Synchrobetatron resonant coupling mechanism in a storage ring

Kouichi Jimbo*

Institute of Advanced Energy, Kyoto University, Uji, 611-0011 Kyoto, Japan

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A clear synchrobetatron resonant coupling of Mg ion beam was observed experimentally in the horizontal laser beam cooling experiment in small laser equipped storage ring. Synchrotron and horizontal betatron motions were intentionally coupled in a rf cavity. Using the Hamiltonian which is composed of coasting, synchrotron and betatron motions, physical mechanism of the coupling is analyzed to explain the observed horizontal betatron tune jump near the synchrobetatron resonant coupling point. There energy exchange between the synchrotron oscillation and the horizontal betatron oscillation was mediated by coasting particles and the freedom of the horizontal direction is connected with the freedom of the longitudinal direction.

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I. INTRODUCTION

Laser cooling techniques have been applied to cool down the longitudinal (orbital) direction of an ion beam. This technique, however, cannot cool the transverse (horizontal and vertical) direction. The synchrobetatron resonant coupling method where the cooling force in the longitudinal direction was extended to the horizontal direction, was proposed to enable the horizontal cooling. At the synchrobetatron resonant coupling, the difference between the fractional part of the betatron tune and the synchrotron tune is negligible (the difference integer resonance condition) [1]. small laser equipped storage ring (S-LSR), as shown in Fig. 1, is a synchrotron-type small storage ring at Kyoto University [2]. The synchrobetatron resonant coupling method was employed in S-LSR, in which the horizontal laser cooling was already observed [3]. Near the synchrobetatron resonant coupling point where the difference integer resonance condition was satisfied, an unexpected tune jump of the horizontal betatron tune was observed. The tune jump had not been recognized in the computer simulation [4]. In this manuscript, a synchrobetatron resonant coupling mechanism is analyzed analytically from the Hamiltonian for an orbiting particle to clarify the physics of the observed tune jump. Then we discuss the horizontal cooling mechanism, which may help to achieve a crystalline beam in future [5].

II. EXPERIMENTAL ARRANGEMENT AND RESULTS

A 40 keV Mg ion beam was injected into S-LSR, which was equipped with a frequency tunable laser system of 280 nm for beam cooling. S-LSR consists of 6 bending magnets, and accordingly it has 6-fold symmetry. The lattice of it was constructed so that magnet error was as small as possible to minimize beam heating. A drift tube (a rf cavity) was designed to bunch the ion beam at harmonic number 100. The frequency of rf wave was 2.52 MHz. It was installed at the location of finite dispersion function (1.1 m) to couple synchrotron and horizontal betatron motions. The horizontal beam size was observed by a CCD camera and a photomultiplier [3]. Table I shows the main parameters of S-LSR.

Betatron and synchrotron tunes were measured precisely to reveal the resonance condition. Figure 2 shows the

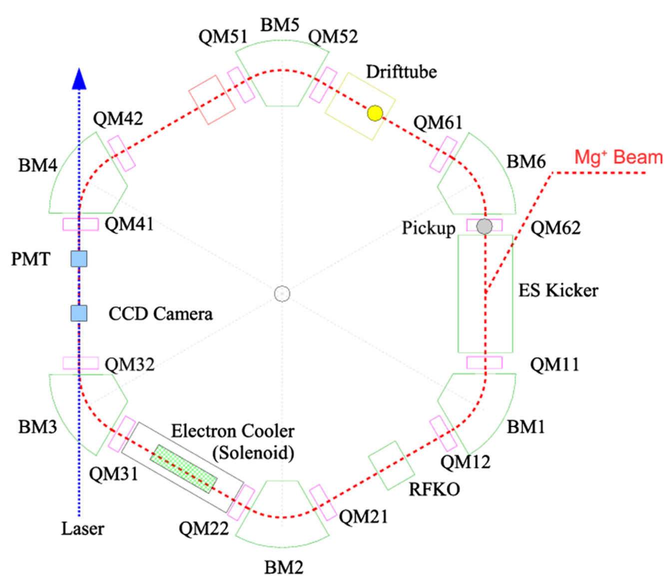

FIG. 1. Layout of the S-LSR It consists of 6 bending magnets BMi (the digit i indicates BM number) and 12 quadrupole magnets QMij (focusing for digit $j = 1$, defocusing for digit $j = 2$, and digit i indicate QM number). The horizontal beam size was observed by CCD camera and PMT (photo multiplier tube).

*jimbo@iae.kyoto-u.ac.jp

TABLE I. Main parameter of S-LSR.

Circumference	22.557 m
Average radius	3.59 m
Length of straight section	1.86 m
Radius of curvature	1.05 m
Revolution frequency	25.192 kHz
Super periodicity	6
Ion species	$^{24}Mg^{+}$
Kinetic beam energy	40 keV

conceptual diagram of tune measurements. The beam was excited by parallel-plate electrodes RFKO connected to a network analyzer (Agilent 4395A). Betatron tunes were adjusted by currents of quadrupole magnets. Sideband behaviors distinguished the horizontal and the vertical betatron tunes. Synchrotron tune, which was varied by the (rf) cavity voltage, was identified in the same way. Tune signals were detected by pickup, amplified by a pre amp (SA-220F5) and analyzed by the network analyzer, in which we observed sidebands as variations of fractional part of tunes [6].

In Fig. 3, the difference integer resonance condition of integer 2 of the synchrobetatron resonant coupling is observed near the resonant point ($\nu_{\beta x} = 2.10$, $\nu_s = 0.10$) where $\nu_{\beta x}$ is the horizontal betatron tune and ν_s is the synchrotron tune. At the resonant point $\bar{\nu} = 0.10$ and the cavity voltage 65 [V] are satisfied ($\bar{\nu}$ is the fractional part of tune). When the square root of the cavity voltage increases, the fractional part of tune of $\nu_{\beta x}$ should stay at a constant value: the fractional part of tune of $\nu_{\beta x}$ should be a constant value. However, it changes discontinuously near the resonant point. There the fractional part of tune of $\nu_{\beta x}$ jumps about 0.015. Apparently the value of $\nu_{\beta x}$ is shifted laterally. When the cavity voltage increases, ν_s scales with the square root of the cavity voltage (the synchrotron tune looks like a straight line). Near the resonant point, however, discontinuity is also observable but clear break from a straight line is not recognized in the both sides of the resonant point.

For a comparison, the vertical betatron tune $\nu_{\beta z}$ ($\nu_{\beta z} = 1.10$) is also shown in Fig. 3. Its fractional value is intentionally reduced a little from $\bar{\nu} = 0.10$ for clear view. The difference integer resonance condition of integer 1 is satisfied between the horizontal and the vertical betatron tunes but there is no coupling mechanism. The vertical betatron tune is constant as it is expected.

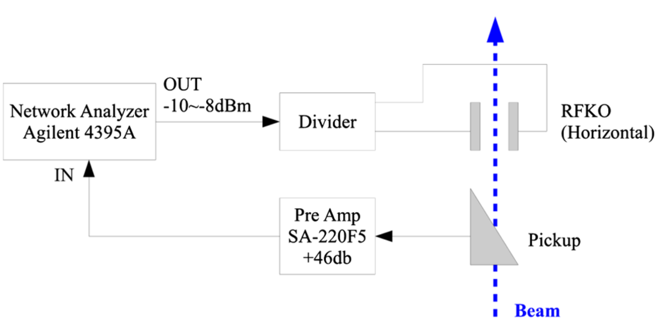

FIG. 2. Conceptual diagram of tune measurement.

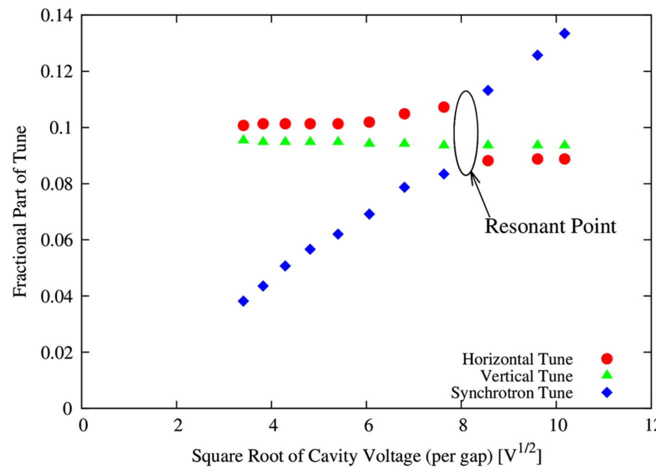

FIG. 3. Synchrotron and horizontal betatron coupling near the resonant point $\bar{\nu} = 0.10$ ($\nu_{\beta x} = 2.10$, $\nu_s = 0.10$, $\nu_{\beta z} = 1.10$) and the cavity voltage 65 [V] where $\bar{\nu}$ is a fractional part of tune. $\nu_{\beta x}$ (red) is the horizontal (betatron) tune, $\nu_{\beta z}$ (green) is the vertical (betatron) tune, ν_s (blue) is the synchrotron tune. There the fractional part of tune of $\nu_{\beta x}$ jumps about 0.015. Apparently the value of $\nu_{\beta x}$ is shifted laterally. The fractional part of tune of $\nu_{\beta z}$ is intentionally reduced a little from $\bar{\nu} = 0.10$ for clear view. The fractional part of tune of $\nu_{\beta z}$ is constant.

III. PHENOMENOLOGICAL ANALYSIS

A. The Hamiltonian composed of coasting, betatron and synchrotron motions

The Hamiltonian E for a particle under Lorentz force in a storage ring becomes [7]

$$E = c\left\{m^2c^2 + \frac{(p_s - qA_s)^2}{(1+\frac{x}{\rho})^2} + (p_x - qA_x)^2 + (p_z - qA_z)^2\right\}^{\frac{1}{2}}, \tag{1}$$

where ρ is the radius of curvature, q is the elementary charge, c is the velocity of light, and m is the particle mass. We neglect torsion and the scalar potential. In the right-handed curvilinear coordinate system (x, s, z), $A_{x,s,z}$ is the vector potential and $p_{x,s,z}$ is the canonical momentum. Here s is orbit length. For a positive value of orbital momentum p_s, which is time independent, the particle is moving in a counterclockwise direction. And x and p_x are horizontal coordinate and momentum around the reference closed orbit. We neglect vertical motion and put $z = 0$ and $p_z = 0$. We further assume $A_x = A_z = 0$ for magnets. Then $\nu_\beta = \nu_{\beta x}$: "betatron tune" means "horizontal betatron tune." The energy E and the momentum p of the particle satisfy

$$p = \sqrt{\frac{E^2}{c^2} - m^2c^2}, \tag{2}$$

Where $p^2 = {p_x}^2 + {p_s}^2$. We obtain

$$-p_s = -\left(1+\frac{x}{\rho}\right)\left\{\frac{E^2}{c^2} - m^2c^2 - p_x^2\right\}^{\frac{1}{2}} - qA_s. \tag{3}$$

Now (x, p_x) and $(t, -E)$ become canonical variables. Since p_x is much smaller than p_s, we have

$$-p_s \approx -\left(1+\frac{x}{\rho}\right)p + \frac{1}{2p}\left(1+\frac{x}{\rho}\right)p_x^2 - qA_s, \tag{4}$$

$$A_s = -B_0x + \frac{1}{2}B_1(x^2 - z^2) - \frac{B_0}{2\rho}x^2 + \cdots + A_{\rm rf}, \tag{5}$$

where $A_{\rm rf}$ is the vector potential of the rf cavity, $B_0 = \frac{p_0}{q\rho}$ is the magnetic field of the main dipole, $B_1 = \frac{\partial B_z}{\partial x}$ is the quadrupole gradient function and $K_x = \frac{1}{\rho^2} - \frac{B_1}{B_0\rho}$.

Equation (4) turns to be

$$-p_s \approx -\left(1+\frac{x}{\rho}\right)p + \left(1+\frac{x}{\rho}\right)\left(\frac{p_x^2}{2p}\right) + qB_0x - \frac{qB_1}{2}x^2 + \frac{qB_0}{2\rho}x^2 + \cdots - qA_{\rm rf}. \tag{6}$$

We assume that a reference particle of the momentum p_0 is circling the reference closed orbit of the average radius R with velocity βc and energy E_0. Synchronous particles synchronize with the rf wave of angular frequency $\omega_{\rm rf} = h\omega_0$ (h is the harmonic number) and are bunched by the rf wave. We have the revolution frequency ω_0 ($\omega_0 = \frac{d\theta}{dt}$) and the orbit angle θ ($\theta = \frac{s}{R}$). Equation (6) is further simplified by taking s or θ as an independent variable. Define $E = E_0 + \Delta E$ and $p = p_0 + \Delta p$. Δp is the momentum deviation and ΔE is the energy deviation from the reference particle.

We also have $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $\frac{ds}{dt} = \beta c$, $E_0 = m\gamma c^2$ and $p_0 = m\gamma\beta c$. We obtain

$$\frac{\Delta p}{p_0} \approx \frac{\Delta E}{\beta^2 E_0} - \frac{1}{2\gamma^2}\left(\frac{\Delta E}{\beta^2 E_0}\right)^2, \tag{7}$$

Where $\frac{\Delta p}{p_0}$ is the fractional (momentum) deviation and $\frac{\Delta E}{\beta^2 E_0}$ is the rationalized fractional (energy) deviation. From Eq. (6),

$$-\frac{p_s}{p_0} \approx -\left\{1 + \left(\frac{\Delta p}{p_0}\right)\right\} - \frac{x}{\rho}\left(\frac{\Delta p}{p_0}\right) + \left(\frac{p_x^2}{2{p_0}^2}\right)\left(1+\frac{x}{\rho}\right) + \frac{1}{2}K_x x^2 - \frac{qA_{\rm rf}}{p_0}. \tag{8}$$

The energy gain for the synchronous particle in the rf cavity is given by

$$\frac{dE_0}{ds} = \frac{\omega_0}{\beta c}\frac{dE_0}{dt} = \frac{\omega_0}{2\pi\beta c}qV\sin\psi_S = \frac{\sin\psi_S}{2\pi R}qV, \tag{9}$$

where V is the effective rf cavity voltage seen by particles per passage [8]. ψ_S is the phase angle for the synchronous particle with respect to the rf cavity voltage.

The rf vector potential, which faces h bunches, is

$$A_{\rm rf} = \frac{V}{\omega_0}\cos(\omega_0 t + \phi_0)\sum_n \delta\left(s - s_0 - 2\pi n\frac{R}{h}\right)$$
$$\Rightarrow \frac{hV}{2\pi R\omega_0}\cos\left\{\omega_0 t - \frac{s}{R} + \left(\phi_0 + \frac{s_0}{R}\right)\right\}, \tag{10}$$

where δ is a delta function and ϕ_0 is the initial phase at the rf cavity located at s_0. Since $\phi_0 + \frac{s_0}{R}$ should be an integer multiple of 2π, we can neglect these terms [9].

E_0 has finite s-dependence of 1st order. Then $\frac{\Delta E}{\beta^2 E_0}$ has s-dependence of 2nd order, and we can neglect s-dependence of it. From Eq. (7), s-dependence of $\frac{\Delta p}{p_0}$ is neglected. p_0 is s-independent but E_0 is s-dependent since $A_{\rm rf}$ is s-dependent.

Define a symbol of rationalized fractional deviation $\delta \equiv \frac{\Delta E}{\beta^2 E_0}$. Keeping up to 2nd order in Eq. (8), the Hamiltonian H, which is a constant of motion, is obtained

$$H = -\frac{p_s}{p_0}$$
$$= -(1+\delta) + \frac{1}{2\gamma^2}\delta^2 - \frac{x}{\rho}\delta + \left(\frac{p_x^2}{2{p_0}^2}\right) + \frac{1}{2}K_x x^2 - \frac{qA_{\rm rf}}{p_0}. \tag{11}$$

We transform the coordinate system of $-E$ onto $-\Delta E$ then to $-\delta$. That is

$$(x, p_x;\, t, -E) \to (\bar{x}, \bar{p}_x;\, \bar{t}, -\Delta E) \to \left(\bar{x}, \frac{\bar{p}_x}{p_0};\, \phi, -\delta\right).$$

Let us define the generating function F_2 for a canonical transformation [9] as

$$F_2\left(x, \frac{\bar{p}_x}{p_0};\, t, -\delta\right) = (x - D\delta)\left(\frac{\bar{p}_x}{p_0}\right) - \left(\frac{E_0}{p_0} + \beta c\delta\right)t + D'x\delta - \frac{1}{2}DD'\delta^2 \tag{12}$$

The prime denotes differentiation with respect to s. Around the off-momentum closed orbit, $\bar{x}$ and $\bar{p}_x$ are horizontal coordinate and horizontal momentum. $\bar{t}$ and ϕ are time and the phase of off-momentum particle. We obtain

$$x = \bar{x} + D\delta, \qquad \frac{p_x}{p_0} = \frac{\bar{p}_x}{p_0} + D'\delta,$$
$$t = \bar{t} - \frac{D}{\beta c}\frac{\bar{p}_x}{p_0} + \frac{D'}{\beta c}\bar{x}, \qquad \phi = \omega_0 \bar{t} - \frac{s}{R},$$

and the dispersion function D satisfies

$$D'' + K_x D = \frac{1}{\rho}. \tag{13}$$

Since

$$\frac{\partial F_2}{\partial s} = -\frac{1}{p_0}\frac{\partial E_0}{\partial s} t - D'\delta\left(\frac{\bar{p}_x}{p_0}\right) + D''\delta x - (D'D' + DD'')\delta^2, \tag{14}$$

we have a new Hamiltonian $\bar{H}$, which is also a constant of motion, as

$$\begin{aligned}\bar{H} &= H + \frac{\partial F_2}{\partial s}\\ &= -(1+\delta) - \frac{1}{2}\left(\frac{D}{\rho} - \frac{1}{\gamma^2}\right)\delta^2 - \frac{1}{p_0}\frac{\partial E_0}{\partial s}\left(\bar{t} - \frac{D}{\beta c}\left(\frac{\bar{p}_x}{p_0}\right)\right.\\ &\quad \left. + \frac{D'}{\beta c}\bar{x}\right) - \frac{qA_{\mathrm{rf}}}{p_0} + \left\{\frac{1}{2}\left(\frac{\bar{p}_x}{p_0}\right)^2 + \frac{1}{2}K_x \bar{x}^2\right\}.\end{aligned} \tag{15}$$

From Eqs. (9) and (10),

$$\begin{aligned}&\frac{dE_0}{ds}\left\{\bar{t} - \frac{D}{\beta c}\left(\frac{\bar{p}_x}{p_0}\right) + \frac{D'}{\beta c}\bar{x}\right\}\\ &\quad = p_0 \frac{\sin\psi_S hqV}{2\pi\beta^2 E_0}\left\{\phi - \omega_0 \frac{D}{\beta c}\left(\frac{\bar{p}_x}{p_0}\right) + \omega_0 \frac{D'}{\beta c}\bar{x}\right\},\end{aligned} \tag{16}$$

$$\begin{aligned}qA_{\mathrm{rf}} &= \frac{p_0 hqV}{2\pi\beta^2 E_0}\cos\left(\omega_0 t - \frac{s}{R}\right)\\ &= p_0 \frac{hqV}{2\pi\beta^2 E_0}\cos\left\{\phi - \omega_0 \frac{D}{\beta c}\left(\frac{\bar{p}_x}{p_0}\right) + \omega_0 \frac{D'}{\beta c}\bar{x}\right\}.\end{aligned} \tag{17}$$

Averaging over one revolution around the ring of circumference $C = 2\pi R$ [10]

$$\frac{D}{\rho} - \frac{1}{\gamma^2} \to \eta = \frac{1}{C}\int_{\mathrm{cir}}\left(\frac{D}{\rho} - \frac{1}{\gamma^2}\right) ds, \tag{18}$$

where η is the phase slip factor. The Hamiltonian turns to be

$$\begin{aligned}\bar{H} &= -(1+\delta) + \frac{1}{2}(-\eta)\delta^2 - \frac{\sin\psi_S hqV}{2\pi\beta^2 E_0}(\phi + \phi_D)\\ &\quad - \frac{hqV}{2\pi\beta^2 E_0}\cos(\phi + \phi_D) + \frac{1}{2}\left(\frac{\bar{p}_x}{p_0}\right)^2 + \frac{1}{2}K_x \bar{x}^2,\end{aligned} \tag{19}$$

where $\phi_D = -\frac{D}{R}\left(\frac{\bar{p}_x}{p_0}\right) + \frac{D'}{R}\bar{x}$, which is very small.

Adding to $\bar{H}$ the following arbitrary term for convenience: $(\psi_S \sin\psi_S + \cos\psi_S)\frac{hqV}{2\pi\beta^2 E_0}$,

Putting $\phi_S = \psi_S - \phi_D$, we have the Hamiltonian for an orbiting particle

$$\begin{aligned}\bar{H} &= -(1+\delta) + \frac{1}{2}(-\eta)\delta^2 - \frac{hqV}{2\pi\beta^2 E_0}\{\cos(\phi + \phi_D)\\ &\quad - \cos(\phi_s + \phi_D) + (\phi - \phi_s)\sin(\phi_s + \phi_D)\}\\ &\quad + \frac{1}{2}\left(\frac{\bar{p}_x}{p_0}\right)^2 + \frac{1}{2}K_x \bar{x}^2.\end{aligned} \tag{20}$$

The Hamiltonian $\bar{H}$ is composed of coasting, synchrotron and betatron motions.

B. The fractional deviation divided into coasting and synchrotron motions

The (rationalized) fractional deviation δ consists of two components. The fractional deviation caused by the coasting motion δ_C (DC component) and that caused by the synchrotron motion δ_S (oscillating component): $\delta = \delta_C + \delta_S$. We will show that most of synchronous particles oscillate in sinusoidal manner (the synchrotron oscillation) since no oscillating particle leaves the storage ring immediately. From Eq. (20),

$$\begin{aligned}\bar{H} &= -(1 + \delta_C + \delta_S) + \frac{1}{2}\left(\frac{\bar{p}_x}{p_0}\right)^2 + \frac{1}{2}K_x \bar{x}^2\\ &\quad + \frac{1}{2}(-\eta)(\delta_C + \delta_S)^2 - \frac{hqV}{2\pi\beta^2 E_0}\{\cos(\phi + \phi_D)\\ &\quad - \cos(\phi_s + \phi_D) + (\phi - \phi_s)\sin(\phi_s + \phi_D)\}.\end{aligned} \tag{21}$$

We obtain Hamilton's equations of motion for (ϕ, δ_S) from $\bar{H}$

$$\frac{d\phi}{d\theta} = \frac{\partial \bar{H}}{\partial \delta_S} = -1 + (-\eta)(\delta_C + \delta_S). \tag{22}$$

Putting $\phi \to \phi_S$, we can differentiate the following equation as

$$\frac{d\delta_S}{d\theta} = -\frac{\partial \bar{H}}{\partial \phi} = -\frac{hqV}{2\pi\beta^2 E_0}\{\sin(\phi+\phi_D) - \sin(\phi_S+\phi_D)\}$$
$$\approx -\frac{hqV\cos(\phi_s+\phi_D)}{2\pi\beta^2 E_0}(\phi-\phi_S). \tag{23}$$

From Eqs. (22) and (23),

$$\frac{d^2\delta_S}{d\theta^2} = -\frac{hqV\cos(\phi_S+\phi_D)}{2\pi\beta^2 E_0}\frac{d\phi}{d\theta}$$
$$= -\frac{hqV\cos(\phi_S+\phi_D)}{2\pi\beta^2 E_0}\{-1+(-\eta)(\delta_C+\delta_S)\}. \tag{24}$$

Then

$$\frac{d^2}{d\theta^2}(\delta_S-\delta_0) = -\nu_s^2(\delta_S-\delta_0). \tag{25}$$

We obtain the following equations

$$\delta_S - \delta_0 = \hat{\delta}\cos\{\nu_s(\theta-\theta_0)\}. \tag{26}$$

$$\nu_s^2 = \frac{\omega_s^2}{\omega_0^2} = \frac{hqV|\eta\cos(\phi_S+\phi_D)|}{2\pi\beta^2 E_0}. \tag{27}$$

where $\hat{\delta}$ is the amplitude of synchrotron oscillation, $\delta_0 = -\delta_C - \frac{1}{\eta}$ is a properly decided initial condition at s_0, θ_0 is the initial orbit angle at s_0, ν_s is the synchrotron tune and ω_s is the synchrotron frequency.

C. The betatron tune jump proportional to a change of the fractional deviation

We also obtain Hamilton's equations of motion for $(\bar{x}, \frac{\bar{p}_x}{p_0})$ from $\bar{H}$

$$\frac{d\bar{x}}{ds} = \frac{d\bar{H}}{d(\frac{\bar{p}_x}{p_0})} = \left(\frac{\bar{p}_x}{p_0}\right), \tag{28}$$

$$\frac{d(\frac{\bar{p}_x}{p_0})}{ds} = -\frac{d\bar{H}}{d\bar{x}} = -K_x\bar{x}. \tag{29}$$

We have

$$\frac{d^2\bar{x}}{ds^2} + K_x\bar{x} = 0. \tag{30}$$

This is the betatron oscillation around the off-momentum closed orbit. We neglect $\bar{p}_x$ and $\bar{x}$ dependence in ϕ_D. We have the following relation [11]

$$\Delta\nu_x = \frac{1}{4\pi}\int \beta_x \Delta K_x ds = \left(-\frac{1}{4\pi}\int \beta_x K_x ds\right)\Delta\delta_C, \tag{31}$$

where β_x is the horizontal component of betatron function. The betatron tune jump (shift) $\Delta\nu_x$ is proportional to $\Delta\delta_C$, which is the amount of change of the (DC component) fractional deviation.

D. Resonance between synchrotron and betatron motions

In the standard theory of the off- momentum betatron oscillation [12], the horizontal coordinate around the off-momentum closed orbit $\bar{x}$ is defined as $x = \bar{x} + D\delta_C$.

We rewrite it as $x = x' + D\delta_C$. Now x' is the horizontal coordinate around the off-momentum closed orbit in the standard theory, and $\bar{x}$ is our horizontal coordinate around the off-momentum closed orbit. Then $x = \bar{x} + D(\delta_C+\delta_S) = x' + D\delta_C$. We have

$$x' = \bar{x} + D\,\delta_S. \tag{32}$$

Now $\bar{x}$ represents x' plus the synchrotron oscillation effect. Substituting Eq. (32) into (30), we have

$$\frac{d^2x'}{ds^2} + K_x x' = \frac{\delta_S}{\rho}. \tag{33}$$

Putting $\theta_0 = 0$ and $\delta_0 = 0$, we substitute Eq. (26) into Eq. (33),

$$\frac{d^2x'}{ds^2} + K_x x' = \frac{\hat{\delta}}{\rho}\cos(\nu_s\theta). \tag{34}$$

We perform Floquet transformation to Eq. (34). The closed orbit displacement x'_{co} is given as follows [13]

$$x'_{\mathrm{co}}(s) = \frac{\nu_\beta\sqrt{\beta_x}}{2\sin(\pi\nu_\beta)}\int_\theta^{\theta+2\pi}\beta_x^{\frac{3}{2}}\frac{\hat{\delta}}{\rho}\cos\{\nu_s(\varphi-\varphi_0)\}$$
$$\times\cos\{\nu_\beta(\pi+\theta-\varphi)\}d\varphi. \tag{35}$$

After integration

$$x'_{\mathrm{co}}(s) = \frac{\sqrt{\beta_x}}{2}\sum_{\ell=-\infty}^{\infty} f(\ell)e^{i\ell\phi}\left\{\frac{\nu_\beta(\nu_\beta+\nu_s)}{(\nu_\beta+\nu_s)^2-\ell^2}\right.$$
$$\left.+\frac{\nu_\beta(\nu_\beta-\nu_s)}{(\nu_\beta-\nu_s)^2-\ell^2}\right\}, \tag{36}$$

where $f(\ell) = \frac{1}{2\pi\rho}\oint \beta_x^{\frac{3}{2}}\hat{\delta}e^{-i\ell\theta}d\theta$ and $\ell =$ integers.

We find the synchrobetatron resonant coupling at $\ell = \nu_\beta \pm \nu_s$ in Eq. (36) ($\ell =$ integers). There are sum integer resonance condition $\nu_\beta + \nu_s$ and difference integer resonance condition $\nu_\beta - \nu_s$.

IV. DISCUSSION

Near the resonant point of the synchrobetatron resonant coupling, unexpected betatron tune jump was observed. We propose a physical explanation on how it occurred.

In this coupling mechanism, no apparent resonant coupling term exists in the Hamiltonian $\bar{H}$. The synchrotron oscillation in the longitudinal direction, however, induces an oscillation in the horizontal direction [See Eq. (33)], which resonate with the off-momentum betatron oscillation.

Let us discuss the tune jump with $\bar{H}$ as a constant of motion

$$\bar{H} = \bar{H}_C + \bar{H}_\beta + \bar{H}_S, \tag{37}$$

where $\bar{H}_C = -(1 + \delta_C + \delta_S)$, $\bar{H}_\beta = \frac{1}{2}(\frac{\bar{p}_x}{p_0})^2 + \frac{1}{2}K_x\bar{x}^2$ and

$$\bar{H}_S = \frac{1}{2}(-\eta)(\delta_C + \delta_S)^2 - \frac{qV}{2\pi h\beta^2 E_0}\{\cos(\phi + \phi_D) - \cos(\phi_s + \phi_D) + (\phi - \phi_s)\sin(\phi_s + \phi_D)\}.$$

$\bar{H}_C$ corresponds to the energy of the off-momentum coasting particle. $\bar{H}_\beta$ corresponds to the energy of the off-momentum betatron oscillation where $\bar{x} = x - D\delta$ and $\frac{\bar{p}_x}{p_0} = \frac{p_x}{p_0} - D'\delta \approx \frac{p_x}{p_0}$ (D' has both $+$ and $-$ components and totally their contribution is very small). $\bar{H}_S$ corresponds to the energy of synchrotron oscillation. As Eq. (31) shows, δ_C is changed near the resonant point. Consider the case δ_C is a small enough positive value. As δ_C decreases, both $\bar{H}_C$ and $\bar{H}_\beta$ increase. Then $\bar{H}_S$ has to decrease. As δ_C increases, $\bar{H}_C$ and $\bar{H}_\beta$ decrease and $\bar{H}_S$ increases.

The orbital momentum $(-p_s)$ of clockwise direction decreases as δ_C decreases and the particle deflects inside from the orbit (the momentum increases as δ_C increases and the particle deflects outside from the orbit). Deflecting insides is equivalent to prolongation of the amplitude of the betatron oscillation and shortening of that of the synchrotron oscillation, which leads to an increase of the betatron oscillation energy and a decrease of the synchrotron oscillation energy when the synchrobetatron resonant coupling condition is satisfied and vice versa.

Some amount of synchrotron oscillation energy is exchanged with that of betatron oscillation energy near the resonant point where a coasting particle mediates their energy exchange. $\Delta\delta_C$ stands for the strength of energy exchange and brings about the observed betatron tune jump near the resonant point. There the freedom of the horizontal direction is connected with the freedom of the longitudinal direction when $\Delta\delta_C \neq 0$. In other words, an amount of $\Delta\delta_C$ unites these two freedoms.

Generally two freedoms are connected via collisions (for an example, particles in magnetically confined plasma). In a S-LSR experiment, collisions among particles are neglected. The synchrobetatron resonant coupling takes the role of collisions near the resonant point.

If the longitudinal (synchrotron) component of a beam is cooled, the transverse (betatron) component of the beam is also cooled since the longitudinal and the transverse freedoms are connected near the resonant point. This result is what we observed in our cooling experiment [3].

In the future we would like to find the way to control the strength of energy exchange so that we can cool the transverse direction of the beam more efficiently in the experiment.

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