Theoretical study of a waveguide THz free electron laser and comparisons with simulations

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In a so-called waveguide free electron laser (FEL) for THz radiations, an extremely small aperture (∼mm) waveguide is used to confine angularly wide spread radiation fields from a low energy electron beam into a small area. This confinement increases the interaction between the electron beam and the radiation fields to achieve a much higher FEL gain. The radiation fields propagate inside the waveguide as waveguide modes, not like a light flux in a free space FEL. This characteristic behavior of the radiation fields makes intuitive understanding of the waveguide FEL difficult. We developed a three-dimensional waveguide FEL theory to calculate a gain of THz waveguide FEL including the effects of the energy spread, the beam size and the betatron oscillations of an electron beam, and effects of a rectangular waveguide. The FEL gain can be calculated as a function of frequency by solving the dispersion relation. Theoretical gains are compared with simulation results for a waveguide FEL with a planar undulator similar to the KAERI one. Good agreements are obtained.

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I. INTRODUCTION

The radiation at THz frequency provides great tools to analyze molecular structures and chemical compounds by moderately exciting molecular oscillations and activating the interaction between molecules. However, the lack of a powerful radiation source has been a major bottleneck for advances of THz sciences and technologies.

In 1986, Electron Laser Facility [1] generated 35 GHz laser. In 1998, Israeli Tandem Electrostatic Accelerator Free-Electron Laser realized the radiation at 100.5 GHz [2]. Recently, the so-called waveguide free electron laser (FEL) technology has emerged as an effective power source for THz radiation. At KAERI [3,4], they successfully operate a compact THz waveguide FEL driven by a magnetron-based microtron and a high-performance planar undulator.

The main differences of the waveguide FEL from a free-space FEL are that it uses a very low energy electron beam (of the order of several MeV) and a small cross sectional waveguide (of the order of several mm). The undulator radiation from such a low energy beam spreads out angularly with a large spread on the order of $1/\gamma$, where γ is the Lorentz factor. By using a small cross sectional waveguide, the THz radiation can be confined into a small area to increase the interaction between the electron beam and the radiation fields and thus the FEL gain as well. In a waveguide FEL, the radiation field propagates inside the waveguide as waveguide modes, not like a light flux in a free space FEL. This characteristic behavior of the radiation field in a waveguide FEL makes intuitive understanding of the waveguide FEL difficult.

Many analytical studies have been done for calculations of FEL gain in free-space [5] as well as in waveguides [6]. One theoretical work for the waveguide FEL is done by Y. Pinhasi and A. Gover [6]. In their theory, the FEL gain for a waveguide with a few cm aperture size is obtained by the direct calculation of the amplitude of the radiation fields excited by the beam with no energy or angular spreads.

On the other hand, Chin *et al.* [7] developed a three-dimensional theory of small-signal high gain FEL in free space. In their work, the gain is obtained by solving the dispersion relation based on the Maxwell-Vlasov equations. The crux of this theory is that they combine the Maxwell-Vlasov equations into a single integral equation for the electron beam distribution, not for the radiation field. In this way, the beam parameters such as the energy spread appear more explicitly in the final form. The results are found to be consistent with the ones obtained by Moore [8] and Yu *et al.* [9].

In the present paper, Chin *et al.*'s theory [7] is generalized in order to cope with the waveguide modes. The beam is assumed to be surrounded by a rectangular chamber. The theory includes the effects of the energy spread, the beam size and the betatron oscillations of an electron beam.

Theoretical results of the FEL gain are compared with the simulation code developed by KAERI. This simulation code can handle only a planar undulator in a very flat rectangular chamber or a helical undulator in a circular chamber. So, for numerical comparisons, we derive a dispersion relation for an infinitely wide waveguide in the horizontal direction.

In Sec. II, we explain the outline of the derivation of a dispersion relation for a planar undulator in a rectangular chamber, starting from the Vlasov equation. The FEL gain can be calculated as a function of frequency by solving the dispersion relation. The boundary condition the radiation fields should satisfy on the surface of the perfectly conductive chamber is considered in the derivation. The dispersion relation for the waveguide FEL reproduces the previous one for the free-space FEL by extending the gap sizes of the waveguide to infinity. For comparison with simulation results, the dispersion relation for the undulator in two infinitely long flat plates is derived by extending the gap with of the rectangular chamber into infinity. In Sec. III, theoretical gains are compared with simulation results for different parameters. The paper is concluded in Sec. IV.

Some details of the derivation of the dispersion relation are described in the Appendices. In Appendix A, we describe the scope of approximations in the present theory. In Appendix B, we introduce a formal expression of the radiation fields in a rectangular waveguide. In Appendix C, the Hamiltonian formalism is introduced to construct the Vlasov equation. In Appendix D, expressions of the radiation fields as a function of the solution of the Vlasov equation are derived. In Appendix E, we calculate the energy change of the beam by the radiation fields, which are needed in the Vlasov equation. By summarizing all results, the Vlasov equation is finally converted to the dispersion relation in Appendix F.

II. FORMULATION TO CALCULATE A FEL GAIN

In this paper, we deal with a waveguide FEL with a planar undulator where the peak wiggler parameter K is of the order of 1, and the Lorentz γ of a beam is of the order of 10. In such a case where $K/\gamma \ll 1$, all terms of $O(K^2/\gamma^2)$ or higher can be neglected. As a result, the radiation fields far from an electron beam are dominated by transverse electric (TE) modes, and contributions of transverse magnetic (TM) modes are negligibly small [10] (see Appendix A). Consequently, the longitudinal component of the vector potential as well as the scalar potential can be neglected as good approximations. Unless higher harmonic generations of the radiation fields are issues, these approximations significantly simplify the formulation and are consistent with conventional FEL theories [10].

Based on the Hamiltonian formalism where the longitudinal coordinate z is chosen as an independent variable, the Vlasov equation is given by

$$\frac{\partial f}{\partial z}+\frac{d\vec{x}_\beta}{dz}\frac{\partial f}{\partial \vec{x}_\beta}+\frac{d\vec{p}_\beta}{dz}\frac{\partial f}{\partial \vec{p}_\beta}+\frac{d\tau}{dz}\frac{\partial f}{\partial \tau}+\frac{d\gamma}{dz}\frac{\partial f}{\partial \gamma}=0, \quad (1)$$

where $\vec{x}_\beta$ and $\vec{p}_\beta$ are the betatron variables and their canonical momenta, τ is the arrival time difference of the electron at the position z relative to that of the reference electron, and $f(\vec{x}_\beta, \vec{p}_\beta, \tau, \gamma; z)$ is the electron distribution function, which is normalized as

$$\int_1^\infty d\gamma \int_{-\infty}^\infty d\tau \int_{-\infty}^\infty d^2\vec{p}_\beta \int_{-\infty}^\infty d^2\vec{x}_\beta f(\vec{x}_\beta, \vec{p}_\beta, \tau, \gamma; z) = N. \quad (2)$$

Here, N is the total number of electrons in the beam.

Using the perturbation method, the distribution function can be decomposed as

$$f = f_0 + f_1, \quad (3)$$

where f_0 and f_1 are the unperturbed and the perturbed parts, respectively.

Consequently, the Vlasov equation can be divided as

$$\frac{\partial f_1}{\partial z}+\vec{p}_\beta\frac{\partial f_1}{\partial \vec{x}_\beta}-k_\beta^2\vec{x}_\beta\frac{\partial f_1}{\partial \vec{p}_\beta}+\frac{d\tau}{dz}\frac{\partial f_1}{\partial \tau}+\frac{d\gamma}{dz}\frac{\partial f_0}{\partial \gamma}=0, \quad (4)$$

for the perturbed part and

$$\frac{\partial f_0}{\partial z}+\vec{p}_\beta\frac{\partial f_0}{\partial \vec{x}_\beta}-k_\beta^2\vec{x}_\beta\frac{\partial f_0}{\partial \vec{p}_\beta}+\frac{d\tau}{dz}\frac{\partial f_0}{\partial \tau}=0, \quad (5)$$

for the unperturbed parts, where k_β is the betatron wave number, which is given by Eq. (C22) [11]. One solution for Eq. (5) is given by

$$f_0 = f_{0\perp}(\vec{x}_\beta^2 + \vec{p}_\beta^2/k_\beta^2) f_{0\parallel}(\gamma), \quad (6)$$

where we assume that f_0 is uniform in the longitudinal direction. The total bunch length is $\hat{\tau}$, and we assume that it is much larger than the wavelength of the FEL light. Equations (4)–(6) are valid only within this bunch length.

The transverse current density $\vec{J}_\perp$ is described in terms of the density distribution of the betatron orbit $\rho_1(\vec{x}_\beta, \tau; z)$ as [7]

$$\vec{J}_\perp = e\frac{d\vec{x}}{dz}\rho_1(\vec{x}_\beta, \tau; z), \quad (7)$$

where e is the electron charge and $\vec{x}$ is the total transverse trajectory of the electron including the wiggler motion $\vec{x}_w$, and the contribution from a scalar potential is neglected. The density $\rho_1(\vec{x}_\beta, \tau; z)$ is expressed as

$$\rho_1(\vec{x}_\beta, \tau; z) = \int_1^\infty d\gamma \int_{-\infty}^{\infty} d^2\vec{p}_\beta f_1(\vec{x}_\beta, \vec{p}_\beta, \tau, \gamma; z), \quad (8)$$

where τ is given by

$$\tau = t - \frac{z}{\bar{v}_r} + \frac{1}{8k_{wz}c}\left(\frac{K}{\tilde{\gamma}}\right)^2 \sin 2k_{wz}z, \quad (9)$$

$$\tilde{\gamma} = \sqrt{\gamma^2 - 1}, \quad (10)$$

$$\frac{1}{\bar{v}_r} = \frac{1}{c}\left(1 + \frac{k_{wz}}{k_1}\right), \quad (11)$$

$$k_1 = \frac{2k_{wz}(\gamma_r^2 - 1)}{1 + \frac{K^2}{2}}, \quad (12)$$

and γ_r is the resonant Lorentz-γ of the reference electron. Here, $\bar{v}_r$ shows the average velocity over one wiggler period, c is speed of light and the wave number k_{wz} is 2π divided by the respective wiggler period length λ_w. Notice that the modulations of the longitudinal motion and thus, that of the longitudinal current density are proportional to $K^2/\tilde{\gamma}^2$. In the scope of the present theory, they can be neglected.

The Fourier transform of ρ_1 on the transverse plane is defined as

$$\rho_1(\vec{x}'_\beta, \tau'; z') = \int_{-\infty}^{\infty} d\omega' e^{-i\omega'\tau'} \times \sum_{n_x, n_y \geq 1}^{\infty} \sin\frac{n_x\pi(x'_w(z') + x'_\beta + \frac{a}{2})}{a} \times \sin\frac{n_y\pi(y_w(z') + y'_\beta + \frac{b}{2})}{b} \rho_{\omega'}(n_x, n_y, z'), \quad (13)$$

and those of the function $\rho_{\omega'}(n_x, n_y, z')$ in the longitudinal direction are introduced as

$$\rho_{\omega' q'}(n_x, n_y) = \int_{-\infty}^{\infty} dz' e^{-iq'z'} \rho_{\omega'}(n_x, n_y, z'), \quad (14)$$

$$\rho_{\omega'}(n_x, n_y, z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dq' e^{iq'z'} \rho_{\omega' q'}(n_x, n_y), \quad (15)$$

where i is the imaginary unit, and we assume that the waveguide is placed in $-a/2 \leq x \leq a/2$ and $-b/2 \leq y \leq b/2$.

By retaining the fast oscillating parts in the transverse motion, Eq. (7) is approximated as [7]

$$\vec{J}_\perp \simeq e\frac{d\vec{x}_w}{dz}\rho_1(\vec{x}_\beta, \tau; z). \quad (16)$$

The transverse current density $\vec{J}_\perp$ is successfully expressed by the density distribution of the betatron orbit $\rho_1(\vec{x}_\beta, \tau; z)$.

Here, the final term in Vlasov Eq. (4) is proportional to the energy change by the radiation fields $\vec{A}_R$, which is given by Eq. (C12), and is approximated as

$$\frac{d\gamma}{dz} = -\frac{e}{m_e c^2}\frac{d\vec{x}_w}{dz}\frac{\partial\vec{A}_R}{\partial t}, \quad (17)$$

by retaining the fast oscillating motion, where m_e is the mass of electron.

The vector potential $\vec{A}_R$ for the radiation field satisfies the inhomogeneous wave equation

$$\nabla^2\vec{A}_R - \frac{1}{c^2}\frac{\partial^2\vec{A}_R}{\partial t^2} = -\mu_0\vec{J}_\perp(\vec{r}, t), \quad (18)$$

where $\mu_0 = Z_0/c$, $Z_0 = 120\pi\ \Omega$ is the impedance of free space. The solution $\vec{A}_R = \vec{A}^w$ is expressed as

$$A_x^w = e\mu_0\int_{-\infty}^{\infty} d\omega e^{-i\omega\tau}\int_{-\infty}^{\infty}\frac{dq}{2\pi}e^{iqz}\frac{1}{4ab}\sum_{m,n=-\infty}^{\infty} e^{i\frac{m\pi x}{a} + i\frac{n\pi y}{b}}\frac{4\pi^2}{4ab}\sum_{n_x,n_y=-\infty}^{\infty}\Theta_{\omega q}(n_x, n_y)H^w_{x,\omega q}(n_x, n_y; m, n, z), \quad (19)$$

where the function $\Theta_{\omega' q'}(n_x, n_y)$ is introduced as

$$\Theta_{\omega' q'}(n_x, n_y) = \int_{-\infty}^{\infty} dz' e^{-iq'z'}\Theta_{\omega'}(n_x, n_y, z'), \quad (20)$$

$$\Theta_{\omega'}(n_x, n_y, z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dq' e^{iq'z'}\Theta_{\omega' q'}(n_x, n_y), \quad (21)$$

$$\Theta_{\omega'}(n_x,n_y,z') = \frac{ab}{4\pi^2} e^{i\frac{n_x\pi}{2}+i\frac{n_y\pi}{2}}(1-\delta_{n_x,0})(1-\delta_{n_y,0}) \begin{cases} -\rho_{\omega'}(n_x,n_y,z'), & \text{for } (n_x\geq 0)\cap(n_y\geq 0), \\ \rho_{\omega'}(n_x,-n_y,z'), & \text{for } (n_x\geq 0)\cap(n_y\leq 0), \\ \rho_{\omega'}(-n_x,n_y,z'), & \text{for } (n_x\leq 0)\cap(n_y\geq 0), \\ -\rho_{\omega'}(-n_x,-n_y,z'), & \text{for } (n_x\leq 0)\cap(n_y\leq 0), \end{cases} \tag{22}$$

where $\delta_{n,m}$ is Kronecker-δ. The function $H^w_{x,\omega q}(n_x,n_y;m,n,z)$ is given by Eq. (D7).

It should be noticed that the function $\Theta_{\omega q}(n_x,n_y)$ appears in $d\gamma/dz$. As Eqs. (8), (21) and (22) show, the function $\Theta_{\omega q}(n_x,n_y)$ depends on the perturbed distribution function f_1. This indicates that Vlasov equation (4) finally converts to a dispersion relation.

Finally, we obtain the dispersion relation:

$$1+\beta_{0,0}M^{0,0,0}_{0,0,0}=0, \tag{23}$$

for a hollow beam:

$$f_{0\perp}(r^2)=\frac{1}{\pi^2R_0^4k_\beta^2}\delta\left(1-\frac{r^2}{R_0^2}\right), \tag{24}$$

where $\delta(x)$ is the δ-function,

$$\beta_{0,0}=2i\frac{kk_w}{k_1\gamma_r}\int_1^\infty d\gamma\frac{f_{0\parallel}(\gamma)}{(iq+2i\frac{k}{k_1}k_w\frac{(\gamma-\gamma_r)}{\gamma_r}-i\frac{1}{2}kk_\beta^2R_0^2)^2}, \tag{25}$$

$$M^{0,0,0}_{0,0,0}=\frac{\pi^2}{8\pi^3a^2b^2}\sum_{n_x,n_y=-\infty}^{\infty}\sum_{\tilde m,\tilde n=-\infty}^{\infty}\bar P^w_{\omega q}(0,n_x,n_y;\tilde m,\tilde n)\frac{n_x}{|n_x|}\frac{n_y}{|n_y|}e^{i\frac{n_x\pi}{2}+i\frac{n_y\pi}{2}}(1-\delta_{n_x,0})(1-\delta_{n_y,0})(e^{i\frac{|n_y|\pi}{2}}-e^{-i\frac{|n_y|\pi}{2}})$$
$$\times\frac{J_1\left(\sqrt{\frac{\tilde m^2\pi^2}{a^2}+\frac{\tilde n^2\pi^2}{b^2}}R_0\right)J_1\left(\sqrt{\frac{|n_x|^2\pi^2}{a^2}+\frac{|n_y|^2\pi^2}{b^2}}R_0\right)}{\left(\sqrt{\frac{\tilde m^2\pi^2}{a^2}+\frac{\tilde n^2\pi^2}{b^2}}R_0\right)\left(\sqrt{\frac{|n_x|^2\pi^2}{a^2}+\frac{|n_y|^2\pi^2}{b^2}}R_0\right)}, \tag{26}$$

$J_n(z)$ is the Bessel function [12], R_0 is a transverse beam size, $r=\sqrt{r_x^2+r_y^2}$ $\bar P^w_{\omega q}(m',n_x,n_y;\tilde m,\tilde n)$ and the betatron wave number k_β are given by Eqs. (F24) and (C22), respectively. Here, we introduce the polar coordinates (r_x,ϕ_x) and (r_y,ϕ_y) in the transverse plane as

$$x_\beta=r_x\cos\phi_x,\qquad \frac{p_{\beta_x}}{k_\beta}=r_x\sin\phi_x, \tag{27}$$

$$y_\beta=r_y\cos\phi_y,\qquad \frac{p_{\beta_y}}{k_\beta}=r_y\sin\phi_y. \tag{28}$$

A. Reproduction of the previous results

Let us check if the present theory can reproduce the free space FEL theory derived by Chin *et al.* [7] by taking the limit of infinitely large waveguide.

It is convenient to introduce the variables:

$$k_x=\frac{\tilde m\pi}{a},\qquad k_y=\frac{\tilde n\pi}{b}, \tag{29}$$

$$k'_x=\frac{n_x\pi}{a},\qquad k'_y=\frac{n_y\pi}{b}. \tag{30}$$

In the limit of small amplitude of the wiggler motion, $r_w\to 0$, $\bar P^w_{\omega q}(0,n_x,n_y;\tilde m,\tilde n)$ becomes

$$\bar P^w_{\omega q}(0,n_x,n_y;\tilde m,\tilde n)\simeq\frac{r_e[e^{-i\frac{|n_x|\pi}{2}}-e^{i\frac{|n_x|\pi}{2}}]}{4c}\left(\frac{K}{\gamma}\right)^2\sum_{\tilde m'=-\infty}^{\infty}\left\{J_{\tilde m'}\left[\frac{\omega}{8k_{wz}c}\left(\frac{K}{\gamma}\right)^2\right]+J_{\tilde m'+1}\left[\frac{\omega}{8k_{wz}c}\left(\frac{K}{\gamma}\right)^2\right]\right\}^2$$
$$\times\left[\frac{1}{\frac{i\omega}{\bar v_r}+i(2\tilde m'+1)k_{wz}+iq-i\sqrt{\frac{\omega^2}{c^2}-\frac{\tilde m^2\pi^2}{a^2}-\frac{\tilde n^2\pi^2}{b^2}}}-\frac{1}{\frac{i\omega}{\bar v_r}+i(2\tilde m'+1)k_{wz}+iq+i\sqrt{\frac{\omega^2}{c^2}-\frac{\tilde m^2\pi^2}{a^2}-\frac{\tilde n^2\pi^2}{b^2}}}\right]$$
$$\times 16\pi^4\delta(-k_x+k'_x)\delta(-k_y+k'_y), \tag{31}$$

where r_e is the classical radius of electron, the factor:

$$\frac{\sin\frac{(-\tilde{m}+n_x)\pi}{2}\sin\frac{(-\tilde{n}+n_y)\pi}{2}}{\frac{(-\tilde{m}+n_x)\pi}{2}\frac{(-\tilde{n}+n_y)\pi}{2}}, \tag{32}$$

is replaced by

$$\frac{4\pi^2}{ab}\delta(-k_x+k'_x)\delta(-k_y+k'_y). \tag{33}$$

After this manipulation, Eq. (26) becomes

$$\begin{aligned} M^{0,0,0}_{0,0,0} \simeq & -\int_{-\infty}^{\infty} dk_x dk_y \frac{r_e}{c}\left(\frac{K}{\gamma}\right)^2 \sum_{\tilde{m}'=-\infty}^{\infty}\left\{J_{\tilde{m}'}\left[\frac{\omega}{8k_{wz}c}\left(\frac{K}{\gamma}\right)^2\right]+J_{\tilde{m}'+1}\left[\frac{\omega}{8k_{wz}c}\left(\frac{K}{\gamma}\right)^2\right]\right\}^2 \\ & \times\left[\frac{1}{\frac{i\omega}{c}(1+\frac{k_{wz}}{k_1})+i(2\tilde{m}'+1)k_{wz}+iq-i\sqrt{\frac{\omega^2}{c^2}-\frac{\tilde{m}^2\pi^2}{a^2}-\frac{\tilde{n}^2\pi^2}{b^2}}}-\frac{1}{\frac{i\omega}{c}(1+\frac{k_{wz}}{k_1})+i(2\tilde{m}'+1)k_{wz}+iq+i\sqrt{\frac{\omega^2}{c^2}-\frac{\tilde{m}^2\pi^2}{a^2}-\frac{\tilde{n}^2\pi^2}{b^2}}}\right] \\ & \times\frac{1}{2\pi}\times\frac{J_1^2\left(\sqrt{\frac{|n_x|^2\pi^2}{a^2}+\frac{|n_y|^2\pi^2}{b^2}}R_0\right)}{\left(\sqrt{\frac{|n_x|^2\pi^2}{a^2}+\frac{|n_y|^2\pi^2}{b^2}}R_0\right)^2}, \end{aligned} \tag{34}$$

where the summations:

$$\sum_{\tilde{m},\tilde{n}=-\infty}^{\infty}, \qquad \sum_{n_x,n_y=-\infty}^{\infty}, \tag{35}$$

are replaced by the integrations:

$$\frac{ab}{\pi^2}\int_{-\infty}^{\infty} dk_x dk_y, \qquad \frac{ab}{\pi^2}\int_{-\infty}^{\infty} dk'_x dk'_y. \tag{36}$$

By sustaining only $\tilde{m}'=-1$ term, Eq. (34) is simplified as

$$M^{0,0,0}_{0,0,0} \simeq -\frac{r_e}{cR_0^2}\left(\frac{K}{\gamma}\right)^2\left\{J_0\left[\frac{\omega}{8k_{wz}c}\left(\frac{K}{\gamma}\right)^2\right]-J_1\left[\frac{\omega}{8k_{wz}c}\left(\frac{K}{\gamma}\right)^2\right]\right\}^2\int_0^{\frac{\pi}{2}}(kR_0)^2\theta d\theta\frac{1}{[iq+ik_{wz}\frac{(k-k_1)}{k_1}+i\frac{\theta^2}{2k}]}\frac{J_1^2(k\theta R_0)}{(k\theta R_0)^2}. \tag{37}$$

Following Ref. [7], if the distribution function is given by the Gaussian function as

$$f_{0\|}(\gamma)=\frac{N}{\hat{\tau}}\frac{e^{-\frac{(\gamma-\gamma_r)^2}{2\gamma_r^2\sigma_\gamma^2}}}{\sqrt{2\pi}\sigma_\gamma\gamma_r}, \tag{38}$$

where $\hat{\tau}$ is the electron bunch length in time unit, σ_γ is the rms energy spread, Eq. (25) is approximated as

$$\beta_{0,0} \simeq 2i\frac{kk_w}{k_1\gamma_r}\frac{N}{\hat{\tau}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dt\frac{e^{-\frac{t^2}{2}}}{(iq+2i\frac{k}{k_1}k_w\sigma_\gamma t - i\frac{1}{2}kk_\beta^2R_0^2)^2}, \tag{39}$$

for small σ_γ, which extends the lower bound of the integration to minus infinity.

In combination with Eqs. (37) and (39), the dispersion relation Eq. (23) is rewritten as

$$1 = 2i\frac{k}{k_1}\frac{(2\rho k_{wz})^3}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dt\frac{e^{-\frac{t^2}{2}}}{(iq+2i\frac{k}{k_1}k_w\sigma_\gamma t - i\frac{1}{2}kk_\beta^2R_0^2)^2} \times \int_0^{\frac{\pi}{2}}(kR_0)^2\theta d\theta\frac{1}{(iq+ik_{wz}\frac{(k-k_1)}{k_1}+i\frac{\theta^2}{2k})}\frac{J_1^2(k\theta R_0)}{(k\theta R_0)^2}, \tag{40}$$

where the Pierce parameter ρ is defined as

$$(2\rho k_{wz})^3 = \pi r_e\frac{N}{c\hat{\tau}\pi R_0^2}\left(\frac{K}{\gamma_r}\right)^2\frac{k_{wz}}{\gamma_r}\left\{J_0\left[\frac{\omega}{8k_{wz}c}\left(\frac{K}{\gamma_r}\right)^2\right] - J_1\left[\frac{\omega}{8k_{wz}c}\left(\frac{K}{\gamma_r}\right)^2\right]\right\}^2, \tag{41}$$

for the planar undulator, which is identical to Eq. (95) in Ref. [7]. In the scope of the present theory, the second term in the bracket in Eq. (41) should be neglected, because it is higher order one for K/γ_r.

The dispersion relation [given by Eq. (40)] is identical to Eq. (94) in Ref. [7], when the unperturbed part of the electron beam is given by the hollow beam [defined in Eq. (24)].

B. Dispersion relation for the case of infinitely wide ($a \to \infty$) waveguide

The simulation code developed at KAERI assumes a uniform distribution for electron energy. For later numerical comparisons, we also derive an explicit form of the dispersion relation for the uniform energy distribution: let us consider the case that the distribution function is given by the uniform one as

$$f_{0\|}(\gamma) = \frac{N}{\hat{\tau}(\gamma_2-\gamma_1)}, \tag{42}$$

where γ_1 and γ_2 are the upper and the lower limits of the Lorentz-γ of the beam. Equation (25) is calculated as

$$\beta_{0,0} = \frac{N}{\hat{\tau}(\gamma_2-\gamma_1)}\left[\frac{1}{(iq+2i\frac{k}{k_1}k_w\frac{(\gamma_1-\gamma_r)}{\gamma_r} - i\frac{1}{2}kk_\beta^2R_0^2)} - \frac{1}{(iq+2i\frac{k}{k_1}k_w\frac{(\gamma_2-\gamma_r)}{\gamma_r} - i\frac{1}{2}kk_\beta^2R_0^2)}\right]. \tag{43}$$

By sustaining the first order terms for $K/\tilde{\gamma}_r$, the dispersion relation is finally simplified as

$$\begin{aligned}
1 &- \frac{i\pi N}{\hat{\tau}(\gamma_2-\gamma_1)}\left[\frac{1}{(iq+2i\frac{k}{k_1}k_w\frac{(\gamma_1-\gamma_r)}{\gamma_r} - \frac{i}{2}kk_\beta^2R_0^2)} - \frac{1}{(iq+2i\frac{k}{k_1}k_w\frac{(\gamma_2-\gamma_r)}{\gamma_r} - \frac{i}{2}kk_\beta^2R_0^2)}\right]\frac{r_e}{b\frac{c^2}{\omega}}\left(\frac{K}{\tilde{\gamma}_r}\right)^2\sum_{n_y=-\infty}^{\infty}\sum_{\tilde{n}=-\infty}^{\infty}(e^{in_y\pi}-1) \\
&\times\left[\frac{\sin\frac{(-\tilde{n}+n_y)\pi}{2}}{\frac{(-\tilde{n}+n_y)\pi}{2}} - \frac{e^{i\tilde{n}\pi}\sin\frac{(\tilde{n}+n_y)\pi}{2}}{\frac{(\tilde{n}+n_y)\pi}{2}}\right] \\
&\times\sum_{m=0}^{\infty}\left[\frac{\{-[\frac{\omega}{c}(1+\frac{k_{wz}}{k_1})+k_{wz}+q]^2-\frac{\tilde{n}^2\pi^2}{b^2}+\frac{\omega^2}{c^2}+\frac{n_y^2\pi^2}{b^2}\}^m\{-[\frac{\omega}{c}(1+\frac{k_{wz}}{k_1})+k_{wz}+q]^2+\frac{\omega^2}{c^2}\}^{\frac{2m+1}{2}}}{\sqrt{[\frac{\omega}{c}(1+\frac{k_{wz}}{k_1})+k_{wz}+q]^2+\frac{\tilde{n}^2\pi^2}{b^2}-\frac{\omega^2}{c^2}}\sqrt{-[\frac{\omega}{c}(1+\frac{k_{wz}}{k_1})+k_{wz}+q]^2+\frac{\omega^2}{c^2}}}\right. \\
&\times\frac{R_0^{4m}J_{1+2m}\left(\sqrt{\{-2[\frac{\omega}{c}(1+\frac{k_{wz}}{k_1})+k_{wz}+q]^2+2\frac{\omega^2}{c^2}-\frac{\tilde{n}^2\pi^2}{b^2}+\frac{n_y^2\pi^2}{b^2}\}R_0^2}\right)}{2^{2m+1}m!(m+1)![\{-2[\frac{\omega}{c}(1+\frac{k_{wz}}{k_1})+k_{wz}+q]^2+2\frac{\omega^2}{c^2}-\frac{\tilde{n}^2\pi^2}{b^2}+\frac{n_y^2\pi^2}{b^2}\}R_0^2]^{\frac{1+2m}{2}}} \\
&+\frac{\{-[\frac{\omega}{c}(1+\frac{k_{wz}}{k_1})-k_{wz}+q]^2-\frac{\tilde{n}^2\pi^2}{b^2}+\frac{\omega^2}{c^2}+\frac{n_y^2\pi^2}{b^2}\}^m\{-[\frac{\omega}{c}(1+\frac{k_{wz}}{k_1})-k_{wz}+q]^2+\frac{\omega^2}{c^2}\}^{\frac{2m+1}{2}}}{\sqrt{(\frac{\omega}{c}(1+\frac{k_{wz}}{k_1})-k_{wz}+q)^2+\frac{\tilde{n}^2\pi^2}{b^2}-\frac{\omega^2}{c^2}}\sqrt{-(\frac{\omega}{c}(1+\frac{k_{wz}}{k_1})-k_{wz}+q)^2+\frac{\omega^2}{c^2}}} \\
&\left.\times\frac{R_0^{4m}J_{1+2m}\left(\sqrt{\{-2[\frac{\omega}{c}(1+\frac{k_{wz}}{k_1})-k_{wz}+q]^2+2\frac{\omega^2}{c^2}-\frac{\tilde{n}^2\pi^2}{b^2}+\frac{n_y^2\pi^2}{b^2}\}R_0^2}\right)}{2^{2m+1}m!(m+1)![\{-2[\frac{\omega}{c}(1+\frac{k_{wz}}{k_1})-k_{wz}+q]^2+2\frac{\omega^2}{c^2}-\frac{\tilde{n}^2\pi^2}{b^2}+\frac{n_y^2\pi^2}{b^2}\}R_0^2]^{\frac{1+2m}{2}}}\right] = 0.
\end{aligned} \tag{44}$$

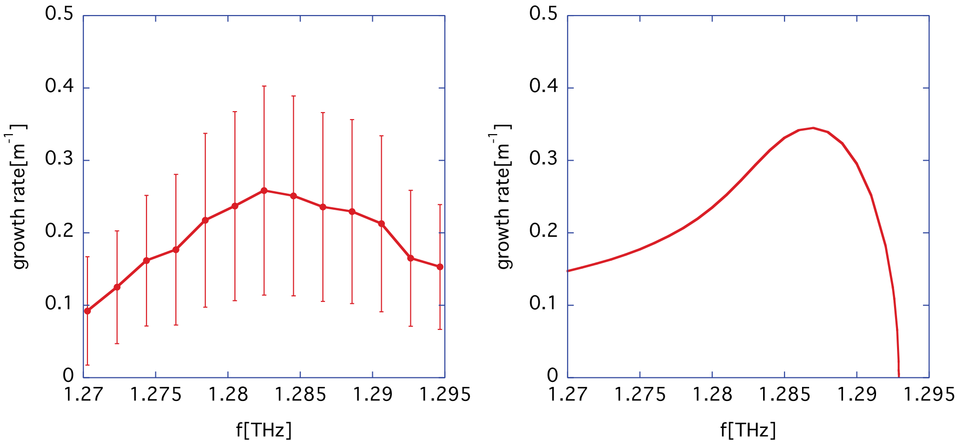

FIG. 1. The simulation (left) and the theoretical (right) results of the beam growth rate for $R_0 = 0.5$ mm, $\gamma_1 = 9.98043$ and $\gamma_2 = 10.0196$.

The beam growth rate is given by the imaginary part of q as a function of frequency f. Its double provides a FEL gain.

III. COMPARISON OF THE FEL GAIN WITH SIMULATION RESULTS

Let us numerically calculate the growth rate of a beam whose energy (Lorentz γ) distributes uniformly between γ_1 to γ_2. The growth rate of the beam is theoretically obtained by solving Eq. (44) as a function of a frequency. The parameters are given as follows: the number of particles per a bunch $N = 6.25 \times 10^6$, the total bunch length $\tau = 20$ ps, the vertical size of the chamber $b = 2$ mm with infinite a, the wiggler period length $\lambda_w = 25$ mm and the K-value $K = 1$. The parameters are similar to those of the planar-type Terahertz FEL in KAERI.

Simulations are done including the space charge effects. The beam growth rate is obtained by calculating the radiation power as a function of the electron flight time. We found that simulation results largely depend on the initial distribution of electron energy (generated by random generators). We take average values of the growth rate over 20 simulations for each set of parameters. We also calculate the standard deviation of results from the average values, shown by error bars in the figures to follow.

Figure 1 shows the results with $R_0 = 0.5$ mm, $\gamma_1 = 9.98043$ and $\gamma_2 = 10.0196$, which correspond to the total energy spread $\Delta E/E$ of about 0.4%. The left and the right figures show the simulation and the theoretical results, respectively. The maximum growth rate (a half of the FEL gain) is obtained at about 1.285 THz in both results. They show a good agreement within the error bars. The TE_{01} mode is excited in the simulation. By artificially extracting the component with mode $\tilde{n}$ and n_y from Eq. (44), the dominant excitation mode can be identified. The theory also shows that the dominant waveguide mode is the TE_1 mode.

The previous studies [3,13] derive formulas for the resonant frequency from the two conditions. One is the resonance condition between the electron and the radiation fields of the pth harmonic:

$$q = \frac{\omega}{c}\left(1 + \frac{k_{wz}}{k_1}\right) - pk_{wz}, \tag{45}$$

where $k_{wz} = 2\pi/\lambda_w$ and λ_w is the wiggler period length. The parameter k_1 is defined by Eq. (12) and introduced to incorporate the modification of the longitudinal velocity of the beam. The other is the dispersion relation of the waveguide:

$$q = \sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}. \tag{46}$$

By combining these two conditions, we can derive a formula for the resonant frequency:

$$f_{\pm} = \frac{c}{2\pi}\left[\frac{pk_{wz}(1 + \frac{k_{wz}}{k_1}) \pm \sqrt{p^2k_{wz}^2(1 + \frac{k_{wz}}{k_1})^2 - \frac{k_{wz}}{k_1}(\frac{k_{wz}}{k_1} + 2)(p^2k_{wz}^2 + \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2})}}{\frac{k_{wz}}{k_1}(\frac{k_{wz}}{k_1} + 2)}\right], \tag{47}$$

where the signs + and − correspond to the Doppler up and Doppler down shifted frequencies of the radiation fields emitted in the forward and the backward directions in the rest frame of electron, respectively. If we use the formula (47) for the parameters used in the simulation in Fig. 1, and set $p = n = 1$ and $m = 0$, the estimated resonant frequency becomes $f_+ = 1.29$ THz. This frequency is in a good agreement with the theoretical and the simulation results shown in Fig. 1. Though the formula (47) is very simple, it provides a remarkably accurate estimate of the resonant frequency.

One should notice that there is no resonance between the electron beam and the radiation fields if the argument of the square root in the formula (47) is negative. In other words, the waveguide sizes a and b must satisfy the following condition for a FEL to lase:

$$\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \le p^2 k_{wz}^2 \left[\frac{k_1(1+\frac{k_{wz}}{k_1})^2}{k_{wz}(2+\frac{k_{wz}}{k_1})} - 1\right] \simeq \frac{p^2 k_{wz} k_1}{2}\left(1 - \frac{k_{wz}}{2k_1}\right) \quad \text{for } k_1 \gg k_{wz}. \tag{48}$$

For the present parameters, the vertical waveguide size b must exceed 1.5 mm for lasing.

Let us see the dependence of the growth rate and the resonant frequency on the vertical waveguide size b in more details. The red and the blue lines in Fig. 2 show the theoretical results of the dependence of the growth rate and the resonant frequency on the vertical waveguide size b, respectively. The growth rate has a peak around $b = 1.6$ mm. This waveguide size corresponds to the condition that the argument of the square root in Eq. (47) is close to zero.

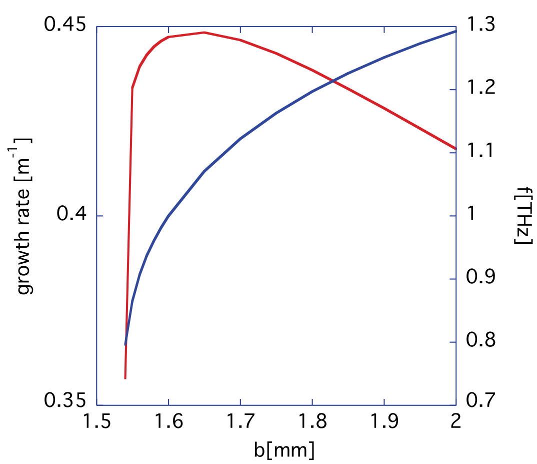

FIG. 2. The dependence of the growth rate (red) and the resonant frequency (blue) on the waveguide size b with $R_0 = 1$ μm, $\gamma_r = 10$ and $\Delta E/E \simeq 0.4\%$. The red and the blue curves are read by using the scale markings on the left and the right vertical axes, respectively.

Let us consider the physical meaning of this condition. The group velocity of a waveguide mode is given by a derivative of ω by q. From Eq. (46), we have

$$\frac{d\omega}{dq} = \frac{c^2}{\omega}\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}. \tag{49}$$

By substituting Eq. (47) into (49) and using the condition that the argument of the square root in Eq. (47) is zero,

$$p^2 k_{wz}^2\left(1+\frac{k_{wz}}{k_1}\right)^2 - \frac{k_{wz}}{k_1}\left(\frac{k_{wz}}{k_1}+2\right) \times \left(p^2 k_{wz}^2 + \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right) = 0, \tag{50}$$

the group velocity is simplified as

$$\frac{d\omega}{dq} = \frac{c}{(1+\frac{k_{wz}}{k_1})}, \tag{51}$$

which is identical to the average velocity of the beam $\bar{v}_r$, given by Eq. (11).

At this grazing point, two resonant waveguide modes emerge into one. Thus, we can conclude that the maximum FEL gain is obtained when the group velocity of the waveguide mode is equal (or close) to the average beam velocity. In this condition, there is no slippage between the FEL light and the electron beam and thus the maximum saturation power will be also obtained at zero (or small) cavity detuning (i.e. the roundtrip length of the cavity between two mirrors is equal to the electron bunch spacing in an oscillator).

Next, let us see the dependence of the beam growth rate on the beam size. Figure 3 shows the theoretical results of the dependence of the beam growth rate on the beam size

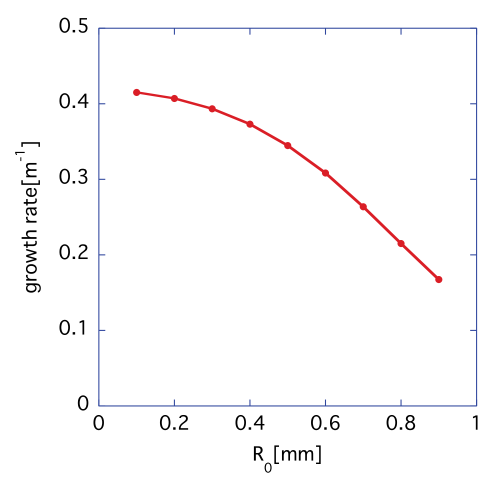

FIG. 3. The dependence of the growth rate on the beam size R_0 with $\gamma_r = 10$ and $\Delta E/E \simeq 0.4\%$.

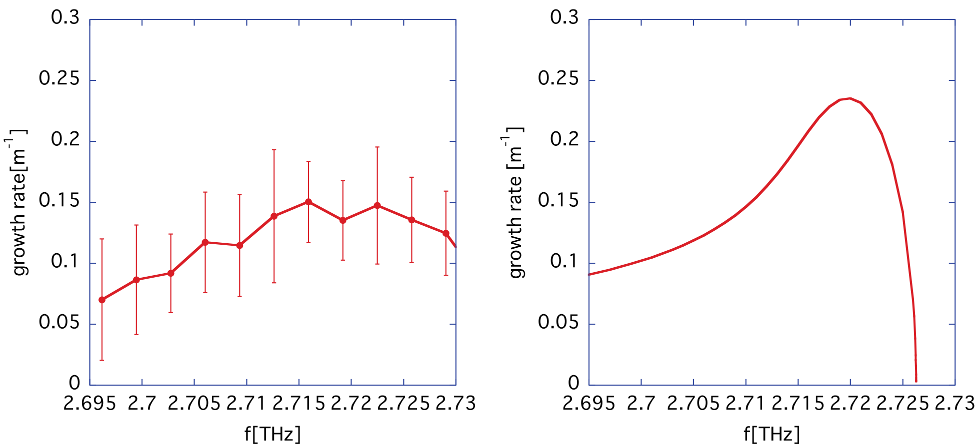

FIG. 4. The simulation (left) and the theoretical (right) results of the beam growth rate for $R_0 = 0.5$ mm, $\gamma_1 = 13.7011$ and $\gamma_2 = 13.7392$.

R_0. The respective points correspond to the values for the beam size from $R_0 = 0.1$ mm to $R_0 = 0.9$ mm with 0.1 mm interval. The growth rate seems to saturate at zero beam size. The growth rate (a half of the FEL gain) slowly decreases as the beam size increases by losing the coherence of the radiation. Even at $R_0 = 0.5$ mm at which the beam occupies a half of the vertical waveguide size of 2 mm, the growth rate still attains about 80% of the ideal saturated gain at zero beam size.

Next, let us increase the beam energy to get THz radiation at twice higher frequency. Figure 4 shows the results with $\gamma_1 = 13.7011$ and $\gamma_2 = 13.7392$, where $\Delta E/E \simeq 0.3\%$. The peak frequency shifts to 2.72 THz. The agreement between the simulation and the theoretical results is good overall. However, the agreement is less than the previous result, due to choice of a smaller energy spread.

To see the dependency of the growth rate on the energy spreads, we calculate the peak growth rate for different energy spread (all other parameters are fixed). The results for $\gamma_r = 10$ are shown in Fig. 5. The theoretical and the simulation results are shown by the blue and the red lines, respectively. The agreement between the theory and the simulation is good overall, in particular at large energy spread. But, the simulation results start to deviate from the theoretical ones at small energy spread region (less than 0.3%).

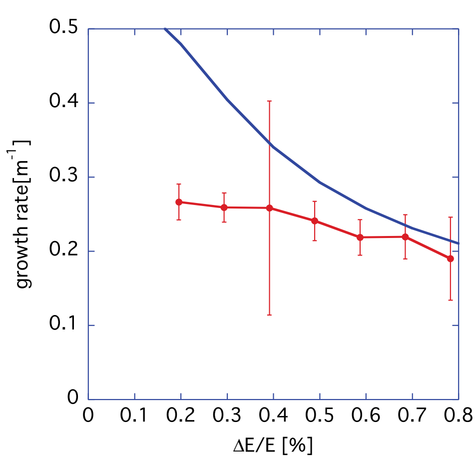

FIG. 5. The energy spread $\Delta E/E$ dependence of the maximum beam growth rate with $\gamma_r = 10$. The simulation results (red) start to deviate from the theoretical ones (blue) in a small energy spread region.

We believe the reason of this deviation as follows. When the initial energy spread is too small in a simulation, it will be quickly enlarged by the space charge effects and the interaction between the beam and the radiation field. Thus, the actual energy spreads during the simulations in the small initial energy spread region are larger than the initial ones. As a result, the growth rate becomes smaller.

IV. SUMMARY

We have developed the three-dimensional theory of a waveguide FEL for THz radiation by expanding the method shown in Ref. [7] to include effects of a rectangular chamber. The radiation fields are calculated by solving the inhomogeneous wave equations with the boundary condition. Once the distribution function of electron energy is given, the Maxwell-Vlasov equation gives the dispersion relation. The beam growth rate (a half of the FEL gain) can be calculated by solving the dispersion relation as a function of the frequency. The present theory can reproduce the result of Ref. [7] for free space by taking the limit of infinitely large waveguide. The theory predicts that the maximum FEL gain is obtained when the waveguide size is optimized so that the group velocity of the waveguide mode is close to the average velocity of an electron beam. This zero slippage condition also implies that the maximum saturation power will be obtained at zero (or small) cavity detuning.

KAERI develops a simulation code for Terahertz FEL, where two parallel plates are inserted in a planar undulator. The reliability of the theory was investigated by comparing the results with the simulation results. The comparisons were done by taking the horizontal size of the waveguide to infinity in the theory.

The numerical comparisons show good agreements. The simulation shows smaller gains with small initial energy spreads than the theory, but this may be explained by quick dilution of the initial energy spread (when it is too small) due to the space-charge effect and the interaction between the beam and the radiation field.

We hope that the present theory will provide a useful tool for design and understanding of a waveguide FEL and advances in the THz sciences and technologies. The *Mathematica* [14] input file to compute the FEL gain is available from the authors.

ACKNOWLEDGMENTS

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APPENDIX A: THE RADIATION FIELD INSIDE TWO PARALLEL PLATES WAVEGUIDE

In Ref. [10], Amir *et al.* analyzed the incoherent emission from an undulating electron beam in the presence of metallic boundaries with the gap height b. The electric and the magnetic fields, when a single electron at point $\boldsymbol{r}'(\boldsymbol{t}')$ moves with a velocity $\boldsymbol{\beta}(\boldsymbol{t}')$, are given by

$$\boldsymbol{E}(\boldsymbol{r},t)=\frac{1}{2\pi}\int\boldsymbol{E}_{\omega}(\boldsymbol{r})e^{-i\omega t}d\omega, \tag{A1}$$

$$\boldsymbol{B}(\boldsymbol{r},t)=\frac{1}{2\pi}\int\boldsymbol{B}_{\omega}(\boldsymbol{r})e^{-i\omega t}d\omega, \tag{A2}$$

where

$$\boldsymbol{E}_{\omega}=ike\frac{2\pi i}{b}\sum_{m=1}^{\infty}\int dt'\left(\overleftrightarrow{\boldsymbol{I}}-\frac{1}{k^2}\boldsymbol{\nabla}\boldsymbol{\nabla}'\right)H_0^{(1)}(k_{\parallel}\rho)\sin\frac{m\pi}{b}y\sin\frac{m\pi}{b}y'\boldsymbol{\beta}(t')e^{i\omega t'}, \tag{A3}$$

$$\boldsymbol{B}_{\omega}=-\frac{2\pi ie}{b}\sum_{m=1}^{\infty}\int dt'\boldsymbol{\nabla}\times\left[\boldsymbol{\beta}(t')H_0^{(1)}(k_{\parallel}\rho)\sin\frac{m\pi}{b}y\sin\frac{m\pi}{b}y'\right]e^{i\omega t'}, \tag{A4}$$

$\overleftrightarrow{I}$ is identity matrix, $k=\sqrt{k_x^2+k_y^2+k_z^2}=\omega/c$, $k_{\parallel}=\sqrt{k^2-k_y^2}$, $H_0^{(1)}(z)$ is the Hankel function of the first kind [12] and $\rho\equiv\sqrt{(x-x')^2+(z-z')^2}$ (not the Pierce parameter) only in this appendix.

If we focus on the far field and consider only the leading terms of the order of K/γ, they are simplified as,

$$\boldsymbol{E}_{\omega}=ike\frac{2}{b}\sqrt{2\pi}e^{i\pi/4}\sum_{m=1}^{\infty}\int dt'\frac{e^{ik_{\parallel}\rho}}{\sqrt{k_{\parallel}\rho}}\left[\hat{\boldsymbol{x}}\left(\beta_x-\beta_z\frac{k_{\parallel}}{k}\sin\xi\right)\sin\frac{m\pi}{b}y+i\hat{\boldsymbol{y}}\frac{m\pi}{kb}\beta_z\cos\frac{m\pi}{b}y\right]\sin\frac{m\pi}{b}y'e^{i\omega t'}+O(\rho^{-\frac{3}{2}},\gamma^{-2},K^2), \tag{A5}$$

$$\begin{aligned}\boldsymbol{B}_{\omega}=-\frac{2ke}{b}\sqrt{2\pi}e^{i\pi/4}\sum_{m=1}^{\infty}\int dt'\frac{e^{ik_{\parallel}\rho}}{\sqrt{k_{\parallel}\rho}}&\left[\hat{\boldsymbol{x}}\frac{m\pi}{kb}\cos\frac{m\pi}{b}y+i\hat{\boldsymbol{y}}\left(\beta_x\sin\frac{m\pi}{b}y-\beta_z\frac{k_{\parallel}}{k}\sin\xi\sin\frac{m\pi}{b}y\right)\right.\\&\left.-\hat{\boldsymbol{z}}\frac{m\pi}{kb}\beta_x\cos\frac{m\pi}{b}y\right]\sin\frac{m\pi}{b}y'e^{i\omega t'}+O(\rho^{-\frac{3}{2}},\gamma^{-2},K^2),\end{aligned} \tag{A6}$$

(typos in Eq. (3.8) in Ref. [10] are corrected), where $\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},\hat{\boldsymbol{z}}$ are the unit vectors in the direction of the x, y, and z of Cartesian coordinate, and ξ is the angle from the axis in the (x,z) plane, given by

$$x=\rho\sin\xi, \tag{A7}$$

$$z=\rho\cos\xi, \tag{A8}$$

and it is related to the cosine of an emission angle θ as

$$\cos\theta = \frac{k_\parallel}{k}\cos\xi. \tag{A9}$$

Thus, the particle couples mainly to TE modes.

In Ref. [10], the cgs unit is used and the vector $\boldsymbol{A}$ and the scalar ϕ potentials are introduced as

$$\boldsymbol{B} = \mathrm{rot}\boldsymbol{A}, \tag{A10}$$

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \mathrm{grad}\phi. \tag{A11}$$

We can explicitly demonstrate that only the transverse components of the vector potential:

$$A_{\omega,x} = -e\frac{2}{b}\sqrt{2\pi}e^{i\pi/4}\sum_m \int dt' \frac{e^{ik_\parallel\rho}}{\sqrt{k_\parallel\rho}}\left(\beta_x - \beta_z\frac{k_\parallel}{k}\sin\xi\right) \times \sin\frac{m\pi}{b}y\sin\frac{m\pi}{b}y' e^{i\omega t'}, \tag{A12}$$

$$A_{\omega,y} = -e\frac{2}{b}\sqrt{2\pi}e^{i\pi/4}\sum_m \int dt' \frac{e^{ik_\parallel\rho}}{\sqrt{k_\parallel\rho}} i\frac{m\pi}{kb}\beta_z\cos\frac{m\pi}{b}y \times \sin\frac{m\pi}{b}y' e^{i\omega t'}, \tag{A13}$$

reproduce Eqs. (A5) and (A6) by assuming $\theta \approx 1/\gamma$.

APPENDIX B: FORMAL SOLUTION FOR THE RADIATION FIELDS

Let us consider an undulator, where a nonrelativistic electron beam wiggles in a rectangular waveguide with the gap width a and the gap height b. The vector potential $\vec{A}_R$ for the radiation field satisfies the inhomogeneous wave equation

$$\nabla^2\vec{A}_R - \frac{1}{c^2}\frac{\partial^2\vec{A}_R}{\partial t^2} = -\mu_0\vec{J}_\perp(\vec{r},t), \tag{B1}$$

where c is velocity of light, $\vec{J}_\perp(\vec{r},t)$ is the transverse current density of the electron beam, $\mu_0 = Z_0/c$, $Z_0 = 120\pi\ \Omega$ is the impedance of free space, $\vec{r}$ is the three-dimensional vector $\vec{r} = (\vec{x}, z)$, and t is time when the electron of concern arrives at the position z.

The solution of Eq. (B1) is formally given by

$$\vec{A}_R = \mu_0\int_{-\infty}^{\infty} d^3\vec{r}' \int_{-\infty}^{\infty} dt' \vec{G}(r,t|r',t')\vec{J}_\perp(r',t'). \tag{B2}$$

Here, the Green function $\vec{G}(r,t|r',t')$ satisfies

$$\nabla^2\vec{G}(r,t|r',t') - \frac{1}{c^2}\frac{\partial^2\vec{G}(r,t|r',t')}{\partial t^2} = -\vec{I}\delta(r-r')\delta(t-t'), \tag{B3}$$

where $\vec{I}$ is the unit matrix and $\delta(x)$ is the δ-function. The Green function should satisfy the boundary condition determined by the waveguide shape.

Since we assume that the waveguide is placed in $-a/2 \le x \le a/2$ and $-b/2 \le y \le b/2$, the Green function $\vec{G}(r,t|r',t')$ satisfies the boundary condition as

$$\left.\frac{\partial G_{x,x}(r,t|r',t')}{\partial x}\right|_{x=-a/2,a/2} = 0, \quad G_{x,x}(r,t|r',t')|_{y=-b/2,b/2} = 0, \tag{B4}$$

$$G_{y,y}(r,t|r',t')|_{x=-a/2,a/2} = 0, \quad \left.\frac{\partial G_{y,y}(r,t|r',t')}{\partial y}\right|_{y=-b/2,b/2} = 0, \tag{B5}$$

$$G_{z,z}(r,t|r',t')|_{x=-a/2,a/2} = 0, \quad G_{z,z}(r,t|r',t')|_{y=-b/2,b/2} = 0, \tag{B6}$$

where the waveguide is assumed to be made of perfectly conductive material. The solution is given by [13]

$$G_{x,x}(r,t|r',t') = \frac{1}{4ab}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\sum_{m,n=-\infty}^{\infty}F^{x,x}_{m,n}(z|z')\Big[e^{i\frac{m\pi(x-x')}{a}+i\frac{n\pi(y-y')}{b}} - e^{i\frac{m\pi(x-x')}{a}+i\frac{n\pi(y+y'+b)}{b}} + e^{i\frac{m\pi(x+x'+a)}{a}+i\frac{n\pi(y-y')}{b}} - e^{i\frac{m\pi(x+x'+a)}{a}+i\frac{n\pi(y+y'+b)}{b}}\Big], \tag{B7}$$

$$G_{y,y}(r,t|r',t') = \frac{1}{4ab}\int_{-\infty}^{\infty}d\omega\frac{e^{-i\omega(t-t')}}{2\pi}\sum_{m,n=-\infty}^{\infty}F^{y,y}_{m,n}(z|z')\Big[e^{i\frac{m\pi(x-x')}{a}+i\frac{n\pi(y-y')}{b}} + e^{i\frac{m\pi(x-x')}{a}+i\frac{n\pi(y+y'+b)}{b}} - e^{i\frac{m\pi(x+x'+a)}{a}+i\frac{n\pi(y+y'+b)}{b}} - e^{i\frac{m\pi(x+x'+a)}{a}+i\frac{n\pi(y-y')}{b}}\Big], \tag{B8}$$

$$G_{z,z}(r,t|r',t') = \frac{1}{4ab}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\sum_{m,n=-\infty}^{\infty}F_{m,n}^{z,z}(z|z')[e^{i\frac{m\pi(x-x')}{a}+i\frac{m\pi(y-y')}{b}} - e^{i\frac{m\pi(x-x')}{a}+i\frac{m\pi(y+y'+b)}{b}} - e^{i\frac{m\pi(x+x'+a)}{a}+i\frac{m\pi(y-y')}{b}} + e^{i\frac{m\pi(x+x'+a)}{a}+i\frac{m\pi(y+y'+b)}{b}}], \tag{B9}$$

where i is the imaginary unit,

$$F_{m,n}^{x,x}(z|z') = \frac{ie^{i|z-z'|\sqrt{\frac{\omega^2}{c^2}-\frac{m^2\pi^2}{a^2}-\frac{n^2\pi^2}{b^2}}}}{2\sqrt{\frac{\omega^2}{c^2}-\frac{m^2\pi^2}{a^2}-\frac{n^2\pi^2}{b^2}}}, \tag{B10}$$

$$F_{m,n}^{y,y}(z|z') = \frac{ie^{i|z-z'|\sqrt{\frac{\omega^2}{c^2}-\frac{m^2\pi^2}{a^2}-\frac{n^2\pi^2}{b^2}}}}{2\sqrt{\frac{\omega^2}{c^2}-\frac{m^2\pi^2}{a^2}-\frac{n^2\pi^2}{b^2}}}, \tag{B11}$$

$$F_{m,n}^{z,z}(z|z') = \frac{ie^{i|z-z'|\sqrt{\frac{\omega^2}{c^2}-\frac{m^2\pi^2}{a^2}-\frac{n^2\pi^2}{b^2}}}}{2\sqrt{\frac{\omega^2}{c^2}-\frac{m^2\pi^2}{a^2}-\frac{n^2\pi^2}{b^2}}}. \tag{B12}$$

In order to obtain explicit solutions for the radiation fields, the transverse current density $\vec{J}_\perp$ needs to be given. It can be calculated by Eq. (16) via the electron distribution function, which is a solution of the Vlasov equation.

APPENDIX C: HAMILTONIAN FORMALISM

Let us introduce the Hamiltonian formalism to derive equations of motion for electrons in a planar undulator. When the longitudinal coordinate z is chosen as an independent variable, the Hamiltonian is identical to p_z:

$$\begin{aligned} p_z &= [m_e^2c^2\gamma^2 - m_e^2c^2 - (p_x - eA_x)^2 - (p_y - eA_y)^2]^{\frac{1}{2}} \\ &= \left[\frac{H^2}{c^2} - m_e^2c^2 - (p_x - eA_x)^2 - (p_y - eA_y)^2\right]^{\frac{1}{2}}, \end{aligned} \tag{C1}$$

where m_e is the mass of electron, e is the electron charge, p_x and p_y are the transverse components of momenta for the electron, and A_x and A_y are the transverse components of the total vector potential in the x and y directions, respectively. The vector potential $\vec{A}$ consists of the wiggler fields $\vec{A}_w$ and the radiation fields $\vec{A}_R$ introduced in the previous section. For a small transverse displacement of the beam, the wiggler fields $\vec{A}_w$ are approximated as

$$\begin{aligned} \vec{A}_w = \frac{Km_eck_{wz}}{ek_{wy}}\Bigg[&\vec{i}_x\frac{k_{wy}}{k_{wz}}\left(1+\frac{1}{2}k_{wx}^2x^2+\frac{1}{2}k_{wy}^2y^2\right) \\ &\times\sin k_{wz}z - \vec{i}_y\frac{k_{wx}}{k_{wz}}k_{wx}k_{wy}xy\sin k_{wz}z\Bigg], \end{aligned} \tag{C2}$$

$$k_{wz} = \sqrt{k_{wx}^2 + k_{wy}^2}, \tag{C3}$$

for the planar undulator, where K is the peak wiggler parameter, the wave number k_{wz} is 2π divided by the wiggler period length λ_w, and $\vec{i}_x$ and $\vec{i}_y$ are the unit vectors in the x and y directions, respectively.

The corresponding equations of motion for the electron are given by

$$\frac{dx}{dz} = -\frac{\partial p_z}{\partial p_x}, \qquad \frac{dp_x}{dz} = \frac{\partial p_z}{\partial x}, \tag{C4}$$

$$\frac{dy}{dz} = -\frac{\partial p_z}{\partial p_y}, \qquad \frac{dp_y}{dz} = \frac{\partial p_z}{\partial y}, \tag{C5}$$

$$\frac{dt}{dz} = \frac{\partial p_z}{\partial H}, \qquad \frac{dH}{dz} = -\frac{\partial p_z}{\partial t}. \tag{C6}$$

Under the condition:

$$(p_x - eA_x)^2 + (p_y - eA_y)^2 \ll m_e^2c^2\tilde{\gamma}^2, \tag{C7}$$

where $\tilde{\gamma}$ is introduced as

$$\tilde{\gamma} = \sqrt{\gamma^2 - 1}, \tag{C8}$$

linear parts are dominant, and, thus, we obtain

$$\frac{dx}{dz} \simeq \frac{(p_x - eA_x)}{m_ec\tilde{\gamma}}, \qquad \frac{dp_x}{dz} \simeq -\frac{e^2A_x}{m_ec\tilde{\gamma}}\frac{\partial A_x}{\partial x} - \frac{e^2A_y}{m_ec\tilde{\gamma}}\frac{\partial A_y}{\partial x}, \tag{C9}$$

$$\frac{dy}{dz} \simeq \frac{(p_y - eA_y)}{m_ec\tilde{\gamma}}, \qquad \frac{dp_y}{dz} \simeq -\frac{e^2A_x}{m_ec\tilde{\gamma}}\frac{\partial A_x}{\partial y} - \frac{e^2A_y}{m_ec\tilde{\gamma}}\frac{\partial A_y}{\partial y}, \tag{C10}$$

$$\frac{dt}{dz} \simeq \frac{1}{c}\left(1 + \frac{1}{2\gamma^2} + \frac{(p_x - eA_x)^2}{2m_e^2c^2\tilde{\gamma}^2} + \frac{(p_y - eA_y)^2}{2m_e^2c^2\tilde{\gamma}^2}\right), \tag{C11}$$

$$m_ec^2\frac{d\gamma}{dz} \simeq -e\frac{dx}{dz}\frac{\partial A_x}{\partial t} - e\frac{dy}{dz}\frac{\partial A_y}{\partial t}. \tag{C12}$$

The total transverse trajectory of the electron $\vec{x}$ including the wiggler motion $\vec{x}_w$ is given by

$$\vec{x} = \vec{x}_w + \vec{x}_\beta, \tag{C13}$$

$$\vec{x}_w = \vec{i}_x r_w \cos k_{wz} z, \tag{C14}$$

$$r_w = \frac{K}{\tilde{\gamma} k_{wz}}, \tag{C15}$$

where $\vec{x}_\beta$ is the betatron motion.

Let us define the transverse betatron variables x_β and y_β and their canonical momenta $p_{\beta,x}$ and $p_{\beta,y}$ by averaging out the transverse electron motion over the fast wiggling motion:

$$x_\beta = \frac{1}{\lambda_w}\int_z^{z+\lambda_w} x dz, \quad p_{\beta,x} = \frac{1}{\lambda_w}\int_z^{z+\lambda_w} \frac{p_x}{m_e c\tilde{\gamma}} dz, \tag{C16}$$

$$y_\beta = \frac{1}{\lambda_w}\int_z^{z+\lambda_w} y dz, \quad p_{\beta,y} = \frac{1}{\lambda_w}\int_z^{z+\lambda_w} \frac{p_y}{m_e c\tilde{\gamma}} dz. \tag{C17}$$

Accordingly, Hamilton equations are finally simplified by modifying Eqs. (C9)–(C11) as

$$\frac{dx_\beta}{dz} \simeq p_{\beta,x}, \qquad \frac{dy_\beta}{dz} \simeq p_{\beta,y}, \tag{C18}$$

$$\frac{d}{dz} p_{\beta,x} \simeq -k_\beta^2 x_\beta, \qquad \frac{d}{dz} p_{\beta,y} \simeq -k_\beta^2 y_\beta, \tag{C19}$$

$$\begin{aligned}\frac{d\tau}{dz} &\equiv \frac{dt}{dz} - \frac{dt_r^p}{dz}\\ &\simeq \frac{1}{c}\left[-\left(\frac{1}{\gamma_r^2}+\frac{K^2}{2\gamma_r^2}\right)\frac{(\gamma-\gamma_r)}{\gamma_r}+\frac{p_{\beta,x}^2+p_{\beta,y}^2}{2}+\frac{k_\beta^2(x_\beta^2+y_\beta^2)}{2}\right]\\ &= \frac{1}{c}\left[-\frac{2k_{wz}}{k_1}\frac{(\gamma-\gamma_r)}{\gamma_r}+\frac{p_{\beta,x}^2+p_{\beta,y}^2}{2}+\frac{k_\beta^2(x_\beta^2+y_\beta^2)}{2}\right],\end{aligned} \tag{C20}$$

$$\begin{aligned}\frac{dt_r^p}{dz} &\simeq \frac{1}{c}\left(1+\frac{2+K^2}{4\tilde{\gamma}_r^2}-\frac{K^2}{4\tilde{\gamma}_r^2}\cos 2k_{wz}z\right)\\ &= \frac{1}{c}\left(1+\frac{k_{wz}}{k_1}-\frac{K^2}{4\tilde{\gamma}_r^2}\cos 2k_{wz}z\right) \equiv \frac{1}{\bar{v}_r} - \frac{K^2}{4c\tilde{\gamma}_r^2}\cos 2k_{wz}z,\end{aligned} \tag{C21}$$

where $\bar{v}_r$ shows the average velocity over one wiggler period λ_w, the index r means the variables for the reference electron, and the betatron wave number k_β is given by

$$k_\beta = \frac{Kk_{wx}}{\sqrt{2}\tilde{\gamma}} = \frac{Kk_{wy}}{\sqrt{2}\tilde{\gamma}}, \tag{C22}$$

(in this paper, the betatron focusing assumed to be equal in the x and y directions, for simplicity), γ_r is the resonant Lorentz-γ of the reference electron and $\tilde{\gamma}_r = \sqrt{\gamma_r^2 - 1}$. The resonant radiation wave number k_1 is introduced as

$$k_1 = \frac{2k_{wz}\tilde{\gamma}_r^2}{(1+\frac{K^2}{2})}, \tag{C23}$$

for the planar undulator.

APPENDIX D: DESCRIPTION OF THE VECTOR POTENTIAL VIA BEAM DISTRIBUTION FUNCTION

By inserting Eqs. (B7)–(B9) and (16) into Eq. (B2) and changing the volume element from $d^3\vec{r}'dt'$ to $d^2\vec{x}_\beta dz' d\tau'$, the vector potential $\vec{A}_R = \vec{A}^w$ for the radiation fields is expressed by Eqs. (19) by using Eqs. (13)–(15) and (20)–(22). The function $H^w_{x,\omega q}(n_x, n_y; m, n, z)$ in Eq. (19) is given by

$$\begin{aligned}H^w_{x,\omega q}(n_x,n_y;m,n,z) = -i\frac{Ke^{-i\omega\frac{z}{\bar{v}_r}+i\omega\frac{1}{8k_{wz}c}(\frac{K}{\tilde{\gamma}})^2\sin 2k_{wz}z - iqz}}{2\tilde{\gamma}\sqrt{\frac{\omega^2}{c^2}-\frac{m^2\pi^2}{a^2}-\frac{n^2\pi^2}{b^2}}}\left(e^{iz\sqrt{\frac{\omega^2}{c^2}-\frac{m^2\pi^2}{a^2}-\frac{n^2\pi^2}{b^2}}}I^{(1)}_{\omega,q,w}+e^{-iz\sqrt{\frac{\omega^2}{c^2}-\frac{m^2\pi^2}{a^2}-\frac{n^2\pi^2}{b^2}}}I^{(2)}_{\omega,q,w}\right)\\ \times ab\left[\frac{\sin\frac{(-m+n_x)\pi}{2}\sin\frac{(-n+n_y)\pi}{2}}{\frac{(-m+n_x)\pi}{2}\frac{(-n+n_y)\pi}{2}} - \frac{e^{in\pi}\sin\frac{(-m+n_x)\pi}{2}\sin\frac{(n+n_y)\pi}{2}}{\frac{(-m+n_x)\pi}{2}\frac{(n+n_y)\pi}{2}}\right.\\ \left.+\frac{e^{im\pi}\sin\frac{(m+n_x)\pi}{2}\sin\frac{(-n+n_y)\pi}{2}}{\frac{(m+n_x)\pi}{2}\frac{(-n+n_y)\pi}{2}} - \frac{e^{i(m+n)\pi}\sin\frac{(m+n_x)\pi}{2}\sin\frac{(n+n_y)\pi}{2}}{\frac{(m+n_x)\pi}{2}\frac{(n+n_y)\pi}{2}}\right],\end{aligned} \tag{D1}$$

where the functions $I^{(1)}_{\omega,q,w}$ and $I^{(2)}_{\omega,q,w}$ are introduced as

$$I^{(1)}_{\omega,q,w} = \int_{-\frac{L}{2}}^{z} dz' e^{i\omega\frac{z'}{\bar{v}_r}-i\omega\frac{1}{8k_{wz}c}(\frac{K}{\tilde{\gamma}})^2\sin 2k_{wz}z'+iqz'-iz'\sqrt{\frac{\omega^2}{c^2}-\frac{m^2\pi^2}{a^2}-\frac{n^2\pi^2}{b^2}}}\sin k_{wz}z', \tag{D2}$$

$$I^{(2)}_{\omega,q,w} = \int_z^{\frac{L}{2}} dz' e^{i\omega\frac{z'}{\bar{v}_r} - i\omega\frac{1}{8k_{wz}c}(\frac{K}{\tilde{\gamma}})^2 \sin 2k_{wz}z' + iqz' + iz'\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}} \sin k_{wz}z', \tag{D3}$$

where L is a regularization parameter, which will be removed in the final form of the dispersion relation (44) because it is taken to infinity in Eq. (E11). It is noticeable that the information of the beam distribution is confined in the function $\Theta_{\omega q}(n_x, n_y)$ in Eq. (19).

The expression of the function $H^w_{x,\omega q}(n_x, n_y; m, n, z)$ is simplified by using the expansion formula for the Bessel function [11]:

$$e^{ik_x r_h \sin k_w z'} = \sum_{\sigma=-\infty}^{\infty} (-1)^\sigma J_\sigma(k_x r_h) e^{-i\sigma k_w z'}. \tag{D4}$$

The z'-integration in Eqs. (D2) and (D3) is carried out with the result,

$$I^{(1)}_{\omega q,w} = \sum_{\sigma=-\infty}^{\infty} \frac{(-1)^\sigma}{2i} J_\sigma\left[\frac{\omega}{8k_{wz}c}\left(\frac{K}{\tilde{\gamma}}\right)^2\right] \frac{e^{i\omega\frac{z}{\bar{v}_r} + i(2\sigma+1)k_{wz}z + iqz - iz\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}} - e^{-i\omega\frac{L}{2\bar{v}_r} - i(2\sigma+1)k_{wz}\frac{L}{2} - iq\frac{L}{2} + i\frac{L}{2}\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}}}{\frac{i\omega}{\bar{v}_r} + i(2\sigma+1)k_{wz} + iq - i\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}}$$
$$- \sum_{\sigma=-\infty}^{\infty} \frac{(-1)^\sigma}{2i} J_\sigma\left[\frac{\omega}{8k_{wz}c}\left(\frac{K}{\tilde{\gamma}}\right)^2\right] \frac{e^{i\omega\frac{z}{\bar{v}_r} + i(2\sigma-1)k_{wz}z + iqz - iz\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}} - e^{-i\omega\frac{L}{2\bar{v}_r} - i(2\sigma-1)k_{wz}\frac{L}{2} - iq\frac{L}{2} + i\frac{L}{2}\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}}}{\frac{i\omega}{\bar{v}_r} + i(2\sigma-1)k_{wz} + iq - i\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}}, \tag{D5}$$

$$I^{(2)}_{\omega q,w} = \sum_{\sigma=-\infty}^{\infty} \frac{(-1)^\sigma}{2i} J_\sigma\left[\frac{\omega}{8k_{wz}c}\left(\frac{K}{\tilde{\gamma}}\right)^2\right] \left[\frac{e^{i\omega\frac{L}{2\bar{v}_r} + i\frac{(2\sigma+1)k_{wz}L}{2} + iq\frac{L}{2} + i\frac{L}{2}\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}} - e^{i\omega\frac{z}{\bar{v}_r} + i(2\sigma+1)k_{wz}z + iqz + iz\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}}}{\frac{i\omega}{\bar{v}_r} + i(2\sigma+1)k_{wz} + iq + i\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}}\right.$$
$$\left. - \frac{e^{i\omega\frac{L}{2\bar{v}_r} + i\frac{(2\sigma-1)k_{wz}L}{2} + iq\frac{L}{2} + i\frac{L}{2}\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}} - e^{i\omega\frac{z}{\bar{v}_r} + i(2\sigma-1)k_{wz}z + iqz + iz\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}}}{\frac{i\omega}{\bar{v}_r} + i(2\sigma-1)k_{wz} + iq + i\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}}\right]. \tag{D6}$$

Finally, we obtain the expression of the function $H^w_{x,\omega q}(n_x, n_y; m, n, z)$ as

$$H^w_{x,\omega q}(n_x,n_y;m,n,z) = -\sum_{p=-\infty}^{\infty} \frac{iKJ_p\left[\frac{\omega}{8k_{wz}c}(\frac{K}{\tilde{\gamma}})^2\right]}{2\tilde{\gamma}\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}} \sum_{\sigma=-\infty}^{\infty} \frac{(-1)^{\sigma+p} e^{i(2\sigma+1-2p)k_{wz}z}}{2i} \left\{J_\sigma\left[\frac{\omega}{8k_{wz}c}\left(\frac{K}{\tilde{\gamma}}\right)^2\right] + J_{\sigma+1}\left[\frac{\omega}{8k_{wz}c}\left(\frac{K}{\tilde{\gamma}}\right)^2\right]\right\}$$
$$\times \left[\frac{1 - e^{(-\frac{i\omega}{\bar{v}_r} - i(2\sigma+1)k_{wz} - iq + i\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}})(\frac{L}{2}+z)}}{\frac{i\omega}{\bar{v}_r} + i(2\sigma+1)k_{wz} + iq - i\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}} - \frac{1 - e^{(\frac{i\omega}{\bar{v}_r} + i(2\sigma+1)k_{wz} + iq + i\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}})(\frac{L}{2}-z)}}{\frac{i\omega}{\bar{v}_r} + i(2\sigma+1)k_{wz} + iq + i\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}}\right]$$
$$\times ab\left[\frac{\sin\frac{(-m+n_x)\pi}{2}\sin\frac{(-n+n_y)\pi}{2}}{\frac{(-m+n_x)\pi}{2}\frac{(-n+n_y)\pi}{2}} - \frac{e^{in\pi}\sin\frac{(-m+n_x)\pi}{2}\sin\frac{(n+n_y)\pi}{2}}{\frac{(-m+n_x)\pi}{2}\frac{(n+n_y)\pi}{2}}\right.$$
$$\left. + \frac{e^{im\pi}\sin\frac{(m+n_x)\pi}{2}\sin\frac{(-n+n_y)\pi}{2}}{\frac{(m+n_x)\pi}{2}\frac{(-n+n_y)\pi}{2}} - \frac{e^{i(m+n)\pi}\sin\frac{(m+n_x)\pi}{2}\sin\frac{(n+n_y)\pi}{2}}{\frac{(m+n_x)\pi}{2}\frac{(n+n_y)\pi}{2}}\right]. \tag{D7}$$

Now, the radiation field from the planar undulator can be explicitly calculated by using Eqs. (19) and (D7).

APPENDIX E: THE ENERGY CHANGE OF THE BEAM BY THE RADIATION FIELDS

Since we obtain the expression of the radiation fields, let us calculate the energy change of the beam by the radiation fields to complete the Vlasov equation. The energy change by the radiation fields is given by Eq. (C12), and it is approximated by Eq. (17).

By substituting Eqs. (C14) and (19) into Eq. (17), the energy change by the radiation fields is described as

$$\frac{d\gamma}{dz} = -\frac{ie^2\mu_0}{m_e c^2}\frac{K}{\tilde{\gamma}}\frac{1}{4ab}\sum_{m,n=-\infty}^{\infty} e^{i\frac{m\pi x_\beta}{a}+i\frac{n\pi y_\beta}{b}} L_w \int_{-\infty}^{\infty} d\omega e^{-i\omega\tau}\int_{-\infty}^{\infty}\frac{dq}{2\pi}e^{iqz}\frac{4\pi^2}{4ab}\sum_{n_x,n_y=-\infty}^{\infty}\Theta_{\omega q}(n_x,n_y)\omega H^w_{x,\omega q}(n_x,n_y;m,n,z), \quad \text{(E1)}$$

where

$$L_w = \sin k_{wz}z e^{i\frac{m\pi x_w}{a}} = \sin k_{wz}z e^{i\frac{m\pi r_w \cos k_{wz}z}{a}}. \quad \text{(E2)}$$

By using the expansion formula [11]:

$$e^{-ik_y r_h \cos k_w z} = \sum_{n=-\infty}^{\infty}(-i)^n J_n(k_y r_h)e^{-ink_w z}, \quad \text{(E3)}$$

the factor L_w is expressed as

$$L_w = \frac{1}{2i}\sum_{\nu=-\infty}^{\infty} e^{i\nu k_{wz}z}e^{i\frac{\pi}{2}\nu}\left[e^{-i\frac{\pi}{2}}J_{\nu-1}\left(\frac{m\pi r_w}{a}\right) - e^{i\frac{\pi}{2}}J_{\nu+1}\left(\frac{m\pi r_w}{a}\right)\right]. \quad \text{(E4)}$$

By using the formula (D4), Eq. (E1) is rewritten as

$$\frac{d\gamma}{dz} = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}\int_{-\infty}^{\infty}\frac{dq}{2\pi}e^{iqz}\frac{4\pi^2}{4ab}\sum_{n_x,n_y=-\infty}^{\infty}\frac{1}{4ab}\sum_{m,n=-\infty}^{\infty}e^{i\frac{m\pi x_\beta}{a}+i\frac{n\pi y_\beta}{b}}$$
$$\times\sum_{\sigma=-\infty}^{\infty}\sum_{\nu=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}e^{i(\nu+2\sigma-2p+1)k_{wz}z}P^w_{\omega q}(n_x,n_y;m,n,\sigma,\nu,p)\Theta_{\omega q}(n_x,n_y), \quad \text{(E5)}$$

where

$$P^w_{\omega q}(n_x,n_y;m,n,\sigma,\nu,p) = \hat{P}^w_{\omega q}(n_x,n_y;m,n,\sigma,\nu,p)\left[\frac{1-\sum_{\bar{\alpha}=-\infty}^{\infty}A_{\bar{\alpha}}(m,n,\sigma;q)e^{i\bar{\alpha}k_{wz}z}}{\frac{i\omega}{\bar{v}_r}+i(2\sigma+1)k_{wz}+iq-i\sqrt{\frac{\omega^2}{c^2}-\frac{m^2\pi^2}{a^2}-\frac{n^2\pi^2}{b^2}}}\right.$$
$$\left.-\frac{1-\sum_{\bar{\alpha}=-\infty}^{\infty}B_{\bar{\alpha}}(m,n,\sigma;q)e^{i\bar{\alpha}k_{wz}z}}{\frac{i\omega}{\bar{v}_r}+i(2\sigma+1)k_{wz}+iq+i\sqrt{\frac{\omega^2}{c^2}-\frac{m^2\pi^2}{a^2}-\frac{n^2\pi^2}{b^2}}}\right], \quad \text{(E6)}$$

$$\hat{P}^w_{\omega q}(n_x,n_y;m,n,\sigma,\nu,p) = \frac{r_e}{4c\frac{c}{\omega}\sqrt{\frac{\omega^2}{c^2}-\frac{m^2\pi^2}{a^2}-\frac{n^2\pi^2}{b^2}}}\left(\frac{K}{\tilde{\gamma}}\right)^2(-1)^p J_p\left[\frac{\omega}{8k_{wz}c}\left(\frac{K}{\tilde{\gamma}}\right)^2\right]\left[e^{i\frac{\pi}{2}(\nu-1)}J_{\nu-1}\left(\frac{m\pi r_w}{a}\right)\right.$$
$$\left.-e^{i\frac{\pi}{2}(\nu+1)}J_{\nu+1}\left(\frac{m\pi r_w}{a}\right)\right](-1)^\sigma\left\{J_\sigma\left[\frac{\omega}{8k_{wz}c}\left(\frac{K}{\tilde{\gamma}}\right)^2\right]+J_{\sigma+1}\left[\frac{\omega}{8k_{wz}c}\left(\frac{K}{\tilde{\gamma}}\right)^2\right]\right\}$$
$$\times 4\pi^2 ab\left[\frac{\sin\frac{(-m+n_x)\pi}{2}\sin\frac{(-n+n_y)\pi}{2}}{\frac{(-m+n_x)\pi}{2}\frac{(-n+n_y)\pi}{2}}-\frac{e^{in\pi}\sin\frac{(-m+n_x)\pi}{2}\sin\frac{(n+n_y)\pi}{2}}{\frac{(-m+n_x)\pi}{2}\frac{(n+n_y)\pi}{2}}\right.$$
$$\left.+\frac{e^{im\pi}\sin\frac{(m+n_x)\pi}{2}\sin\frac{(-n+n_y)\pi}{2}}{\frac{(m+n_x)\pi}{2}\frac{(-n+n_y)\pi}{2}}-\frac{e^{i(m+n)\pi}\sin\frac{(m+n_x)\pi}{2}\sin\frac{(n+n_y)\pi}{2}}{\frac{(m+n_x)\pi}{2}\frac{(n+n_y)\pi}{2}}\right], \quad \text{(E7)}$$

$$A_{\bar{\alpha}}(m,n,\sigma;q) = \frac{e^{[-i\bar{\alpha}k_{wz} - \frac{i2\omega}{\bar{v}_r} - i2(2\sigma+1)k_{wz} - i2q + i2\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}]\frac{L}{2}} - e^{i\bar{\alpha}k_{wz}\frac{L}{2}}}{L\left[-i\bar{\alpha}k_{wz} - \frac{i\omega}{\bar{v}_r} - i(2\sigma+1)k_{wz} - iq + i\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}\right]}, \tag{E8}$$

$$B_{\bar{\alpha}}(m,n,\sigma;q) = \frac{e^{-i\bar{\alpha}k_{wz}\frac{L}{2}} - e^{[i\bar{\alpha}k_{wz} + \frac{i2\omega}{\bar{v}_r} + i2(2\sigma+1)k_{wz} + i2q + i2\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}]\frac{L}{2}}}{L\left[-i\bar{\alpha}k_{wz} - \frac{i\omega}{\bar{v}_r} - i(2\sigma+1)k_{wz} - iq - i\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}\right]}, \tag{E9}$$

$$r_e = \frac{e^2}{4\pi\epsilon_0 m_e c^2}, \tag{E10}$$

we choose the branch $i\sqrt{\omega^2/c^2 - m^2\pi^2/a^2 - n^2\pi^2/b^2} = -\sqrt{m^2\pi^2/a^2 + n^2\pi^2/b^2 - \omega^2/c^2}$ for $\omega^2/c^2 - m^2\pi^2/a^2 - n^2\pi^2/b^2 < 0$, r_e is the classical radius of electron and ϵ_0 is dielectric constant of vacuum. Finally, the regularization parameter L can be taken to infinity so that Eq. (E6) is simplified as

$$P^w_{\omega q}(n_x,n_y;m,n,\sigma,\nu,p) = \hat{P}^w_{\omega q}(n_x,n_y;m,n,\sigma,\nu,p)\left[\frac{1}{\frac{i\omega}{\bar{v}_r} + i(2\sigma+1)k_{wz} + iq - i\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}} - \frac{1}{\frac{i\omega}{\bar{v}_r} + i(2\sigma+1)k_{wz} + iq + i\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}}\right]. \tag{E11}$$

It should be noticed that the function $\Theta_{\omega q}(n_x,n_y)$ appears in $d\gamma/dz$ [Eq. (E5)]. As Eqs. (8), (21), and (22) show, the function $\Theta_{\omega q}(n_x,n_y)$ depends on the perturbed distribution function f_1. In this way, Vlasov equation (4) finally converts to a dispersion relation.

APPENDIX F: DISPERSION RELATION

The dispersion relation is obtained by transforming Eq. (4). After substituting Eq. (E5) into Eq. (4), the Vlasov equation (4) is expressed as

$$\left(iq - i\omega\frac{d\tau}{dz}\right)f_{\omega,q}(\vec{x}_\beta,\vec{p}_\beta,\gamma) + \vec{p}_\beta\frac{\partial f_{\omega,q}(\vec{x}_\beta,\vec{p}_\beta,\gamma)}{\partial\vec{x}_\beta} - k_\beta^2\vec{x}_\beta\frac{\partial f_{\omega,q}(\vec{x}_\beta,\vec{p}_\beta,\gamma)}{\partial\vec{p}_\beta} + \frac{\partial f_0}{\partial\gamma}\frac{4\pi^2}{4ab}\sum_{n_x,n_y=-\infty}^{\infty}\frac{1}{4ab}\sum_{m,n=-\infty}^{\infty}e^{i\frac{m\pi x_\beta}{a} + i\frac{n\pi y_\beta}{b}} \times \sum_{\sigma=-\infty}^{\infty}\sum_{\nu=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}e^{i(\nu+2\sigma-2p+1)k_{wz}z}P^w_{\omega q}(n_x,n_y;m,n,\sigma,\nu,p)\Theta_{\omega q}(n_x,n_y) = 0, \tag{F1}$$

where the functions $f_{\omega,q'}$ and f_1 are related as

$$f_1(\vec{x}_\beta,\vec{p}_\beta,\tau,\gamma;z) = \int_{-\infty}^{\infty}d\omega\frac{e^{-i\omega\tau}}{2\pi}\int_{-\infty}^{\infty}dq'\frac{e^{iq'z}}{2\pi}f_{\omega,q'}(\vec{x}_\beta,\vec{p}_\beta,\gamma). \tag{F2}$$

Using Eqs. (8), (14)–(22), and (F2), the function $\Theta_{\omega,q}(n_x,n_y)$ is enable to be associated with the function $f_{\omega,q}(\vec{x}'_\beta,\vec{p}_\beta,\gamma)$ as

$$\Theta_{\omega q}(n_x,n_y) = -\frac{n_x}{|n_x|}\frac{n_y}{|n_y|}\frac{ab}{4\pi^2}e^{i\frac{n_x\pi}{2} + i\frac{n_y\pi}{2}}(1-\delta_{n_x,0})(1-\delta_{n_y,0})\rho_{\omega q}(|n_x|,|n_y|), \tag{F3}$$

where $\delta_{m,n}$ is Kronecker-δ,

$$\rho_{\omega,q}(n_x,n_y)=\frac{4}{2\pi ab}\int_{-\infty}^{\infty}dze^{-iqz}\int_{-\frac{a}{2}-x_w(z)}^{\frac{a}{2}-x_w(z)}dx_\beta\int_{-\frac{b}{2}}^{\frac{b}{2}}dy_\beta\sin\frac{n_x\pi[x_\beta+x_w(z)+\frac{a}{2}]}{a}\sin\frac{n_y\pi(y_\beta+\frac{b}{2})}{b}\times\int_1^{\infty}d\gamma\int_{-\infty}^{\infty}d^2\vec{p}_\beta\int_{-\infty}^{\infty}dq'\frac{e^{iq'z}}{2\pi}f_{\omega,q'}(\vec{x}_\beta,\vec{p}_\beta,\gamma). \quad \text{(F4)}$$

In order to proceed the analysis, let us introduce the polar coordinate in the transverse plane as Eqs. (27) and (28), and expand $f_{\omega,q}$ by using the azimuthal angle ϕ_x and ϕ_y as

$$f_{\omega,q}(\vec{x}_\beta,\vec{p}_\beta,\gamma)=\sum_{m,n=-\infty}^{\infty}F_{\omega,q}^{(m,n)}(r_x,r_y,\gamma)e^{im\phi_x}e^{in\phi_y}. \quad \text{(F5)}$$

Since the value of $\sqrt{r_x^2+r_y^2}$ is typically smaller than $\min\{a/2-x_w,b/2\}$, the substitution of Eqs. (27), (28), and (F5) into Eq. (F4) approximates Eq. (F4) as

$$\rho_{\omega,q}(n_x,n_y)\simeq-\int_{-\infty}^{\infty}dze^{-iqz}\sum_{m,n=-\infty}^{\infty}\frac{2\pi k_\beta^2}{ab}\int_0^{\infty}dr_xr_x\int_0^{\infty}dr_yr_y[e^{i\frac{n_x\pi}{2}}e^{i\frac{n_x\pi x_w(z)}{a}}i^{-|m|}(-1)^{|m|}-e^{-i\frac{n_x\pi}{2}}e^{-i\frac{n_x\pi x_w(z)}{a}}i^{-|m|}]\times[e^{i\frac{n_y\pi}{2}}i^{-|n|}(-1)^{|n|}-e^{-i\frac{n_y\pi}{2}}i^{-|n|}]\int_{-\infty}^{\infty}dq'\frac{e^{iq'z}}{2\pi}\int_1^{\infty}d\gamma J_{|m|}\left(\frac{n_x\pi r_x}{a}\right)J_{|n|}\left(\frac{n_y\pi r_y}{b}\right)F_{\omega,q'}^{(m,n)}(r_x,r_y,\gamma), \quad \text{(F6)}$$

where the formula [11]:

$$\int_0^{2\pi}\frac{d\phi}{2\pi}e^{il\phi-ix\cos\phi}=i^{-|l|}J_{|l|}(x), \quad \text{(F7)}$$

is used.

Hence, the Vlasov equation (F1) is rewritten in the Fourier space, as

$$\left[iq-i\omega\frac{d\tau}{dz}-ik_\beta(\bar{m}+\bar{n})\right]F_{\omega,q}^{(\bar{m},\bar{n})}(r_x,r_y,\gamma)=-f_{0\perp}(r^2)\frac{\partial f_{0\parallel}(\gamma)}{\partial\gamma}\frac{4\pi^2}{4ab}\sum_{n_x,n_y=-\infty}^{\infty}\frac{1}{4ab}\sum_{m,n=-\infty}^{\infty}i^{|\bar{m}|}J_{|\bar{m}|}\left(\frac{m\pi r_x}{a}\right)i^{|\bar{n}|}J_{|\bar{n}|}\left(\frac{n\pi r_y}{b}\right)\times\sum_{\sigma=-\infty}^{\infty}\sum_{\nu=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}e^{i(\nu+2\sigma-2p+1)k_{wz}z}\hat{P}_{\omega q}^{w}(n_x,n_y;m,n,\sigma,\nu,p)\left[\frac{1}{\frac{i\omega}{\bar{v}_r}+i(2\sigma+1)k_{wz}+iq-i\sqrt{\frac{\omega^2}{c^2}-\frac{m^2\pi^2}{a^2}-\frac{n^2\pi^2}{b^2}}}-\frac{1}{\frac{i\omega}{\bar{v}_r}+i(2\sigma+1)k_{wz}+iq+i\sqrt{\frac{\omega^2}{c^2}-\frac{m^2\pi^2}{a^2}-\frac{n^2\pi^2}{b^2}}}\right]\frac{n_x}{|n_x|}\frac{n_y}{|n_y|}\frac{ab}{4\pi^2}e^{i\frac{n_x\pi}{2}+i\frac{n_y\pi}{2}}(1-\delta_{n_x,0})(1-\delta_{n_y,0})\times\int_{-\infty}^{\infty}dze^{-iqz}\sum_{m',n'=-\infty}^{\infty}\frac{2\pi k_\beta^2}{ab}\int_0^{\infty}dr'_xr'_x\int_0^{\infty}dr'_yr'_yi^{-|m'|-|n'|}[e^{i\frac{|n_x|\pi}{2}}e^{i\frac{|n_x|\pi x_w(z)}{a}}(-1)^{|m'|}-e^{-i\frac{|n_x|\pi}{2}}e^{-i\frac{|n_x|\pi x_w(z)}{a}}]\times[e^{i\frac{|n_y|\pi}{2}}(-1)^{|n'|}-e^{-i\frac{|n_y|\pi}{2}}]\int_{-\infty}^{\infty}dq'\frac{e^{iq'z}}{2\pi}\int_1^{\infty}d\gamma J_{|m'|}\left(\frac{|n_x|\pi r'_x}{a}\right)J_{|n'|}\left(\frac{|n_y|\pi r'_y}{b}\right)F_{\omega,q'}^{(m',n')}(r'_x,r'_y,\gamma), \quad \text{(F8)}$$

where we use the relations

$$\vec{p}_\beta\frac{\partial}{\partial\vec{x}_\beta}-k_\beta^2\vec{x}_\beta\frac{\partial}{\partial\vec{p}_\beta}=-k_\beta\left(\frac{\partial}{\partial\phi_x}+\frac{\partial}{\partial\phi_y}\right), \quad \text{(F9)}$$

$$f_0=f_{0\perp}(r^2)f_{0\parallel}(\gamma), \quad \text{(F10)}$$

$$r = \sqrt{r_x^2 + r_y^2}. \tag{F11}$$

Here, the betatron variables and their canonical momenta $\vec{x}_\beta, \vec{p}_\beta$ are converted by Eqs. (27) and (28).

The z-dependence in Eq. (F8) is eliminated by retaining the slowing varying terms. Finally, Eq. (F8) is simplified as

$$\left[iq - i\omega\frac{d\tau}{dz} - ik_\beta(\bar{m}+\bar{n})\right]F_{\omega,q}^{(\bar{m},\bar{n})}(r_x, r_y, \gamma) = -f_{0\perp}(r^2)\frac{\partial f_{0\parallel}(\gamma)}{\partial\gamma}\int_1^\infty d\gamma \sum_{m',n'=-\infty}^{\infty}\int_0^\infty dr'_x r'_x \int_0^\infty dr'_y r'_y \times \int_{-\infty}^{\infty} dq' K_{\omega,q,q'}^{(\bar{m},\bar{n},m',n')}(r'_x, r'_y|r_x, r_y)F_{\omega,q'}^{(m',n')}(r'_x, r'_y, \gamma), \tag{F12}$$

where

$$K_{\omega,q,q'}^{(\bar{m},\bar{n},m',n')}(r'_x, r'_y|r_x, r_y) = i^{|\bar{m}|+|\bar{n}|-(|m'|+|n'|)}(2\pi k_\beta)^2\frac{4\pi^2}{4ab}\sum_{n_x,n_y=-\infty}^{\infty}\frac{1}{4ab}\sum_{m,n=-\infty}^{\infty}\sum_{\sigma=-\infty}^{\infty}\sum_{\nu=-\infty}^{\infty} \times \sum_{p=-\infty}^{\infty} e^{i(\nu+2\sigma-2p+1)k_{wz}z}\hat{P}_{\omega q}^{w}(n_x, n_y; m, n, \sigma, \nu, p)\left[\frac{1}{\frac{i\omega}{\bar{v}_r} + i(2\sigma+1)k_{wz} + iq - i\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}} - \frac{1}{\frac{i\omega}{\bar{v}_r} + i(2\sigma+1)k_{wz} + iq + i\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}}\right]\frac{n_x}{|n_x|}\frac{n_y}{|n_y|}\frac{ab}{4\pi^2}e^{i\frac{n_x\pi}{2}+i\frac{n_y\pi}{2}}(1-\delta_{n_x,0})(1-\delta_{n_y,0}) \times \int_{-\infty}^{\infty} dz\frac{e^{-i(q-q')z}}{2\pi}\frac{1}{2\pi ab}[e^{i\frac{|n_x|\pi}{2}}e^{i\frac{|n_x|\pi x_w(z)}{a}}(-1)^{|m'|} - e^{-i\frac{|n_x|\pi}{2}}e^{-i\frac{|n_x|\pi x_w(z)}{a}}][e^{i\frac{|n_y|\pi}{2}}(-1)^{|n'|} - e^{-i\frac{|n_y|\pi}{2}}] \times J_{|\bar{m}|}\left(\frac{m\pi r_x}{a}\right)J_{|\bar{n}|}\left(\frac{n\pi r_y}{b}\right)J_{|m'|}\left(\frac{|n_x|\pi r'_x}{a}\right)J_{|n'|}\left(\frac{|n_y|\pi r'_y}{b}\right), \tag{F13}$$

$$\simeq i^{|\bar{m}|+|\bar{n}|-(|m'|+|n'|)}(2\pi k_\beta)^2\frac{4\pi^2}{4ab}\sum_{n_x,n_y=-\infty}^{\infty}\frac{1}{4ab}\sum_{m,n=-\infty}^{\infty}\sum_{\sigma=-\infty}^{\infty}\sum_{\nu=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}J_{\nu+2\sigma-2p+1}\left(\frac{|n_x|\pi r_w}{a}\right)P_{\omega q}^{w}(n_x, n_y; m, n, \sigma, \nu, p) \times \frac{n_x}{|n_x|}\frac{n_y}{|n_y|}\frac{ab}{4\pi^2}e^{i\frac{n_x\pi}{2}+i\frac{n_y\pi}{2}}(1-\delta_{n_x,0})(1-\delta_{n_y,0})\int_{-\infty}^{\infty}dz\frac{e^{-i(q-q')z}}{2\pi}\frac{1}{2\pi ab}[e^{i\frac{|n_x|\pi}{2}}(-i)^{-\nu-2\sigma+2p-1}(-1)^{|m'|} - e^{-i\frac{|n_x|\pi}{2}}(-i)^{\nu+2\sigma-2p+1}][e^{i\frac{|n_y|\pi}{2}}(-1)^{|n'|} - e^{-i\frac{|n_y|\pi}{2}}]J_{|\bar{m}|}\left(\frac{m\pi r_x}{a}\right)J_{|\bar{n}|}\left(\frac{n\pi r_y}{b}\right)J_{|m'|}\left(\frac{|n_x|\pi r'_x}{a}\right)J_{|n'|}\left(\frac{|n_y|\pi r'_y}{b}\right). \tag{F14}$$

It is convenient to remove the γ dependence from Eq. (F12). For this purpose, let us introduce the function $R_{\omega,q}^{m,n}(r_x, r_y)$ as

$$R_{\omega,q}^{m,n}(r_x, r_y) = \int_1^\infty d\gamma F_{\omega,q}^{m,n}(r_x, r_y, \gamma). \tag{F15}$$

Accordingly, Eq. (F12) becomes the dispersion relation as

$$R_{\omega,q}^{\bar{m},\bar{n}}(r_x, r_y) = -f_{0\perp}(r^2)\int_1^\infty d\gamma\frac{\frac{\partial f_{0\parallel}(\gamma)}{\partial\gamma}}{[iq - i\omega\frac{d\tau}{dz}(r,\gamma) - ik_\beta(\bar{m}+\bar{n})]} \times \sum_{m',n'=-\infty}^{\infty}\int_0^\infty dr'_x r'_x\int_0^\infty dr'_y r'_y\int_{-\infty}^{\infty}dq' K_{\omega,q,q'}^{(\bar{m},\bar{n},m',n')}(r'_x, r'_y|r_x, r_y)R_{\omega,q'}^{m',n'}(r'_x, r'_y). \tag{F16}$$

The dispersion relation (F16) is solved as follows. First, let us expand the function $R^{m,n}_{\omega,q}(r_x, r_y)$ as

$$R^{m,n}_{\omega,q}(r_x, r_y) = W_\perp(r^2) \sum_{k=0}^{\infty} a_k^{(m,n)}(q) f_k^{(|m|,|n|)}(r_x, r_y) r_x^{|m|} r_y^{|n|}, \tag{F17}$$

where the weight function $W_\perp(r^2)$ is described as

$$W_\perp(r^2) = C f_{0\perp}(r^2), \tag{F18}$$

and C is the normalization constant. The function $f_k^{(|m|,|n|)}(r_x, r_y)$ satisfies the orthogonality condition:

$$\int_0^\infty dr_x r_x^{2|m|+1} \int_0^\infty dr_y r_y^{2|n|+1} f_k^{(|m|,|n|)}(r_x, r_y) f_l^{(|m|,|n|)}(r_x, r_y) W_\perp(r^2) = \delta_{k,l}. \tag{F19}$$

The product of the Bessel function, which appears in Eq. (F14), can be expanded by $f_k^{(|m|,|n|)}(r_x, r_y)$, as

$$J_{|m|}(k_x r_x) J_{|n|}(k_y r_y) = \sum_{k=0}^{\infty} C_{|m|,|n|,k}(k_x, k_y) f_k^{(|m|,|n|)}(r_x, r_y) r_x^{|m|} r_y^{|n|}, \tag{F20}$$

where

$$C_{|m|,|n|,l}(k_x, k_y) = \int_0^\infty dr_x \int_0^\infty dr_y r_x^{|m|+1} r_y^{|n|+1} J_{|m|}(k_x r_x) J_{|n|}(k_y r_y) f_l^{(|m|,|n|)}(r_x, r_y) W_\perp(r^2). \tag{F21}$$

By substituting Eq. (F20) into Eq. (F14), Eq. (F14) is simplified as

$$\begin{aligned} K^{(m,n,m',n')}_{\omega,q,q'}(r'_x, r'_y | r_x, r_y) = {} & i^{|m|+|n|-(|m'|+|n'|)} (2\pi k_\beta)^2 \frac{4\pi^2}{4ab} \sum_{n_x,n_y=-\infty}^{\infty} \frac{1}{4ab} \sum_{\tilde{m},\tilde{n}=-\infty}^{\infty} \bar{P}^w_{\omega q}(m', n_x, n_y; \tilde{m}, \tilde{n}) \\ & \times \frac{n_x}{|n_x|} \frac{n_y}{|n_y|} \frac{ab}{4\pi^2} e^{i\frac{n_x\pi}{2} + i\frac{n_y\pi}{2}} (1-\delta_{n_x,0})(1-\delta_{n_y,0}) \frac{\delta(q-q')}{2\pi ab} [e^{i\frac{|n_y|\pi}{2}}(-1)^{|n'|} - e^{-i\frac{|n_y|\pi}{2}}] \\ & \times \sum_{l=0}^{\infty} C_{|m|,|n|,l}\left(\frac{\tilde{m}\pi}{a}, \frac{\tilde{n}\pi}{b}\right) f_l^{(|m|,|n|)}(r_x, r_y) r_x^{|m|} r_y^{|n|} \\ & \times \sum_{j=0}^{\infty} C_{|m'|,|n'|,j}\left(\frac{|n_x|\pi}{a}, \frac{|n_y|\pi}{b}\right) f_j^{(|m'|,|n'|)}(r'_x, r'_y) r_x'^{|m'|} r_y'^{|n'|}, \end{aligned} \tag{F22}$$

where

$$
\begin{aligned}
\bar{P}^{w}_{\omega q}(m', n_x, n_y; \tilde{m}, \tilde{n}) = {}& \frac{r_e}{4c\frac{c}{\omega}\sqrt{\frac{\omega^2}{c^2}-\frac{\tilde{m}^2\pi^2}{a^2}-\frac{\tilde{n}^2\pi^2}{b^2}}}\left(\frac{K}{\tilde{\gamma}}\right)^2 4\pi^2 ab\left[\frac{\sin\frac{(-\tilde{m}+n_x)\pi}{2}\sin\frac{(-\tilde{n}+n_y)\pi}{2}}{\frac{(-\tilde{m}+n_x)\pi}{2}\frac{(-\tilde{n}+n_y)\pi}{2}} - \frac{e^{i\tilde{n}\pi}\sin\frac{(-\tilde{m}+n_x)\pi}{2}\sin\frac{(\tilde{n}+n_y)\pi}{2}}{\frac{(-\tilde{m}+n_x)\pi}{2}\frac{(\tilde{n}+n_y)\pi}{2}}\right.\\
&\left.+\frac{e^{i\tilde{m}\pi}\sin\frac{(\tilde{m}+n_x)\pi}{2}\sin\frac{(-\tilde{n}+n_y)\pi}{2}}{\frac{(\tilde{m}+n_x)\pi}{2}\frac{(-\tilde{n}+n_y)\pi}{2}} - \frac{e^{i(\tilde{m}+\tilde{n})\pi}\sin\frac{(\tilde{m}+n_x)\pi}{2}\sin\frac{(\tilde{n}+n_y)\pi}{2}}{\frac{(\tilde{m}+n_x)\pi}{2}\frac{(\tilde{n}+n_y)\pi}{2}}\right]\\
&\times\sum_{\sigma=-\infty}^{\infty}\left[\frac{1}{\frac{i\omega}{\tilde{v}_r}+i(2\sigma+1)k_{wz}+iq-i\sqrt{\frac{\omega^2}{c^2}-\frac{\tilde{m}^2\pi^2}{a^2}-\frac{\tilde{n}^2\pi^2}{b^2}}} - \frac{1}{\frac{i\omega}{\tilde{v}_r}+i(2\sigma+1)k_{wz}+iq+i\sqrt{\frac{\omega^2}{c^2}-\frac{\tilde{m}^2\pi^2}{a^2}-\frac{\tilde{n}^2\pi^2}{b^2}}}\right]\\
&\times\left\{J_\sigma\left[\frac{\omega}{8k_{wz}c}\left(\frac{K}{\tilde{\gamma}}\right)^2\right]+J_{\sigma+1}\left[\frac{\omega}{8k_{wz}c}\left(\frac{K}{\tilde{\gamma}}\right)^2\right]\right\}\sum_{p=-\infty}^{\infty}J_p\left[\frac{\omega}{8k_{wz}c}\left(\frac{K}{\tilde{\gamma}}\right)^2\right]\\
&\times\left[-e^{i\frac{|n_x|\pi}{2}}(-1)^{|m'|}J_{2\sigma-2p+2}\left(\left|\frac{\tilde{m}\pi r_w}{a}+\frac{|n_x|\pi r_w}{a}\right|\right) - e^{i\frac{|n_x|\pi}{2}}(-1)^{|m'|}J_{2\sigma-2p}\left(\left|\frac{\tilde{m}\pi r_w}{a}+\frac{|n_x|\pi r_w}{a}\right|\right)\right.\\
&\left.+e^{-i\frac{|n_x|\pi}{2}}J_{2\sigma-2p+2}\left(\left|\frac{\tilde{m}\pi r_w}{a}-\frac{|n_x|\pi r_w}{a}\right|\right)+e^{-i\frac{|n_x|\pi}{2}}J_{2\sigma-2p}\left(\left|\frac{\tilde{m}\pi r_w}{a}-\frac{|n_x|\pi r_w}{a}\right|\right)\right]. \qquad \text{(F23)}
\end{aligned}
$$

When we retain the first order terms for $K/\tilde{\gamma}$, $\bar{P}^{w}_{\omega q}(m', n_x, n_y; \tilde{m}, \tilde{n})$ is approximated as

$$
\begin{aligned}
\bar{P}^{w}_{\omega q}(m', n_x, n_y; \tilde{m}, \tilde{n}) \simeq {}& \frac{r_e}{4c\frac{c}{\omega}\sqrt{\frac{\omega^2}{c^2}-\frac{\tilde{m}^2\pi^2}{a^2}-\frac{\tilde{n}^2\pi^2}{b^2}}}\left(\frac{K}{\tilde{\gamma}}\right)^2 4\pi^2 ab\left[\frac{\sin\frac{(-\tilde{m}+n_x)\pi}{2}\sin\frac{(-\tilde{n}+n_y)\pi}{2}}{\frac{(-\tilde{m}+n_x)\pi}{2}\frac{(-\tilde{n}+n_y)\pi}{2}} - \frac{e^{i\tilde{n}\pi}\sin\frac{(-\tilde{m}+n_x)\pi}{2}\sin\frac{(\tilde{n}+n_y)\pi}{2}}{\frac{(-\tilde{m}+n_x)\pi}{2}\frac{(\tilde{n}+n_y)\pi}{2}}\right.\\
&\left.+\frac{e^{i\tilde{m}\pi}\sin\frac{(\tilde{m}+n_x)\pi}{2}\sin\frac{(-\tilde{n}+n_y)\pi}{2}}{\frac{(\tilde{m}+n_x)\pi}{2}\frac{(-\tilde{n}+n_y)\pi}{2}} - \frac{e^{i(\tilde{m}+\tilde{n})\pi}\sin\frac{(\tilde{m}+n_x)\pi}{2}\sin\frac{(\tilde{n}+n_y)\pi}{2}}{\frac{(\tilde{m}+n_x)\pi}{2}\frac{(\tilde{n}+n_y)\pi}{2}}\right]\\
&\times\left\{\left[\frac{1}{\frac{i\omega}{\tilde{v}_r}+ik_{wz}+iq-i\sqrt{\frac{\omega^2}{c^2}-\frac{\tilde{m}^2\pi^2}{a^2}-\frac{\tilde{n}^2\pi^2}{b^2}}} - \frac{1}{\frac{i\omega}{\tilde{v}_r}+ik_{wz}+iq+i\sqrt{\frac{\omega^2}{c^2}-\frac{\tilde{m}^2\pi^2}{a^2}-\frac{\tilde{n}^2\pi^2}{b^2}}}\right]\right.\\
&\times\left\{-e^{i\frac{|n_x|\pi}{2}}(-1)^{|m'|}\left[J_2\left(\left|\frac{\tilde{m}\pi r_w}{a}+\frac{|n_x|\pi r_w}{a}\right|\right)+J_0\left(\left|\frac{\tilde{m}\pi r_w}{a}+\frac{|n_x|\pi r_w}{a}\right|\right)\right]\right.\\
&\left.+e^{-i\frac{|n_x|\pi}{2}}\left[J_2\left(\left|\frac{\tilde{m}\pi r_w}{a}-\frac{|n_x|\pi r_w}{a}\right|\right)+J_0\left(\left|\frac{\tilde{m}\pi r_w}{a}-\frac{|n_x|\pi r_w}{a}\right|\right)\right]\right\}\\
&+\left[\frac{1}{\frac{i\omega}{\tilde{v}_r}-ik_{wz}+iq-i\sqrt{\frac{\omega^2}{c^2}-\frac{\tilde{m}^2\pi^2}{a^2}-\frac{\tilde{n}^2\pi^2}{b^2}}} - \frac{1}{\frac{i\omega}{\tilde{v}_r}-ik_{wz}+iq+i\sqrt{\frac{\omega^2}{c^2}-\frac{\tilde{m}^2\pi^2}{a^2}-\frac{\tilde{n}^2\pi^2}{b^2}}}\right]\\
&\times\left\{-e^{i\frac{|n_x|\pi}{2}}(-1)^{|m'|}\left[J_0\left(\left|\frac{\tilde{m}\pi r_w}{a}+\frac{|n_x|\pi r_w}{a}\right|\right)+J_2\left(\left|\frac{\tilde{m}\pi r_w}{a}+\frac{|n_x|\pi r_w}{a}\right|\right)\right]\right.\\
&\left.\left.+e^{-i\frac{|n_x|\pi}{2}}\left(J_0\left(\left|\frac{\tilde{m}\pi r_w}{a}-\frac{|n_x|\pi r_w}{a}\right|\right)+J_2\left(\left|\frac{\tilde{m}\pi r_w}{a}-\frac{|n_x|\pi r_w}{a}\right|\right)\right]\right\}\right\}. \qquad \text{(F24)}
\end{aligned}
$$

By inserting Eqs. (F17) and (F22) into Eq. (F16), multiplying it by $f_k^{(|m|,|n|)}(r_x, r_y)r_x^{|m|+1}r_y^{|n|+1}$ and integrating it over r_x and r_y, the final expression of the matrix form for the dispersion relation is derived as

$$
a_k^{(m,n)} + \sum_{m',n'=-\infty}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\beta^{m',n'}_{m,n,l,j}M^{m,n,l}_{m',n',k}a_j^{(m',n')} = 0, \qquad \text{(F25)}
$$

where

$$\beta^{m',n'}_{m,n,l,j} = \int_1^\infty d\gamma \int_0^\infty dr_x \int_0^\infty dr_y \frac{W_\perp(r^2) f_l^{(|m'|,|n'|)}(r_x,r_y) f_j^{(|m'|,|n'|)}(r_x,r_y) r_x^{2|m'|+1} r_y^{2|n'|+1}}{(iq - i\omega \frac{d\tau}{dz}(r,\gamma) - ik_\beta(m+n))} \frac{\partial f_{0\parallel}(\gamma)}{\partial\gamma}, \tag{F26}$$

$$M^{m,n,l}_{m',n',k} = i^{|m|+|n|-(|m'|+|n'|)} \frac{(2\pi k_\beta)^2}{C} \frac{4\pi^2}{4ab} \sum_{n_x,n_y=-\infty}^{\infty} \frac{1}{4ab} \sum_{\tilde m,\tilde n=-\infty}^{\infty} \bar P^w_{\omega q}(m', n_x, n_y; \tilde m, \tilde n) \frac{n_x}{|n_x|} \frac{n_y}{|n_y|} \frac{ab}{4\pi^2} e^{i\frac{n_x\pi}{2} + i\frac{n_y\pi}{2}} (1-\delta_{n_x,0})(1-\delta_{n_y,0})$$
$$\times \frac{1}{2\pi ab} (e^{i\frac{|n_y|\pi}{2}}(-1)^{|n'|} - e^{-i\frac{|n_y|\pi}{2}}) C_{|m|,|n|,k}\left(\frac{\tilde m\pi}{a}, \frac{\tilde n\pi}{b}\right) C_{|m'|,|n'|,l}\left(\frac{|n_x|\pi}{a}, \frac{|n_y|\pi}{b}\right), \tag{F27}$$

which gives the existence condition of the eigenvalue q for any amplitude $a_k^{(m,n)}$ as a function of ω. The beam growth rate is given by the imaginary part of q. Its double provides a FEL gain.

1. Dispersion relation for a hollow beam case

In order to calculate the beam growth rate (the half of the FEL gain), we have to make models for the unperturbed part of the distribution function f_0. For example, let us consider a hollow beam:

$$f_{0\perp}(r^2) = \frac{1}{\pi^2 R_0^4 k_\beta^2} \delta\left(1 - \frac{r^2}{R_0^2}\right), \tag{F28}$$

where R_0 is the transverse beam size. The weight function $W_\perp(r^2)$ is chosen to be

$$W_\perp(r^2) = \delta\left(1 - \frac{r^2}{R_0^2}\right). \tag{F29}$$

The normalization constant C in Eq. (F18) is given by

$$C = \pi^2 R_0^4 k_\beta^2. \tag{F30}$$

Perturbations on the hollow beam take place only at $r = R_0$. As a result, $R^{m,n}_{\omega,q}$ has the characteristic as

$$R^{m,n}_{\omega,q} \propto \delta\left(1 - \frac{r^2}{R_0^2}\right) r_x^{|m|} r_y^{|n|}. \tag{F31}$$

Comparing Eq. (F17) with Eq. (F31), we find that the function $f_k^{(|m|,|n|)}(r_x, r_y)$ is nonzero constant only for $k = 0$ and vanishes otherwise.

By using Eq. (F19), we obtain

$$f_0^{(|m|,|n|)} = \frac{1}{R_0^{2+|m|+|n|}} \left[\frac{2}{\alpha_{|m|,|n|}}\right]^{\frac{1}{2}}, \tag{F32}$$

where

$$\alpha_{|m|,|n|} = \int_0^{\frac{\pi}{2}} \cos^{2|m|+1}\theta \sin^{2|m|+1}\theta d\theta$$
$$= \frac{(2|m|)!!(2|n|)!!}{(2|m|+2|n|+2)!!}. \tag{F33}$$

In this case, Eq. (F21) can be simplified as

$$C_{|m|,|n|,0}(k_x, k_y) = f_0^{(|m|,|n|)} \int_0^\infty dr_x \int_0^\infty dr_y r_x^{|m|+1} r_y^{|n|+1} J_{|m|}(k_x r_x) J_{|n|}(k_y r_y) \delta\left(1 - \frac{r^2}{R_0^2}\right)$$
$$= \frac{R_0^2}{\sqrt{2\alpha_{|m|,|n|}}} \frac{J_{|m|+|n|+1}(\sqrt{k_x^2 + k_y^2} R_0)}{(\sqrt{k_x^2 + k_y^2} R_0)^{|m|+|n|+1}} (k_x R_0)^{|m|} (k_y R_0)^{|n|}, \tag{F34}$$

where Eq. (F32) and

$$\int_0^{\frac{\pi}{2}} d\theta \sin^{\mu+1}\theta \cos^{\nu+1}\theta J_\mu(a\sin\theta) J_\nu(b\cos\theta) = \frac{a^\nu b^\mu}{(a^2+b^2)^{(\mu+\nu+1)/2}} J_{\mu+\nu+1}\left(\sqrt{a^2+b^2}\right), \quad \text{for } \Re\mu, \Re\nu > -1, \tag{F35}$$

are used.

Equations (F26) and (F27) can be finally given by

$$\beta^{m',n'}_{m,n,0,0} = \beta_{m,n} = \int_1^\infty d\gamma \frac{\frac{\partial f_{0\|}(\gamma)}{\partial\gamma}}{(iq - i\omega\frac{d\tau}{dz}(r=R_0,\gamma) - ik_\beta(m+n))}$$
$$= \int_1^\infty d\gamma \frac{\frac{\partial f_{0\|}(\gamma)}{\partial\gamma}}{[iq + 2i\frac{k}{k_1}k_w\frac{(\gamma-\gamma_r)}{\gamma_r} - i\frac{1}{2}kk_\beta^2R_0^2 - ik_\beta(m+n)]}$$
$$= 2i\frac{kk_w}{k_1\gamma_r}\int_1^\infty d\gamma \frac{f_{0\|}(\gamma)}{[iq + 2i\frac{k}{k_1}k_w\frac{(\gamma-\gamma_r)}{\gamma_r} - i\frac{1}{2}kk_\beta^2R_0^2 - ik_\beta(m+n)]^2}, \tag{F36}$$

$$M^{m,n,0}_{m',n',0} = i^{|m|+|n|-(|m'|+|n'|)}\frac{(2\pi k_\beta)^2}{C}\frac{4\pi^2}{4ab}\sum_{n_x,n_y=-\infty}^{\infty}\frac{1}{4ab}$$
$$\times\sum_{\tilde m,\tilde n=-\infty}^{\infty}\bar P^w_{\omega q}(m',n_x,n_y;\tilde m,\tilde n)\frac{n_x}{|n_x|}\frac{n_y}{|n_y|}\frac{ab}{4\pi^2}e^{i\frac{n_x\pi}{2}+i\frac{n_y\pi}{2}}(1-\delta_{n_x,0})(1-\delta_{n_y,0})$$
$$\times\frac{1}{2\pi ab}(e^{i\frac{|n_y|\pi}{2}}(-1)^{|n'|} - e^{-i\frac{|n_y|\pi}{2}})\frac{R_0^2}{\sqrt{2\alpha_{|m|,|n|}}}\frac{J_{|m|+|n|+1}\left(\sqrt{\frac{\tilde m^2\pi^2}{a^2}+\frac{\tilde n^2\pi^2}{b^2}}R_0\right)}{\left(\sqrt{\frac{\tilde m^2\pi^2}{a^2}+\frac{\tilde n^2\pi^2}{b^2}}R_0\right)^{|m|+|n|+1}}\left(\frac{\tilde m\pi}{a}R_0\right)^{|m|}\left(\frac{\tilde n\pi}{b}R_0\right)^{|n|}\frac{R_0^2}{\sqrt{2\alpha_{|m'|,|n'|}}}$$
$$\times\frac{J_{|m'|+|n'|+1}\left(\sqrt{\frac{|n_x|^2\pi^2}{a^2}+\frac{|n_y|^2\pi^2}{b^2}}R_0\right)}{\left(\sqrt{\frac{|n_x|^2\pi^2}{a^2}+\frac{|n_y|^2\pi^2}{b^2}}R_0\right)^{|m'|+|n'|+1}}\left(\frac{|n_x|\pi}{a}R_0\right)^{|m'|}\left(\frac{|n_y|\pi}{b}R_0\right)^{|n'|}. \tag{F37}$$

The simplified dispersion relation for the hollow beam is expressed as

$$a_0^{(m,n)} + \beta_{m,n}\sum_{m',n'=-\infty}^{\infty}M^{m,n,0}_{m',n',0}a_0^{(m',n')} = 0. \tag{F38}$$

a. In the case of infinitely wide ($a \to \infty$) waveguide

In some of planar undulators for waveguide FELs, the horizontal size of the waveguide far exceeds the vertical size. In this case, the rectangular waveguide is basically identical to two parallel plates.

For the infinitely wide waveguide, we obtain

$$M^{m,n,0}_{m',n',0} = \frac{i^{|m|+|n|-(|m'|+|n'|)}}{b}\sum_{n_y=-\infty}^{\infty}\int_{-\infty}^{\infty}dk_x\sum_{\tilde n=-\infty}^{\infty}\frac{r_e}{4c\frac{c}{\omega}}\left(\frac{K}{\tilde\gamma}\right)^2\left[\frac{\sin\frac{(-\tilde n+n_y)\pi}{2}}{\frac{(-\tilde n+n_y)\pi}{2}} - \frac{e^{i\tilde n\pi}\sin\frac{(\tilde n+n_y)\pi}{2}}{\frac{(\tilde n+n_y)\pi}{2}}\right]$$
$$\times\sum_{\sigma=-\infty}^{\infty}\left\{\frac{2i}{-[\frac{\omega}{\bar v_r}+(2\sigma+1)k_{wz}+q]^2+\frac{\omega^2}{c^2}-k_x^2-\frac{\tilde n^2\pi^2}{b^2}}\right\}\left\{J_\sigma\left[\frac{\omega}{8k_{wz}c}\left(\frac{K}{\tilde\gamma}\right)^2\right]+J_{\sigma+1}\left[\frac{\omega}{8k_{wz}c}\left(\frac{K}{\tilde\gamma}\right)^2\right]\right\}^2$$
$$\times(1-\delta_{n_y,0})[e^{in_y\pi}(-1)^{|n'|}-1]\frac{J_{|m|+|n|+1}\left(\sqrt{k_x^2+\frac{\tilde n^2\pi^2}{b^2}}R_0\right)}{\sqrt{2\alpha_{|m|,|n|}}\left(\sqrt{k_x^2+\frac{\tilde n^2\pi^2}{b^2}}R_0\right)^{|m|+|n|+1}}(k_xR_0)^{|m|}\left(\frac{\tilde n\pi}{b}R_0\right)^{|n|}$$
$$\times\frac{J_{|m'|+|n'|+1}\left(\sqrt{\frac{n_x^2\pi^2}{a^2}+\frac{|n_y|^2\pi^2}{b^2}}R_0\right)}{\sqrt{2\alpha_{|m'|,|n'|}}\left(\sqrt{k_x^2+\frac{|n_y|^2\pi^2}{b^2}}R_0\right)^{|m'|+|n'|+1}}(k_xR_0)^{|m'|}\left(\frac{n_y\pi}{b}R_0\right)^{|n'|}. \tag{F39}$$

Equation (F39) is approximated as

$$M^{0,0,0}_{0,0,0} = \frac{ir_e}{b\frac{c^2}{\omega}}\left(\frac{K}{\tilde{\gamma}}\right)^2 \sum_{\tilde{n}=-\infty}^{\infty}\sum_{n_y=-\infty}^{\infty}\sum_{\sigma=-\infty}^{\infty}(e^{in_y\pi}-1)\left[\frac{\sin\frac{(-\tilde{n}+n_y)\pi}{2}}{\frac{(-\tilde{n}+n_y)\pi}{2}} - \frac{e^{i\tilde{n}\pi}\sin\frac{(\tilde{n}+n_y)\pi}{2}}{\frac{(\tilde{n}+n_y)\pi}{2}}\right]\left\{J_\sigma\left[\frac{\omega}{8k_{wz}c}\left(\frac{K}{\tilde{\gamma}}\right)^2\right] + J_{\sigma+1}\left[\frac{\omega}{8k_{wz}c}\left(\frac{K}{\tilde{\gamma}}\right)^2\right]\right\}^2$$
$$\times\int_{-\infty}^{\infty}dk_x \frac{J_1\left(\sqrt{k_x^2+\frac{\tilde{n}^2\pi^2}{b^2}}R_0\right)J_1\left(\sqrt{k_x^2+\frac{n_y^2\pi^2}{b^2}}R_0\right)}{\{-[\frac{\omega}{\bar{v}_r}+(2\sigma+1)k_{wz}+q]^2-k_x^2-\frac{\tilde{n}^2\pi^2}{b^2}+\frac{\omega^2}{c^2}\}\left(\sqrt{k_x^2+\frac{\tilde{n}^2\pi^2}{b^2}}R_0\right)\left(\sqrt{k_x^2+\frac{n_y^2\pi^2}{b^2}}R_0\right)}$$
$$= -\frac{ir_e}{b\frac{c^2}{\omega}}\left(\frac{K}{\tilde{\gamma}}\right)^2 \sum_{\tilde{n}=-\infty}^{\infty}\sum_{n_y=-\infty}^{\infty}\sum_{\sigma=-\infty}^{\infty}(e^{in_y\pi}-1)\left[\frac{\sin\frac{(-\tilde{n}+n_y)\pi}{2}}{\frac{(-\tilde{n}+n_y)\pi}{2}} - \frac{e^{i\tilde{n}\pi}\sin\frac{(\tilde{n}+n_y)\pi}{2}}{\frac{(\tilde{n}+n_y)\pi}{2}}\right]\left\{J_\sigma\left[\frac{\omega}{8k_{wz}c}\left(\frac{K}{\tilde{\gamma}}\right)^2\right]\right.$$
$$\left. + J_{\sigma+1}\left[\frac{\omega}{8k_{wz}c}\left(\frac{K}{\tilde{\gamma}}\right)^2\right]\right\}^2 \frac{\pi}{\sqrt{[\frac{\omega}{\bar{v}_r}+(2\sigma+1)k_{wz}+q]^2+\frac{\tilde{n}^2\pi^2}{b^2}-\frac{\omega^2}{c^2}}\sqrt{-[\frac{\omega}{\bar{v}_r}+(2\sigma+1)k_{wz}+q]^2+\frac{\omega^2}{c^2}}R_0^2}$$
$$\times\frac{1}{\sqrt{-(\frac{\omega}{\bar{v}_r}+(2\sigma+1)k_{wz}+q)^2-\frac{\tilde{n}^2\pi^2}{b^2}+\frac{\omega^2}{c^2}+\frac{n_y^2\pi^2}{b^2}}}\sum_{m=0}^{\infty}$$
$$\times\left(\frac{\sqrt{\{-[\frac{\omega}{\bar{v}_r}+(2\sigma+1)k_{wz}+q]^2+\frac{\omega^2}{c^2}\}\{-[\frac{\omega}{\bar{v}_r}+(2\sigma+1)k_{wz}+q]^2-\frac{\tilde{n}^2\pi^2}{b^2}+\frac{\omega^2}{c^2}+\frac{n_y^2\pi^2}{b^2}\}}R_0^2}{2}\right)^{2m+1}$$
$$\times\frac{J_{1+2m}\left(\sqrt{\{-2[\frac{\omega}{\bar{v}_r}+(2\sigma+1)k_{wz}+q]^2+2\frac{\omega^2}{c^2}-\frac{\tilde{n}^2\pi^2}{b^2}+\frac{n_y^2\pi^2}{b^2}\}R_0^2}\right)}{m!(m+1)![(-2(\frac{\omega}{\bar{v}_r}+(2\sigma+1)k_{wz}+q)^2+2\frac{\omega^2}{c^2}-\frac{\tilde{n}^2\pi^2}{b^2}+\frac{n_y^2\pi^2}{b^2})R_0^2]^{\frac{1+2m}{2}}}, \tag{F40}$$

for $m = n = m' = n' = 0$, where we use Macdonald's integral representation:

$$J_\nu(z)J_\nu(\zeta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} I_\nu\left(\frac{z\zeta}{t}\right)\exp\left(\frac{t}{2}-\frac{z^2+\zeta^2}{2t}\right)\frac{dt}{t}, \tag{F41}$$

for $\Re\nu > -1$, $c > 0$, $|arg(z \pm \zeta)| < \pi/4$, and the expansion formula of the modified Bessel function $I_\nu(z)$ [12]

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty}\frac{(z/2)^{2n}}{n!\Gamma(\nu+n+1)}. \tag{F42}$$

In the derivation, the k_x-integration is performed by picking up residues in the complex plane.

When the distribution function is given by the uniform one as Eq. (42), Eq. (F36) becomes Eq. (43). Combining Eqs. (F40) and (43), the dispersion relation Eq. (F38) is finally expressed as Eq. (44).

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