Moment constraints in beams with discrete and continuous rotational symmetry

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This study presents a general analysis of how the transverse rotational symmetry of a beam imposes equality constraints among transverse beam moments. Efficient analytic methods are developed to derive symmetry-imposed constraints among kth order moments in beams with continuous or n-fold rotational symmetry for arbitrary k and n. The formalism also enables one to construct symmetry arguments based on how discrete and continuous rotational symmetries manifest themselves differently in terms of beam moments. Three case studies on beams with continuous, twofold, and threefold rotational symmetries are conducted. We prove that, regardless of their triangulation, beams with threefold rotational symmetry (e.g., from electron cyclotron resonance ion source) always have the same rms properties as beams with cylindrical symmetry. These counterintuitive results derived purely from symmetry considerations have clarified beam dynamics at the Facility for Rare Isotope Beams.

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I. INTRODUCTION

In theoretical analysis, a beam is commonly assumed to have inherited the rotational symmetry of the beam line, which often exists in the form of twofold rotational symmetry (e.g., quadrupoles) or cylindrical symmetry (e.g., solenoids and einzel lens). The possession of any nontrivial rotational symmetry imposes conditions on the beam's phase space distribution, and a question to be asked is how or whether these conditions are manifested in the transverse beam moments. Two familiar cases are (i) for beams with cylindrical symmetry, second order moments obey a set of equality constraints [1]; and (ii) for beams with twofold rotational symmetry, no constraint is imposed upon second order moments. In this paper, we present a general analysis of how rotational symmetries constrain beam moments and explore its implications. The analysis applies to moments of arbitrary order k and any rotational symmetry, both continuous and discrete with arbitrary order n.

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A. Notation

We denote rotational symmetries by the commonly adopted notation for their respective symmetry groups. A beam with n-fold rotational symmetry is said to have C_n symmetry or is called a C_n beam. For beams with continuous rotational symmetry, the symbol SO(2) is used and the symmetry is occasionally referred to as cylindrical symmetry.

It should be emphasized that, in this paper, we never use "axisymmetry" or "axisymmetric" as a stand-alone term to refer to continuous rotational symmetry. In the discussion on the relationship between discrete and continuous rotational symmetry in Sec. IV B, we will define the concept of "k-th order axisymmetry." To avoid confusion, the term "axisymmetry" will only be used as part of a phrase in that context.

B. Organization

This paper is organized as follows: Section II presents a general discussion on how rotational symmetry imposes equality constraints among transverse beam moments of the same order. These constraints can be derived efficiently using analytic methods developed in Sec. III. Section IV distills symmetry arguments from the formalism by examining how discrete and continuous rotational symmetries manifest themselves differently in terms of beam moments. Beams with SO(2), C_2, and C_3 symmetries are analyzed in

Sec. V using the tools and arguments constructed in the previous sections. The analysis of C_3 beams contains highly counterintuitive results which have implications on both beam transport and rms characterization of the transverse phase space. The results are applied to electron cyclotron resonance (ECR) ion sources and help elucidate beam dynamics in the front end of the Facility for Rare Isotope Beams (FRIB) [2]. Section VI concludes this study with an outlook on future work.

II. EFFECT OF SYMMETRY ON BEAM MOMENTS

Define transverse beam moments

$$\langle x^{b_1} x'^{b_2} y^{b_3} y'^{b_4} \rangle \equiv \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,x',y,y') x^{b_1} x'^{b_2} y^{b_3} y'^{b_4} dx dx' dy dy'}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,x',y,y') dx dx' dy dy'}$$

where $F(x,x',y,y')$ is the distribution function in four-dimensional transverse phase space with transverse positions x, y and transverse angles x', y', and $b_1, b_2, b_3, b_4 \in \mathbb{Z}_{\geq 0}$. The moment is said to be of kth order with $k = b_1 + b_2 + b_3 + b_4$. Note that the moments are calculated assuming each variable has zero mean, i.e., all four first order moments vanish. This corresponds physically to the beam centroid following the design (reference) orbit and must be true for beams with rotational symmetry, as will be proved in subsequent arguments.

For a beam with n-fold rotational symmetry (i.e., C_n), the distribution function $F(x,x',y,y')$ is invariant under a rotation by $\theta = 2\pi/n$. This implies all transverse beam moments remain unchanged under the transformation:

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}. \tag{1}$$

For an SO(2) beam (i.e., beam with continuous rotational symmetry), the invariance of F and beam moments holds for any rotation angle θ. This is the only piece of information that we need to extract from the rotational symmetry of the beam, with no other assumption made or implied. The rest of this paper explores its consequences and applications.

Given a beam distribution F with rotational symmetry, its beam moments must satisfy certain constraints in order for them to be invariant under the corresponding rotational transformations. Since rotations amount to linear transformations among phase space coordinates, one expects the constraints to arise from systems of linear equations among beam moments of the same order. Formally, denote (x,x',y,y') as (x_1,x_2,x_3,x_4) and given the rotation matrix:

$$\mathbf{R} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix}, \tag{2}$$

where θ is a rotation angle of the symmetry, the relation

$$\langle x_1^{b_1} x_2^{b_2} x_3^{b_3} x_4^{b_4} \rangle = \left\langle \left(\sum_{j=1}^{4} R_{1j} x_j\right)^{b_1} \left(\sum_{j=1}^{4} R_{2j} x_j\right)^{b_2} \left(\sum_{j=1}^{4} R_{3j} x_j\right)^{b_3} \left(\sum_{j=1}^{4} R_{4j} x_j\right)^{b_4} \right\rangle \tag{3}$$

holds for every $b_1, b_2, b_3, b_4 \in \mathbb{Z}_{\geq 0}$. Equation (3) can be rewritten in the form

$$\langle x_1^{b_1} x_2^{b_2} x_3^{b_3} x_4^{b_4} \rangle = \sum_{a_1}\sum_{a_2}\sum_{a_3}\sum_{a_4} C(b_1,b_2,b_3,b_4,a_1,a_2,a_3,a_4) \langle x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4} \rangle \tag{4}$$

where $a_1, a_2, a_3, a_4 \in \mathbb{Z}_{\geq 0}$ and $a_1 + a_2 + a_3 + a_4 = b_1 + b_2 + b_3 + b_4$. Each coefficient $C(b_1,b_2,b_3,b_4,a_1,a_2,a_3,a_4)$ is a polynomial of matrix elements of $\mathbf{R}$, which means it is either zero or a polynomial of degree k in $\sin\theta$ and $\cos\theta$.

Thus, the set of all kth order beam moments in Eq. (4) for non-negative integers b_j forms a linear system:

$$\begin{pmatrix} \langle x_1^k x_2^0, x_3^0, x_4^0 \rangle \\ \langle x_1^{k-1} x_2^1, x_3^0, x_4^0 \rangle \\ \langle x_1^{k-1} x_2^0, x_3^1, x_4^0 \rangle \\ \vdots \\ \langle x_1^0 x_2^0, x_3^1, x_4^{k-1} \rangle \\ \langle x_1^0 x_2^0, x_3^0, x_4^k \rangle \end{pmatrix} = \mathbf{A}(\theta) \begin{pmatrix} \langle x_1^k x_2^0, x_3^0, x_4^0 \rangle \\ \langle x_1^{k-1} x_2^1, x_3^0, x_4^0 \rangle \\ \langle x_1^{k-1} x_2^0, x_3^1, x_4^0 \rangle \\ \vdots \\ \langle x_1^0 x_2^0, x_3^1, x_4^{k-1} \rangle \\ \langle x_1^0 x_2^0, x_3^0, x_4^k \rangle \end{pmatrix} \tag{5}$$

that describes relations among beam moments resulting from invariance under rotation by angle θ. $\mathbf{A}(\theta)$ is a matrix whose elements equal the respective coefficients $C(b_1, b_2, b_3, b_4, a_1, a_2, a_3, a_4)$ in Eq. (4) which depend on θ, and so the relations change with the order of rotational symmetry.

While Eq. (5) is a general result that already contains the properties of beam moments under rotational symmetry, the expressions are complicated because most elements in $\mathbf{A}(\theta)$ are nonzero. It is difficult to know whether or how the equations in Eq. (5) can be manipulated to obtain simple equality constraints that are practically useful. One trick to achieving some simplification exploits the fact that Eq. (5) holds for all $\theta = 2m\pi/n$ with $0 \leq m < n$. Therefore, for a beam with C_n symmetry, a sum over the set of θ that preserves symmetry can be taken to obtain

$$\begin{pmatrix} \langle x_1^k x_2^0, x_3^0, x_4^0 \rangle \\ \langle x_1^{k-1} x_2^1, x_3^0, x_4^0 \rangle \\ \langle x_1^{k-1} x_2^0, x_3^1, x_4^0 \rangle \\ \vdots \\ \langle x_1^0 x_2^0, x_3^1, x_4^{k-1} \rangle \\ \langle x_1^0 x_2^0, x_3^0, x_4^k \rangle \end{pmatrix} = \frac{1}{n}\sum_{m=0}^{n-1} \mathbf{A}\left(\frac{2m\pi}{n}\right) \begin{pmatrix} \langle x_1^k x_2^0, x_3^0, x_4^0 \rangle \\ \langle x_1^{k-1} x_2^1, x_3^0, x_4^0 \rangle \\ \langle x_1^{k-1} x_2^0, x_3^1, x_4^0 \rangle \\ \vdots \\ \langle x_1^0 x_2^0, x_3^1, x_4^{k-1} \rangle \\ \langle x_1^0 x_2^0, x_3^0, x_4^k \rangle \end{pmatrix}. \tag{6}$$

Each coefficient contains summations of polynomials P with two trigonometric variables:

$$\sum_{m=0}^{n-1} P\left[\sin\left(\frac{2m\pi}{n}\right), \cos\left(\frac{2m\pi}{n}\right)\right], \tag{7}$$

which may be reducible. For $k = 2$, this method invokes the same identities as the alternative proof presented in the Appendix B and enables the derivation of simple constraints among second order beam moments. The $k = 2$ results suggest that simple equality constraints exist for $k > 2$, but the method of Eq. (6) is difficult to generalize to $k > 2$ because the elements of $\mathbf{A}$ [i.e., coefficients $C(b_1, b_2, b_3, b_4, a_1, a_2, a_3, a_4)$ of Eq. (4)] grow exceedingly complicated as the moment order k increases.

Here we take a step back and note that Eq. (5) can be interpreted as the action of an element of a rotation group R on the space of polynomials of degree k. Hence, the search for constraints is fundamentally a problem in representation theory where efforts can be greatly reduced if one finds a simple representation of the symmetry group. For example, one may exploit the properties of spherical functions and such treatment will readily afford generalizations to higher-dimensional rotational symmetries such as SO(3).

With this in mind, we employ an efficient method to derive moment constraints specifically for SO(2) or C_n symmetries in Sec. III.

III. DERIVATION OF MOMENT CONSTRAINTS FROM SYMMETRY

Having developed the intuition on the relationship between rotational symmetry and beam moments in Sec. II, we devised a more abstract and efficient method to derive clean expressions of the equality constraints. Complex numbers are used extensively by exploiting the fact that a 2D rotational transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{8}$$

can be expressed using complex numbers in polar form as follows:

$$w \mapsto e^{i\theta} w \tag{9}$$

for $w = x + iy$ with $i \equiv \sqrt{-1}$, and

$$e^{i\theta} = \cos\theta + i\,\sin\theta \tag{10}$$

by Euler's formula.

The method we develop below was inspired by the treatment on symmetry properties of transfer maps in Chapter 5 of Ref. [3] and is readily applicable to moments of arbitrary order in beams with any transverse rotational symmetry.

To begin, we define two complex conjugate pairs composed of transverse phase space coordinates:

$$\begin{pmatrix} w \\ \bar{w} \\ w' \\ \overline{w'} \end{pmatrix} \equiv \begin{pmatrix} x + iy \\ x - iy \\ x' + iy' \\ x' - iy' \end{pmatrix}. \tag{11}$$

From Eq. (12), this is how they transform upon a coordinate rotation by θ:

$$\begin{pmatrix} w \\ \bar{w} \\ w' \\ \overline{w'} \end{pmatrix} \mapsto \begin{pmatrix} e^{i\theta} & 0 & 0 & 0 \\ 0 & e^{-i\theta} & 0 & 0 \\ 0 & 0 & e^{i\theta} & 0 \\ 0 & 0 & 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} w \\ \bar{w} \\ w' \\ \overline{w'} \end{pmatrix}. \tag{12}$$

In analogy to Sec. II, we construct complex moments $\langle w^{a_1} \bar{w}^{a_2} w'^{a_3} \overline{w'}^{a_4} \rangle$ where $a_1, a_2, a_3, a_4 \in \mathbb{Z}_{\geq 0}$ and $a_1 + a_2 + a_3 + a_4 = k$. The real and imaginary parts of $\langle w^{a_1} \overline{w}^{a_2} w'^{a_3} \overline{w'}^{a_4} \rangle$ each comprise sums of physical kth order beam moments. Upon a rotation by θ, the complex moment undergoes the following transformation in accordance with Eq. (12):

$$\langle w^{a_1} \bar{w}^{a_2} w'^{a_3} \overline{w'}^{a_4} \rangle \mapsto e^{i(a_1 - a_2 + a_3 - a_4)\theta} \langle w^{a_1} \bar{w}^{a_2} w'^{a_3} \overline{w'}^{a_4} \rangle. \tag{13}$$

If rotation by θ is a symmetry of the beam, $\langle w^{a_1}\bar{w}^{a_2}w'^{a_3}\overline{w'}^{a_4}\rangle$ remains unchanged upon the transformation which gives

$$\langle w^{a_1}\bar{w}^{a_2}w'^{a_3}\overline{w'}^{a_4}\rangle = e^{i(a_1-a_2+a_3-a_4)\theta}\langle w^{a_1}\bar{w}^{a_2}w'^{a_3}\overline{w'}^{a_4}\rangle. \quad (14)$$

Equation (14) is the key equation that efficiently generates equality constraints among beam moments due to symmetry. For every 4-tuple (a_1, a_2, a_3, a_4) where $e^{i(a_1-a_2+a_3-a_4)\theta} \neq 1$, $\langle w^{a_1}\bar{w}^{a_2}w'^{a_3}\overline{w'}^{a_4}\rangle = 0$ must follow; this gives

$$\mathrm{Re}(\langle w^{a_1}\bar{w}^{a_2}w'^{a_3}\overline{w'}^{a_4}\rangle) = 0, \quad (15)$$

$$\mathrm{Im}(\langle w^{a_1}\bar{w}^{a_2}w'^{a_3}\overline{w'}^{a_4}\rangle) = 0. \quad (16)$$

For a C_n beam (i.e., beam with n-fold rotational symmetry), each 4-tuple (a_1, a_2, a_3, a_4) that satisfies the following conditions:

Condition III 1. $a_1 + a_2 + a_3 + a_4 = k$.
Condition III 2. $(a_1 + a_3) - (a_2 + a_4) > 0$.
Condition III 3. $(a_1 + a_3) - (a_2 + a_4) \neq 0 \pmod{n}$.

gives two unique constraints on the kth order transverse moments of the beam, in the form of Eqs. (15) and (16). Condition III. 1 is merely a restatement of the moment being kth order, while condition III. 3 asserts $e^{i(a_1-a_2+a_3-a_4)\theta} \neq 1$. Condition III. 2 removes redundant constraints for the following reason. Suppose a 4-tuple (a_1, a_2, a_3, a_4) fulfills conditions III. 1 and III 3 but not III. 2:

$$\langle w^{a_1}\bar{w}^{a_2}w'^{a_3}\overline{w'}^{a_4}\rangle = 0 \quad (a_1 + a_3) - (a_2 + a_4) < 0. \quad (17)$$

Its complex conjugate

$$\langle w^{a_2}\bar{w}^{a_1}w'^{a_4}\overline{w'}^{a_3}\rangle = 0 \quad (a_2 + a_4) - (a_1 + a_3) > 0 \quad (18)$$

is equivalent to and contains the same information as

$$\langle w^{b_1}\bar{w}^{b_2}w'^{b_3}\overline{w'}^{b_4}\rangle = 0 \quad (b_1 + b_3) - (b_2 + b_4) > 0. \quad (19)$$

Therefore, the constraints generated by (a_1, a_2, a_3, a_4), which violates condition III 2, are already covered by another 4-tuple (b_1, b_2, b_3, b_4) that satisfies condition III. 2.

For an SO(2) beam (i.e., beam with continuous rotational symmetry), rotation by any angle θ keeps $\langle w^{a_1}\bar{w}^{a_2}w'^{a_3}\overline{w'}^{a_4}\rangle$ unchanged. Therefore, $e^{i(a_1-a_2+a_3-a_4)\theta} \neq 1$ can always be satisfied for some θ so long as $a_1 - a_2 + a_3 - a_4 \neq 0$. In this case, conditions III. 1 and III. 2 suffice for a 4-tuple (a_1, a_2, a_3, a_4) to impose unique constraints upon kth order transverse moments.

An immediate corollary of these analytic arguments is the fact that any beam with rotational symmetry must have zero means (i.e., first order centroid moments) in all transverse phase space coordinates. For moments of order $k = 1$, when $n \neq 1$, $(a_1, a_2, a_3, a_4) = (1, 0, 0, 0)$ and $(a_1, a_2, a_3, a_4) = (0, 0, 1, 0)$ always satisfy conditions III. 1, III. 2, and III. 3. Therefore, $\langle w\rangle = 0$ and $\langle w'\rangle = 0$ must hold, which imply $\langle x\rangle = \langle y\rangle = 0$ and $\langle x'\rangle = \langle y'\rangle = 0$, respectively.

IV. RELATIONSHIP BETWEEN DISCRETE AND CONTINUOUS ROTATIONAL SYMMETRY

With a general method to derive equality constraints imposed upon beam moments due to the rotational symmetry possessed by the beam, it is illuminating to ask: how do continuous and discrete rotational symmetries manifest themselves differently in beam moment constraints?

This question, which explores deep connections between beam moments and rotational symmetries, was one of the primary motivations for developing the analytic tools in Sec. III. Its answer will also form the basis for robust symmetry arguments that enable simple proofs of certain useful results. In particular, the arguments will be used to derive highly counterintuitive results for C_3 beams in Sec. V C which have wide implications for ECR ion sources in general and helped clarify beam dynamics at the FRIB front end.

A. Number of moment constraints

Let S_{k,C_n} and $S_{k,\mathrm{SO}(2)}$ be the sets of constraints governing kth order moments of a C_n beam and an SO(2) beam, respectively, where each constraint has the form of Eq. (15) or (16). Since C_n is a subgroup of SO(2), every constraint imposed upon a C_n beam due to symmetry must also hold for an SO(2) beam. Therefore,

$$S_{k,C_n} \subseteq S_{k,\mathrm{SO}(2)}. \quad (20)$$

One important question is whether $S_{k,C_n} \subsetneq S_{k,\mathrm{SO}(2)}$ or $S_{k,C_n} = S_{k,\mathrm{SO}(2)}$ for given k and n. This can be answered by comparing the cardinalities of the respective sets.

Define (a) $\chi(k)$ as the number of 4-tuples (a_1, a_2, a_3, a_4) that satisfy conditions III. 1 and III. 2; and (b) $\eta(k, n)$ as the number of 4-tuples (a_1, a_2, a_3, a_4) that satisfy conditions III. 1 and III. 2, but not condition III. 3, for the given n.

We prove in Appendix A 1 that

$$\chi(k) = \begin{cases} \dfrac{(k+3)(k+2)(k+1)}{12} & \text{if } k \text{ is odd,} \\ \dfrac{k(k+2)(2k+5)}{24} & \text{if } k \text{ is even.} \end{cases} \quad (21)$$

$\chi(k)$ is the number of 4-tuples that correspond to unique vanishing $\langle w^{a_1}\bar{w}^{a_2}w'^{a_3}\overline{w'}^{a_4}\rangle$ for an SO(2) beam. Since each vanishing complex moment generates two equations,

$$|S_{k,\mathrm{SO}(2)}| = 2\chi(k), \tag{22}$$

where $|S_{k,\mathrm{SO}(2)}|$, the cardinality of the $S_{k,\mathrm{SO}(2)}$, is the total number of equality constraints governing kth order transverse moments of an SO(2) beam.

Equation (20) implies $2\chi(k)$ is the maximum number of constraints that can be imposed upon kth order moments of a beam due to rotational symmetry. For beams with discrete rotational symmetry, the number of constraints is given by

$$|S_{k,C_n}| = |S_{k,\mathrm{SO}(2)}| - 2\eta(k,n) = 2[\chi(k) - \eta(k,n)], \tag{23}$$

where $\eta(k,n)$ accounts for the fact that some 4-tuples (a_1, a_2, a_3, a_4) that satisfy condition III.1 and condition III. 2 may violate condition III. 3.

Stated explicitly, $\eta(k,n)$ is the number of 4-tuples for which

$$(a_1 + a_3) - (a_2 + a_4) > 0, \quad \text{and}$$
$$(a_1 + a_3) - (a_2 + a_4) = 0 \pmod{n}.$$

$\eta(k,n) = 0$ is quite common and always holds if (a) $n > k$; or (b) k is even and n is odd, or vice versa.

For $k \geq n$ and $k + n = 0 \,(\mathrm{mod}\ 2)$, $\eta(k,n)$ is given by

$$\eta(k,n) = \sum_{j=1}^{\lfloor k/n \rfloor} \left[\frac{1}{2}(k + jn) + 1\right]\left[\frac{1}{2}(k - jn) + 1\right] \tag{24}$$

where $\lfloor k/n \rfloor$ denotes the largest integer $\leq k/n$. The proof is given in Appendix A 2. One special case to be observed is

$$\eta(n,n) = n + 1 \tag{25}$$

which can also be shown by noting that the number of 2-tuples (a_1, a_3) where $a_1 + a_3 = n$ equals $n + 1$.

The question of whether $S_{k,C_n} \subsetneq S_{k,\mathrm{SO}(2)}$ or $S_{k,C_n} = S_{k,\mathrm{SO}(2)}$ can thus be answered by checking whether $\eta(k,n) \neq 0$ or $\eta(k,n) = 0$. The implications for these two possibilities are discussed below.

B. kth order axisymmetry

We define k-th order axisymmetry as a beam property that is related to the beam's kth order moments. A beam has k-th order axisymmetry, or is said to be "k-th order axisymmetric," if its moments obey all constraints in $S_{k,\mathrm{SO}(2)}$, i.e., the set of $2\chi(k)$ constraints obeyed by kth order moments of an SO(2) beam.

kth order axisymmetry merely concerns relationships among kth order moments. Such a property makes no reference to any moments of other orders, so a beam can be kth order axisymmetric but not lth order axisymmetric for any $l \neq k$. It should be emphasized that kth order axisymmetry is a much weaker notion than SO(2), i.e., continuous rotational symmetry, which implies kth order axisymmetry for all k.

When $\eta(k,n) = 0$, $S_{k,C_n} = S_{k,\mathrm{SO}(2)}$ which means that a C_n beam is kth order axisymmetric, since its kth order moments obey the same set of constraints as those of an SO(2) beam. This is true despite the fact that C_n and SO(2) are different rotational symmetries. This equivalence indicates that kth order axisymmetry renders the C_n beam indistinguishable from an SO(2) beam in terms of kth order beam moments. As a result, the C_n beam must share all properties of an SO(2) beam which only depend on kth order moments. For example, the kth order spatial moment $\langle x^k \rangle$ of an SO(2) beam remains the same regardless of the orientation of the transverse coordinate system. Using the indistinguishability argument above, we can immediately conclude that the same statement also holds for any beam that is kth order axisymmetric. A detailed application of this argument can be found in Sec. V C 1.

We also know from the last part of Sec. IV A that, given n, $k = n$ is the smallest k at which $\eta(k,n) \neq 0$ occurs. In other words, a C_n beam is not nth order axisymmetric, but is kth order axisymmetric for all $k < n$. This agrees with the heuristic picture where, the higher the order of discrete rotational symmetry n, the closer the beam is to being SO (2) and the higher the order of moments it takes for deviations to start. From Eq. (25), we know that $S_{k,\mathrm{SO}(2)}$ contains $2(n+1)$ more elements (i.e., moment constraints) than S_{k,C_n}. With n being the lowest moment order at which C_n and SO(2) are distinguishable, the difference between S_{k,C_n} and $S_{k,\mathrm{SO}(2)}$ can be interpreted as a manifestation of the difference between C_n and SO(2) in terms of beam moments. Such a perspective may enable one to employ beam moments to quantify how close a C_n beam is from cylindrical symmetry.

V. THREE CASE STUDIES

To demonstrate the analytic tools and symmetry arguments developed in Secs. III and IV, we conduct case studies on beams with SO(2), C_2, and C_3 symmetries by deriving equality constraints imposed upon their respective second and third order moments.

The analysis on symmetries that are commonly encountered, SO(2) and C_2, obtains results which are in agreement with expectations. The case on C_3 beams, however, contains highly counterintuitive results on both the rms characterization and transport behavior of the beam. The connection to ECR ion sources is made and we discuss how the results have been applied to elucidate beam dynamics in the front end of FRIB.

A. Beams with SO(2) symmetry

A beam has SO(2) symmetry when there is no *a priori* reason for azimuthal angular dependence or if such dependence is assumed to be negligible. SO(2) symmetry would be preserved by transport through cylindrically symmetric elements such as solenoids and einzel lens.

1. Second order moments

For $k = 2$, Eq. (21) gives $\chi(2) = 3$, which means three complex moments satisfy conditions III. 1 and III. 2. They are easily found to be $\langle ww \rangle$, $\langle ww' \rangle$, and $\langle w'w' \rangle$. The equality constraints they impose on transverse beam moments are given as follows:

$$\begin{aligned} \mathrm{Re}\,(\langle ww \rangle) = 0 &\Rightarrow \langle xx \rangle = \langle yy \rangle, \\ \mathrm{Re}\,(\langle ww' \rangle) = 0 &\Rightarrow \langle xx' \rangle = \langle yy' \rangle, \\ \mathrm{Re}\,(\langle w'w' \rangle) = 0 &\Rightarrow \langle x'x' \rangle = \langle y'y' \rangle, \\ \mathrm{Im}\,(\langle ww \rangle) = 0 &\Rightarrow \langle xy \rangle = 0, \\ \mathrm{Im}\,(\langle ww' \rangle) = 0 &\Rightarrow \langle xy' \rangle = -\langle x'y \rangle, \\ \mathrm{Im}\,(\langle w'w' \rangle) = 0 &\Rightarrow \langle x'y' \rangle = 0, \end{aligned} \tag{26}$$

which recover known results (e.g., [1]) effortlessly. The first three equations show that the rms phase space ellipse in $x - x'$ and $y - y'$ is identical, as one should expect from an SO(2) beam. The last three equations say that the only $x - y$ correlation that is allowed to exist arises from the angular momentum of the beam L. L can be expressed in terms of beam moments by

$$\frac{L}{p_z} = \langle xy' \rangle - \langle x'y \rangle = 2\langle xy' \rangle = -2\langle x'y \rangle,$$

where p_z is the axial momentum.

2. Third order moments

All third order moments of an SO(2) beam must vanish. In fact, the same conclusion holds for all other odd-order moments as well. This can be seen immediately if one applies a rotation by $\theta = \pi$ which gives

$$\begin{aligned} \langle x^{a_1} x'^{a_2} y^{a_3} y'^{a_4} \rangle &= (-1)^{a_1+a_2+a_3+a_4} \langle x^{a_1} x'^{a_2} y^{a_3} y'^{a_4} \rangle \\ &= -\langle x^{a_1} x'^{a_2} y^{a_3} y'^{a_4} \rangle. \end{aligned}$$

B. Beams with C_2 symmetry

A beam has C_2 symmetry throughout a quadrupole transport line if it has SO(2) or C_2 symmetry as it enters, which is a common assumption in simulations and theoretical studies. A C_2 beam also retains its symmetry in a cylindrically symmetric transport line.

1. Second order moments

C_2 symmetry does not impose any constraint among second order moments. Since $\chi(2) = 3$ and $\eta(2,2) = 3$, we know from Eq. (23) that $|S_{2,C_2}| = 0$. Given the ubiquity of C_2 beams, this result should come as no surprise. While C_2 beams may rarely be considered from a symmetry perspective, any other result would have already become common knowledge.

2. Third order moments

All third order moments of a C_2 beam vanish due to the same argument as Sec. V A 2. Alternatively, we can prove the same result using the fact that $\eta(3,2) = 0$ (because $k = 3$ and $n = 2$ have different parities). $\eta(3,2) = 0$ means a C_2 beam is third order axisymmetric, so its third order moments must follow the same set of constraints as that of an SO(2) beam.

C. Beams with C_3 symmetry

Although much less common than C_2 and SO(2), one can imagine a beam having C_3 symmetry if a sextupole exists in an otherwise cylindrically symmetric configuration. One notable example is ECR ion sources whose cylindrical plasma chambers have external applied fields from both solenoids and sextupoles for confinement purposes. Simulation results of an ECR beam at the extraction plane are shown in Fig. 1 where, due to the strong sextupole field of the ECR, the beam has a triangulated spatial density profile that would remain unchanged upon a rotation by $2\pi/3$. Note that the dominant triangulation is shown over the plasma chamber extent but is still manifest within the extraction aperture.

We will derive counterintuitive results about beam moments of C_3 beams and discuss their implications for beam transport and phase space characterization. We show how these analytic results can be utilized to clarify beam dynamics in the front end of FRIB.

1. Second order moments

Following the argument in Sec. IV B, since $k = 2 < n = 3$, we know that $\eta(2,3) = 0$, which means a C_3 beam is second order axisymmetric, i.e., it is indistinguishable from a beam with SO(2) symmetry in terms of second order moments. In addition to the immediate conclusion that a C_3 beam must obey Eq. (26), the indistinguishability also entails the two propositions:

Proposition V. 1 A C_3 beam obeys Eq. (26) regardless of the orientation of the transverse coordinate system.

Proposition V. 2 Values of all second order moments remain unchanged regardless of the orientation of the transverse coordinate system.

Proposition V. 2 obviously implies Proposition V 1, and we make the distinction to illustrate the importance of the concept of second order axisymmetry. Let $u(\varphi) = x \cos \varphi + y \sin \varphi$ be a rotated coordinate. Analytic methods in Sec. III only prove Proposition V. 1, which says that, for all φ,

$$\langle u(\varphi)u(\varphi) \rangle = \langle u(\varphi + \phi)u(\varphi + \phi) \rangle \tag{27}$$

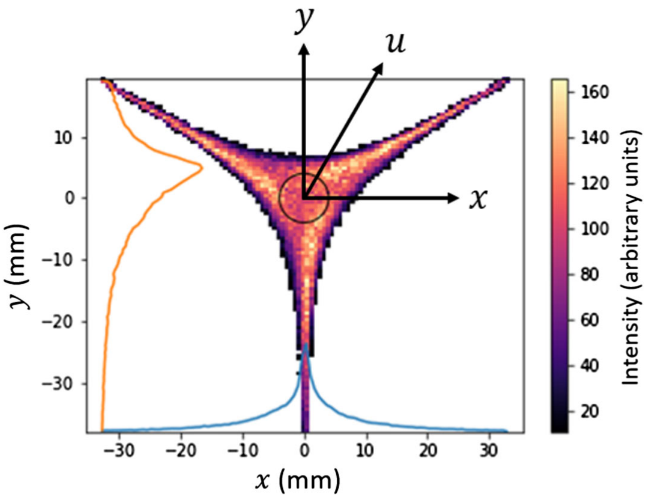

FIG. 1. Simulated spatial distribution of Ar^{9+} ions at the extraction plane of the Artemis [4] ECR ion source at FRIB. The inner circle indicates the extraction aperture. Subsequent analysis will show that, despite severe triangulation, the rms envelope and emittance of the beam is the same along any transverse direction. Simulation results are courtesy of the code developed by Vladirmir Mironov and his colleagues at Joint Institute for Nuclear Research [5,6].

holds when $\phi = \pm\pi/2$. Note that Proposition V 1 has nothing to say concerning other values of ϕ. On the other hand, second order axisymmetry means the beam's second order moments have the exact same properties as those of an SO(2) beam. This equivalence implies Proposition V. 2 and guarantees that Eq. (27) holds for any ϕ. If we do not invoke symmetry arguments, it is still possible to prove Proposition V. 2 from Proposition V. 1 by expanding $u(\varphi + \phi)$, but the approach is much more tedious.

The results we proved above are rather surprising. Consider Fig. 1 where the beam has a strongly triangulated spatial density profile and a u-axis makes angle φ with the x-axis. Although the beam's projected distribution in $u - u'$ phase space varies significantly with φ, all second order phase space moments, and hence the rms phase space ellipse, always remain identical for every such distribution. This conclusion also implies that the azimuthal orientation of the ECR sextupole has no effect on second order moments of the ECR beam.

An alternative proof of the fact that Eq. (26) holds for a beam with threefold rotational symmetry is provided in Appendix B. The treatment employs a less elegant, yet more direct, formalism that invokes trigonometric identities upon rotations in cylindrical coordinates. The proof is applicable to any n-fold rotational symmetry where $n > 2$.

2. Third order moments

For $k = 3$, we know from Eqs. (21) and (25) that $\chi(3) = 10$ and $\eta(3,3) = 4$, respectively. These two quantities tell us that ten complex moments satisfy conditions III. 1 and III. 2 while four among them violate condition III. 3 when $k = n = 3$. The six complex moments that also satisfy condition III. 3 must vanish when the beam has C_3 symmetry and impose 12 conditions on third order moments. This result is also captured by Eq. (23) which states that

$$|S_{3,C_3}| = 2[\chi(3) - \eta(3,3)] = 12. \tag{28}$$

The four complex moments that provide no information are

$$\langle www\rangle, \langle www'\rangle, \langle ww'w'\rangle, \quad \text{and} \quad \langle w'w'w'\rangle.$$

They have no reason to vanish because $e^{3i\theta} = 1$ for $\theta = 2\pi/3$.

The six complex moments that vanish give

$$\begin{aligned}
\mathrm{Re}(\langle ww\bar{w}\rangle) = 0 &\Rightarrow \langle xxx\rangle = -\langle xyy\rangle,\\
\mathrm{Im}(\langle ww\bar{w}\rangle) = 0 &\Rightarrow \langle yyy\rangle = -\langle xxy\rangle,\\
\mathrm{Re}(\langle ww\overline{w'}\rangle) = 0 &\Rightarrow 2\langle xyy'\rangle = \langle x'yy\rangle - \langle xxx'\rangle,\\
\mathrm{Im}(\langle ww\overline{w'}\rangle) = 0 &\Rightarrow 2\langle xx'y\rangle = \langle xxy'\rangle - \langle yyy'\rangle,\\
\mathrm{Re}(\langle w\bar{w}w'\rangle) = 0 &\Rightarrow \langle xxx'\rangle = -\langle x'yy\rangle,\\
\mathrm{Im}(\langle w\bar{w}w'\rangle) = 0 &\Rightarrow \langle xxy'\rangle = -\langle yyy'\rangle,\\
\mathrm{Re}(\langle \bar{w}w'w'\rangle) = 0 &\Rightarrow \langle xx'x'\rangle = -\langle xy'y'\rangle,\\
\mathrm{Im}(\langle \bar{w}w'w'\rangle) = 0 &\Rightarrow \langle x'x'y\rangle = -\langle yy'y'\rangle,\\
\mathrm{Re}(\langle ww'\overline{w'}\rangle) = 0 &\Rightarrow 2\langle x'yy'\rangle = \langle xy'y'\rangle - \langle xx'x'\rangle,\\
\mathrm{Im}(\langle ww'\overline{w'}\rangle) = 0 &\Rightarrow 2\langle xx'y'\rangle = \langle x'x'y\rangle - \langle yy'y'\rangle,\\
\mathrm{Re}(\langle w'w'\overline{w'}\rangle) = 0 &\Rightarrow \langle x'x'x'\rangle = -\langle x'y'y'\rangle,\\
\mathrm{Im}(\langle w'w'\overline{w'}\rangle) = 0 &\Rightarrow \langle y'y'y'\rangle = -\langle x'x'y'\rangle.
\end{aligned}$$

The 12 constraints can be further simplified into the following eight equations on third order moments:

$$\begin{aligned}
\langle xxx\rangle &= -\langle xyy\rangle,\\
\langle yyy\rangle &= -\langle xxy\rangle,\\
\langle xyy'\rangle &= \langle x'yy\rangle = -\langle xxx'\rangle,\\
\langle xx'y\rangle &= \langle xxy'\rangle = -\langle yyy'\rangle,\\
\langle x'yy'\rangle &= \langle xy'y'\rangle = -\langle xx'x'\rangle,\\
\langle xx'y'\rangle &= \langle x'x'y\rangle = -\langle yy'y'\rangle,\\
\langle x'x'x'\rangle &= -\langle x'y'y'\rangle,\\
\langle y'y'y'\rangle &= -\langle x'x'y'\rangle.
\end{aligned} \tag{29}$$

Note that the number of constraints is consistent with previous results. We know from Sec. VA 2 that all third order moments of an SO(2) beam vanish. To make all equations in Eq. (29) equal zero requires eight additional constraints, which is indeed the number provided by the four complex moments that vanish for an SO(2) beam but not for a C_3 beam.

3. Implications for beam transport

A comparison between moment constraints for C_3 beams against those for SO(2) and C_2 beams reveals interesting implications for beam transport. Although the approximation of linear optics is often made, beam transport inevitably involves nonlinear terms in the transfer map. Consider a beam with some specific rotational symmetry entering a beam line with the nonlinear map:

$$x_l = \sum_{i=1}^{4} R_{li} x_i + \sum_{i=1}^{4}\sum_{j=1}^{4} T_{lij} x_i x_j + \mathcal{O}(x_i^3), \qquad (30)$$

where subscripts can all attain integer values from 1 to 4 with l, m denoting final phase space coordinates and i, j, k denoting initial phase space coordinates. R_{li} and T_{lij} are coefficients of first and second order terms in the map, respectively. Second order moments of the beam exiting the beam line are given by

$$\langle x_l x_m \rangle = \sum_{i=1}^{4}\sum_{j=1}^{4} R_{li} R_{mj} \langle x_i x_j \rangle + \sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{4} (R_{li} T_{mjk} + R_{mi} T_{ljk}) \langle x_i x_j x_k \rangle + \mathcal{O}(\langle x_i^4 \rangle) \qquad (31)$$

which have a dependence on initial moments higher than second order due to nonlinear terms in the map. For C_2 and SO(2) beams, all of their third order moments vanish, so the leading correction to linear optics arises from fourth order moments. For C_3 beams, however, third order moments are probably nonzero and can affect the final second order moments. Hence, in terms of the order of the residual terms that are neglected, linear optics is a better approximation when the incoming beam is C_2 or SO(2) than when it has C_3 symmetry.

Note that C_2 and SO(2) are the most commonly occurring rotational symmetries in beam lines due to the ubiquity of quadrupoles and solenoids. This fact may render linear optics a surprisingly poor assumption when it is applied to C_3 beams. The assumption may be poor because second order terms in the transfer map act more strongly on the rms evolution of C_3 beams than on that of C_2 or SO(2) beams, and the poorness may be surprising because most of our experience is built upon the less error-prone cases of C_2 and SO(2).

4. Application to FRIB front end

The FRIB front end constitutes a concrete example which demonstrates how results derived solely from symmetry arguments can provide insight on beam dynamics and commissioning activities. A schematic of the FRIB front end [7] is shown in Fig. 2. As is common among heavy ion accelerators, the FRIB front end consists of an ECR ion source which has C_3 symmetry, followed by a transport line with SO(2) symmetry. In the ideal case, the accelerator considered as a whole has C_3 symmetry up to dipole fringe fields illustrated by the green curves.

One question can be asked which is central to the quality and transport of an ECR beam: does $\varepsilon_x = \varepsilon_y$ before the dipole? Under conventional assumptions, the question is purely empirical and $\varepsilon_x \neq \varepsilon_y$ before the dipole would be deemed an unsurprising phenomenon caused by the beam's triangulated spatial density profile and $x - y$ coupling in solenoids. However, theoretical results from Sec. V C 1 have invalidated such an explanation.

Consider the statements:

P: The beam has C_3 symmetry

Q: The beam has the same emittance along any transverse direction

We proved in Sec. V C 1 that $P \Rightarrow Q$. This proposition is a consequence of symmetry alone, so it holds even in the presence of radial field nonlinearities, chromatic aberrations due to beam energy spread, and multispecies space charge with arbitrary intensity and species composition. Furthermore, the proposition applies to both the entire multispecies beam and each individual species therein.

The contrapositive of $P \Rightarrow Q$, i.e., $\neg Q \Rightarrow \neg P$, points to an insightful fact: there is only one fundamental cause for $\varepsilon_x \neq \varepsilon_y$ in a cylindrically symmetric beam line downstream of an ECR ion source and it is broken symmetry. This clarifies the beam dynamics and imparts greater meaning to the question of whether $\varepsilon_x = \varepsilon_y$ before the dipole.

At the FRIB front end, phase space diagnostics are located downstream of the dipole shown in Fig. 2 and their measurements on the target species often show $\varepsilon_x \neq \varepsilon_y$ [9]. While it is possible for ε_x to increase due to dispersion generated by the dipole, and for ε_y to couple to ε_x via space charge, the sign and magnitude of the difference between ε_x and ε_y were found to depend on the solenoid strengths. Since $\varepsilon_y > \varepsilon_x$ also occurred, dispersion cannot be the sole reason for $\varepsilon_x \neq \varepsilon_y$ downstream of the dipole, and we conclude that $\varepsilon_x \neq \varepsilon_y$ upstream of nonzero dipole fields as depicted in Fig. 2.

We can then deduce from $\neg Q \Rightarrow \neg P$ that the beam does not have C_3 symmetry when it enters the dipole. This motivates an investigation on how the symmetry is broken. Three possible causes are (i) imperfect elements; (ii) misalignments; and (iii) neutralization effects from backflowing electrons. The first two causes are related to the properties of applied fields from the ECR ion source and transport line. Among them, our colleagues at FRIB determined the first solenoid was a likely culprit—it was

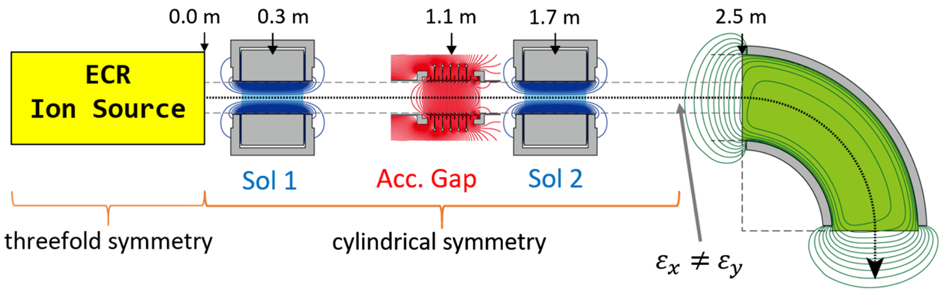

FIG. 2. Schematic of the ARTEMIS beam line at the FRIB front end. Ideally, the ECR ion source has C_3 symmetry while the transport line consisting of two solenoids and an electrostatic acceleration gap is cylindrically symmetric. This image is based on the original from Ref. [8].

later found via 3D magnet simulations [10] to have strong multipole fields due to a nonoptimal design of the current leads. Discussion on replacing the solenoid ensued. The third possibility, electron neutralization, concerns the backflow of electrons from residual gas ionization in downstream regions where no nontrivial rotational symmetry exists. The extent to which these electrons break the C_3 symmetry of self-fields remains to be studied.

VI. CONCLUSION

We performed a general analysis of how the rotational symmetry of a beam imposes equality constraints among transverse beam moments. These constraints can be derived efficiently using analytic methods developed in this paper, and the formalism enabled us to conclude that, for some k and n, SO(2) and C_n symmetries can impose identical constraints on kth order moments. Such a relationship between discrete and continuous rotational symmetries led to the concept of k-th order axisymmetry which can be used to construct elegant symmetry arguments.

Three case studies were conducted to demonstrate the utility of these analytic tools. The cases on SO(2) and C_2 beams exhibited an effortless derivation of familiar results. The case on C_3 beams featured counterintuitive results that were both theoretically interesting and practically important. Their successful application to the FRIB front end constituted a concrete example on how arguments derived purely from symmetry considerations can provide insight on beam dynamics and facilitate commissioning activities at a large project.

All theoretical results in this paper explore the consequences of perfect rotational symmetry. Two extensions to this work will be interesting. First, it is worthwhile to incorporate mirror symmetry into the analysis and study its interplay with rotational symmetry. Second, one can investigate the effects of small errors (e.g., misalignments and multipole fields) that break the rotational symmetry of a beam line. The existing framework is already relevant, for one can deduce by transposition that beam line imperfections exist when phase space measurements deviate from analytic predictions. This was done in the FRIB example, and it would be beneficial to look for other applications which can employ similar reasoning. However, while these conclusions are instructive, they are limited to qualitative statements. More investigations are needed to quantitatively analyze how beam moment constraints change when the rotational symmetry is slightly violated and to examine whether the source and degree of violation can be inferred from phase space diagnostics.

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APPENDIX A: COUNTING THE NUMBER OF MOMENT CONSTRAINTS

In this Appendix, we count the number of 4-tuples (a_1, a_2, a_3, a_4) that satisfy or violate conditions III. 1–III. 3 for given k and n to prove Eqs. (21) and (24).

1. Proof of Eq. (21)

We first define (a) $\xi(k)$ as the number of 4-tuples (a_1, a_2, a_3, a_4) that satisfy condition III. 1; and (b) $\hat{\chi}(k)$ as the number of 4-tuples (a_1, a_2, a_3, a_4) that satisfy conditions III. 1 and $(a_1 + a_3) - (a_2 + a_4) = 0$.

$\xi(k)$ is the number of distinct kth order moments, which applies equally to real moments $\langle x^{a_1} x'^{a_2} y^{a_3} y'^{a_4} \rangle$

and complex moments $\langle w^{a_1}\bar{w}^{a_2}w'^{a_3}\overline{w'}^{a_4}\rangle$. Akin to the number of microstates under Bose-Einstein statistics [11], $\xi(k)$ is equivalent to the total number of ways to distribute k indistinguishable balls into four distinguishable boxes, which gives

$$\xi(k) = \binom{k+3}{k} = \frac{(k+3)!}{3!k!} = \frac{(k+3)(k+2)(k+1)}{3\times 2\times 1}. \tag{A1}$$

To find $\chi(k)$, we use the fact that the number of 4-tuples satisfying $(a_1+a_3)-(a_2+a_4)>0$ and $(a_1+a_3)-(a_2+a_4)<0$ are equal. Therefore,

$$\chi(k) = \frac{1}{2}[\xi(k) - \hat{\chi}(k)]. \tag{A2}$$

For odd k, $(a_1+a_3)-(a_2+a_4)=0$ cannot occur. For even k, $(a_1+a_3)-(a_2+a_4)=0$ occurs whenever $a_1+a_3=a_2+a_4=k/2$. The number of 2-tuples (a_1,a_3) such that $a_1+a_3=k/2$ equals $1+k/2$, likewise for (a_2,a_4). Since (a_1,a_3) and (a_2,a_4) are independent, we obtain

$$\hat{\chi}(k) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ (\frac{k}{2}+1)^2 & \text{if } k \text{ is even.} \end{cases} \tag{A3}$$

Summarizing the results in Eqs. (A1)–(A3) gives Eq. (21) in Sec. IV A. More specifically, when k is even,

$$\begin{aligned} \chi(k) &= \frac{1}{2}\left[\frac{(k+3)(k+2)(k+1)}{3\times 2\times 1} - \left(\frac{k}{2}+1\right)^2\right] \\ &= \frac{k(k+2)(2k+5)}{24}. \end{aligned} \tag{A4}$$

2. Proof of Eq. (24)

To find $\eta(k,n)$, we have to count the number of 4-tuples that satisfy conditions III.1 and III.2 but violate condition III.3. In other words, we have to count the number of 4-tuples that satisfy the two following equations:

$$a_1+a_2+a_3+a_4 = k, \tag{A5}$$

$$(a_1+a_3)-(a_2+a_4) = jn, \tag{A6}$$

where $j \in \{j \in \mathbb{Z}_{>0} | j \le k/n\}$. Note that it is impossible for j to be larger than k/n because $(a_1+a_3)-(a_2+a_4) \le k$.

For a given j, Eqs. (A5) and (A6) are true if and only if

$$a_1+a_3 = \frac{1}{2}(k+jn), \tag{A7}$$

$$a_2+a_4 = \frac{1}{2}(k-jn). \tag{A8}$$

There are $[\frac{1}{2}(k+jn)+1]$ 2-tuples (a_1,a_3) which satisfy Eq. (A7). Similarly, $[\frac{1}{2}(k-jn)+1]$ 2-tuples (a_2,a_4) satisfy Eq. (A8). Hence, the total number of 4-tuples that satisfy Eqs. (A5) and (A6) for a given j is

$$\left[\frac{1}{2}(k+jn)+1\right]\left[\frac{1}{2}(k-jn)+1\right]. \tag{A9}$$

Summing over the number of 4-tuples for all possible values of j, we get

$$\eta(k,n) = \sum_{j=1}^{\lfloor k/n \rfloor}\left[\frac{1}{2}(k+jn)+1\right]\left[\frac{1}{2}(k-jn)+1\right] \tag{A10}$$

where $\lfloor k/n \rfloor$ denotes the largest integer $\le k/n$.

APPENDIX B: ALTERNATIVE PROOF OF CONSTRAINTS GOVERNING SECOND ORDER MOMENTS OF C_n BEAMS

This Appendix presents an alternative, more direct, proof of the fact that, for all $n>2$, the second order moments of a beam with n-fold rotational symmetry (i.e., C_n) obey the same constraints as those of an SO(2) beam, i.e.,

$$\begin{aligned} \langle xx\rangle &= \langle yy\rangle, \\ \langle xx'\rangle &= \langle yy'\rangle, \\ \langle x'x'\rangle &= \langle y'y'\rangle, \\ \langle xy\rangle &= 0, \\ \langle x'y'\rangle &= 0, \\ \langle xy'\rangle &= -\langle x'y\rangle. \end{aligned} \tag{B1}$$

This is true regardless of the orientation of the coordinate axis on the transverse plane. Methods outlined here would become increasingly cumbersome when applied to higher-order moments.

1. Spatial density and local velocity

Define $F(r,\theta,v_r,v_\theta)$ as the beam distribution function in 4D transverse phase space where r and θ are the usual cylindrical polar spatial coordinates and v_r and v_θ denote the corresponding angles in a cylindrical polar representation. The usual transverse particle angles (i.e., transverse velocity normalized by v_z, $x'=v_x/v_z$) are related to v_r and v_θ by

$$\begin{aligned} x' &= v_r\cos\theta - v_\theta\sin\theta, \\ y' &= v_r\sin\theta + v_\theta\cos\theta. \end{aligned}$$

A conveniently normalized spatial distribution function $f(r,\theta)$ with $\int_0^{2\pi}\int_0^\infty f(r,\theta) r dr d\theta = 1$ is defined as

$$f(r,\theta) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(r,\theta,v_r,v_\theta) dv_r dv_\theta}{\int_0^{2\pi}\int_0^\infty\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(r,\theta,v_r,v_\theta) r dr d\theta dv_r dv_\theta}.$$

The local average flow angle of particles within the distribution is denoted $\vec{V}(r,\theta) = V_r(r,\theta)\hat{r} + V_\theta(r,\theta)\hat{\theta}$ (components vary locally as a function of r and θ) and is obtained by averaging over angular degrees of freedom:

$$\vec{V}(r,\theta) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(r,\theta,v_r,v_\theta)(v_r\hat{r}(\theta) + v_\theta\hat{\theta}(\theta)) dv_r dv_\theta}{\int_0^{2\pi}\int_0^\infty\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(r,\theta,v_r,v_\theta) r dr d\theta dv_r dv_\theta}.$$

For a beam with n-fold rotational symmetry, its spatial density and local flow angle of the particle distribution must remain unchanged upon a coordinate rotation by angle $\phi = 2j\pi/n \,\forall\, j \in \mathbb{Z}$, i.e.,

$$f(r,\theta) = f\left(r, \theta + \frac{2j\pi}{n}\right), \tag{B2}$$

$$V_r(r,\theta) = V_r\left(r, \theta + \frac{2j\pi}{n}\right), \tag{B3}$$

$$V_\theta(r,\theta) = V_\theta\left(r, \theta + \frac{2j\pi}{n}\right). \tag{B4}$$

A sufficient, but not necessary, condition for these constraints to hold is the even stronger statement that $F(r,\theta,v_r,v_\theta)$ satisfies

$$F(r,\theta,v_r,v_\theta) = F\left(r, \theta + \frac{2j\pi}{n}, v_r, v_\theta\right) \quad \forall\; j \in \mathbb{Z}.$$

In that case, Eqs. (B2) and (B3) can be verified from

$$\begin{aligned} f(r,\theta) &= \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(r,\theta,v_r,v_\theta) dv_r dv_\theta}{\int_0^{2\pi}\int_0^\infty\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(r,\theta,v_r,v_\theta) r dr d\theta dv_r dv_\theta} \\ &= \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(r,\theta+\frac{2j\pi}{n},v_r,v_\theta) dv_r dv_\theta}{\int_0^{2\pi}\int_0^\infty\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(r,\theta+\frac{2j\pi}{n},v_r,v_\theta) r dr d\theta dv_r dv_\theta} \\ &= f\left(r,\theta+\frac{2j\pi}{n}\right), \end{aligned}$$

and

$$\begin{aligned} V_r(r,\theta) &= \hat{r}(\theta)\cdot\vec{V}(r,\theta) \\ &= \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(r,\theta,v_r,v_\theta) v_r dv_r dv_\theta}{\int_0^{2\pi}\int_0^\infty\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(r,\theta,v_r,v_\theta) r dr d\theta dv_r dv_\theta} \\ &= \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(r,\theta+\frac{2n\pi}{3},v_r,v_\theta) v_r dv_r dv_\theta}{\int_0^{2\pi}\int_0^\infty\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(r,\theta+\frac{2n\pi}{3},v_r,v_\theta) r dr d\theta dv_r dv_\theta} \\ &= V_r\left(r,\theta+\frac{2n\pi}{3}\right). \end{aligned} \tag{B5}$$

Similarly, $V_\theta(r,\theta) = V_\theta(r,\theta+\frac{2n\pi}{3})$.

2. Useful trigonometric identities

In this section, several trigonometric identities are proved in anticipation of their utility in subsequent arguments. The first identity is

Identity B.1

$$\sum_{j=1}^{n}\cos\left(\theta+\frac{4j\pi}{n}\right) = \sum_{j=1}^{n}\sin\left(\theta+\frac{4j\pi}{n}\right) = 0 \quad \forall\; n > 2.$$

Identity B. 1 can be efficiently proven in a complex number representation with Euler's formula $e^{ix} = \cos x + i \sin x$ with $i \equiv \sqrt{-1}$:

$$\sum_{j=1}^{n}\exp\left[i\left(\theta+\frac{4j\pi}{n}\right)\right] = \exp\left(\frac{4\pi i}{n}\right)\sum_{j=1}^{n}\exp\left[i\left(\theta+\frac{4j\pi}{n}\right)\right].$$

For $\exp(\frac{4\pi i}{n}) \neq 1$, this implies that

$$\sum_{j=1}^{n}\exp\left[i\left(\theta+\frac{4j\pi}{n}\right)\right] = 0.$$

The fact that both the real and imaginary parts of the lhs must vanish individually gives us Identity B. 1. Note that when $n = 2$, the proof does not work because $\exp(4\pi i/2) = 1$.

Next we define sums:

$$S_2 \equiv \sum_{j=1}^{n}\cos\left(\theta+\frac{2j\pi}{n}\right)\sin\left(\theta+\frac{2j\pi}{n}\right), \tag{B6}$$

$$S_3 \equiv \sum_{j=1}^{n}\cos^2\left(\theta+\frac{2j\pi}{n}\right), \tag{B7}$$

$$S_4 \equiv \sum_{j=1}^{n}\sin^2\left(\theta+\frac{2j\pi}{n}\right), \tag{B8}$$

to prove two corollaries of Identity B. 1 as follows:

Identity B.2 $S_2 = 0 \;\forall\; n > 2$.

Identity B.3 $S_3 = S_4 \;\forall\; n > 2$.

Identity B.2 can be proven using the trigonometric identity $\sin x \cos x = \frac{1}{2}\sin(2x)$ to rewrite S_2 as

$$S_2 \equiv \sum_{j=1}^{n} \cos\left(\theta + \frac{2j\pi}{n}\right) \sin\left(\theta + \frac{2j\pi}{n}\right) = \frac{1}{2}\sum_{j=1}^{n} \sin\left(2\theta + \frac{4j\pi}{n}\right).$$

It then follows from Identity 1 with $\theta \to 2\theta$ that $S_2 = 0\ \forall\ n > 2$.

Similarly, Identity B.3 can be proven using the trigonometric identities $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$ and $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$ and adding in zero symmetrically:

$$\begin{aligned} S_3 &\equiv \sum_{j=1}^{n} \cos^2\left(\theta + \frac{2j\pi}{n}\right) \\ &= \frac{n}{2} + \frac{1}{2}\sum_{j=1}^{n} \cos\left(2\theta + \frac{4j\pi}{n}\right) \\ &= \frac{n}{2} + 0 && \text{by Identity B.1} \\ &= \frac{n}{2} - \frac{1}{2}\sum_{j=1}^{n} \cos\left(2\theta + \frac{4j\pi}{n}\right) && \text{by Identity B.1} \\ &= \sum_{j=1}^{n} \sin^2\left(\theta + \frac{2j\pi}{n}\right) \\ &\equiv S_4. \end{aligned}$$

3. Proof of beam moment constraints

In this section, we prove the second order axisymmetric beam moment constraints in Eq. (B1) hold for a beam with C_n symmetry by the following trick. For such a beam, all moments remain the same when the coordinate system rotates by $2j\pi/n$ where j is an integer. By summing over $j = 1$ to $j = n$, the trigonometric identities derived in Sec. B 2 can be applied to simplify resulting trigonometric functions and show that the second order axisymmetric moment constraints hold. All constraints are proven for completeness, but with decreasing levels of detail in each successive subproof as the underlying manipulations are analogous.

a. Proof of $\langle xy\rangle = 0$

To prove $\langle xy\rangle = 0$, we note that

$$\langle xy\rangle = \int_0^{2\pi}\int_0^{\infty} r^2 \cos\theta \sin\theta f(r,\theta)\, r dr d\theta.$$

By symmetry, for all integer j,

$$\begin{aligned} \langle xy\rangle &= \int_0^{2\pi}\int_0^{\infty} r^2 \cos\left(\theta + \frac{2j\pi}{n}\right) \sin\left(\theta + \frac{2j\pi}{n}\right) f\left(r, \theta + \frac{2j\pi}{n}\right) r dr d\theta \\ &= \int_0^{2\pi}\int_0^{\infty} r^2 \cos\left(\theta + \frac{2j\pi}{n}\right) \sin\left(\theta + \frac{2j\pi}{n}\right) f(r,\theta) r dr d\theta. \end{aligned} \tag{B9}$$

Using Eq. (B9), we can write

$$\begin{aligned} n\langle xy\rangle &= \sum_{j=1}^{n}\int_0^{2\pi}\int_0^{\infty} r^2 \cos\left(\theta + \frac{2j\pi}{n}\right) \sin\left(\theta + \frac{2j\pi}{n}\right) f(r,\theta) r dr d\theta \\ \langle xy\rangle &= \frac{1}{n}\int_0^{2\pi}\int_0^{\infty} r^2 S_2 f(r,\theta) r dr d\theta \\ &= 0 \quad \text{by Identity B.2.} \end{aligned}$$

b. Proof of $\langle xx\rangle = \langle yy\rangle$

$$\begin{aligned} \langle xx\rangle &= \int_0^{2\pi}\int_0^{\infty} r^2 \cos^2\theta f(r,\theta) r dr d\theta \\ &= \frac{1}{n}\int_0^{2\pi}\int_0^{\infty} r^2 S_3 f(r,\theta) r dr d\theta, \\ \langle yy\rangle &= \int_0^{2\pi}\int_0^{\infty} r^2 \sin^2\theta f(r,\theta) r dr d\theta \\ &= \frac{1}{n}\int_0^{2\pi}\int_0^{\infty} r^2 S_4 f(r,\theta) r dr d\theta. \end{aligned}$$

$S_3 = S_4$ from Identity B. 3. Therefore, $\langle xx\rangle = \langle yy\rangle$.

c. Proof of $\langle xx'\rangle = \langle yy'\rangle$

$$\begin{aligned}\langle xx'\rangle &= \int_0^{2\pi}\int_0^{\infty} r\cos\theta[V_r(r,\theta)\cos\theta - V_\theta(r,\theta)\sin\theta]f(r,\theta)rdrd\theta\\ &= \frac{1}{n}\int_0^{2\pi}\int_0^{\infty}[S_3V_r(r,\theta) - S_2V_\theta(r,\theta)]f(r,\theta)r^2drd\theta,\\ \langle yy'\rangle &= \int_0^{2\pi}\int_0^{\infty} r\sin\theta[V_r(r,\theta)\sin\theta + V_\theta(r,\theta)\cos\theta]f(r,\theta)rdrd\theta\\ &= \frac{1}{n}\int_0^{2\pi}\int_0^{\infty}[S_4V_r(r,\theta) + S_2V_\theta(r,\theta)]f(r,\theta)r^2drd\theta.\end{aligned}$$

$S_3 = S_4$ from Identity B.3 and $S_2 = 0$ from Identity B.2. Therefore, $\langle xx'\rangle = \langle yy'\rangle$.

Henceforth, we will suppress angular arguments of $V_r(r,\theta)$, $V_\theta(r,\theta)$, and $f(r,\theta)$ to further abbreviate.

d. Proof of $\langle xy'\rangle = -\langle x'y\rangle$

$$\begin{aligned}\langle xy'\rangle &= \int_0^{2\pi}\int_0^{\infty} r\cos\theta[V_r\sin\theta + V_\theta\cos\theta]frdrd\theta\\ &= \frac{1}{n}\int_0^{2\pi}\int_0^{\infty} r[S_2V_r - S_3V_\theta]frdrd\theta,\\ \langle x'y\rangle &= \int_0^{2\pi}\int_0^{\infty} r\sin\theta[V_r\cos\theta - V_\theta\sin\theta]frdrd\theta\\ &= \frac{1}{n}\int_0^{2\pi}\int_0^{\infty} r[S_2V_r + S_4V_\theta]frdrd\theta.\end{aligned}$$

$S_3 = S_4$ from Identity B.3 and $S_2 = 0$ from Identity B.2. Therefore, $\langle xy'\rangle = -\langle x'y\rangle$.

e. Proof of $\langle x'x'\rangle = \langle y'y'\rangle$

$$\begin{aligned}\langle x'x'\rangle &= \int_0^{2\pi}\int_0^{\infty}[V_r\cos\theta - V_\theta\ \sin\ \theta]^2frdrd\theta\\ &= \frac{1}{n}\int_0^{2\pi}\int_0^{\infty}[S_3V_r^2 - 2S_2V_rV_\theta + S_4V_\theta^2]frdrd\theta,\\ \langle y'y'\rangle &= \int_0^{2\pi}\int_0^{\infty}[V_r\sin\theta + V_\theta\cos\theta]^2frdrd\theta\\ &= \frac{1}{n}\int_0^{2\pi}\int_0^{\infty}[S_4V_r^2 + 2S_2V_rV_\theta + S_3V_\theta^2]frdrd\theta.\end{aligned}$$

$S_3 = S_4$ from Identity B.3 and $S_2 = 0$ from Identity B.2. Therefore, $\langle x'x'\rangle = \langle y'y'\rangle$.

f. Proof of $\langle x'y'\rangle = 0$

$$\begin{aligned}\langle x'y'\rangle &= \int_0^{2\pi}\int_0^{\infty}[V_r\cos\theta - V_\theta\sin\theta][V_r\sin\theta + V_\theta\cos\theta]frdrd\theta\\ &= \frac{1}{n}\int_0^{2\pi}\int_0^{\infty}\{S_2[V_r^2 + V_\theta^2] + V_rV_\theta[S_3 - S_4]\}frdrd\theta.\end{aligned}$$

$S_3 = S_4$ from Identity B.3 and $S_2 = 0$ from Identity B.2. Therefore, $\langle x'y'\rangle = 0$.

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