta E_x| \sim |\vec{E}| \cdot \left(\frac{1}{\gamma \cdot |\vec{\mathbf{k}}| \cdot a_z} \cdot \frac{|\mathbf{k}_x|}{|\mathbf{k}_z|} + \frac{1}{|\vec{\mathbf{k}}| \cdot a_x}\right); \qquad |\delta E_x| \sim |\vec{E}| \cdot \left(\frac{1}{|\vec{\mathbf{k}}| \cdot a_x} + \frac{1}{|\vec{\mathbf{k}}| \cdot a_y} \cdot \frac{|\mathbf{k}_x|}{|\mathbf{k}_y|}\right). \tag{B14}$$

Now, let's introduce the following definitions:

$$\text{And} \quad \begin{array}{lll} \varepsilon_x = \frac{1}{|\vec{\mathbf{k}}| \cdot a_x}; & \varepsilon_y = \frac{1}{|\vec{\mathbf{k}}| \cdot a_y}; & \varepsilon_z = \frac{1}{\gamma \cdot |\vec{\mathbf{k}}| \cdot a_z}; \\ r_{xy} = \frac{|\mathbf{k}_x|}{|\mathbf{k}_y|}; & r_{xz} = \frac{|\mathbf{k}_x|}{|\mathbf{k}_z|}; & r_{yz} = \frac{|\mathbf{k}_y|}{|\mathbf{k}_z|}; \end{array} \tag{B15}$$

and rewrite (B13) and (B14) as

$$|\delta E_x| \sim |\vec{E}| \cdot (\varepsilon_x + \frac{\varepsilon_y}{r_{xy}} + \frac{\varepsilon_z}{r_{xz}}); \qquad |\delta E_y| \sim |\vec{E}| \cdot (\varepsilon_x \cdot r_{xy} + \varepsilon_y + \frac{\varepsilon_z}{r_{yz}});$$
$$|\delta E_z| \sim |\vec{E}| \cdot (\varepsilon_x \cdot r_{xz} + \varepsilon_y \cdot r_{yz} + \varepsilon_z);$$
$$|\delta E_y| \sim |\vec{E}| \cdot (\varepsilon_z \cdot r_{yz} + \varepsilon_y); \qquad |\delta E_y| \sim |\vec{E}| \cdot \left(\varepsilon_y + \frac{\varepsilon_x}{r_{xy}}\right);$$
$$|\delta E_z| \sim |\vec{E}| \cdot (\varepsilon_z + \frac{\varepsilon_y}{r_{yz}}); \qquad |\delta E_z| \sim |\vec{E}| \cdot \left(\varepsilon_z + \frac{\varepsilon_x}{r_{xz}}\right);$$
$$|\delta E_x| \sim |\vec{E}| \cdot (\varepsilon_z r_{xz} + \varepsilon_x); \qquad |\delta E_x| \sim |\vec{E}| \cdot (\varepsilon_x + \varepsilon_y r_{xy}). \tag{B16}$$

And finally, the combination of all estimations results in the following:

$$|\delta E_x| \sim |\vec{E}| \cdot \min\left(\varepsilon_x + \frac{\varepsilon_y}{r_{xy}} + \frac{\varepsilon_z}{r_{xz}}, \varepsilon_x + \varepsilon_z \cdot r_{xz}, \varepsilon_x + \varepsilon_y \cdot r_{xy}\right) \le \varepsilon_x + \varepsilon_y + \varepsilon_z;$$
$$|\delta E_y| \sim |\vec{E}| \min\left(\varepsilon_x \cdot r_{xy} + \varepsilon_y + \frac{\varepsilon_z}{r_{yz}}, \varepsilon_y + \varepsilon_z \cdot r_{yz}, \frac{\varepsilon_x}{r_{xy}} + \varepsilon_y\right) \le \varepsilon_x + \varepsilon_y + \varepsilon_z;$$
$$|\delta E_z| \sim |\vec{E}| \min\left(\varepsilon_x \cdot r_{xz} + \varepsilon_y \cdot r_{yz} + \varepsilon_z, \frac{\varepsilon_y}{r_{yz}} + \varepsilon_z, \frac{\varepsilon_x}{r_{xz}} + \varepsilon_z\right) \le \varepsilon_x + \varepsilon_y + \varepsilon_z; \tag{B17}$$

where we took into account that $\min(r, r^{-1}) \le 1,\ \forall\ r \ge 0$. It means that

$$|\vec{\mathbf{k}}| \cdot a_x \gg 1; \qquad |\vec{\mathbf{k}}| \cdot a_y \gg 1; \qquad \gamma_o \cdot |\vec{\mathbf{k}}| \cdot a_z \gg 1;$$
$$a_{1,2} \cdot \sqrt{(\beta_o \cdot \gamma_o)^2 \cdot ({k_1}^2 + {k_2}^2) + {k_3}^2} \gg \beta_o \cdot \gamma_o;$$
$$a_3 \cdot \sqrt{(\beta_o \cdot \gamma_o)^2 ({k_1}^2 + {k_2}^2) + {k_3}^2} \gg 1. \tag{B18}$$

are sufficient conditions in the comoving frame for Eq. (B7) to be a valid approximation for the electric field.

Lorentz transformation (B2) changes these conditions to the lab frame as follows:

$$a_{x,y} \cdot \sqrt{{\vec{k}_\perp}^2 + \frac{{\vec{k}_z}^2}{\gamma^2}} \gg 1; \qquad a_z \cdot \sqrt{\gamma^2 \cdot {\vec{k}_\perp}^2 + {\vec{k}_z}^2} \gg 1. \tag{B19}$$

These conditions are most important in the case of the longitudinal density modulation

$$\vec{k}_{\mathrm{lab}} = \hat{z} \cdot k_{//}; \qquad k_{//} \gg \max\left(\frac{\gamma}{a_{x,y}}, \frac{1}{a_z}\right), \tag{B20}$$

which means that transverse beam size can play an important role in determining the applicability of this important approximation.

Figure 3 provides intuitive illustrations of applicability and violations of conditions (B19).

These findings have the following foundation: (1) In the comoving frame of the beam, where we are evaluating the electric field, the length of the bunch is increased by a factor of γ, while the value of the longitudinal component of k vector, k_z, is reduced by a factor of γ. In contrast, transverse components remain unchanged. (2) In a plane

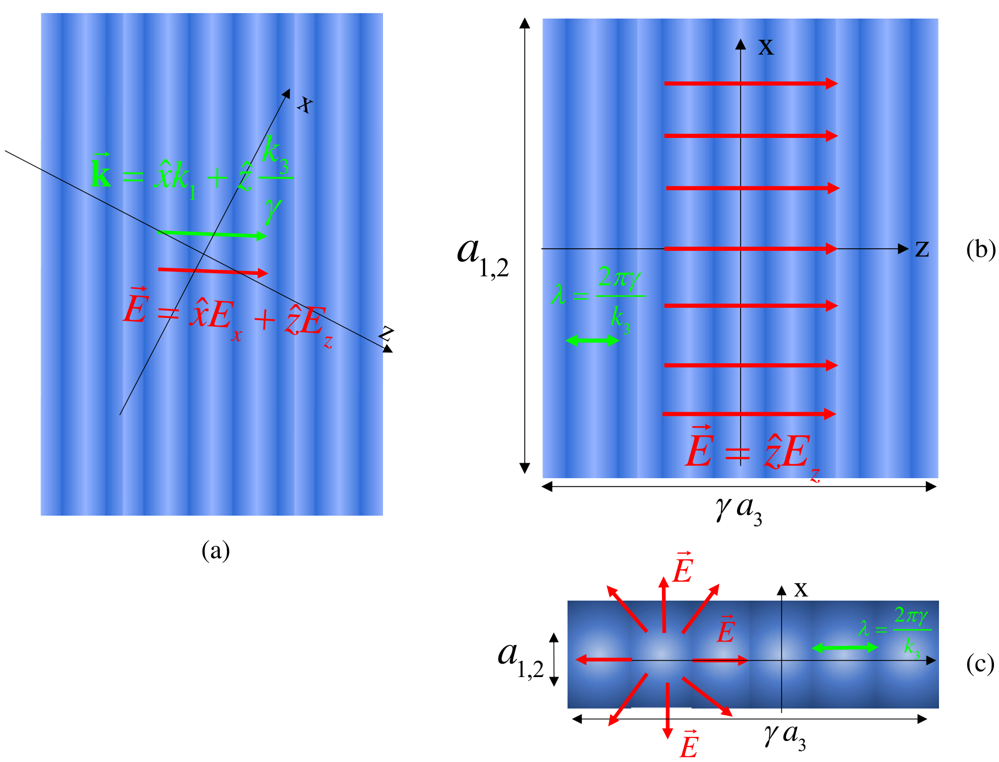

FIG. 3. Geometrical explanation of proportionality between the components of electric field, $\vec{E}$, and wave vector, $\vec{\mathbf{k}}$. Blue color represents a periodic density modulation in the direction of $\vec{\mathbf{k}}$-vector. (b) and (c) illustrate electric field structures (red arrows) in two cases: (b) when the condition (B19), $|a_{1,2} \cdot k_3| \gg \gamma_o$ is satisfied and electric field has a plane-wave structure, and (c) when the condition (B19) is violated $a_{1,2} \sim \lambda = \frac{2\pi \cdot \gamma}{k_3}$, and the electric field is no longer parallel to vector k.

geometry of a periodic density modulation, the strength of the electric field components is directly proportional to the value of the corresponding components of the k vector in the comoving frame—see illustration in Fig. 3. This is a direct result of solving the electrostatic equation

$$\mathrm{div}\vec{E} = 4\pi \cdot \tilde{\rho}_{\vec{\mathbf{k}}} \cdot \cos(\vec{\mathbf{k}} \cdot \vec{r}) \Rightarrow \vec{E} = \vec{\mathbf{k}} \cdot \frac{4\pi \cdot \tilde{\rho}_{\vec{\mathbf{k}}}}{|\vec{\mathbf{k}}|^2} \cdot \sin(\vec{\mathbf{k}} \cdot \vec{r}).$$

(3) It means that if a component of the $\boldsymbol{k}$ vector is zero, it is also true for the electric field component. For example, with the longitudinal modulation with $\vec{\mathbf{k}}_\perp = 0$, the conditions (B19) are reduced to

$$|a_3 \cdot k_3| \gg 1; \qquad |a_{1,2} \cdot k_3| \gg \beta_o\gamma_o;$$

where the first condition is quite natural—and intuitive—requiring that there will be multiple oscillations in the longitudinal direction for the Fourier components to stay uncoupled. The second condition is less obvious: it comes from the requirement that the Fourier components of the electric field repeat the structure of the Fourier components of density modulation, as can be seen in Figs. 3(a) and 3(b). When this condition is violated, simple proportionality relations between the density modulation and the electric field brakes and it results in the coupling of the Fourier harmonics. Such coupling turns the problem under consideration into an unsolvable problem.

While we illustrated the importance of both conditions in Eq. (B19) for longitudinal modulation, the same considerations are valid for transverse directions.

APPENDIX C: EXPRESSION FOR CHARGE AND CURRENT DENSITY MODULATION IN LABORATORY FRAME

Solving Maxwell's equations requires the knowledge of the charge and current densities as functions of coordinates and time:

$$\begin{aligned}
&\mathrm{div}\vec{B} = 0; \qquad \mathrm{div}\vec{E} = 4\pi \cdot \rho; \\
&\mathrm{curl}\vec{E} = -\frac{1}{c} \cdot \frac{\partial \vec{B}}{\partial t}; \qquad \mathrm{curl}\vec{B} = \frac{1}{c} \cdot \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} \cdot \vec{j}; \\
&\rho(\vec{r}, t) = e \cdot \int f(\vec{r}, \vec{\mathrm{v}}, t) \cdot d\vec{\mathrm{v}}^3; \\
&\vec{j}(\vec{r}, t) = e \cdot \int \vec{\mathrm{v}} \cdot f(\vec{r}, \vec{\mathrm{v}}, t) \cdot d\vec{\mathrm{v}}^3, \qquad \text{(C1)}
\end{aligned}$$

where $f(\vec{r}, \vec{v}, t)$ is the particle distribution function in the $(\vec{r}, \vec{\mathrm{v}})$ configuration space. Using s as an independent variable makes the connection between ρ, $\vec{j}$ and the phase-space distribution function $f(q, P, s)$ nontrivial, where (q, P) is the conjugate Canonical set of coordinates and momenta. This Appendix is dedicated to establishing such a connection and finding the corresponding 4-potential of the EM field.

Let's introduce an instantaneous Cartesian coordinate system with the z axis along the reference trajectory at $s = s_o$ [see Eq. (1) in the main text]:

$$\begin{aligned}
&\hat{x} = \hat{e}_1(s_o); \quad \hat{y} = \hat{e}_2(s_o); \quad \hat{z} = \hat{e}_3(s_o) = \left.\frac{d\vec{r}_o(s)}{ds}\right|_{s=s_o}; \\
&\frac{d\hat{e}_1}{ds} = K(s) \cdot \hat{e}_3 - \kappa(s) \cdot \hat{e}_2; \qquad \frac{d\hat{e}_2}{ds} = \kappa(s) \cdot \hat{e}_1; \\
&\vec{r} = \vec{r}_o(s_o) + \hat{x} \cdot x + \hat{y} \cdot y + \hat{z} \cdot z \equiv \vec{r}_o(s) + \hat{e}_1 q_1 + \hat{e}_2(s) q_2; \\
&q_1 = \hat{e}_1(s) \cdot [\hat{x} \cdot x + \hat{y} \cdot y + \vec{r}_o(s_o) - \vec{r}_o(s)]; \\
&q_2 = \hat{e}_2(s) \cdot [\hat{x} \cdot x + \hat{y} \cdot y + \vec{r}_o(s_o) - \vec{r}_o(s)]; \\
&q_3 = c(t_o(s) - t). \qquad \text{(C2)}
\end{aligned}$$

With the following ratios in the vicinity of $\vec{r}_o(s)$:

$$\begin{aligned}
&dx = dq_1 + q_2\kappa(s_o)ds; \qquad dy = dq_2 - q_1\kappa(s_o)ds; \\
&dz = [1 + q_1 K(s_o)]ds; \qquad dq_3 = \frac{ds}{\beta_o(s_o)} - cdt, \qquad \text{(C3)}
\end{aligned}$$

where we used Eq. (1) for the reference trajectory. At a fixed s:

$$\begin{aligned}
&ds = 0 \to dz = 0; \\
&dq_1 = dx; \qquad dq_1 = dy; \qquad dq_3 = -cdt. \qquad \text{(C4)}
\end{aligned}$$

Number of particles confined in an infinitesimal volume dq^3 at a fixed $s = s_o$ is defined as

$$\begin{aligned}
dn = |dq^3| \cdot \int \tilde{f}(q, P, s_o) \cdot dP^3 = c \cdot dx \cdot dy \cdot dt \\
\cdot \int \tilde{f}(q, P, s_o) \cdot dP^3 \qquad \text{(C5)}
\end{aligned}$$

which is identical to the number of particles passing through an elementary area $dx \cdot dy$ located at $s = s_o$ in time interval dt:

$$\begin{aligned}
&\vec{j}(\vec{r}, t) = \int \vec{\mathrm{v}} \cdot f(\vec{r}, \vec{\mathrm{v}}, t) \cdot d\vec{\mathrm{v}}^3; \qquad dn = d\vec{a} \cdot \vec{j} \cdot dt; \\
&d\vec{a} = \hat{z} dxdy; dn = dx \cdot dy \cdot dt \int \mathrm{v}_z \cdot f(\vec{r}, \vec{\mathrm{v}}, t) \cdot d\vec{\mathrm{v}}^3, \qquad \text{(C6)}
\end{aligned}$$

resulting in

$$\int \tilde{f}(q, P, s) \cdot dP^3 = \frac{1}{c} \int \mathrm{v}_z \cdot f(\vec{r}, \vec{\mathrm{v}}, t) \cdot d\vec{\mathrm{v}}^3. \qquad \text{(C7)}$$

Using paraxial approximation $v_z = v_o + \delta v$; $|\delta v| \ll v_o$, and neglecting δv in the integral (C7), we can express ρ, $\vec{j}$ using the phase-space distribution function as

$$\rho(\vec{r},t) \cong \frac{\rho(q,s)}{\beta_o(s)}; \qquad \vec{j}(\vec{r},t) \cong \hat{z}\cdot c\cdot \rho(q,s); \qquad \rho(q,s) = e\cdot \int \tilde{f}(q,P,s)\cdot dP^3. \tag{C8}$$

Applying Fourier transformation[14]

$$g_{\vec{k}} \equiv g(k_1,k_2,k_3,s) = \int g(\vec{q},s)\cdot e^{-i\vec{k}\cdot\vec{q}}\cdot d\vec{q}^3$$

to (C8), we obtain expressions for ρ, $\vec{j}$ at fixed s:

$$\rho_{\vec{k}} = e\cdot \frac{\tilde{f}_{\vec{k}}}{\beta_o}; \qquad \vec{j}_{\vec{k}} = e\cdot\hat{z}\cdot c\cdot \tilde{f}_{\vec{k}}; \qquad \tilde{f}_{\vec{k}} \equiv \tilde{f}(\vec{k},s) = \int e^{-i\vec{k}\cdot\vec{q}}\cdot \tilde{f}(q,P,s)\cdot dP^3\cdot dq^3;$$
$$\rho = \frac{1}{(2\pi)^3}\cdot\frac{1}{\beta_o}\cdot\int \tilde{f}_{\vec{k}}\cdot e^{i\vec{k}\cdot\vec{q}}\cdot d\vec{k}^3; \qquad \vec{j} = \frac{e\cdot\hat{z}\cdot c}{(2\pi)^3}\int \tilde{f}_{\vec{k}}\cdot e^{i\vec{k}\cdot\vec{q}}\cdot d\vec{k}^3. \tag{C9}$$

Let's calculate Fourier harmonic of the density

$$\rho(\vec{\mathbf{k}},\omega) = \int \rho\cdot e^{-i(\vec{\mathbf{k}}\cdot\vec{r}-\omega\cdot t)}\cdot d\vec{r}^3\cdot dt = \frac{e}{(2\pi)^3}\cdot\int d\vec{k}^3\cdot\int \frac{\tilde{f}_{\vec{k}}(s)}{\beta_o(s)}e^{i(\vec{k}\cdot\vec{q}-\vec{\mathbf{k}}\cdot\vec{r}+\omega t)}\cdot d\vec{r}^3\cdot dt. \tag{C10}$$

Using $q_1 = x$, $q_2 = y$, and combining terms in the exponent:

$$\vec{k}\cdot\vec{q} - \vec{\mathbf{k}}\cdot\vec{r} + \omega\cdot t = (k_1 - \mathbf{k}_x)\cdot x + (k_2 - \mathbf{k}_y)\cdot y + (\omega - k_3 c)\cdot t + k_3\cdot c\cdot t_o(s) - \mathbf{k}_z\cdot z; \tag{C11}$$

makes expression integrable:

$$\frac{1}{(2\pi)^3}\int\int e^{i\cdot(k_1-\mathbf{k}_x)\cdot x}e^{i\cdot(k_2-\mathbf{k}_y)\cdot y}e^{-i\cdot(\omega-ck_3)\cdot t}\cdot dx\cdot dy\cdot dt = \delta(k_1-\kappa_x)\cdot\delta(k_2-\kappa_y)\cdot\delta(\omega-ck_3);$$
$$\iint \delta(k_1-\mathbf{k}_x)\cdot\delta(k_2-\mathbf{k}_y)\cdot\delta(\omega-c\mathbf{k}_3)\cdot\tilde{f}(k_1,k_2,k_3,s)\cdot dk^3 = \frac{1}{c}\cdot\tilde{f}\left(\mathbf{k}_x,\mathbf{k}_y,\frac{\omega}{c},s\right);$$
$$\rho(\vec{\mathbf{k}},\omega) = \frac{e}{c}\cdot\int\frac{\tilde{f}(\mathbf{k}_x,\mathbf{k}_y,\frac{\omega}{c},s)}{\beta_o(s)}\cdot e^{i\cdot(\omega\cdot t_o(s)-\mathbf{k}_z\cdot z)}\cdot dz. \tag{C12}$$

At this point, we can use our assumption that the scale of the variation of the accelerator parameters (such as trajectory curvature, β_o, etc.) is much larger than that of the modulation. In addition, we assume that the evolution of the density modulation as a function of s is also much slower than the fast oscillating term $e^{i\cdot\omega\cdot t_o(s)}$. This assumption will allow us to move $\tilde{f}$ and β_o outside of the integral and also to expand the arrival time of the reference particle with respect to the azimuth $\bar{s}$, where we locate the origin of the z axis:

$$t_o(s) \cong t_o(\bar{s}) + \frac{z}{c\cdot\beta_o(\bar{s})}, \tag{C13}$$

and arrive at the final relation between the Fourier components in two systems of coordinates:

$$\rho(\vec{\boldsymbol{k}},\omega) = \frac{2\pi\cdot e}{c}\cdot\tilde{f}(\boldsymbol{k}_x,\boldsymbol{k}_y,\beta_o\cdot\boldsymbol{k}_z,\bar{s})\cdot\delta(k_o-\beta_o\cdot\boldsymbol{k}_z)e^{i\cdot\beta_o\cdot\boldsymbol{k}_z\cdot c\cdot t_o(\bar{s})}; \qquad k_o = \frac{\omega}{c};$$
$$\vec{j}(\vec{\boldsymbol{k}},\omega) = \hat{z}\cdot c\cdot\beta_o\cdot\rho(\vec{\boldsymbol{k}},\omega), \tag{C14}$$

where we used $\delta\left(\frac{k_o}{\beta_o} - \vec{\mathbf{k}}_z\right) = \beta_o\cdot\delta(k_o - \beta_o\cdot\vec{\mathbf{k}}_z)$ and singularity of Dirac's δ-function:

[14]In this Appendix, we use interchangeably both the compact, $g_{\vec{k}}$, and detailed, $g(k_1,k_2,k_3,s)$, notation for the Fourier components defined in the accelerator coordinates.

$$g(x) \cdot \delta(x-y) = g(y) \cdot \delta(x-y).$$

To find 4-potential induced by such perturbation, we can use the Lorenz gauge $\frac{\partial\varphi}{c\partial t} + \mathrm{div}\vec{A} = 0$ providing for the separation of equations for each component of 4-potential [37][15]:

$$\frac{\partial^2\varphi}{c^2 \cdot \partial t^2} - \Delta\varphi = 4\pi \cdot \rho; \qquad \frac{\partial^2\vec{A}}{c^2 \cdot \partial t^2} - \Delta\vec{A} = 4\pi \cdot \vec{j},$$

which can be Fourier transformed to:

$$\varphi(\vec{\mathbf{k}},\omega) \cdot e^{i(\vec{\mathbf{k}}\cdot\vec{r}-\omega\cdot t)} = \frac{4\pi \cdot \rho(\vec{\mathbf{k}},\omega)}{\vec{\mathbf{k}}^2 - k_o^2} e^{i(\vec{\mathbf{k}}\cdot\vec{r}-\omega\cdot t)} = \frac{8\pi^2 \cdot e}{c} \cdot \frac{\tilde{f}(\mathbf{k}_x, \mathbf{k}_y, \beta_o\mathbf{k}_z, \bar{s})}{\vec{\mathbf{k}}^2 - \beta_o{}^2 \cdot \mathbf{k}_z{}^2} \delta(k_o - \beta_o \cdot \mathbf{k}_z) e^{i(\vec{\mathbf{k}}\cdot\vec{r}+\omega\cdot(t_o(\bar{s})-t))};$$

$$\vec{A}(\vec{\mathbf{k}},\omega) = \vec{z} \cdot \varphi(\vec{\mathbf{k}},\omega) \cdot \frac{k_o}{\mathbf{k}_z} = \vec{z} \cdot \beta_o \cdot \varphi(\vec{\mathbf{k}},\omega). \tag{C15}$$

In the inverse Fourier transform

$$\varphi(\vec{r},t) = \frac{1}{(2\pi)^4} \cdot \int \varphi(\vec{\mathbf{k}},\omega) \cdot e^{i(\vec{\mathbf{k}}\cdot\vec{r}-\omega\cdot t)} \cdot d\omega \cdot d\mathbf{k}^3,$$

δ function makes integral over ω straightforward:

$$\frac{1}{2\pi} \int e^{i\omega(t_o(\bar{s})-t)} \cdot g\left(\vec{\mathbf{k}}, \frac{\omega}{c}\right) \cdot \delta\left(\frac{\omega}{c} - \beta_o \cdot \mathbf{k}_z\right) \cdot d\omega = \frac{c}{2\pi} \cdot g(\vec{\mathbf{k}}, \beta_o.\mathbf{k}_z) e^{i\beta_o\cdot\mathbf{k}_z\cdot\tau(\bar{s})};$$

$$\tau(\bar{s}) = c \cdot (t_o(\bar{s}) - t) \tag{C16}$$

with the remaining integral of

$$\varphi(\vec{r},t) = 4\pi \cdot e \cdot \int \frac{\tilde{f}(\boldsymbol{k}_x, \boldsymbol{k}_y, \beta_o\boldsymbol{k}_z, \bar{s})}{\vec{\boldsymbol{k}}^2 - \beta_o{}^2 \cdot \boldsymbol{k}_z{}^2} e^{i[\vec{\boldsymbol{k}}\cdot\vec{r}+\beta_o\cdot\boldsymbol{k}_z\cdot\tau(\bar{s})]} \cdot \frac{d\boldsymbol{k}^3}{(2\pi)^3}; \qquad \vec{A}(\vec{r},t) = \vec{z} \cdot \beta_o \cdot \varphi(\vec{r},t). \tag{C17}$$

Taking into account expansion (C13), the exponent in (C17) can be expressed using the accelerator coordinates:

$$\vec{\mathbf{k}} \cdot \vec{r} + \beta_o \cdot \mathbf{k}_z \cdot \tau(\bar{s}) = \vec{k} \cdot \vec{q}; \qquad k_{1,2} = \mathbf{k}_{x,y}; \qquad k_3 = \beta_o \cdot \mathbf{k}_z; \tag{C18}$$

and using ratio $\beta_o \cdot d\mathbf{k}^3 = dk^3$, we get an expression connecting the 4-potential and density perturbation in the accelerator coordinates:

$$\varphi(\vec{q},s) \equiv \varphi(\vec{r},t) = 4\pi \cdot e \cdot \beta_o \cdot \gamma_o{}^2 \cdot \int \frac{\tilde{f}_{\vec{k}}}{\gamma_o{}^2 \cdot \beta_o{}^2 \cdot \vec{k}_\perp^2 + k_3{}^2} \cdot e^{i\vec{k}\cdot\vec{q}} \cdot \frac{d\vec{k}^3}{(2\pi)^3}; \qquad \vec{A}(\vec{q},s) = \vec{z} \cdot \beta_o \cdot \varphi(\vec{q},s). \tag{C19}$$

APPENDIX D: PERTURBED HAMILTONIAN

As derived in Appendix C, density perturbation results in an additional 4-potential

$$\delta\varphi^i = \{\delta\varphi, \delta\vec{A}\}; \qquad \delta\vec{A} = \hat{z} \cdot \beta_o \cdot \delta\varphi. \tag{D1}$$

which we will consider to be infinitesimally small: $\delta\varphi \sim O(\varepsilon)$, $\varepsilon \ll 1$. The goal of this Appendix is to define an additional term of the reduced accelerator Hamiltonian (6) resulting from the density perturbation:

[15]The Lorentz gauge can be used for time-dependent component of the EM field, which is of interest in this paper.

$$h^* = -(1+K\cdot q_1)\cdot\sqrt{\frac{(\mathbf{E}_o + c\cdot P_3 - e\cdot\varphi_\perp - e\cdot\delta\varphi)^2}{c^2} - m^2c^2 - \left(P_1 - \frac{e}{c}A_1\right)^2 - \left(P_2 - \frac{e}{c}A_2\right)^2}$$
$$-\frac{e}{c}(1+K\cdot q_1)\cdot(A_z + \beta_o\cdot\delta\varphi) + \kappa\cdot q_1\cdot P_2 - \kappa\cdot q_2\cdot P_1 - \frac{c}{\mathrm{v}_o(s)}\cdot P_3 + q_3\cdot\frac{d}{ds}\left(\mathbf{E}_o(s) + e\cdot\frac{\varphi(\vec{r}_o(s),t)}{c}\right) \quad \text{(D2)}$$

where we used the explicit expression for A_3 component of the vector potential (7). Perturbation of the Hamiltonian is coming only from the first two terms in rhs of (D2).

$$\delta h^* = -(1+K\cdot q_1)\cdot\left(\sqrt{\frac{(\mathbf{E}-e\cdot\delta\varphi)^2}{c^2} - m^2\cdot c^2 - \vec{p}_\perp{}^2} - p_z + \frac{e}{c}\cdot\beta_o\cdot\delta\varphi\right);$$
$$p_z = \sqrt{\frac{\mathbf{E}^2}{c^2} - m^2\cdot c^2 - \vec{p}_\perp{}^2};\ \sqrt{\frac{(\mathbf{E}-e\cdot\delta\varphi)^2}{c^2} - m^2\cdot c^2 - \vec{p}_\perp{}^2} - p_z = -\frac{\mathbf{E}}{c\cdot p_z}\cdot\frac{e}{c}\cdot\delta\varphi + O(\delta\varphi)^2;$$
$$\mathbf{E} = H - e\varphi;\qquad \beta_o = \frac{\mathrm{v}_o}{c};\qquad \beta_z = \frac{\mathrm{v}_z}{c} = \frac{cp_z}{\mathbf{E}};\qquad 1-\beta_o{}^2 = \gamma_o{}^{-2};$$
$$\delta h^* = (1+K\cdot q_1)\cdot\frac{e}{c}\cdot\delta\varphi\left(\frac{c}{\mathrm{v}_z} - \frac{\mathrm{v}_o}{c}\right) = \frac{1}{\beta_o\cdot\gamma_o{}^2}\cdot\frac{e}{c}\cdot\delta\varphi\left\{1 + \gamma_o{}^2\cdot\left(\frac{\beta_o}{\beta_z} - 1\right)\right\}\cdot(1+K\cdot q_1). \quad \text{(D3)}$$

First, in paraxial approximation term, $|Kq_1| \ll 1$ can be dropped. It is also easy to show that the second term in the curly brackets is infinitesimally small in the case of paraxial motion resulting in nonrelativistic motion in the comoving frame:

$$\gamma_o{}^2\cdot\left(\frac{\beta_o}{\beta_z} - 1\right) = \frac{\gamma_o{}^2\cdot\vec{\beta}_\perp{}^2}{\beta_z\cdot(\beta_o+\beta_z)} + \frac{\gamma-\gamma_o}{\gamma\cdot\beta_z\cdot(\beta_o+\beta_z)} \sim \frac{\vec{\beta}_{cm\perp}{}^2}{2} + \frac{\delta\gamma}{2\gamma} \ll 1. \quad \text{(D4)}$$

Specifically, $c\cdot\gamma_o\cdot\vec{\beta}_\perp$ is the transverse velocity in the comoving frame and $\frac{\delta\gamma}{\gamma}$ is the relative energy deviation in the beam. Both of these values are assumed to be infinitesimally small. As a result, the perturbation of the Hamiltonian is reduced to:

$$\delta h^* = \frac{1}{\beta_o\cdot\gamma_o{}^2}\cdot\frac{e}{c}\cdot\delta\varphi = \frac{4\pi\cdot e^2}{c}\cdot\int\frac{\tilde{\rho}_{\vec{k}}}{\gamma_o{}^2\cdot\beta_o{}^2\cdot\vec{k}_\perp^2 + k_z{}^2}\cdot e^{i\cdot\vec{k}\cdot\vec{q}}\cdot\frac{d\vec{k}^3}{(2\pi)^3};$$
$$\tilde{\rho}_{\vec{k}} = \int e^{-i\cdot\vec{k}\cdot\vec{q}}\cdot\tilde{f}(q,p,s)\cdot dp^3\cdot dq^3. \quad \text{(D5)}$$

APPENDIX E: NUMERICAL SOLUTION FOR LINEAR INTEGRAL EQUATION

While it is known that the linear integral equations are relatively easy to solve, for completeness, we describe a simple, by design, step-by-step process for our specific case. More sophisticated methods can be found in Refs. [43,45].

Let's split our accelerator into small segments Δs. We start from $s=0$ and evaluate $\tilde{\rho}_{\vec{k}_o}(0)$, assuming a known infinitesimal perturbation of initial beam density $\tilde{f}_o(\underline{q},\underline{P})$:

$$\tilde{\rho}_o = \tilde{\rho}_{\underline{\vec{k}}}(0) = \int e^{i\underline{\vec{k}}\cdot\vec{q}}\cdot dq^3\cdot dP^3\cdot\tilde{f}_o(q,P) \equiv \int e^{i\underline{\vec{k}}\cdot\vec{q}}\cdot dq^3\cdot dP^3\tilde{f}_o(q,P). \quad \text{(E1)}$$

Next, we will use the transport matrices at $s_1 = \Delta s$ to evaluate all other evolving components: step $i=1$

$$\mathbf{A}_0 = \mathbf{D}_0 = \mathbf{I};\qquad \mathbf{B}_0 = \mathbf{C}_0 = \mathbf{0};\qquad \mathbf{A}_1 = \mathbf{A}(\Delta s);\qquad \mathbf{B}_1 = \mathbf{B}(\Delta s);\qquad \mathbf{C}_1 = \mathbf{C}(\Delta s);\qquad \mathbf{D}_1 = \mathbf{D}(\Delta s);$$
$$\vec{k}_1 = \vec{k}(\Delta s) = \underline{\vec{k}}\cdot\mathbf{A}^{-1}(\Delta s);\qquad \vec{k}_o \equiv \underline{\vec{k}}$$
$$\tilde{\rho}_{\vec{k}_o}(\Delta s) = \frac{1}{\det\mathbf{A}_1}\int e^{i\vec{k}(\Delta s)\cdot\vec{q}}\cdot dq^3\cdot dP^3\cdot\tilde{f}_o(\mathbf{D}_1{}^T\cdot q - \mathbf{B}_1{}^T\cdot p, -\mathbf{C}_1{}^T\cdot q + \mathbf{A}_1{}^T\cdot p);$$
$$\upsilon_1 = \gamma_o(\Delta s)^2\cdot\beta_o(\Delta s)^2\cdot\vec{k}_\perp(\Delta s)^2 + k_3(\Delta s)^2;\qquad u_1 = \vec{k}_1\cdot\overleftrightarrow{\mathbf{B}}_1\cdot\underline{\vec{k}};\qquad u_o = u(0) = 0. \quad \text{(E2)}$$

For a known background initial momentum distribution, one can calculate the Landau damping term

$$\vec{\eta}_1 = \vec{k}_1 \cdot \overleftrightarrow{\mathbf{B}}_1; \quad \vec{\eta}_0 = 0; \quad L_{10} = \int e^{i(\vec{\eta}_1 - \vec{\eta}_o)\cdot\vec{P}} \cdot \underline{f}_o(P) \cdot dP^3 \tag{E3}$$

and calculate density modulation in step 1:

$$\tilde{\rho}_1 = \tilde{\rho}[\Delta s, \vec{k}(\Delta s)] = -\frac{4\pi \cdot n_o \cdot e^2}{c \cdot \det \boldsymbol{A}_0 \cdot v_0} L_{10}(u_1 - u_0) \cdot \Delta s \cdot \tilde{\rho}_o + \tilde{\rho}_{\vec{k}_o}(\Delta s). \tag{E4}$$

Let's assume that we completed step $i = n - 1$ and are going to the next step $s_n = n\Delta s$:

$$\begin{aligned}
\mathbf{A}_n &= \mathbf{A}(n \cdot \Delta s); \qquad \mathbf{B}_n = \mathbf{B}(n \cdot \Delta s); \\
\mathbf{C}_n &= \mathbf{C}(n \cdot \Delta s); \qquad \mathbf{D}_n = \mathbf{D}(n \cdot \Delta s); \\
\vec{k}_n &= \vec{k}(n \cdot \Delta s) = \vec{k}_o \cdot \mathbf{A}^{-1}(n \cdot \Delta s); \\
\tilde{\rho}_{\vec{k}_o}(s_n) &= \frac{1}{\det \mathbf{A}_1} \int e^{i\vec{k}(\Delta s)\cdot\vec{q}} \cdot dq^3 \cdot dp^3 \\
&\quad \cdot \underline{f}_o(\mathbf{D}_n{}^T \cdot q - \mathbf{B}_n{}^T \cdot p, -\mathbf{C}_n{}^T \cdot q + \mathbf{A}_n{}^T \cdot p); \\
v_n &= \gamma_o(n \cdot \Delta s)^2 \cdot \beta_o(n \cdot \Delta s)^2 \cdot \vec{k}_{n\perp}{}^2 + k_{n3}{}^2; \\
u_n &= \vec{k}_n \cdot \overleftrightarrow{\mathbf{B}}_n \cdot \vec{k}_o;
\end{aligned} \tag{E5}$$

and calculate all relevant Landau damping terms for propagation from s_i to s_n;

$$\vec{\eta}_n = \vec{k}_n \cdot \overleftrightarrow{\mathbf{B}}_n; \qquad L_{ni} = \int e^{i(\vec{\eta}_n - \vec{\eta}_i)\cdot\vec{P}} \cdot \underline{f}_o(P) \cdot dP^3 \tag{E6}$$

evaluating the density perturbation as a sum

$$\tilde{\rho}_n = \tilde{\rho}(s_n, \vec{k}_n) = -\frac{4\pi \cdot n_o \cdot e^2 \cdot \Delta s}{c} \cdot \sum_{i=0}^{n-1} \frac{L_{ni} \cdot (u_n - u_i) \cdot \tilde{\rho}_i}{\det \mathbf{A}_i \cdot v_i} + \tilde{\rho}_{\vec{k}_o}(s_n). \tag{E7}$$

This process is iterative with all the information about density evaluation prepared in previous steps. What is shown here is a process with the first order of precision, but it can be improved using higher order procedures.

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