Statistics of relativistic electrons radiating in periodic fields

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We develop a general method for assessing the evolution of the energy spectrum of relativistic electrons that undergo small quantum losses, such as the ionization losses when the electrons pass through matter and the radiation losses in periodic fields. These processes are characterized by a small magnitude of the recoil quantum as compared with the particle's initial energy. We convey the statistical consideration of the radiating electrons and demonstrate that, for a small average number of recoils, the electron energy spectrum can be described as a composition of consecutive convolutions of the recoil spectrum with itself, weighted with a Poisson distribution. In this stage, the electron's spectrum reveals some individual characteristics of the recoil spectrum. Later, the spectrum loses individuality and allows for an approximate description in terms of statistical parameters. This consideration reveals that the width of the electron's spectrum is increasing with the number of recoils according to a power law, with the power index being inverse to the stability parameter, which gradually increases with the number of recoils from one to two. The increase of the spectrum width limits the ability of the beam to generate coherent radiation in the hard x-ray and gamma-ray region.

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I. INTRODUCTION

In a number of processes involving beams of high-energy electrons, such as radiation in periodic structures, ionization losses in matter, etc., the energy degradation of an incident electron is in the form of small portions (recoil quanta), which spectrum is almost independent of the electron's energy. In our previous papers [1,2], we considered the evolution of the spectrum of such an electron beam. It was shown that the spectrum is determined by the parameters of a single recoil and the average number of recoils.

In a small average number of recoils, the electron energy spectrum can be described as a composition of consecutive convolutions of the recoil spectrum with itself weighted with a Poisson distribution. In the diffusion limit (a big number of recoils), the width of the spectrum increases as the square root of the number of recoils. This paper is focused on the dependence of the spectrum width on the number of recoils in the intermediate range.

The paper is organized as follows: In the second section, we present a method of assessing the evolution of the straggling function that describes the distribution of the energy losses in the interim range of the number of recoils. In the third section, we validate the method by comparing it to the known theories at the limiting cases. The fourth section presents the results of the study of the radiating electrons kinetics in short undulators. The fifth section summarizes the results.

II. STATISTICS OF THE RADIATING ELECTRONS

A. Preliminaries

Distinguishing features of the considered system are (i) a big number of ensemble members (electrons in the radiating bunch) $n \sim 10^{10}$ and (ii) a small average number of recoils (defined as the ratio of the energy emitted by the electron to the mean energy of the spectrum of the radiation) $x < 10^4$.

We adopt the assumption that the spectrum of emitting quantum of the radiation-inducing energy loss (recoil), $w(\omega)$, is "physical": It has compact support, $0 \leq \omega_{\min} \leq \omega \leq \omega_{\max} < \infty$ with ω being the energy of the recoil. The spectrum is normalized to unity, $\int w(\omega)\mathrm{d}\omega = 1$. (Here and below, we drop the infinite limits in integration.)

In this paper, we use the reduced energy units: ϵ for the energy in the straggling spectrum and ω for the energy of

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the spectrum of the recoil quantum; both are dimensionless, normalized to the energy unit, e.g., to the charged particle rest energy [2]. We use a convention for the Fourier transform in the form

$$(\mathcal{F}f)(s) = \hat{f}(s) = \int \mathrm{e}^{-2i\pi\omega s} f(\omega)\mathrm{d}\omega$$

with s being the variable in the Fourier transform domain that complements ω (or ϵ). For the inverse Fourier transform $(\mathcal{F}^{-1}f)(s) = \check{f}(s)$, the (−) sign in the exponent of the integrand is replaced with a (+) sign.

A sketch of the straggling electron's trajectory in the plane (x, ϵ) is presented in Fig. 1. The trajectory is composed of free paths of random length, with the mean unit value and the (positive) random jumps having the same probability density distribution $w(\omega)$. Such a process belongs to the subclass of the *subordinate to the compound Poisson process,* which in turn belongs to the α-stable (or Lévy) processes; see, e.g., Ref. [3].

Evolution of the energy spectrum of the electron bunch is described by a transport equation [4,5]:

$$\frac{\partial f(x,\gamma)}{\partial x} = \int_{-\infty}^{\infty} [w(\omega)f(x,\gamma+\omega) - w(\omega)f(x,\gamma)]\mathrm{d}\omega, \quad (1)$$

with γ being the dimensionless particle energy (Lorentz factor).

A solution to (1) in the form of the characteristic function (Fourier transform of the distribution density) [1,2] is

$$\hat{f} = \hat{f}_0 \exp[x(\check{w} - 1)], \quad (2)$$

where f_0 is the initial spectrum. The parameter $x > 0$ is the ensemble average number of the recoils undergone by an electron since entering the driving force [1,2].

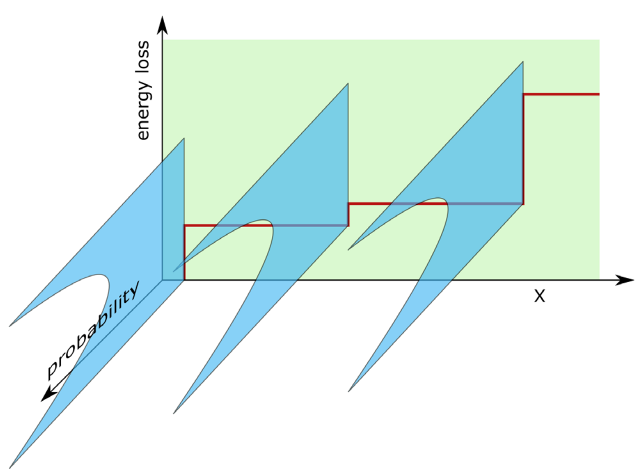

FIG. 1. A sketch of the straggling process. The distribution of recoil magnitudes resembles the dipole radiation spectrum.

Equation (2) may be generalized and simplified due to the assumption of independence of the recoils on the electron energy, as proposed in Ref. [4]. Instead of the beam spectrum, we consider the distribution density of losses (the straggling function [2]). The straggling function, normalized to unity, presents the loss spectrum: Only the particles that have undergone at least one recoil contribute to it.

The characteristic function for the straggling function S_x and its Poisson-weighted expansion are

$$\hat{S}_x = \hat{w}\mathrm{e}^{x(\hat{w}-1)}, \qquad S_x(\epsilon) = \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-x}x^n}{\Gamma(n+1)} F_n(\epsilon),$$
$$F_n = F_{n-1} * w, \qquad F_0 = w, \quad (3)$$

where $*$ stands for the convolution operation. The first three moments of the straggling function—mean, variance, and skewness—read

$$\bar{\epsilon} = (1+x)\bar{\omega}; \quad (4a)$$

$$\mathrm{Var}[\epsilon] \equiv \overline{(\epsilon - \bar{\epsilon})^2} = (1+x)\overline{\omega^2} - \bar{\omega}^2; \quad (4b)$$

$$\mathrm{Sk}[\epsilon] \equiv \overline{(\epsilon - \bar{\epsilon})^3} = (1+x)\overline{\omega^3} - 3\overline{\omega^2}\,\bar{\omega} + 2\bar{\omega}^3. \quad (4c)$$

Here $\bar{\omega}$, $\overline{\omega^2}$, and $\overline{\omega^3}$ are the raw moments of the recoil spectrum $w(\omega)$, $\overline{\omega^n} \equiv \int \omega^n w(\omega)\mathrm{d}\omega$, and the "overline" indicates the ensemble average.

A universal solution for the straggling function (3) allows for an accurate evaluation at the beginning of the process, $x \lesssim 1$ when the series may be limited to a few self-states F_n and, in the opposite limit of the large number of recoils, $x \to \infty$ when the few first moments (4) adequately represent the function. The first moment—mean energy loss—always holds, since it presents the energy conservation law.

B. Statistical properties for a finite number of recoils

From a practical point of view, the most interesting for the physically realizable systems is a medium number of recoils, when a particle, after entering the system, lost a small fraction of its initial energy in the moderate number of recoils, $x\bar{\omega} \ll \gamma_0$, with γ_0 being the initial electron energy.

To evaluate the functional dependency of the spectrum width against the average number of recoils, $\sigma = \sigma(x)$, we compare the distribution (3) with the Lévy α-stable distributions, the only ones attracting the sum of independent identically distributed variables; see Ref. [6].

It should be emphasized that, despite the relevance of the trajectory to the α-stable class, the bunch of such trajectories does not rigorously match the class, since individual trajectories come into the stage at different x, as is depicted

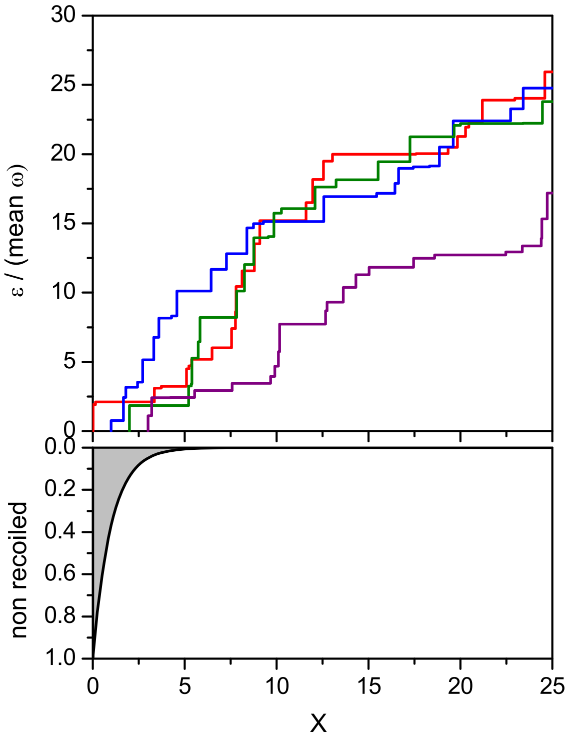

FIG. 2. Four simulated trajectories for the recoils from the dipole radiation emission (top). The density of the nonrecoiled particles is indicated in gray in the bottom panel.

in Fig. 2. Nevertheless, at $x \gg 1$, when almost all of the particles have been recoiled and the width of the distribution exceeds the width of the spectrum of recoil, the bunch of the trajectories is expected to obey a stable law.

The characteristic function of the α-stable process (see, e.g., Ref. [6]) has the general form

$$\hat{\phi}(s) = \exp\{-2\pi i s\mu - |2\pi\sigma s|^{\alpha}[1 - i\beta \mathrm{sgn}(s)\Phi]\}, \quad (5)$$

with

$$\Phi = \begin{cases} \tan(\frac{\pi\alpha}{2}), & \alpha \neq 1, \\ -\frac{2}{\pi}\log|s|, & \alpha = 1. \end{cases}$$

The parameters of the stable distribution are $\alpha \in (0, 2]$ the stability parameter, $\sigma > 0$ the scale parameter, β the skewness parameter, and μ the location parameter.

The model under consideration allows for a reduction of the range of the parameters: The stability parameter should be in the range $\alpha \in (1, 2]$ due to a finite mean of the recoil spectrum; the location parameter is simply equal to the first moment of the straggling function (4a).

A comparison of the characteristic function of straggling (3) with that of the stable distribution (5) leads to two important conclusions: (i) The scaling parameter σ is determined by the real part of the exponent, and (ii) the Fourier transform of a recoil spectrum, in general, may not be of the power form, $\propto |s|^{\alpha}$ with $\alpha = \mathrm{const}$.

We aim the study at the evaluation of the scale parameter of distribution and take into account the similarity theorem, which states that the width of the distribution is inversely proportional to the width of its Fourier transform. Therefore, we suggest evaluating the stability parameter at $s = s_*$, where the real part of the exponent of the characteristic function equals unity:

$$\mathrm{Re}[x(1 - \hat{w}(s_*))] = 1 = |\pi\sigma s_*|^{\alpha}. \quad (6)$$

Here in the square brackets in the left-hand side of Eq. (6), we intentionally omit the small term $\mathrm{Re}[\log \hat{w}(s_*)] \propto -s_*^2$.

From this suggestion, an expression for the stability parameter is readily derived:

$$\alpha = \frac{sD_s \mathrm{Re}[\hat{w}]}{1 - \mathrm{Re}[\hat{w}]}\bigg|_{s=s_*}, \quad (7)$$

where $D_s \equiv \frac{\partial}{\partial s}$ and $s_* = s_*(x) > 0$ is the root of Eq. (6).

Substituting the explicit expression for $\mathrm{Re}[\hat{w}]$,

$$|\pi\sigma s_*|^{\alpha} = x \int w(\omega)[1 - \cos(-2\pi s_*\omega)]\mathrm{d}\omega$$
$$= x \int w(\omega)[\pi s_*\omega]^{\alpha}\mathrm{d}\omega = x m_{\alpha}[w][\pi s_*]^{\alpha},$$

we get a general dependence of the scale parameter—the width of the straggling distribution—on the number of recoils:

$$\sigma(x) = [x m_{\alpha}[w]]^{1/\alpha}, \quad (8)$$

where $m_{\alpha}[w]$ is the raw generalized α moment of the recoil spectrum:

$$m_{\alpha}[w] \equiv \int \omega^{\alpha} w(\omega)\mathrm{d}\omega.$$

Thus, the width of the spectrum increases with the average number of recoils as $\propto x^{1/\alpha(x)}$. The stability parameter $\alpha(x)$, in turn, increases with x from unity to two.

It should be noted that the scale parameter is equal to the half-width of the distribution at $1/e$ of the maximum. At $\alpha \to 2$ when the distribution approaches the normal (Gaussian) distribution, the scale parameter approaches the square root of Gaussian variance divided by 2, $\sigma(x) \to [\overline{(\epsilon - \bar{\epsilon})^2}/2]^{1/2}$.

III. VERIFICATION OF THE METHOD

The two known functional limits of the considered process, $1 \leq \alpha \leq 2$ with $\alpha = 1$ being the Landau distribution and $\alpha = 2$ the Gaussian distribution (Fokker-Plank or diffusive limit), may be considered as benchmarks of the method; see, e.g., Ref. [7].

A. The diffusion limit

As stated by the central limit theorem, the sum of independent identically distributed variables with the finite variance should approach the normal (Gaussian) distribution, which is a limiting case of the stable distributions with $\alpha = 2$. Directly following from Eq. (3), at $x \to \infty$, the real part of the exponent approaches Gaussian:

$$\mathrm{Re}[\hat{w} - 1] \approx 2\pi^2 s^2 \overline{\omega^2}.$$

The same result, $\alpha = 2$, directly stems from Eq. (7), since $s_* \to 0$ when $x \to \infty$.

B. The Landau distribution

A particular case of the stable distributions, the Landau distribution function [4] ($\alpha = 1$, $\beta = 1$), is of special importance, since it has undergone extensive study and experimental validation; see Refs. [8,9]. The process of ionization losses described by the Landau distribution agrees with the assumption of small recoils, whose spectrum is independent of the energy of particles. The problem is the dependence on the energy of the idealized unbound recoil spectrum of $\propto \omega^{-2}$. This spectrum—the Rutherford cross section—cannot be normalized (it has infinite moments).

To avoid the divergence, we consider a truncated recoil spectrum, $0 < a \le \omega \le b < \infty$, then take the limits $a \to 0$, $b \to \infty$, and keep the total energy losses finite. A physical normalized Rutherford cross section (see Refs. [10,11]) reads

$$w_{\mathrm{L}}(\omega) = \frac{\mathrm{sgn}(\omega - a) - \mathrm{sgn}(\omega - b)}{2\omega^2} \frac{ab}{(b-a)}. \tag{9}$$

Its raw moments are finite:

$$\bar{\omega} = \frac{ab}{b-a} \log\left(\frac{b}{a}\right), \qquad \overline{\omega^2} = ab.$$

The Fourier transform for this cross section is

$$\begin{aligned}\hat{w}_{\mathrm{L}}(s) = {}& \frac{1}{(b-a)\sqrt{2\pi}} \times \{b\cos(as) - a\cos(bs) \\ & + sab[\mathrm{Si}(as) - \mathrm{Si}(bs)] - \mathrm{i}[sab(\mathrm{Ci}(as) \\ & - \mathrm{Ci}(bs)) - b\sin(as) + a\sin(bs)]\},\end{aligned} \tag{10}$$

where $\mathrm{Si}(z) = \int_0^z \sin(t)/t\mathrm{d}t$ is the integral sinus and $\mathrm{Ci}(z) = -\int_z^\infty \cos(t)/t\mathrm{d}t$ is the integral cosine.

Explicitly, we have for the model

$$\begin{aligned}\alpha_{\mathrm{L}}(s; a, b) = {}& 1 + \frac{a\cos(2\pi bs) - b\cos(2\pi as)}{ab(b-a)} \\ & + \frac{2\pi sab}{(b-a)}[\mathrm{Si}(2b\pi s) - \mathrm{Si}(2a\pi s)],\end{aligned} \tag{11}$$

with $s = s_*$ being the root of Eq. (6).

The stability parameter (11) has two limits: (i) a, b finite, $x \to \infty$ (accordingly, $s_* \to 0$) and (ii) x finite, $a \to 0$, $b \to \infty$ (Rutherford cross section):

$$\lim_{s\to 0} \alpha_{\mathrm{L}}(s; a, b) = 2, \qquad 0 < a < b < \infty; \tag{12a}$$

$$\lim_{a\to 0, b\to\infty} \alpha_{\mathrm{L}}(s; a, b) = 1, \qquad 0 < s. \tag{12b}$$

Thus, the stability parameter (11) coincides with the Landau distribution at the Rutherford cross section and with the Gaussian distribution at $x \to \infty$ and the finite-moments recoil spectra.

For the physically grounded cases of ionization losses, both the Landau and the Vavilov formulas are valid. The Landau distribution evolves into the Gaussian well beyond the physical region, as is illustrated in Fig. 3 for 50 MeV electrons traversing liquid hydrogen (in this case, the ionization losses are dominant).

As it can be seen from Fig. 3, the Landau distribution adequately describes the evolution of the straggling function. The width of the distribution within the range of validity linearly increases with the mean losses. (Small oscillations in the stability parameter occur because of errors in the numerical computation of the root s_*.)

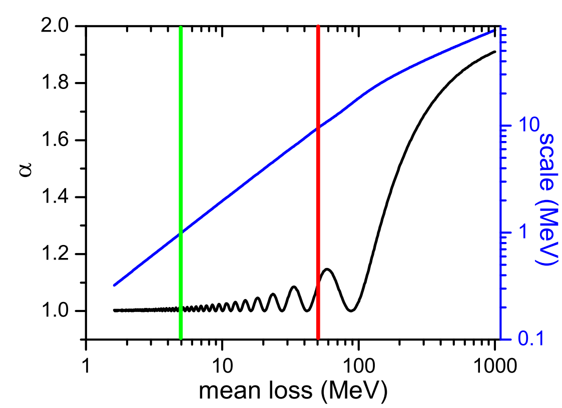

FIG. 3. The stability parameter (black curve) and the scale parameter (blue) against mean energy loss. The vertical green line indicates the limit of validity of the Landau distribution (10% loss; see Ref. [12]), and the red line indicates the physical limit: all the energy radiated out.

IV. RADIATION IN PERIODIC STRUCTURE

As an example of an application of the method to a practical case, we consider the evolution of the straggling function due to emission of the undulator radiation; see, e.g., Ref. [13]. The undulator parameter K is

$$K = \frac{eB\lambda_{\mathrm{u}}}{2\pi m_{\mathrm{e}} c},$$

where B is the magnetic field strength, λ_{u} is the spatial period of the magnetic field, e and m_{e} are the electron charge and the rest mass, respectively, and c is the speed of light.

The evolution of the straggling function for a long undulator and $K \gtrsim 1$ approximates the diffusion process [14]. On the other hand, for the dipole radiation $K \ll 1$ and a short undulator (or the entrance section of a long undulator), $x \lesssim 5$, this function is asymmetric and non-Gaussian [1,15].

Figure 4 represents the straggling function profiles computed in accordance with Eq. (3) for a small average number of recoils. It shows that the straggling function resembles the spectrum of the recoil at $x \ll 1$, and then it gradually spreads out and smoothes, approximating to some degree the Landau distribution.

The stability parameter against the number of recoils, computed in accordance with Eq. (7) for different undulator parameters K, is presented in Fig. 5. As can be seen from the figure, the wider the recoil spectrum, the later the stability parameter approaches the diffusion limit of $\alpha = 2$.

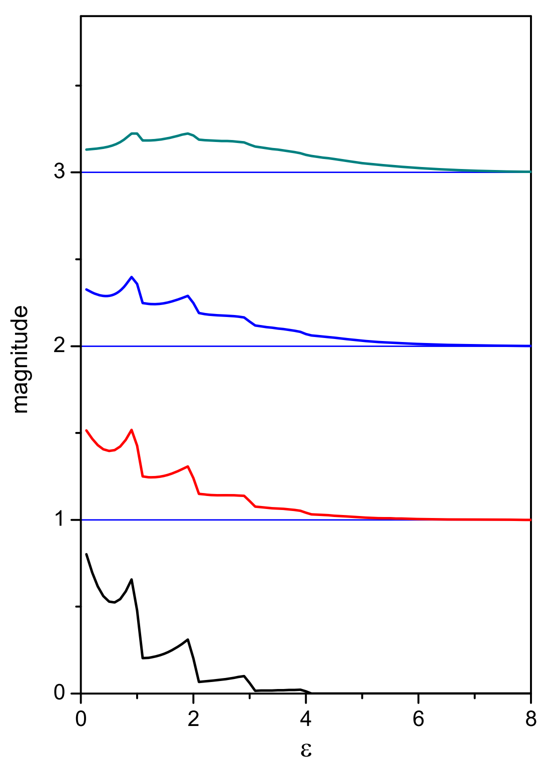

FIG. 4. Straggling distribution function caused by recoils in a helical undulator, $K = 1$, $x = 0.01$, 0.5, 1, 2 (shifted by 0, 1, 2, and 3 from bottom to top, respectively).

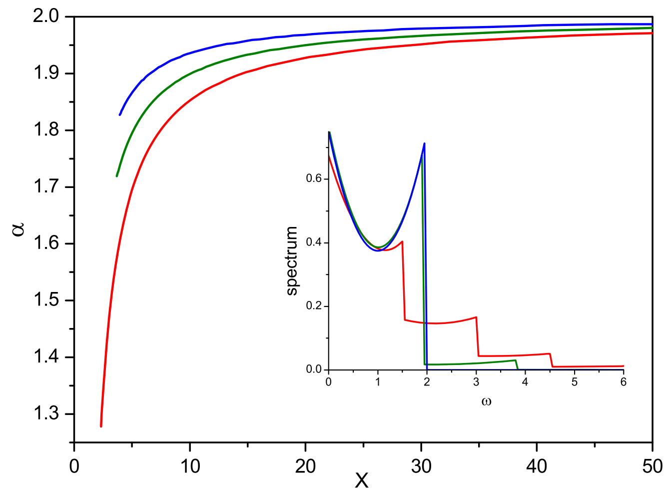

FIG. 5. Stability parameter for $K = 0.01$ (blue), $K = 0.3$ (green), and $K = 1$ (red) vs the average number of recoils. (The inset presents the corresponding recoil spectra.)

When the stability parameter approaches the “diffusion” value of $\alpha = 2$ (still remaining below it), the third centered moment (4c) stays positive and increases with x. We can derive practical information about the mode of the distribution. Making use of Pearson’s skewness for a distribution close to normal (see, e.g., Ref. [16]),

$$\frac{\bar{\epsilon} - \epsilon_{\mathrm{mode}}}{\sigma} = \frac{\mathrm{Sk}[\epsilon]}{2\sigma^3},$$

where ϵ_{mode} is the maximum of the distribution density, we get

$$\begin{aligned}\epsilon_{\mathrm{mode}} &= \bar{\epsilon} - \frac{\mathrm{Sk}[\epsilon]}{2\sigma^2} \\ &= (1+x)\bar{\omega} - \frac{(1+x)\overline{\omega^3} - 3\overline{\omega^2}\,\bar{\omega} + 2\bar{\omega}^3}{2[(1+x)\overline{\omega^2} - \bar{\omega}^2]}.\end{aligned} \tag{13}$$

For a big number of the recoils, $x \to \infty$, the shift of the mode from the mean is almost independent of the number of recoils:

$$\epsilon_{\mathrm{mode}} - \bar{\epsilon} \approx -\frac{\overline{\omega^3}}{2\overline{\omega^2}}.$$

The mode—position of the maximum—is shifted from the mean to smaller energy losses by the constant value, which is determined by the raw moments of the recoil spectrum.

V. SUMMARY

A general dependence of the distribution of energy losses by relativistic electrons due to radiation in periodic structures or ionization losses in matter was analyzed. The straggling function—distribution density of fluctuations—is determined solely by the ensemble-average number of recoils having undergone by the particle since entering the field (or medium in the case of ionization losses) and the spectrum of the recoil.

The straggling function was compared to the Lévy stable process as the only attractor of such processes according to the generalized central limit theorem. The results of this consideration reveal that the width of the electron spectrum is increasing with the number of recoils in accordance with the power law, the power index being inverse to the stability parameter, i.e., linearly with the number of recoils at the beginning of the process, and in proportion to the square root from the number of recoils at the diffusion limit.

An increase of the spectrum width limits the ability of the beam to generate coherent radiation in the hard x-ray and gamma-ray region.

Despite the assumed independence of the recoil spectrum on the electron's energy, a "negligible" (from the electron's point of view) change in this spectrum may play an important role in the reduction of the brightness of sources of hard x rays and gamma rays, which employed relativistic electrons.

It occurs due to the fact that only a small fraction of the spectrum is used: The pinhole fraction of the spectrum has a strong dependency upon the energy spread of the electrons. The attainable width of the pinhole collimated radiation—upper limit of it—will exceed double the electron bunch energy spread [17].

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