Modeling of three-dimensional betatron oscillation and radiation reaction in plasma accelerators

Yulong Liu and Ming Zeng*

Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049, China and University of Chinese Academy of Sciences, Beijing 100049, China

(Received 31 October 2022; accepted 13 February 2023; published 2 March 2023)

Betatron oscillation is a commonly known phenomenon in laser or beam-driven plasma wakefield accelerators. In the conventional model, the plasma wake provides a linear focusing force to a relativistic electron, and the electron oscillates in one transverse direction with the betatron frequency proportional to $1/\sqrt{\gamma}$, where γ is the Lorentz factor of the electron. In this work, we extend this model to three-dimensional by considering the oscillation in two transverse and one longitudinal directions. The long-term equations, with motion in the betatron time scale averaged out, are obtained and compared with the original equations by numerical methods. In addition to the longitudinal and transverse damping due to radiation reaction which has been found before, we show phenomena including the longitudinal phase drift oscillation, betatron phase shift, and betatron polarization change based on our long-term equations. This work can be highly valuable for future plasma-based high-energy accelerators and colliders.

DOI: 10.1103/PhysRevAccelBeams.26.031301

I. INTRODUCTION

The new generation of accelerators, using plasma as the acceleration media, offers a high acceleration gradient in the order of 10–100 GV/m and strong transverse focusing fields [1–3]. Depending on the driver type, the plasma accelerators are named laser wakefield accelerators (LWFAs), which are driven by laser pulses, and plasma wakefield accelerators (PWFAs), which are driven by charged particle beams. When a high-intensity laser pulse ($\gtrsim 10^{18}$ W/cm^2) or a high current particle beam ($\gtrsim 1$ kA) propagates in an underdense plasma, the radiation pressure of the laser or the space charge force of the beam expels all plasma electrons radially, leaving behind a uniform ion channel. The expelled electrons are pulled back by the ion channel and thereby bubble-like plasma wake wave is created. This highly nonlinear three-dimensional (3D) regime has been referred to as the blowout regime [4,5]. In this regime, the wake consists of a longitudinal electric field that is a function of the distance behind the driver, and transversal electromagnetic fields that are proportional to the off-axis distance. Consequently, in addition to the longitudinal acceleration/deceleration, the electrons reside in the wake also perform transverse oscillation, called betatron oscillation (BO), under the action of transverse focusing fields, with the frequency $\omega_\beta = \omega_p \kappa/\sqrt{\gamma}$, where ω_p is the plasma frequency, γ is the relativistic factor of the electron, and κ is the focusing constant which takes $1/\sqrt{2}$ in general [6,7].

Electrons emit synchrotron radiation when performing BO [8–11], which affects the electrons in return. Such an effect is called the radiation reaction (RR) and its classical expression is the Lorentz-Abraham-Dirac (LAD) equation or the Landau-Lifshitz equation [12,13]. Because the RR force is proportional to the classical electron radius $r_e \approx 2.81 \times 10^{15}$ m, it is generally negligible unless under extreme conditions [14,15] or for sufficiently a long interaction time [16]. The BO in a plasma accelerator is another good case for such long interaction time. The radiation leads to the energy loss of electrons and consequently affects the energy-dependent betatron frequency, as well as the other beam properties, such as the energy spread and emittance [17–25].

Although there are many established theories on the long-term RR damping effect of BO, they assert the electron moves in only one plane, which is usually the z-x plane where z is the longitudinal direction and x is one of the transverse directions, thus only linear polarization has been considered. Moreover, these models usually neglect the longitudinal and energy oscillations during one betatron period. In this paper, we establish a 3D BO model with RR effect, which generalizes both linear and elliptical betatron polarization, and also includes the longitudinal and energy oscillations. Long-term equations (LTEs), without

*Corresponding author.
zengming@ihep.ac.cn

TABLE I. The plasma normalization units.

Physical quantities	Variables	Normalization units
Time	t	ω_p^{-1}
Frequency	ω	ω_p
Length	x, y, z	c/ω_p
Velocity	v (or β)	c
Momentum	p	$m_e c$
Angular momentum	L	$m_e c^2/\omega_p$
Electric field	E	$m_e c\omega_p/e$
Magnetic field	B	$m_e\omega_p/e$ (in SI)
Force	f	$m_e c\omega_p$

resolving the betatron period, are derived and verified by numerical methods. The LTEs reproduce the previous results, such as longitudinal and transverse damping due to RR, as the special cases, and meanwhile reveal new phenomena such as betatron phase shift and polarization change.

The rest of this paper is organized as follows: Sec. II gives the original form of the force and shows the equations of motion expressed by the transverse motion only. Section III derives the LTEs by averaging the equations of motion through one betatron period. Section IV discusses the different phenomena in the RR dominant regime and the betatron phase shift dominant regime. Section V numerically verifies the LTEs by comparing them with the code PTracker which solves the equations with the original form of force. Before starting, it is worth noting that some universal notations and calculation rules used throughout the derivation are described in Appendix A. And the plasma normalization units are used throughout the paper, as listed in Table I, where c is the speed of light in vacuum, ω_p is the plasma frequency, e is the elementary charge, and m_e is the electron mass. For example, the time is normalized to ω_p^{-1}, means any time-related quantity such as t in this paper actually means $\omega_p t$ in the un-normalized form.

II. THE ELECTROMAGNETIC FIELDS AND THE EQUATIONS OF MOTION

Consider an electron with $\gamma \gg 1$ is trapped in a plasma wakefield with the longitudinal comoving coordinate $\zeta = z - \beta_w t$, where the wake moves in the $z+$ direction with the phase velocity β_w (normalized to c). Neglecting the interaction between the beam particles, the electromagnetic fields provided by the wake can be modeled as [7]

$$E_z = E_{z0} + \lambda\zeta_1, \tag{1}$$

$$\vec{E}_\perp = \kappa^2(1-\lambda)\vec{r}, \tag{2}$$

$$B_\theta = -\kappa^2\lambda r, \tag{3}$$

where $E_{z0} = E_z|_{\zeta=\langle\zeta\rangle}$, $\lambda = dE_z/d\zeta|_{\zeta=\langle\zeta\rangle}$, and $\vec{r} = (x, y)$ is the transverse offset. The force can be expressed as

$$f_z = -E_{z0} - \lambda\zeta_1 + \kappa^2\lambda(x\beta_x + y\beta_y) + f_z^{\mathrm{rad}}, \tag{4}$$

$$f_x = -\kappa^2(1-\lambda+\lambda\beta_z)x + f_x^{\mathrm{rad}}, \tag{5}$$

$$f_y = -\kappa^2(1-\lambda+\lambda\beta_z)y + f_y^{\mathrm{rad}}, \tag{6}$$

where $\beta_x = \dot{x}$, $\beta_y = \dot{y}$, $\beta_z = \dot{z} = \beta_{z0} + \dot{\zeta}_1$, $\beta_{z0} = \beta_w + \langle\dot{\zeta}\rangle = \langle\beta_z\rangle$, $\vec{f}^{\mathrm{rad}}$ is the RR force, a dot on the top means taking the time derivative, and the subscript 1 means the BO term as described in Appendix A. The formulas of 3D BO with RR can be written in the form of transverse terms only in the limit $r^2 \ll \gamma$, $r^2\gamma \gg 1$, and $r\gamma k_p r_e/2\alpha \ll 1$ (see Appendix B)

$$\dot{\gamma} = -E_{z0}\beta_{z0} + \left(\frac{\lambda\beta_{z0}}{4} + \kappa^2\lambda - \kappa^2\right)(x\beta_x + y\beta_y) - \frac{2}{3}k_p r_e\gamma^2\kappa^4(x^2+y^2), \tag{7}$$

$$\dot{p}_z = -E_{z0} + \lambda\left(\frac{1}{4} + \kappa^2\right)(x\beta_x + y\beta_y) - \frac{2}{3}k_p r_e\gamma^2\kappa^4(x^2+y^2), \tag{8}$$

$$\dot{p}_x = -\kappa^2 x + \frac{\kappa^2\lambda}{2}(\langle\gamma\rangle^{-2} + \beta_x^2 + \beta_y^2)x - \frac{2}{3}k_p r_e\gamma^2\kappa^4(x^2+y^2)\beta_x, \tag{9}$$

$$\dot{p}_y = -\kappa^2 y + \frac{\kappa^2\lambda}{2}(\langle\gamma\rangle^{-2} + \beta_x^2 + \beta_y^2)y - \frac{2}{3}k_p r_e\gamma^2\kappa^4(x^2+y^2)\beta_y, \tag{10}$$

where $k_p = \omega_p/c$, $\vec{p} = \gamma\vec{\beta}$, and

$$\beta_{z0} = 1 - \frac{1}{2}(\langle\gamma\rangle^{-2} + \langle\beta_x^2\rangle + \langle\beta_y^2\rangle) \tag{11}$$

by neglecting high-order terms. One may note the second term in Eqs. (7)–(10), which comes from the oscillation of β_z and the modulation of γ due to transverse oscillation, was neglected in previous works. In the following sections, we show these terms lead to new phenomena.

III. THE LONG-TERM EQUATIONS OF 3D BETATRON OSCILLATION

In this section, we use the same averaging method as Ref. [22]. We first introduce two complex variables

$$U = (x - i\kappa^{-1}\gamma^{\frac{1}{2}}\beta_x)e^{-i\varphi}, \tag{12}$$

$$V = (y - i\kappa^{-1}\gamma^{\frac{1}{2}}\beta_y)e^{-i\varphi}, \tag{13}$$

where

$$\varphi = \int \omega_\beta dt = \kappa \int \gamma^{-\frac{1}{2}} dt \tag{14}$$

is the (fast) betatron phase. Obviously $|U_1| \ll |\langle U\rangle|$ and $|V_1| \ll |\langle V\rangle|$ are satisfied, and we apply the rules in Appendix A often in the following. Because the equations for x and y directions are symmetric, we may derive for x direction only, then exchange x and y, U and V for the y direction. With the help of Eqs. (7) and (9), we may write the time derivative of Eq. (12) as

$$\begin{aligned}\dot{U} = &-i\frac{1}{2}\kappa^{-1}\gamma^{-\frac{1}{2}}E_{z0}\beta_{z0}\beta_x e^{-i\varphi} + i\frac{1}{3}k_p r_e \kappa^3\gamma^{\frac{3}{2}}(x^2+y^2)\beta_x e^{-i\varphi} \\ &+ i\frac{1}{2}\kappa^{-1}\gamma^{-\frac{1}{2}}\left[\frac{\lambda\beta_{z0}}{4} + \kappa^2(\lambda-1)\right](x\beta_x + y\beta_y)\beta_x e^{-i\varphi} \\ &- i\frac{1}{2}\kappa\lambda\gamma^{-\frac{1}{2}}(\langle\gamma\rangle^{-2} + \beta_x^2 + \beta_y^2)xe^{-i\varphi}.\end{aligned} \tag{15}$$

In the following, we omit $\langle\rangle$ on U and V for convenience so that all U and V actually mean $\langle U\rangle$ and $\langle V\rangle$. We perform average on Eq. (15) to obtain (note only the terms with $e^{i0\varphi}$ survive after averaging)

$$\begin{aligned}\dot{U} = &\frac{1}{4}E_{z0}\beta_{z0}\langle\gamma\rangle^{-1}U - \frac{1}{24}k_p r_e\kappa^4\langle\gamma\rangle(|U|^2U + 2|V|^2U - V^2U^*) + i\frac{1}{64}\kappa\lambda\beta_{z0}\langle\gamma\rangle^{-\frac{3}{2}}(|U|^2U + V^2U^*) \\ &- i\frac{1}{16}\kappa^3\langle\gamma\rangle^{-\frac{3}{2}}[(|U|^2 + 2\lambda|V|^2)U - (2\lambda-1)V^2U^*] - i\frac{1}{4}\kappa\lambda\langle\gamma\rangle^{-\frac{5}{2}}U.\end{aligned} \tag{16}$$

By asserting $V = 0$ and omitting the last three terms in Eq. (16), which come from the second terms in Eqs. (7)–(10), we can reproduce Eq. (19) in Ref. [22].

The average of Eq. (7) leads to

$$\langle\dot{\gamma}\rangle = -E_{z0}\beta_{z0} - \frac{1}{3}k_p r_e\kappa^4\langle\gamma\rangle^2(|U|^2 + |V|^2), \tag{17}$$

with the second term reproducing Eq. (B2) in Ref. [19]. E_{z0} is a function of $\langle\zeta\rangle$, which obeys

$$\langle\dot{\zeta}\rangle = \frac{1}{2}\gamma_w^{-2} - \frac{1}{2}\langle\gamma\rangle^{-2} - \frac{1}{4}\kappa^2\langle\gamma\rangle^{-1}(|U|^2 + |V|^2), \tag{18}$$

where $\gamma_w = (1-\beta_w^2)^{-1/2}$ and we have used Eq. (B2).

The above averaged Eqs. (16)–(18) are already enough to predict the long-term behavior of BO. However, the equations for the complex variables are not explicit. To make them more physically meaningful, we split the complex variables into their amplitudes and phases

$$U = |U|e^{i\Phi_x}, \tag{19}$$

$$V = |V|e^{i\Phi_y}, \tag{20}$$

$$\Delta\Phi = \Phi_y - \Phi_x. \tag{21}$$

$|U|$ has the meaning of the BO amplitude in the x direction, and Φ_x is the (slow) betatron phase shift. For the y direction, they are similar. Thus $\Delta\Phi$ is the phase difference between the two directions. By applying $d|U|/dt = (\dot{U}U^* + U\dot{U}^*)/2|U|$ and $\dot{\Phi}_x = (\dot{U}U^* - \dot{U}^*U)/2i|U|^2$, we get

$$\begin{aligned}\frac{d|U|}{dt} = &\frac{1}{4}E_{z0}\beta_{z0}\langle\gamma\rangle^{-1}|U| - \frac{1}{24}k_p r_e\kappa^4\langle\gamma\rangle[|U|^3 + |V|^2|U|(2-\cos 2\Delta\Phi)] \\ &- \frac{1}{16}\kappa\left[\frac{1}{4}\lambda\beta_{z0} - \kappa^2(1-2\lambda)\right]\langle\gamma\rangle^{-\frac{3}{2}}|V|^2|U|\sin 2\Delta\Phi,\end{aligned} \tag{22}$$

$$\begin{aligned}\dot{\Phi}_x = &\frac{1}{24}k_p r_e\kappa^4\langle\gamma\rangle|V|^2\sin 2\Delta\Phi + \frac{1}{64}\kappa\lambda\beta_{z0}\langle\gamma\rangle^{-\frac{3}{2}}[|U|^2 + |V|^2\cos 2\Delta\Phi] \\ &- \frac{1}{16}\kappa^3\langle\gamma\rangle^{-\frac{3}{2}}[|U|^2 + 2\lambda|V|^2 + (1-2\lambda)|V|^2\cos 2\Delta\Phi] - \frac{1}{4}\kappa\lambda\langle\gamma\rangle^{-\frac{5}{2}},\end{aligned} \tag{23}$$

$$\frac{d\Delta\Phi}{dt} = -\frac{1}{24}k_p r_e\kappa^4\langle\gamma\rangle(|V|^2 + |U|^2)\sin 2\Delta\Phi + \frac{1}{8}\kappa\left[\frac{1}{4}\lambda\beta_{z0} - \kappa^2(1-2\lambda)\right]\langle\gamma\rangle^{-\frac{3}{2}}(|V|^2 - |U|^2)\sin^2\Delta\Phi. \tag{24}$$

Note when doing exchange of U and V for the y direction, one also has to change the $\pm$ sign of $\Delta\Phi$.

To further simplify, we notice Eq. (22) can be rewritten with the help of Eq. (17)

$$\frac{d|U|}{dt} = -\frac{1}{4}\frac{\langle\dot{\gamma}\rangle}{\langle\gamma\rangle}|U| - \frac{1}{8}k_p r_e \kappa^4 \langle\gamma\rangle\left[|U|^3 + \frac{4-\cos 2\Delta\Phi}{3}|V|^2|U|\right] - \frac{1}{16}\kappa\left[\frac{1}{4}\lambda\beta_{z0} - \kappa^2(1-2\lambda)\right]\langle\gamma\rangle^{-\frac{3}{2}}|V|^2|U| \sin 2\Delta\Phi, \quad (25)$$

which reproduces Eq. (66) in Ref. [24] if $V = 0$. Introduce

$$S_x = \kappa\langle\gamma\rangle^{\frac{1}{2}}|U|^2, \quad (26)$$

$$S_y = \kappa\langle\gamma\rangle^{\frac{1}{2}}|V|^2, \quad (27)$$

which have the physical meaning of the areas (divided by π) of the ellipses encircled by the particle trajectory in x-p_x and y-p_y phase spaces. Then Eqs. (17), (18), (22)–(24) can be rewritten as

$$\langle\dot{\gamma}\rangle = -E_{z0}\beta_{z0} - \frac{1}{3}k_p r_e \kappa^3 \langle\gamma\rangle^{\frac{3}{2}}(S_x + S_y), \quad (28)$$

$$\langle\dot{\zeta}\rangle = \frac{1}{2}\gamma_w^{-2} - \frac{1}{2}\langle\gamma\rangle^{-2} - \frac{1}{4}\kappa\langle\gamma\rangle^{-\frac{3}{2}}(S_x + S_y), \quad (29)$$

$$\dot{S}_x = -\frac{1}{4}k_p r_e \kappa^3 \langle\gamma\rangle^{\frac{1}{2}}\left(S_x^2 + \frac{4-\cos 2\Delta\Phi}{3}S_x S_y\right) - \frac{1}{8}\left[\frac{1}{4}\lambda\beta_{z0} - \kappa^2(1-2\lambda)\right]\langle\gamma\rangle^{-2}S_x S_y \sin 2\Delta\Phi, \quad (30)$$

$$\dot{\Phi}_x = \frac{1}{24}k_p r_e \kappa^3 \langle\gamma\rangle^{\frac{1}{2}} S_y \sin 2\Delta\Phi + \frac{1}{64}\lambda\beta_{z0}\langle\gamma\rangle^{-2}(S_x + S_y \cos 2\Delta\Phi) - \frac{1}{16}\kappa^2\langle\gamma\rangle^{-2}[S_x + 2\lambda S_y + (1-2\lambda)S_y \cos 2\Delta\Phi] - \frac{1}{4}\kappa\lambda\langle\gamma\rangle^{-\frac{5}{2}}, \quad (31)$$

$$\frac{d\Delta\Phi}{dt} = -\frac{1}{24}k_p r_e \kappa^3 \langle\gamma\rangle^{\frac{1}{2}}(S_y + S_x)\sin 2\Delta\Phi + \frac{1}{8}\left[\frac{1}{4}\lambda\beta_{z0} - \kappa^2(1-2\lambda)\right]\langle\gamma\rangle^{-2}(S_y - S_x)\sin^2\Delta\Phi. \quad (32)$$

It is generally safe to take $\beta_{z0} = 1$ here. But Eq. (11), or $\beta_{z0} = 1 - \frac{1}{2}[\langle\gamma\rangle^{-2} + \frac{1}{2}\kappa\langle\gamma\rangle^{-3/2}(S_x + S_y)]$, gives a better accuracy. The above long-term equations, Eqs. (28)–(32), show that the BO experiences acceleration (for $E_{z0} < 0$) or deceleration (for $E_{z0} > 0$), radiation damping, longitudinal phase drift, and betatron phase shift. These equations may be used for the long-term behavior of BO without resolving the betatron period.

IV. DISCUSSION ON TWO REGIMES

From Eqs. (30) to (32) we note two regimes. One is the RR dominant regime, where $k_p r_e\langle\gamma\rangle^{5/2} \gg 1$, so that the first terms in Eqs. (30)–(32) dominate. This regime has been discussed before [22], although only for the linearly polarized case $\Delta\Phi = 0$ or π (so that the ratio between S_x and S_y is a constant). The other is the betatron phase shift dominant regime, where $k_p r_e\langle\gamma\rangle^{5/2} \ll 1$, so that the remaining terms in Eqs. (30)–(32) dominate. These terms were previously proposed [24], but the betatron phase shift is first found in the present work.

In the RR dominant regime, an interesting phenomenon is that an elliptical polarization (in the $x - y$ plane) always approaches linear polarization, because $\Delta\Phi$ always approaches the nearest integer multiple of π according to Eq. (32), although the approaching speed is gradually reduced and it takes an infinite long time for a perfect linear polarization. This phenomenon can also be viewed by rotating the x axis to the major axis of the ellipse so that $S_x > S_y$ and $\Delta\Phi = \pm\pi/2$. Define $R = S_y/S_x$ and perform time derivative with the help of Eq. (30)

$$\dot{R} = -\frac{1}{6}k_p r_e \kappa^3 \langle\gamma\rangle^{\frac{1}{2}} R(S_x - S_y) < 0, \quad (33)$$

which suggests that the ellipse monotonically becomes thinner.

The betatron phase shift dominant regime requires a moderate γ, or straightforwardly $k_p r_e \to 0$, which corresponds to very dilute plasma cases, leads to a constant $S \equiv S_x + S_y$. It can be proved that the time integral of Eq. (29) reproduces Eq. (6) in Ref. [23], which is the $\langle\zeta\rangle$ drift, in the case that $\langle\gamma\rangle$ linearly depends on t. In another case that $\langle\zeta\rangle$ drifts around the zero point of E_{z0}, the drift frequency can be obtained by using Eqs. (28) and (29) and asserting $E_{z0} = \lambda\langle\zeta\rangle$

$$\omega_{\langle\zeta\rangle} = \sqrt{\lambda\beta_{z0}(1+\frac{3}{8}\kappa\langle\gamma\rangle^{\frac{1}{2}}S)\langle\gamma\rangle^{-3}}. \quad (34)$$

We also note the angular momentum $L_z = \gamma x\beta_y - \gamma y\beta_x$ and its changing rate

$$\langle L_z\rangle = -S_x^{\frac{1}{2}}S_y^{\frac{1}{2}}\sin\Delta\Phi, \quad (35)$$

$$\langle \dot{L_z}\rangle = -\frac{1}{3}k_p r_e \kappa^3\langle\gamma\rangle^{\frac{1}{2}}(S_x+S_y)\langle L_z\rangle, \quad (36)$$

obey the law of conservation of angular momentum if $\langle L_z\rangle = 0$ initially, or $k_p r_e \to 0$. Especially for $k_p r_e \to 0$, the particle trajectory in the x-y plane generally encircles an ellipse, which has precession leading to the rotation of the ellipse. By applying the perturbation method to a special situation that the major axis of the ellipse coincides with the x axis, the precession frequency can be obtained as

$$\Omega = -\frac{1}{8}\left[\frac{\lambda\beta_{z0}}{4} - \kappa^2(1-2\lambda)\right]\langle\gamma\rangle^{-2}\langle L_z\rangle. \quad (37)$$

V. NUMERICAL COMPARISON OF LONG-TERM EQUATIONS AND THE ORIGINAL ONES

To verify the LTEs, we solve them numerically using the backward-differentiation formulas in the SciPy integration package [26]. Meanwhile, the original equations of motion with the force expressions [Eqs. (4)–(6)] are solved by Runge-Kutta fourth-order method using the code PTracker (PT) [27]. We choose four cases with their parameters and initial values listed in Table II and the comparison results are plotted in Figs. 1–4. Note that Φ_x cannot be obtained directly from PT. Thus, we perform the following treatment to the PT results

$$x\cdot\cos\varphi = \frac{|U|}{2}[\cos\Phi_x + \cos(2\varphi+\Phi_x)], \quad (38)$$

because $x = |U|\cos(\varphi+\Phi_x)$, where φ is obtained by numerical integral based on Eq. (14). Then $\cos\Phi_x$ can be obtained by a low-pass filter. A similar treatment is performed to obtain $\cos\Delta\Phi$, according to

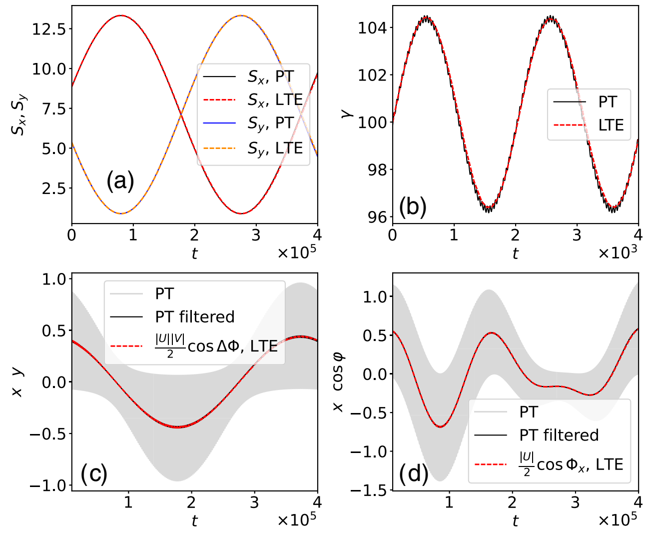

FIG. 1. The numerical comparison of the LTEs and the original equations solved by PTracker in the betatron phase shift dominant regime. (a) S_x and S_y change with the period $\pi/\Omega = 3.89\times10^5$ according to Eq. (37), but S_x+S_y is a constant. (b) γ has oscillation frequencies of $2\omega_\beta \approx 0.14$ due to the BO and $\omega_{\langle\zeta\rangle} \approx 3.11\times10^{-3}$ due to the drift oscillation of $\langle\zeta\rangle$. (c) The gray curve shows xy obtained from PT, and the black curve shows its low-pass filtered result, which is compared with the LTE solution according to Eq. (39). (d) The gray curve shows $x\cdot\cos\varphi$ obtained from PT, and the black curve shows its low-pass filtered result, which is compared with the LTE solution according to Eq. (38).

$$xy = \frac{|U||V|}{2}[\cos\Delta\Phi + \cos(2\varphi+\Phi_x+\Phi_y)]. \quad (39)$$

A case in the betatron phase shift dominant regime is shown in Fig. 1. We see S_x+S_y is a constant, although S_x, S_y, and $\Delta\Phi$ change gradually. The approximate "phase-locking" is chosen, i.e., $\gamma_w \approx \gamma_{z0}$, thus $\langle\zeta\rangle$ oscillates near the zero point of E_z with the drift frequency $\omega_{\langle\zeta\rangle}$ according to Eq. (34). $\langle\gamma\rangle$ oscillates with the same drift frequency as shown in Fig. 1(b).

A second case during the transition of the two regimes is shown in Fig. 2. S_x+S_y is approximately a constant

TABLE II. The cases for comparing PT and LTEs.

	Parameters					Initial values					
Case	E_z	λ	κ	$k_p r_e$	γ_w	$\lvert U\rvert$	$\lvert V\rvert$	Φ_x	Φ_y	$\langle\gamma\rangle$	$\langle\zeta\rangle$
Figure 1	$\lambda\zeta$	$\frac{1}{4}$	$\frac{1}{\sqrt{2}}$	0	14	1.12	0.87	0	$\frac{\pi}{6}$	10^2	−0.05
Figure 2	−0.001	0	$\frac{1}{\sqrt{2}}$	10^{-10}	10^4	1.12	0.87	0	$\frac{\pi}{6}$	10^3	0
Figure 3	$\lambda\zeta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	10^{-10}	10^4	0.2	0.18	0	$\frac{\pi}{2}$	10^3	−0.1
Figure 4	−0.1	0	$\frac{1}{\sqrt{2}}$	10^{-10}	10^4	0.2	0.2	$\frac{\pi}{4}$	π	10^6	0

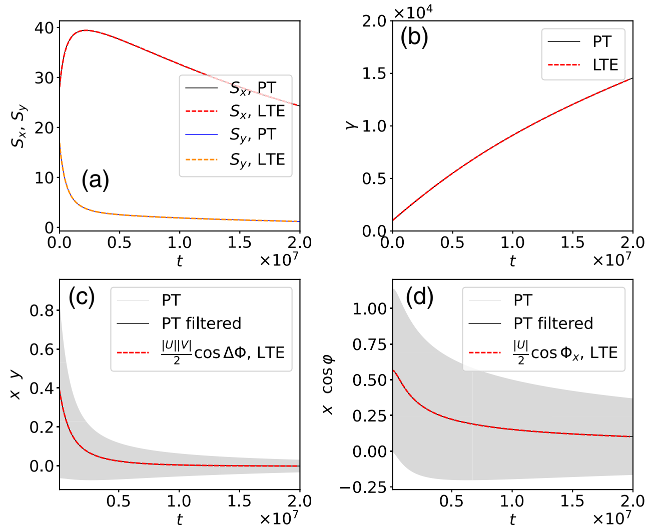

FIG. 2. The numerical comparison of the LTEs and the original equations solved by PTracker in the transition between the betatron phase shift dominant and the RR dominant regimes. (a) $S_x + S_y$ is initially approximately a constant but decreases later. (b) γ increases due to the acceleration field and passes the regime transition at $\gamma = (k_p r_e)^{-2/5} = 10^4$. (c) and (d) show the same treatments as in Figs. 1(c) and 1(d).

initially, and starts to decrease near the regime transition $\gamma = (k_p r_e)^{-2/5}$.

A third case in the RR dominant regime is shown in Fig. 3. The initial values are chosen so that the particle trajectory in the x-y plane is an ellipse with its major axis being on the x axis. As shown in Fig. 3(a), $R = S_y/S_x$ decreases monotonically, as predicted by Eq. (33).

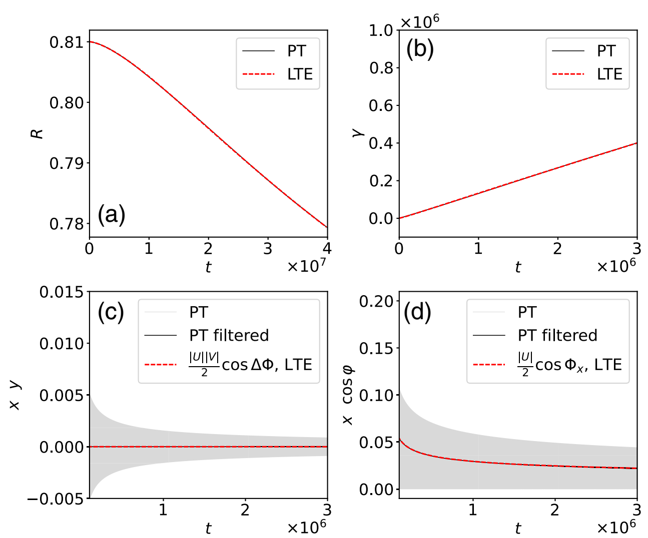

FIG. 3. The numerical comparison of the LTEs and the original equations solved by PTracker in the RR dominant regime. $\Delta\Phi = \pi/2$ and $S_x > S_y$, thus the major axis of the particle trajectory ellipse is on the x axis. (a) $R = S_y/S_x$ decreases with time due to Eq. (33), thus the ellipse is getting thinner. (b) γ increases due to the acceleration field. (c) and (d) show the same treatments as in Figs. 1(c) and 1(d).

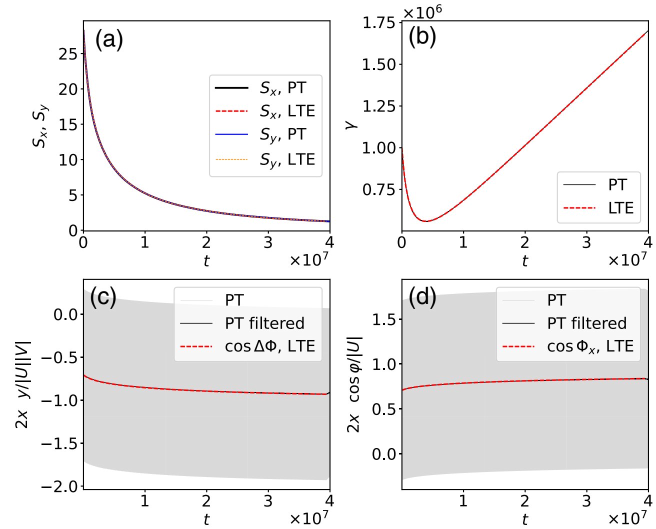

FIG. 4. The numerical comparison of the LTEs and the original equations solved by PTracker in the RR dominant regime. $S_x = S_y$, thus the particle trajectory in the x-y plane is an oblique ellipse. (a) S_x and S_y decrease at the same rate. (b) γ initially decreases with time because the longitudinal RR damping is stronger than the acceleration. Later the RR damping becomes weaker due to the decrease of S_x and S_y, and γ increases. (c) and (d) show the same treatments as in Figs. 1(c) and 1(d), but the oscillation amplitudes are divided so that the changes of $\Delta\Phi$ and Φ_x are clearer. $\Delta\Phi$ is between $\pi/2$ and π, thus $\Delta\Phi$ gradually approaches π.

The last case shown in Fig. 4 is also in the RR dominant regime, but the particle trajectory in the x-y plane is an oblique ellipse. As shown in Fig. 4(c), $\Delta\Phi$ gradually approaches π, which is in accordance with the discussion in Sec. IV.

In all these plots, the results from PT and LTEs show agreement with high accuracy, demonstrating the correctness of LTEs. Because the BO frequency is the highest frequency in our physical process, the LTEs largely reduce the numerical complexity and meanwhile keep the long-term accuracy.

VI. CONCLUSIONS

We have established a three-dimensional betatron oscillation model including radiation reaction to study the long-term behavior of an electron in laser- or beam-driven plasma wakefield. The original equations of motion have been expressed by the transverse oscillation terms as Eqs. (7)–(10) and then averaged in one betatron period to obtain the long-term equations Eqs. (28)–(32). The conditions of our model are $r^2 \ll \gamma$, $r^2\gamma \gg 1$, and $r\gamma k_p r_e/2\alpha \ll 1$, as discussed in Appendix B. Our model, on one hand, reproduces previous results such as longitudinal deceleration and transverse damping, and on the other hand reveals new phenomena such as longitudinal phase drift oscillation, betatron phase shift, and betatron polarization change. Two regimes with distinct behaviors, determined by $k_p r_e \gamma^{5/2}$, are discussed in Sec. IV and are demonstrated by numerical methods in Sec. V. The numerical comparisons of the long-term equations and the original equations of motion show the high accuracy of our model. Although this model is a single-particle model, it can be a very fundamental tool to predict the more practical beam parameters, such as the energy gain, energy spread, and emittance, for future plasma-based high-energy accelerators and colliders [28], if certain initial conditions such as the wakefield parameters and the beam phase space distribution are given.

ACKNOWLEDGMENTS

M. Z. greatly appreciates the fruitful discussion on the averaging method with Igor Kostyukov and Anton Golovanov from the Institute of Applied Physics RAS, Russia. This work is supported by the Research Foundation of the Institute of High Energy Physics, Chinese Academy of Sciences (Grants No. E05153U1 and No. E15453U2).

APPENDIX A: NOTATIONS AND RULES

We give some universal notations and rules in this part. If any variable X, either real or complex, can be expressed as $X = \langle X\rangle + X_1$, where $\langle\rangle$ means taking the average in the betatron period time scale, $\langle X\rangle$ is the slow-varying term and X_1 is the fast-oscillation term with the frequency in the order of the BO frequency, taking average and derivative can permute

$$\left\langle \frac{d}{dt}X \right\rangle = \frac{d}{dt}\langle X\rangle. \tag{A1}$$

We use a dot on the top to express the time derivative if there is no ambiguity. We have the order-of-magnitude estimation

$$\dot{X}_1 \sim \omega_\beta X_1 \sim \gamma^{-\frac{1}{2}} X_1. \tag{A2}$$

If $|X_1| \ll |\langle X\rangle|$, for any power α, we have

$$\langle X^\alpha\rangle = \langle X\rangle^\alpha \left[1 + \mathcal{O}\left(\frac{X_1^2}{\langle X\rangle^2}\right)\right]. \tag{A3}$$

And if another variable $Y = \langle Y\rangle + Y_1$ also has $|Y_1| \ll |\langle Y\rangle|$,

$$\langle XY\rangle = \langle X\rangle\langle Y\rangle \left[1 + \mathcal{O}\left(\frac{X_1 Y_1}{\langle X\rangle\langle Y\rangle}\right)\right]. \tag{A4}$$

If X is a complex, taking average and modulus can permute

$$\langle |X|\rangle = |\langle X\rangle| \left[1 + \mathcal{O}\left(\frac{|X_1|^2}{|\langle X\rangle|^2}\right)\right]. \tag{A5}$$

However, taking modulus and derivative cannot permute.

APPENDIX B: EQUATIONS OF MOTION EXPRESSED BY TRANSVERSE OSCILLATIONS

In Eqs. (4)–(6), the longitudinal and transverse oscillations are coupled. As shown in the following, the longitudinal variables ζ_1 and β_z are dependent variables that can be expressed by the transverse ones.

We treat $\vec{f}^{\mathrm{rad}}$ as a perturbation and omit it first. On one hand, we have

$$\begin{aligned}\gamma^{-2} &= 1 - \beta_z^2 - \beta_x^2 - \beta_y^2 \\ &= \gamma_{z0}^{-2} - 2\beta_{z0}\dot{\zeta}_1 - \beta_x^2 - \beta_y^2 + \mathcal{O}(\dot{\zeta}_1^{\,2}),\end{aligned} \tag{B1}$$

where $\gamma_{z0} = (1 - \beta_{z0}^2)^{-1/2}$. By taking the average, we get

$$\gamma_{z0}^{-2} \approx \langle\gamma\rangle^{-2} + \langle\beta_x^2\rangle + \langle\beta_y^2\rangle. \tag{B2}$$

Write $\gamma = \langle\gamma\rangle + \gamma_1$ in the form

$$\gamma^{-2} = \langle\gamma\rangle^{-2}\left[1 - 2\frac{\gamma_1}{\langle\gamma\rangle} + \mathcal{O}\left(\frac{\gamma_1^2}{\langle\gamma\rangle^2}\right)\right], \tag{B3}$$

we have

$$\gamma_1 \approx \left[\frac{\beta_x^2 - \langle\beta_x^2\rangle}{2} + \frac{\beta_y^2 - \langle\beta_y^2\rangle}{2} + \beta_{z0}\dot{\zeta}_1\right]\langle\gamma\rangle^3. \tag{B4}$$

On the other hand,

$$\dot{\gamma} = -E_{z0}\beta_{z0} - \lambda\beta_{z0}\zeta_1 - \kappa^2(1-\lambda)(x\beta_x + y\beta_y) \tag{B5}$$

by applying $\dot{\gamma} = -\vec{\beta}\cdot\vec{E}$ and Eqs. (1) and (2), or

$$\dot{\gamma}_1 = -\lambda\beta_{z0}\zeta_1 - \kappa^2(1-\lambda)(x\beta_x + y\beta_y). \tag{B6}$$

Note Eq. (A2), Eq. (B4) seems incompatible with Eq. (B6), unless

$$\dot{\zeta}_1 = -\frac{\beta_x^2 - \langle\beta_x^2\rangle}{2} - \frac{\beta_y^2 - \langle\beta_y^2\rangle}{2}, \tag{B7}$$

which leads to

$$\zeta_1 = -\frac{x\beta_x + y\beta_y}{4}, \tag{B8}$$

which is a general form of Eq. (18) in Ref. [24]. Then

$$1 - \beta_z = 1 - \beta_{z0} - \dot{\zeta}_1 = \frac{1}{2}(\langle\gamma\rangle^{-2} + \beta_x^2 + \beta_y^2), \tag{B9}$$

and the formulas of 3D BO with negligible RR are

$$\dot{\gamma} = -E_{z0}\beta_{z0} + \left(\frac{\lambda\beta_{z0}}{4} + \kappa^2\lambda - \kappa^2\right)(x\beta_x + y\beta_y), \quad \text{(B10)}$$

$$\dot{p}_z = -E_{z0} + \lambda\left(\frac{1}{4} + \kappa^2\right)(x\beta_x + y\beta_y), \quad \text{(B11)}$$

$$\dot{p}_x = -\kappa^2 x + \frac{\kappa^2\lambda}{2}(\langle\gamma\rangle^{-2} + \beta_x^2 + \beta_y^2)x, \quad \text{(B12)}$$

$$\dot{p}_y = -\kappa^2 y + \frac{\kappa^2\lambda}{2}(\langle\gamma\rangle^{-2} + \beta_x^2 + \beta_y^2)y. \quad \text{(B13)}$$

From Eq. (B10), we may write

$$\gamma = \langle\gamma\rangle + \left(\frac{\lambda\beta_{z0}}{4} + \kappa^2\lambda - \kappa^2\right)\frac{x^2 - \langle x^2\rangle + \langle y^2\rangle - \langle y^2\rangle}{2}, \quad \text{(B14)}$$

indicating the prerequisite of the above derivation, which has used Eq. (A3), is $r^2 \ll \langle\gamma\rangle$.

Now we consider RR as a perturbation. The LAD equation for the RR four-force is [12]

$$F_\mu^{\text{rad}} = \frac{2}{3}k_p r_e\left[\frac{d^2 P_\mu}{d\tau^2} + \left(\frac{dP_\nu}{d\tau}\frac{dP^\nu}{d\tau}\right)P_\mu\right]. \quad \text{(B15)}$$

with the metric $(1,-1,-1,-1)$, where P_μ is the four-momentum, and τ is the proper time ($d\tau = dt/\gamma$). Using Eqs. (B10)–(B13), we can verify

$$\frac{dP_\nu}{d\tau}\frac{dP^\nu}{d\tau} = \gamma^2(\dot{\gamma}^2 - |\dot{\vec{p}}|^2) \approx -\gamma^2(\dot{p}_x{}^2 + \dot{p}_y{}^2) \quad \text{(B16)}$$

as long as $r^2\gamma^2 \gg 1$. We can also prove that the first term in Eq. (B15) is negligible compared with the second term as long as $r^2\gamma \gg 1$. Finally, the equations of motion expressed by the transverse oscillations are obtained as Eqs. (7)–(10). As it has been discussed in Refs. [24] and [25], this classical RR model is valid as long as $r\gamma k_p r_e/2\alpha \ll 1$, where α is the fine structure constant.

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