Radiation by an electron in an electromagnetic plane wave: The quasiclassical formula and other domains of applications

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The semiclassical Baïer-Katkov (BK) formula is an approximation to the spectrum of the radiation by an ultrarelativistic electron in a nonuniform field, taking into account the recoil and spin effects. The question why it involves only the initial electron trajectory is addressed. When the field is an electromagnetic plane wave, the exact helicity amplitudes in the null-plane frame are derived using Volkov solutions of the Dirac equation and a modified form of the BK formula is obtained. In passing, the application to Compton scattering is outlined and interesting spin properties are noted. For a general field, two conditions of validity of the BK formula are proposed: (1) the field is approximately uniform in the transverse directions, (2) the relative velocity between the field and the electron is close to c. The applications to bremsstrahlung, coherent bremsstrahlung and channeling radiation are discussed.

I. INTRODUCTION

The semiclassical Baïer-Katkov (BK) formula [1,2], written below in the form of Eqs. (24), (27), (28) and 29, predicts an energy-angle spectrum of the radiation by an ultrarelativistic electron in an initial nonuniform field $\mathcal{F}_{\text{in}} = \{\mathbf{E}(t, \mathbf{r}), \mathbf{B}(t, \mathbf{r})\}$. Like the classical radiation formula [3], it involves only the initial classical trajectory of the electron, $\mathbf{r}_i(t)$ (prolongated up to $t = +\infty$) but, in addition, it takes into account the loss of energy-momentum (or *recoil* effect) and predicts a dependence on the electron polarization. These two effects become important when the photon energy ω is comparable to the electron one, $\epsilon \equiv p^0 = \gamma m = (m^2 + \mathbf{p}^2)^{1/2}$.[1] Here are two examples: (i) linear or nonlinear Compton effect when $p_i \cdot k_i \gtrsim m^2$ (p_i and k_i are the initial 4-momenta of the electron and the photon); in this case $\mathcal{F}_{\text{in}}$ is the classical field describing the initial photon or *coherent state* of photons; (ii) synchrotron radiation in the strong field regime, $\chi \equiv \gamma F/m^2 \gtrsim 1$, where $\mathbf{F} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ is the Lorentz force. In the uniform field approximation, the natural photon energy cutoff $\omega_c = \chi \epsilon/(1 + \chi)$ is comparable to ϵ. The BK formula generalizes the "magnetic bremsstrahlung" formula [see, e.g., Eq. (90.22) of [4]] to the case of nonuniform fields. The BK formula has been applied, with good enough agreement with experiment [5–8], in Monte Carlo simulations of channeling radiation or coherent radiation in bent crystals at high energy (e.g., 50 GeV for axial channeling in germanium, 10 GeV in tungsten). Here the strong field parameter is $\chi \equiv \max\{e\gamma|\mathbf{E}(\mathbf{r}_T)|/m^2\} \gtrsim 1$, where $\mathbf{E}(\mathbf{r}_T)$ is the Lindhard continuous field.

The fact that the BK formula does not involve a final electron trajectory $\mathbf{r}_f(t)$ is surprising, since the true quantum amplitude involves both initial and final wave functions, as in Eq. (14) below. On the other hand, $\mathbf{r}_f(t)$ is not uniquely defined. If the emission is considered as a local process which occurs at a definite time t_{em}, $\mathbf{r}_f(t)$ should start at the point $\mathbf{r}_i(t_{\text{em}})$[2] and, due to the recoil effect, separate from $\mathbf{r}_i(t)$ for $t > t_{\text{em}}$. But in reality the photon source is the *whole* initial trajectory [see Eq. (23) or (29)]. For a given $\mathbf{k}$ one can only define a *family* of final trajectories $\mathbf{r}_f(t; t_{\text{em}})$. The BK formula does not involve this family explicitly, as if $\mathbf{r}_f(t; t_{\text{em}})$ was fully determined by $\mathbf{r}_i(t)$ for $t > t_{\text{em}}$, whatever the field $\mathcal{F}_{\text{in}}(t, \mathbf{r})$ is around $\mathbf{r}_i(t)$.

In this article we show that the BK formula becomes exact, after a replacement of ϵ by $p^+/2 \equiv (\epsilon + p^z)/2$, when the field $\mathcal{F}_{\text{in}}(t, \mathbf{r})$ is a plane wave. For this purpose we will use the Volkov solutions of the Dirac equation, to which we associate a classical electron trajectory. In passing we analyze the helicity dependence of the radiation amplitude, give the Stokes parameters of the photon and apply the exact formula to Compton scattering. Then we

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[1]"γ" can also designate a photon. We use units where $\hbar = c = 1$; $\alpha = e^2/(4\pi) \simeq 1/137$.

[2]Here we neglect a *side slipping* [9,10] of the electron, necessary to conserve energy and momentum.

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will give examples where the BK formula is accurate. We finally discuss its use in channeling radiation.

II. VOLKOV STATES

Let $\mathcal{F}_{\text{in}}(t, \mathbf{r}) = \{\mathbf{E}(\zeta), \mathbf{B}(\zeta)\}$ be a plane electromagnetic wave packet moving in the $-\hat{\mathbf{z}}$ direction. $\zeta \equiv X^+ = t + z$ and $\tau \equiv X^- = t - z$ are the lightlike coordinates of a space-time point X, while $X_T = (x, y)$ are the transverse coordinates. $\mathcal{F}_{\text{in}}$ can be derived from a vector potential $\mathbf{A}_T(\zeta) = (A^x(\zeta), A^y(\zeta))$:

$$\mathbf{E}(\zeta) = d\mathbf{A}_T(\zeta)/d\zeta, \qquad \mathbf{B}(\zeta) = -\hat{\mathbf{z}} \times \mathbf{E}(\zeta). \quad (1)$$

The Dirac equation for the electron,

$$[i\partial_t + \boldsymbol{\alpha} \cdot (i\nabla - e\mathbf{A}_T) - m\beta]\Psi(X) = 0, \quad (2)$$

admits the Volkov solutions (see [11] or Chapter IV in [4]), which are eigenstates of $p^+ = p^0 + p^z = i\partial_t - i\partial_z$ and $\mathbf{P}_T = -i\nabla_T$. They can be factorized in the form

$$\Psi(X) = \exp\{-ip^+\tau/2 + i\mathbf{P}_T \cdot \mathbf{X}_T\}\psi(\zeta). \quad (3)$$

Then Eq. (2) reduces to

$$\begin{aligned}[(1 - \alpha_z)p^+/2 + i(1 + \alpha_z)\partial_\zeta \\ - \boldsymbol{\alpha}_T \cdot \mathbf{p}_T(\zeta) - m\beta]\psi(\zeta) = 0, \quad (4)\end{aligned}$$

where $\mathbf{p}_T(\zeta) = \mathbf{P}_T + e\mathbf{A}_T(\zeta)$ is the (gauge-invariant but nonconserved) *mechanical* transverse momentum, $\mathbf{P}_T$ being the (conserved) *canonical* one. We use a representation of Dirac matrices where

$$\alpha_z = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \qquad \beta = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix},$$

$$\boldsymbol{\alpha}_T = \begin{pmatrix} \mathbf{0} & \sigma_z\boldsymbol{\sigma}_T \\ \boldsymbol{\sigma}_T\sigma_z & \mathbf{0} \end{pmatrix}, \quad (5)$$

obtained from the spinorial representation [4] by permuting the second and fourth components of the spinors. Thus $\psi = \binom{\phi}{\chi}$, where the Pauli spinors ϕ and χ are the $\alpha_z = +1$ and $\alpha_z = -1$ components respectively. Equation (4) splits in

$$2ip^+\partial_\zeta\phi(\zeta) = [m^2 + \mathbf{p}_T^2(\zeta)]\phi(\zeta), \quad (6)$$

$$p^+\chi(\zeta) = [m - \sigma_z\boldsymbol{\sigma}_T \cdot \mathbf{p}_T(\zeta)]\phi(\zeta). \quad (7)$$

The solution of the first equation is

$$\phi(\zeta) = \phi(0)\exp\left\{-i\int_0^\zeta d\zeta' \frac{m^2 + \mathbf{p}_T^2(\zeta')}{2p^+}\right\}. \quad (8)$$

Equations (3), (7) and (8) form the Volkov solution. We will consider the Volkov states $|p^+, \mathbf{P}_T, \lambda\rangle$, where $\phi(0) = |\lambda\rangle$ is the basic eigenvector of σ_z with eigenvalue 2λ. $\lambda = \pm 1/2$. is the helicity in the null plane or $p^+ \to \infty$ frame. These states form a continuous complete basis with the orthogonality relation,

$$\begin{aligned}\langle p^+, \mathbf{P}_T, \lambda|p'^+, \mathbf{P}'_T, \lambda'\rangle \\ = \delta_{\lambda\lambda'}J^+ \times (2\pi)^3\delta(p^+ - p'^+)\delta(\mathbf{P}_T - \mathbf{P}'_T), \quad (9)\end{aligned}$$

where $J^+/2 = (J^0 + J^z)/2 = dN_e/(d^2\mathbf{X}_T d\tau)$ is the flux density crossing a hyperplane of constant X^+. With our normalization,

$$J^+ \equiv \Psi^\dagger(X)(1 + \alpha_z)\Psi(X) = 2\phi^\dagger(0)\phi(0) = 2. \quad (10)$$

A. Associated classical trajectories

Let us consider a classical electron world line $X_{\text{cl}}(\zeta) = [t_{\text{cl}}(\zeta), \mathbf{r}_{\text{cl}}(\zeta)]$, parametrized by ζ instead of t. In the field $\mathcal{F}_{\text{in}}$, p^+ is conserved due to the second equation of (1). The canonical transverse momentum $\mathbf{P}_T = \mathbf{p}_{\text{cl},T} - e\mathbf{A}_T$ is also conserved due to the invariance of the four-potential in the transverse directions. Thus $\mathbf{p}_{\text{cl},T}(\zeta) = \mathbf{P}_T + e\mathbf{A}_T(\zeta)$. From the mechanical constraint $p^2 = m^2$ we have also $p^- = p^-(\zeta) = [m^2 + \mathbf{p}_T^2(\zeta)]/p^+$. Using $dX_{\text{cl}}^\mu/d\zeta = p^\mu/p^+$, we deduce

$$\mathbf{r}_{\text{cl},T}(\zeta) = \mathbf{r}_{\text{cl},T}(0) + \int_0^\zeta d\zeta' \frac{\mathbf{p}_{\text{cl},T}(\zeta')}{p^+}, \quad (11)$$

$$\tau_{\text{cl}}(\zeta) = \tau_{\text{cl}}(0) + \int_0^\zeta d\zeta' \frac{m^2 + \mathbf{p}_{\text{cl},T}^2(\zeta')}{(p^+)^2}. \quad (12)$$

To this trajectory corresponds the Volkov states $|p^+, \mathbf{P}_T, \pm 1/2\rangle$. Conversely, for one Volkov state we can associate an infinite family (or "bundle") of classical trajectories, all parallel but distinguished by their $\mathbf{r}_{\text{cl}}(0)$. Since $\mathbf{p}_{\text{cl},T}(\zeta)$ is equal to the eigenvalue of the operator $\mathbf{p}_T(\zeta) = \mathbf{P}_T - \mathbf{A}_T(\zeta)$ of the Volkov state, we will omit the subscript "cl" in the former.

III. RADIATIVE TRANSITION BETWEEN VOLKOV STATES

The spectrum of the radiation by an electron in a monochromatic plane wave was calculated by Ritus [12] using the Volkov states. Here we consider the more general case where the wave is not monochromatic. The radiative transition between states $|i\rangle = |p_i^+, \mathbf{P}_{iT}, \lambda_i\rangle$ and $|f\rangle = |p_f^+, \mathbf{P}_{fT}, \lambda_f\rangle$ yields the differential photon spectrum per electron

$$\frac{dN(\mathbf{k})}{d^3\mathbf{k}/\omega} \equiv \frac{dN(\mathbf{k})}{d^2\mathbf{k}_T dk^+/k^+}$$
$$= \frac{\alpha}{4\pi^2} \sum_{\hat{\boldsymbol{\varepsilon}},\lambda_f} |\langle\lambda_f|a(\mathbf{k},\hat{\boldsymbol{\varepsilon}})|\lambda_i\rangle|^2 \qquad (13)$$

with $\omega = |\mathbf{k}|$ and

$$\langle\lambda_f|a(\mathbf{k},\hat{\boldsymbol{\varepsilon}})|\lambda_i\rangle = \int_{-\infty}^{+\infty} (d\zeta/2) \exp(ik^-\zeta/2)$$
$$\times \psi_f^\dagger(\zeta)(\hat{\boldsymbol{\varepsilon}}^* \cdot \boldsymbol{\alpha})\psi_i(\zeta). \qquad (14)$$

$\psi(\zeta)$ was defined in (3) and normalized with (10); $\hat{\boldsymbol{\varepsilon}}$ is the polarization vector of the photon ($\hat{\boldsymbol{\varepsilon}} \perp \mathbf{k}, \hat{\boldsymbol{\varepsilon}}^* \cdot \hat{\boldsymbol{\varepsilon}} = 1$).

The derivation of Eqs. (13) and (14) is given in the Appendix A. For a given photon momentum $\mathbf{k}$ the electron final state is fully fixed, apart from λ_f, by the conservation laws

$$p_f^+ = p_i^+ - k^+, \qquad \mathbf{P}_{fT} = \mathbf{P}_{iT} - \mathbf{k}_T \qquad (15)$$

which result from the invariance of $\mathcal{F}_{\text{in}}$ under translations in τ and X_T.

Let us place ourselves in a frame where $\mathbf{k}_T = \mathbf{0}$,[3] using the Lorentz transformation "on the null plane"

$$\mathbf{X}_T \Rightarrow \mathbf{X}_T + \boldsymbol{\theta}X^+,$$
$$X^- \Rightarrow X^- + 2\boldsymbol{\theta} \cdot \mathbf{X}_T + \boldsymbol{\theta}^2 X^+,$$
$$\psi(\zeta) \Rightarrow \begin{pmatrix} 1 & 0 \\ \boldsymbol{\theta} \cdot \boldsymbol{\sigma}\sigma_z & 1 \end{pmatrix}\psi(\zeta), \qquad (16)$$

which leaves p^+, ζ, $\mathcal{F}_{\text{in}}$ and the spinor ϕ invariant. We specify the kinematical variables in this frame with a *check* (ˇ) accent. Then, $\check{k}^- = 0$, $\check{\mathbf{P}}_{fT} = \check{\mathbf{P}}_{iT} \equiv \check{\mathbf{P}}_T$ and $\check{\mathbf{p}}_{fT}(\zeta) = \check{\mathbf{p}}_{iT}(\zeta) \equiv \check{\mathbf{p}}_T(\zeta)$. Equation (14) rewrites

$$\langle\lambda_f|a(\mathbf{k},\hat{\boldsymbol{\varepsilon}})|\lambda_i\rangle = \int_{-\infty}^{+\infty} \frac{d\zeta}{2} \langle\lambda_f|Q(\hat{\boldsymbol{\varepsilon}},\zeta)|\lambda_i\rangle e^{i\xi(\zeta)} \qquad (17)$$

with

$$\xi(\zeta) = \frac{k^+}{2}\int_0^\zeta d\zeta' \frac{m^2 + \check{\mathbf{p}}_T^2(\zeta')}{p_i^+ p_f^+} \qquad (18a)$$

$$= Ck \cdot [X_{i,\text{cl}}(\zeta) - X_{i,\text{cl}}(0)], \qquad (18b)$$

$$C = p_i^+/p_f^+, \qquad (19)$$

$$Q(\hat{\boldsymbol{\varepsilon}},\zeta) = \sigma_z\hat{\boldsymbol{\varepsilon}}^* \cdot \boldsymbol{\sigma}\frac{m + \boldsymbol{\sigma} \cdot \check{\mathbf{p}}_T(\zeta)\sigma_z}{p_i^+}$$
$$+ \frac{m + \sigma_z\boldsymbol{\sigma} \cdot \check{\mathbf{p}}_T(\zeta)}{p_f^+}\hat{\boldsymbol{\varepsilon}}^* \cdot \boldsymbol{\sigma}\sigma_z. \qquad (20)$$

Expression (18a) of the phase $\xi(\zeta)$ is derived with the help of Eq. (8). Using (12) and $k^+\tau/2 = k \cdot X$, it becomes (18b). Thus, although we use a full quantum treatment, the phase in the integrand of the amplitude $\langle\lambda_f|a(\mathbf{k},\hat{\boldsymbol{\varepsilon}})|\lambda_i\rangle$ is expressed in terms of a classical trajectory associated to the initial Volkov state.

We choose photons of helicity $\Lambda = \pm 1$, of polarization vector $\hat{\boldsymbol{\varepsilon}} = \hat{\boldsymbol{\varepsilon}}(\Lambda) = (\hat{\mathbf{x}} + i\Lambda\hat{\mathbf{y}})/\sqrt{2}$. Representing helicities only by their signs, Eq. (20) gives

$$\langle-|Q(+,\zeta)|-\rangle = \langle+|Q(-,\zeta)|+\rangle^*$$
$$= \sqrt{2}[\check{p}_x(\zeta) - i\check{p}_y(\zeta)]/p_i^+, \qquad (21a)$$

$$\langle+|Q(+,\zeta)|+\rangle = \langle-|Q(-,\zeta)|-\rangle^*$$
$$= \sqrt{2}[\check{p}_x(\zeta) - i\check{p}_y(\zeta)]/p_f^+, \qquad (21b)$$

$$\langle-|Q(+,\zeta)|+\rangle = -\langle+|Q(-,\zeta)|-\rangle$$
$$= \sqrt{2}(m/p_f^+ - m/p_i^+), \qquad (21c)$$

$$\langle+|Q(+,\zeta)|-\rangle = \langle-|Q(-,\zeta)|+\rangle = 0. \qquad (21d)$$

Gathering Eqs. (17) and (21a)–(21d) we finally obtain

$$\langle\Lambda/2|a(\mathbf{k},\Lambda)|\Lambda/2\rangle = C\hat{\boldsymbol{\varepsilon}}^*(\Lambda) \cdot \mathbf{J}, \qquad (22a)$$

$$\langle-\Lambda/2|a(\mathbf{k},\Lambda)|-\Lambda/2\rangle = \hat{\boldsymbol{\varepsilon}}^*(\Lambda) \cdot \mathbf{J}, \qquad (22b)$$

$$\langle-\Lambda/2|a(\mathbf{k},\Lambda)|\Lambda/2\rangle = \Lambda\frac{C-1}{\sqrt{2}}I, \qquad (22c)$$

$$\langle\Lambda/2|a(\mathbf{k},\Lambda)|-\Lambda/2\rangle = 0, \qquad (22d)$$

where I and $\mathbf{J}$ are expressed in terms of a classical trajectory associated to the initial state:

$$\begin{bmatrix} I \\ \mathbf{J} \end{bmatrix} = \int_{-\infty}^{+\infty} \frac{d\zeta}{p_i^+} \begin{bmatrix} m \\ \check{\mathbf{p}}_T(\zeta) \end{bmatrix} e^{i\xi(\zeta)}, \qquad (23)$$

$\xi(\zeta)$ being given in (18a).

Averaging Eq. (13) over the initial helicity and summing over the final ones gives the *unpolarized* spectrum,

$$\frac{dN(\mathbf{k})}{d^3\mathbf{k}/\omega} = \frac{\alpha}{8\pi^2}\{(C-1)^2|I|^2 + (C^2+1)\mathbf{J} \cdot \mathbf{J}^*\}. \qquad (24)$$

The factor $C = p_i^+/p_f^+ > 1$ is the recoil correction. The classical radiation formulas are obtained by putting $C = 1$,

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[3]This frame is, e.g., the Breit frame of the emitted photon and a photon of the initial wave. Other $\mathbf{k}_T = \mathbf{0}$ frames are related to it by a boost along $\hat{\mathbf{z}}$.

$p_f^+ = p_i^+$ in (18a)–(24). As an example of application of Eqs. (22) we give the Stokes parameters of the photon for unpolarized electrons:

$$\mathcal{P}_1 = \mathcal{N}^{-1} C(|J_x|^2 - |J_y|^2),$$
$$\mathcal{P}_2 = \mathcal{N}^{-1}(C^2 + 1)\mathrm{Im}(J_x J_y^*),$$
$$\mathcal{P}_3 = 2\mathcal{N}^{-1} C \mathrm{Re}(J_x J_y^*),$$
$$\mathcal{N} = (1/2)[(C-1)^2 |I|^2 + (C^2+1)\mathbf{J}\cdot\mathbf{J}^*] \quad (25)$$

($\mathcal{P}_1$ and $\mathcal{P}_3$ = degree of polarizations along $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}} + \hat{\mathbf{y}}$ respectively, $\mathcal{P}_2$ = mean helicity).

A. Notes on the spin properties

For the hardest photons we have $p_f^+ \ll p_i^+$, thus the amplitudes (21b) and (21c) are dominating over (21a): the photon helicity takes the sign of the initial electron one, similarly to the bremsstrahlung case.

The *helicity-flip* amplitude (21c) vanishes for $m = 0$ due to chiral invariance.

In the classical theory the photon is fully polarized; one can check that Eq. (25) gives $\mathcal{P}_1^2 + \mathcal{P}_2^2 + \mathcal{P}_3^2 = 1$ for $C = 1$. For $C > 1$ the polarization is only partial: due to the coupling between the electron and photon spins, part of the entropy (lack of information) on the former affects the later.

We can apply Eqs. (13)–(25) to Compton scattering (see Appendix B). One obtains unconventional but simple expressions (B6) and (B7) for the polarized cross section. The denominators $p_i^+ = (s - m^2)/(2\omega_i)$ and $p_f^+ = (m^2 - u)/(2\omega_i)$ of (21a) and (21b) come from the s and u poles respectively. The absence of the s pole in (21b) may be related to the too large total helicity $\Lambda + \lambda_f = \pm 3/2$ in the s-channel final state. A similar remark holds for the absence of the u pole in (21a). In (21d) both u and s poles are blocked. The null result in this case is nontrivial since it does not ensue from P and T invariances.

B. Generalization to a $\mathbf{k}_T \neq 0$ frame

The $\mathbf{k}_T = 0$ frame simplifies the calculations but for applications it is more convenient to use a frame not linked to $\mathbf{k}$, for instance a $\mathbf{p}_T = 0$ frame, where generally $\mathbf{k}_T \neq 0$. The $\mathbf{k}_T = 0$ frame is deduced from it by the Lorentz transformation (16) with $\boldsymbol{\theta} = -\mathbf{k}_T/k^+$. In this transformation,

$$\mathbf{p}_{iT} \Rightarrow \check{\mathbf{p}}_T \equiv \mathbf{p}_{i,T} - (p_i^+/k^+)\mathbf{k}_T, \quad (26)$$

to be used in formulas (18a), (20), (21a), (21b) and (23).

IV. COMPARISON WITH THE BAÏER-KATKOV FORMULA

The unpolarized BK formula, as written in, e.g. [5,13], takes the form of Eq. (24) but with

$$C = \epsilon_i/\epsilon_f \quad (27)$$

$$\xi(t) \simeq \frac{\omega}{2} \int_0^t dt' \frac{m^2 + \mathbf{p}_\perp^2(t')}{\epsilon_i \epsilon_f}. \quad (28)$$

$$\begin{bmatrix} I \\ \mathbf{J} \end{bmatrix} = \int_{-\infty}^{+\infty} \frac{dt}{\epsilon_i} \begin{bmatrix} m \\ \mathbf{p}_\perp(t) \end{bmatrix} e^{i\xi(t)}, \quad (29)$$

instead of (19), (18a) and (23). $\epsilon_i = \epsilon_f + \omega$ is the initial electron energy. The BK formula is used for the radiation of an ultrarelativistic electron in an arbitrary nonhomogenous field $\mathcal{F}_{\mathrm{in}} = \{\mathbf{E}(t, \mathbf{r}), \mathbf{B}(t, \mathbf{r})\}$, which is not necessarily a plane wave, e.g., a static field. $\mathbf{p}_\perp$ instead of $\mathbf{p}_T$ is the component perpendicular to $\mathbf{k}$, not to a z axis related to $\mathcal{F}_{\mathrm{in}}$. To summarize, the BK formula is obtained from (18a), (19), (23) and (24) by parametrizing the trajectory with t instead of ζ and making the replacements $\check{\mathbf{p}}_T \Rightarrow \mathbf{p}_\perp$ and

$$k^+ \Rightarrow 2\omega, \qquad p^+ \Rightarrow 2\epsilon, \qquad d\zeta \Rightarrow 2dt. \quad (30)$$

Similarly, (22a)–(22d) correspond to equations (18a)–(18d) of [10].

If $\mathcal{F}_{\mathrm{in}}$ is a free plane wave, the BK formula becomes exact in the ultrarelativistic limit $\epsilon_i \to \infty$. This is clear in the $\mathbf{k}_T = 0$ case, where $\mathbf{p}_\perp \equiv \check{\mathbf{p}}_T$, $k^+ \equiv 2\omega$ and $p^+/\epsilon = d\zeta/dt \to 2$. To generalize this result in the $\mathbf{k}_T \neq \mathbf{0}$ case, we cannot invoke a Lorentz transformation because the BK formula is not Lorentz invariant. However, assuming

$$\epsilon_i \omega \gg p_i \cdot k, \quad (31)$$

the 4-vectors p_i and k are nearly collinear and we still have $k^+ d\zeta'/(p_i^+ p_f^+) \simeq \omega dt'/(\epsilon_i \epsilon_f)$ for Eq. (18a), $p_i^+/p_f^+ \simeq \epsilon_i/\epsilon_f$ for Eq. (19) and $d\zeta/p_i^+ \simeq dt/\epsilon_i$ for Eq. (23). It remains to compare $\mathbf{p}_\perp$ to $\check{\mathbf{p}}_T$ defined in (26), measured in their respective transverse planes. The direction of motion of the incoming wave being $-\hat{\mathbf{z}}$, we define $\hat{\mathbf{z}}' = \mathbf{k}/|\mathbf{k}|$ and write $\check{\mathbf{p}}_T = \check{p}_x \hat{\mathbf{x}} + \check{p}_y \hat{\mathbf{y}}$ and $\mathbf{p}_\perp = p_{x'} \hat{\mathbf{x}}' + p_y \hat{\mathbf{y}}$, where $\hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}}'/|\hat{\mathbf{z}} \times \hat{\mathbf{z}}'|$ is a common y axis, $\hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{z}}$ and $\hat{\mathbf{x}}' = \hat{\mathbf{y}} \times \hat{\mathbf{z}}'$. Then $p_y = \check{p}_y$ and some calculations gives

$$p_{x'} - \check{p}_x = (p \cdot k)k_x/(\omega k^+). \quad (32)$$

In the ultrarelativistic limit, $k^+ \to \infty$ while $p \cdot k$ and k_T/ω stay finite, therefore $p_{x'} - \check{p}_x \to 0$. Thus we have $(p_{x'}, p_y) \to (\check{p}_x, \check{p}_y)$, which completes the proof of the exactness of the BK formula, in the ultrarelativistic limit, for an electron crossing a free plane wave.

V. CASES OF GOOD ACCURACY

In the case treated in Sec. III the final trajectory $\mathbf{r}_f(\zeta)$ is fully determined, up to a space-time translation at constant

ζ, by the initial one $\mathbf{r}_i(\zeta)$ and $\mathbf{k}$. In the $\mathbf{k}_T = 0$ frame we have indeed $\mathbf{p}_{fT}(\zeta) = \mathbf{p}_{iT}(\zeta)$ and from Eqs. (11) and (12) one has

$$\mathbf{r}_{fT}(\zeta) - \mathbf{r}_{fT}(0) = C[\mathbf{r}_{iT}(\zeta) - \mathbf{r}_{iT}(0)],$$
$$\tau_f(\zeta) - \tau_f(0) = C^2[\tau_i(\zeta) - \tau_i(0)]. \qquad (33)$$

This explains why the amplitudes (22) and (23) involves only $\mathbf{r}_i(\zeta)$, more precisely $\mathbf{p}_T(\zeta) = p^+ d\mathbf{r}_{iT}/d\zeta$ [there is an equivalent formula involving only $\mathbf{r}_f(\zeta)$]. Two properties of $\mathcal{F}_{\text{in}}$ play a crucial role: (i) it is a wave packet traveling at the light velocity; both trajectories are forced to traverse it entirely; (ii) this wave is *planar*, so that $\mathbf{r}_i(\zeta)$ and $\mathbf{r}_f(\zeta)$ probe the same field.

These properties are not postulated in the original derivation of the BK formula [1]. However the first one is approximately realized for an ultrarelativistic electron in a static field. Indeed, in the initial electron frame, the field becomes almost equivalent to a radiation field (this is the basis of the equivalent photon method). As for the second property, it is also approximately realized if the field does not vary too fast in the transverse directions, more precisely, if the trajectories $\mathbf{r}_i(t)$ and $\mathbf{r}_f(t; t_{\text{em}})$ explore approximately the same field.

Here are two examples where the BK formula is applicable, at least for $\gamma \gg 1$: (i) synchrotron radiation; (ii) coherent bremsstrahlung on atomic planes, provided that ϵ_{iT} and $\epsilon_{fT} \gg U_0$, where $\epsilon_T = \epsilon\psi^2/2$ is the transverse energy, ψ is the angle between the atomic planes and the electron velocity outside the crystal and U_0 is the height (for e^+) or depth (for e^-) of the Lindhard potential.

A. Application to bremsstrahlung on a single atom

In the Born approximation, or if the *coherence length* l_c is much bigger than the atom size r_a, the differential bremsstrahlung cross section takes the factorized form

$$\frac{\omega d\sigma}{d^3\mathbf{k} d^2\mathbf{q}_\perp}(\mathbf{p}_i, \mathbf{p}_f, \mathbf{k}) = \frac{d\sigma_{\text{el}}}{d^2\mathbf{q}_\perp} \times \frac{dN(\mathbf{k})}{d^3\mathbf{k}/\omega}(\mathbf{p}_i, \mathbf{p}_f, \mathbf{k}); \qquad (34)$$

$\mathbf{q} = \mathbf{p}_f + \mathbf{k} - \mathbf{p}_i$ is the momentum transfer; σ_{el} is the elastic cross section; l_c is given by

$$\frac{1}{l_c} \equiv -q_\parallel = \frac{2\omega m^2}{p_i^+ p_f^+} - \frac{\mathbf{p}_{i,\perp}^2}{p_i^+} + \frac{\mathbf{p}_{f,\perp}^2}{p_f^+} \sim \frac{\omega}{\gamma_i \gamma_f}. \qquad (35)$$

The $\perp$ or $\parallel$ components are relative to $\mathbf{k}$; here $p^\pm \equiv p^0 \pm p_\parallel$. The last factor of (34) is independent of the atomic potential. It can be calculated with the BK formula applied to the broken trajectory $\mathbf{r}_i(t)$ made of two semi-infinite straight lines respectively parallel to $\mathbf{p}_i$ and $\mathbf{p}'_{\text{el}}$ defined by

$$\mathbf{p}'_{\text{el}} = |\mathbf{p}_i| \frac{\mathbf{p}_f + \mathbf{k}}{|\mathbf{p}_f + \mathbf{k}|}. \qquad (36)$$

Introducing the vectors $\mathbf{w} = \mathbf{p}_{i,\perp}/m$, $\mathbf{w}' = \mathbf{p}'_{\text{el},\perp}/m$, Eq. (23) gives

$$I = \frac{2\gamma}{C\omega}\left(\frac{1}{1 + \mathbf{w}^2} - \frac{1}{1 + \mathbf{w}'^2}\right),$$
$$\mathbf{J} = \frac{2\gamma}{C\omega}\left(\frac{\mathbf{w}}{1 + \mathbf{w}^2} - \frac{\mathbf{w}'}{1 + \mathbf{w}'^2}\right). \qquad (37)$$

The cross section obtained from (34), (24) and (37) is equivalent to the QED one in the Born approximation [see Eqs. (93.7) of [4]], in the ultrarelativistic limit. In pure QED, $\mathbf{w}'$ is defined as $\mathbf{p}_{f\perp}/m$ but at high energy we have $\mathbf{p}'_{\text{el}} \simeq \mathbf{p}_f + \mathbf{k}$ so that $\mathbf{p}'_{\text{el},\perp} \simeq \mathbf{p}_{f\perp}$.

Beyond the Born approximation and for $l_c \lesssim r_a$, the bremsstrahlung amplitude may be calculated in the eikonal approximation (see e.g. [14]). For a given impact parameter $\mathbf{b}$ and in a frame where both the atom and the electron are relativistic, the eikonal wave functions are close to the Volkov ones in the equivalent radiation field. The associated classical trajectories $\mathbf{r}_i(t)$ and $\mathbf{r}_f(t; t_{\text{em}})$ are practically not separated in the region where the field is important. It seems therefore that the BK formula applies to this case. This has been done in [15] for scalar electron and in [16] to study a correction to the factorization theorem.

B. The case of channeling radiation

The condition of almost uniformity of the field in the transverse directions is not satisfied in channeling radiation. A warning fact is that the spectral lines predicted by the BK formula differ significantly at large ω from those given by the quantized transverse energy levels [10]. The discrepancy is especially strong in the *free-to-bound* transitions, because the initial and final trajectories explore quite different fields. Nevertheless the BK formula is used in Monte Carlo simulations of channeling and above-barrier radiations at high energy [5–8]. In [5,6], for example, the trajectory is divided in pieces containing about one or two channeling oscillations and Eq. (29) is integrated on each piece, prolonged upstream and downstream by two semi-infinite lines of uniform motions. Each piece prolonged in this way is equivalent to the trajectory of an electron crossing a thin crystal. Integrations on the semi-infinite lines introduce a spurious infrared divergence, which however keeps the total radiated energy finite and for which a phenomenological cutoff can be fixed by sum rules [17].

The segmentation prevents large separation between $\mathbf{r}_i(t)$ and $\mathbf{r}_f(t; t_{\text{em}})$ in the equivalent thin crystal, so that these two trajectories explore nearly the same fields, which is the main condition of validity of the BK formula. On the

other hand it discards the interferences between emission from different pieces, so that the spectral peaks generated by a periodic trajectory are blurred. This is a drawback especially in the case of planar channeling of positrons at medium ($\lesssim 10$ GeV) energy, where channeling radiation is quasimonochromatic. However in axial channeling at high energy the spectral peaks become dense enough and the segmentation method just makes a local average of the spectrum. Anyway, at least in the channeling regime where $\omega_{max} \ll \epsilon$, the total radiated energy is not affected by the segmentation because it is given by the Liénard formula,

$$W = \frac{2\alpha}{3m^2} \int_{-\infty}^{\infty} dt (F_{\parallel}^2[\mathbf{r}(t)] + \gamma^2 \mathbf{F}_{\perp}^2[\mathbf{r}(t)]), \quad (38)$$

$\mathbf{F}(\mathbf{r})$ being the force acting on the electron, $\parallel$ and $\perp$ being relative to the electron velocity.

From what we learned in Sec. VA, incoherent bremsstrahlung and its interference with channeling radiation are automatically included if incoherent scatterings are introduced as kinks of the trajectory. This was done in [5–8], where incoherent scatterings were generated at the differential rate $(d^2\sigma_{el}/d^2\mathbf{q}) \times (\text{Debye-Waller factor}) \times \rho(\mathbf{r})$, where $\rho(\mathbf{r})$ is the local density of scattering centers. The simulation results obtained with the above methods in [5,6,18,19] are in good enough agreement with experiments.

VI. CONCLUSION

Using the Volkov states we have given an exact derivation of the formula for the photon spectrum emitted by an electron crossing an electromagnetic plane wave, both in the polarized and unpolarized cases. This formula is similar to the Baïer-Katkov formula, but written in terms of lightlike coordinates. In fact the BK formula converges to it in the ultrarelativistic limit.

We explained why the exact formula for an incident plane wave involves only the initial trajectory. The BK formula also has this property, indicating a strong condition for its validity, namely that in the initial rest frame of the electron the field can be approximated by an electromagnetic plane wave; in particular this field should be homogeneous enough in the transverse directions, so that, after emitting a photon, the electron explores approximately the same field as if its motion was not affected by any recoil. This condition is realized in synchrotron radiation, *beamstrahlung* in future high-luminosity e^+e^- or e^-e^- colliders, *Compton back scattering* on intense focused laser beam and coherent bremsstrahlung at transverse energies $\gg |U_0|$.

The BK formula also applies to bremsstrahlung on a single atom. It gives a simple derivation, via Eqs. (34), (24) and (37), of the differential cross section in the Born approximation, but can be applied to the non-Born, non-factorized regime as well. Concerning channeling radiation, the condition of the nearly uniform field in the

transverse directions is not fulfilled, but the BK formula can nevertheless be used to get a smooth photon spectrum, after segmenting the trajectories in pieces containing only a few oscillations.

In passing we have noted interesting spin properties and derived unconventional but simple expressions of the Compton amplitudes. Using crossing symmetry, most of our derivations and conclusions can be transposed to the Baïer-Katkov formula for e^+e^- pair creation in external fields. Due to its simplicity and intuitive character, the Baïer-Katkov formula is an invaluable tool in theory and phenomenology of high-energy electromagnetic processes.

APPENDIX A: DERIVATION OF EQS. (13) AND (14)

The photon spectrum in the radiative transition from state $|i\rangle$ to state $|f\rangle$ can be written in a covariant way:

$$dN(\hat{\boldsymbol{\varepsilon}})_{fi} = \frac{d^3\mathbf{k}}{2k^0(2\pi)^3} \left| \int d^4X j_{fi}^{\mu}(X) A_{\mu}^*(X) \right|^2, \quad (A1)$$

where $j_{fi}^{\mu}(X) = -e\bar{\Psi}_f(X)\gamma^{\mu}\Psi_i(X)$ is the field source, $A_{\mu}(X) = \varepsilon^{\mu} \exp(-ik_{\nu}X^{\nu})$ the photon wave function and ε^{μ} the polarization 4-vector. Using $d^4X = d^2\mathbf{X}_T d\tau d\zeta/2$, the gauge $\varepsilon^{\mu} = (0, \hat{\boldsymbol{\varepsilon}})$ and $\Psi(X)$ of the form (3), Eq. (A1) becomes

$$dN_{fi}(\hat{\boldsymbol{\varepsilon}}) = \frac{\alpha d^3\mathbf{k}}{k^0(2\pi)^2} \times \mathcal{I}$$
$$\times \left| \int \frac{d\zeta}{2} e^{ik^-\zeta/2} \psi_f{}^{\dagger}(\zeta)(\hat{\boldsymbol{\varepsilon}}^* \cdot \boldsymbol{\alpha})\psi_i(\zeta) \right|^2 \quad (A2)$$

with

$$\mathcal{I} = \int d\tau d\tau' \exp\{-iq^+(\tau' - \tau)/2\}$$
$$\times \int d^2\mathbf{X}_T d^2\mathbf{X}'_T \exp\{i\mathbf{q}_T \cdot (\mathbf{X}'_T - \mathbf{X}_T)\} \quad (A3)$$

and $q^+ = p_f^+ + k^+ - p_i^+$, $\mathbf{q}_T = \mathbf{P}_{fT} + \mathbf{k}_T - \mathbf{P}_{iT}$. Integrating only on τ' and $\mathbf{X}'_T$ gives

$$\mathcal{I} = 2(2\pi)^3 \delta(q^+)\delta^2(\mathbf{q}_T) \int d\tau d^2\mathbf{X}_T. \quad (A4)$$

The delta functions express the conservation laws (15). The remaining integral in (A4) is the total area × duration of the incident wave front. It is infinite, as the time-integrated total flux $(J_i^+/2) \int d\tau d^2\mathbf{X}_T$ of initial electron across the wave front, with $J_i^+ = 2$ according to (10). Thus, removing the infinite factor $\int d\tau d^2\mathbf{X}_T$ and keeping

$$\mathcal{I} = 2(2\pi)^3 \delta(p_f^+ + k^+ - p_i^+)\delta^2(\mathbf{P}_{fT} + \mathbf{k}_T - \mathbf{P}_{iT}), \quad (A5)$$

the expression (A2) becomes the differential photon yield *per incident electron*.

We now sum over the final electron state $|f\rangle$, using the completeness relation

$$\sum_f |f\rangle\langle f| = \sum_{\lambda_f} \int \frac{dp_f^+ d^2\mathbf{P}_{fT}}{J_f^+ (2\pi)^3} |p_f^+, \mathbf{P}_{fT}, \lambda_f\rangle\langle p_f^+, \mathbf{P}_{fT}, \lambda_f|,$$

(A6)

associated to the orthogonality relation (9), with $J_f^+ = 2$. The integrations over p_f^+ and $\mathbf{P}_{fT}$ are trivial owing to (A5). Equations (13) and (14) are thus obtained.

APPENDIX B: APPLICATION TO COMPTON SCATTERING

We consider the reaction $e_i^-(P_i) + \gamma_i(k_i) \rightarrow e_f^-(P_f) + \gamma(k)$, where $k \equiv k_f$. The incoming photon can be described by the classical field

$$\mathbf{A}_T(\zeta) = \mathcal{A}\text{Re}[\hat{\boldsymbol{\varepsilon}}_i \exp(-i\omega_i\zeta)]$$

(B1)

with $\hat{\boldsymbol{\varepsilon}}_i \cdot \hat{\boldsymbol{\varepsilon}}_i^* = 1$. We first place ourselves a frame where $\mathbf{P}_{iT} = \mathbf{0}$, $\mathbf{P}_{f,T} = -\mathbf{k}_T$, $\mathbf{p}_{iT}(\zeta) = e\mathbf{A}_T(\zeta)$, e.g., the c.m. frame or the e_i^- rest frame. Applying (26),

$$\check{\mathbf{P}}_T = -(p_i^+/k^+)\mathbf{k}_T,$$

(B2a)

$$\check{\mathbf{p}}_T(\zeta) = \check{\mathbf{P}}_T + e\mathbf{A}_T(\zeta).$$

(B2b)

Calculating (18) and (23) to first order in $e\mathcal{A}$, we obtain

$$\begin{bmatrix} I \\ \mathbf{J} \end{bmatrix} = 2\pi\delta\left(\frac{m^2 + \check{\mathbf{P}}_T^2}{2p_i^+ p_f^+}k^+ - \omega_i\right)\begin{bmatrix} I' \\ \mathbf{J}' \end{bmatrix}$$

(B3)

with

$$\begin{bmatrix} I' \\ \mathbf{J}' \end{bmatrix} = \frac{-e\mathcal{A}k^+}{p_i^+}\begin{bmatrix} cm\check{\mathbf{P}}_T \cdot \hat{\boldsymbol{\varepsilon}}_i \\ c(\check{\mathbf{P}}_T \cdot \hat{\boldsymbol{\varepsilon}}_i)\check{\mathbf{P}}_T - \hat{\boldsymbol{\varepsilon}}_i/2 \end{bmatrix}$$

(B4)

and $c = k^+/[2\omega_i p_i^+ p_f^+]$. Integrating (13) with (22) and (23) over k^+ gives the differential photon spectrum $dN(\mathbf{k})/d^2\mathbf{k}_T$ of the scattered photon. This is done in analogy with (A3) and (A4): the square of the $2\pi\delta(\cdots)$ factor of (B3) is replaced by $2\pi\delta(\cdots) \times \int d\zeta$. Integrating the delta function over k^+ (noting $p_f^+ = p_i^+ - k^+$) yields a factor $2(p_f^+)^2/(m^2 + \check{\mathbf{P}}_T^2)$. Removing $\int d\zeta$ and dividing by the incident photon density

$$dN_{\text{phot}}/(d^2\mathbf{X}_T d\zeta) = \mathcal{A}^2\omega_i/2,$$

(B5)

we obtain the fully polarized Compton cross section in the form

$$\frac{d\sigma^{\text{pol}}}{d^2\mathbf{w}} = \frac{4\alpha^2}{m^2} \times \frac{|\langle\hat{\boldsymbol{\varepsilon}}, \lambda_f|M(\tau)|\hat{\boldsymbol{\varepsilon}}_i, \lambda_i\rangle|^2}{C^3(1 + \mathbf{w}^2)^4},$$

(B6)

where $\mathbf{w} = \check{\mathbf{P}}_T/m$ and $C = (p_i \cdot k_i)/(p_i \cdot k)$ is linked to $\mathbf{w}$ by $(C - 1)(1 + \mathbf{w}^2) = 2p_i \cdot k_i/m^2$. $\mathbf{w}^2$ ranges from 0 for backward scattering to ∞ for forward scattering. Introducing the complex variable $\underline{w} = w_x + iw_y$, the matrix elements of M for circular polarizations R or L of the photons and helicities $\pm 1/2$ of the electrons have the simple forms

$$\langle L, -|M|L, -\rangle = \langle R, +|M|R, +\rangle^* = C\underline{w}^2,$$

$$\langle L, +|M|L, +\rangle = \langle R, -|M|R, -\rangle^* = \underline{w}^2,$$

$$\langle R, -|M|L, +\rangle = \langle R, -|M|R, +\rangle^* = (C - 1)\underline{w},$$

$$\langle L, +|M|L, -\rangle = \langle L, +|M|R, -\rangle^* = (1 - C)\underline{w},$$

$$\langle R, +|M|L, +\rangle = \langle L, -|M|R, -\rangle = -C,$$

$$\langle R, -|M|L, -\rangle = \langle L, +|M|R, +\rangle = -1,$$

$$\langle R, +|M|L, -\rangle = \langle L, -|M|R, +\rangle = 0.$$

(B7)

The unpolarized Compton cross section

$$\frac{2\alpha^2}{m^2}\int d^2\mathbf{w}\frac{(1 + C^2)(1 + |\mathbf{w}|^4) + 2(C - 1)^2\mathbf{w}^2}{C^3(1 + \mathbf{w}^2)^4}$$

(B8)

agrees the standard result found, e.g., in [4].

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