Statistics of work done in a degenerate parametric amplification process

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We study statistics of work done by two classical electric-field pumps (two-photon and one-photon resonant pumps) on a quantum optical oscillator. We compute the moment generating function for the energy change of the oscillator, interpreted as work done by the classical drives on the quantum oscillator starting out in a thermalized Boltzmann state. The moment generating function is inverted, analytically when only one of the pumps is turned on and numerically when both the pumps are turned on, to get the probability function for the work. The resulting probability function for the work done by the classical drive is shown to satisfy transient detailed and integral work fluctuation theorems. Interestingly, we find that, in order for the work distribution function to satisfy the fluctuation theorem in the presence of both the drivings, the relative phases of drivings need to be shifted by π, which is related to the broken time-reversal symmetry of the Hamiltonian.

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I. INTRODUCTION

Work done by external forces on isolated mesoscopic systems, unlike their macroscopic counterparts [1], is subject to fluctuations [2–4] due to the smallness of the system size. These fluctuations could be due to the uncertainty of the initial state and due to the quantum nature of evolution and measurement process. Despite the noisy nature of the work done by the external force on the nanoscale system, these fluctuations exhibit a symmetry property which links the frequency of a certain amount of work done by the external force on the system to the frequency of the same amount of work extracted by the external drive. Further, these symmetry properties of the probability function for work done by the external force on the system, termed as work-fluctuation theorems, comprise one of the first classes of fluctuation theorems discovered for out-of-equilibrium systems [5,6]. This fluctuation relation states that, for a driven system, the probability that an external force extracts a certain amount of work from the system is finite but exponentially suppressed compared to the probability that exactly the same amount of work is performed on the system. In this sense fluctuation theorems promote the inequality of the second law of thermodynamics (for the dissipated work) to an equality [7]. These results are universal in the sense that only ingredients that are sufficient to establish these relationships are the equilibrium canonical nature of the initial state and the microscopic reversibility of the underlying dynamics, which are insensitive to the nature of microscopic details of the system [2–4]. Nevertheless, the probability function for work done by external force on the system is not universal and depends on the microscopic details of the system. The work distribution function has been computed for a variety of situations for both classical [4,8] and quantum systems [9–24]. Experimental measurement of work distribution and subsequent demonstration of Jarzynski-Crooks fluctuation theorems for classical systems are well established [4,8,25–30]. Because for quantum systems there is no work operator, it was initially confusing to define work in the quantum case [3,31–33]. Subsequently, a two-point measurement protocol was proposed [3,32–35] to define work in a single realization. This was crucial for proving quantum versions of fluctuation theorems [2,3,32,33]. It has been challenging to implement two-point measurement protocol experimentally. However, there have been some recent theoretical proposals of experiments [36–40] and implementations [41–44] of work measurements in quantum systems either directly through two-point measurement protocol or indirectly.

In this paper we consider the two-point measurement protocol to study work statistics in generating displaced squeezed thermal states of a quantum optical oscillator by driving the optical oscillator starting in the Gibbs state by two classical pumps resonant with two-photon and one-photon transition. The scenario where only a two-photon resonant pump is on corresponds to the standard degenerate parametric amplification process. This paper is motivated by recent proposals [45–54] of using squeezed thermal reservoirs in quantum heat engines to surpass standard Carnot efficiency. Squeezed thermal states of light [55–57] can be realized using the well-established parametric amplification process [58–70]. We assume that the oscillator is isolated from the environment during the driving process. We thus interpret the change in the energy of the oscillator as work performed by the classical drives. We compute the work distribution function for this process. The work distribution function is shown to satisfy the quantum version of the Jarzynski-Crooks fluctuation theorem [5,32–35,71–73]. We note that work statistics has been studied in generating the squeezed thermal state of a harmonic oscillator in Refs. [10,11], where classical driving is modeled through temporal modulation of harmonic oscillator frequency and analytic results for work statistics were obtained approximately under limiting conditions. In Ref. [9], work statistics has been studied in generating a displaced thermal state of a harmonic oscillator. In this paper we consider a general process where a quantum optical oscillator is driven by two classical pumps, one resonant with the two-photon transition and the other resonant with the one-photon transition. We obtain an exact result for the moment generating function for work and show that interference between the two drivings affects the work statistics in a nontrivial way and plays an important role in satisfying the fluctuation theorem. In the limit when only one of the pumps is on, we analytically invert the moment generating functions to get the work distribution function. Interestingly, when both the pumps are on, the most probable work may shift to negative values with time for small relative phase difference between the drives. Further, the probability distributions of work for time-forward and time-reversed processes are different. This is due to the broken time-reversal symmetry of the Hamiltonian. We note that time-reversal symmetry-breaking Hamiltonians do not always guarantee that the probabilities of stochastic quantities for time-forward and time-reversed processes are different. For example, in model systems like two-terminal noninteracting electronic Aharnov-Bohm ring junctions [74,75], although the Hamiltonians break the time-reversal symmetry, the statistics of charge and heat flux are the same for time-forward and time-reversed processes. Moreover, results presented here can be tested experimentally using classical optical simulation of quantum dynamics as proposed in Ref. [40] and experimentally implemented in Ref. [43].

In the next section, we describe the model system and obtain the generating function for work done within the two-point measurement scheme and verify fluctuation theorems for work. The generating function is inverted to compute the probability distribution function in Sec. III and cumulants of work are analyzed. We conclude in Sec. IV.

II. GENERATING FUNCTION FOR THE WORK

We consider a quantum optical process of driving an isolated optical oscillator by two classical pumps [76]. We assume that one of the classical pumps has the same frequency as that of the optical oscillator ($\frac{\epsilon}{\hbar}$) and the second pump has twice the frequency of the optical oscillator. We treat the coupling between classical pumps and quantum optical oscillator within the rotating wave approximation. We assume that the initial state of the optical oscillator is the thermal state (i.e., the oscillator is kept in contact with the thermal reservoir till time zero). Note that the driving is nonadiabatic with respect to the natural frequency of the oscillator. The Hamiltonian describing the evolution of the quantum optical oscillator driven by classical pumps (written in the interaction picture) [76–78] is

$$\hat{H} = i\hbar[z_1 b^\dagger - z_1^* b] + \frac{i\hbar}{2}[z_2 b^\dagger b^\dagger - z_2^* bb], \tag{1}$$

where z_1 and z_2 are the products of the coupling constant and electric-field amplitude of the one-photon and two-photon resonant classical pumps, respectively, and b and $b^\dagger$ are the annihilation and creation operators for the quantum optical oscillator, respectively. A schematic of this model system is given in Fig. 1.

We interpret the energy change of the quantum oscillator as the work done by the classical pumps on it [32,33,79,80].

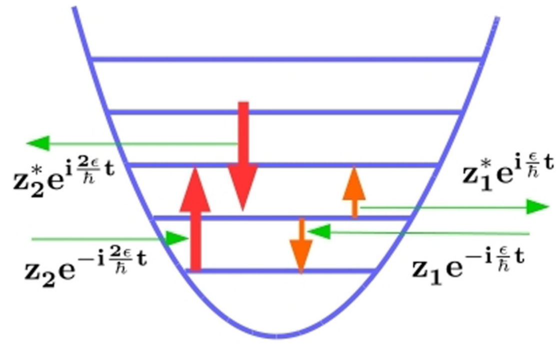

FIG. 1. Schematic of the model system. (Downward) upward thick red and thin orange arrows indicate the two-photon and one-photon (de-)excitation of a quantum optical oscillator by classical drives with amplitudes z_2 and z_1, respectively.

With these assumptions, the work done by the classical drives is proportional to the number of quanta ($n \in \mathbb{Z}$) of energy exchanged with the quantum oscillator. The probability function for number of quanta of work done by the classical drives on the quantum optical oscillator in time t is obtained within the two-point measurement protocol [2,3,34,35,73], consisting of the following process.

(i) Initially, at time zero, after disconnecting the oscillator from its thermal reservoir, a measurement of the number of excitations in the oscillator is performed in the sense of von Neumann's strong projective measurement, obtaining the result $n_i \in \mathbb{W}$ with probability $\langle n_i|\rho(0)|n_i\rangle$; $\rho(0) = \frac{e^{-\beta\epsilon b^\dagger b}}{\mathrm{Tr}[e^{-\beta\epsilon b^\dagger b}]}$ is the initial density matrix of the optical oscillator, which is assumed to be at the thermal state. The state of the system right after measurement is described by the eigenket $|n_i\rangle$ (here $b^\dagger b|n_i\rangle = n_i|n_i\rangle$).

(ii) Subsequently the oscillator is driven by the classical drives for time t. The state of the oscillator at time t is given (in the interaction picture) as $e^{-\frac{i}{\hbar}\hat{H}t}|n_i\rangle$.

(iii) At time t, the external drives are turned off and a second projective measurement of the number of excitations in the oscillator is performed, obtaining the result $n_f \in \mathbb{W}$ with probability $|\langle n_f|e^{-\frac{i}{\hbar}\hat{H}t}|n_i\rangle|^2$ (here $b^\dagger b|n_f\rangle = n_f|n_f\rangle$).

(iv) This process is repeated *ad infinitum* and the probability of the number of quanta of work done by the classical drives on the quantum optical oscillator during time t is constructed using the following self-evident expression:

$$P[n;t] = \sum_{n_i,n_f=0}^{\infty} \delta_{n,n_f-n_i} \left|\langle n_f|e^{-\frac{i}{\hbar}\hat{H}t}|n_i\rangle\right|^2 \langle n_i|\rho(0)|n_i\rangle. \tag{2}$$

Here $\delta_{m,n}$ is the Kronecker delta function. This probability distribution function can be expressed in terms of the moment generating function for the work done ($\mathcal{Z}[\chi;t]$) as

$$P[n;t] = \int_0^{2\pi} \frac{d\chi}{2\pi} \mathcal{Z}[\chi;t] e^{-i\chi n}, \tag{3}$$

where $\mathcal{Z}[\chi;t]$ is given as [2,3]

$$\mathcal{Z}[\chi;t] = \mathrm{Tr}[\mathcal{U}_\chi(t,0)\rho(0)\mathcal{U}_{-\chi}(0,t)]. \tag{4}$$

Here, $\mathcal{U}_\chi(t_1,t_2)=e^{-\frac{i}{\hbar}\hat{H}_\chi(t_1-t_2)}$ is the twisted evolution operator in the interaction picture with the counting field dependent Hamiltonian given as

$$\hat{H}_\chi = i\hbar\big[z_1 e^{i\frac{\chi}{2}}b^\dagger - z_1^* e^{-i\frac{\chi}{2}}b\big]+\frac{i\hbar}{2}[z_2 e^{i\chi}b^\dagger b^\dagger - z_2^* e^{-i\chi}bb]. \tag{5}$$

In order to compute $\mathcal{Z}[\chi;t]$, it is convenient to work with the Weyl generating function, $\mathcal{G}[\zeta,\zeta^*;t]$, defined (in the interaction picture) as [76,81,82]

$$\mathcal{G}_\chi[\zeta,\zeta^*;t]=\mathrm{Tr}\big[e^{i[\zeta^* b+\zeta b^\dagger]}\mathcal{U}_\chi(t,0)\rho(0)\mathcal{U}_{-\chi}(0,t)\big], \tag{6}$$

and then $\mathcal{Z}[\chi;t]=\mathcal{G}_\chi[0,0;t]$. Using standard techniques from quantum optics literature [76,81,82], it can be shown that [using Eqs. (4) and (5)] $\mathcal{G}_\chi[\zeta,\zeta^*;t]$ satisfies the following evolution equation:

$$\frac{\partial}{\partial t}\mathcal{G}_\chi[\zeta,\zeta^*;t]=\left[\frac{1}{2}\begin{pmatrix}\zeta^*\\ \zeta\\ \frac{\partial}{\partial\zeta^*}\\ \frac{\partial}{\partial\zeta}\end{pmatrix}^T\begin{pmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{B}^T & \mathcal{C}\end{pmatrix}\begin{pmatrix}\zeta^*\\ \zeta\\ \frac{\partial}{\partial\zeta^*}\\ \frac{\partial}{\partial\zeta}\end{pmatrix}+\begin{pmatrix}d_1\\ d_2\end{pmatrix}^T\begin{pmatrix}\zeta^*\\ \zeta\\ \frac{\partial}{\partial\zeta^*}\\ \frac{\partial}{\partial\zeta}\end{pmatrix}\right]\mathcal{G}_\chi[\zeta,\zeta^*;t], \tag{7}$$

where V^T represents transpose of a vector V and

$$\mathcal{A}=\begin{pmatrix}-i\frac{z_2}{2}\sin(\chi) & 0\\ 0 & -i\frac{z_2^*}{2}\sin(\chi)\end{pmatrix},\quad \mathcal{B}=\begin{pmatrix}0 & z_2\cos(\chi)\\ z_2^*\cos(\chi) & 0\end{pmatrix},\quad \mathcal{C}=\begin{pmatrix}-2iz_2^*\sin(\chi) & 0\\ 0 & -2iz_2\sin(\chi)\end{pmatrix}, \tag{8}$$

$$d_1=\left(iz_1\cos\left(\tfrac{\chi}{2}\right)\quad iz_1^*\cos\left(\tfrac{\chi}{2}\right)\right)^T,\quad d_2=\left(2z_1^*\sin\left(\tfrac{\chi}{2}\right)\quad 2z_1\sin\left(\tfrac{\chi}{2}\right)\right)^T. \tag{9}$$

The above parabolic partial differential equation, Eq. (7), has to be solved along with the initial condition $\mathcal{G}[\zeta,\zeta^*;t]|_{t=0}=\mathrm{Tr}[e^{i[\zeta^* b+\zeta b^\dagger]}\rho(0)]=e^{-\frac{1}{2}\binom{\zeta^*}{\zeta}^T D\binom{\zeta^*}{\zeta}}$ with 2×2 matrix D defined as $D_{ij}=(1-\delta_{ij})\big(f+\frac{1}{2}\big)$, where $f=(e^{\beta\epsilon}-1)^{-1}$. The solution of Eq. (7) is obtained as (see the Appendix)

$$\mathcal{G}_\chi[\zeta,\zeta^*;t]=\int_{\bar{\zeta}\in\mathbb{C}}d^2\bar{\zeta}\,\mathbf{G}_\chi[\zeta,\zeta^*;t|\bar{\zeta},\bar{\zeta}^*;0]\mathcal{G}[\bar{\zeta},\bar{\zeta}^*;0], \tag{10}$$

with

$$\begin{aligned}&\mathbf{G}_\chi[\zeta,\zeta^*;t|\bar{\zeta},\bar{\zeta}^*;0]\\&=\frac{e^{\int_0^t dt_1[\mathcal{U}_{21}(t_1)^T d_1-\mathcal{U}_{11}(t_1)^T d_2]^T\int_{t_1}^t dt_2[\mathcal{U}_{22}(t_2)^T d_1-\mathcal{U}_{12}(t_2)^T d_2]}}{\pi\sqrt{-\det[\mathcal{U}_{21}(t)]}}e^{-\frac{1}{2}\binom{\zeta^*}{\zeta}^T\mathcal{U}_{12}(t)\mathcal{U}_{22}(t)^{-1}\binom{\zeta^*}{\zeta}}e^{\binom{\bar{\zeta}^*}{\bar{\zeta}}^T\int_0^t dt_1[\mathcal{U}_{22}(t_1)^T d_1-\mathcal{U}_{12}(t_1)^T d_2]}\\&\quad\times e^{-\frac{1}{2}\left\{\binom{\zeta^*}{\zeta}-\mathcal{U}_{22}(t)\left[\binom{\bar{\zeta}^*}{\bar{\zeta}}+\int_0^t dt_1[\mathcal{U}_{21}(t_1)^T d_1-\mathcal{U}_{11}(t_1)^T d_2]\right]\right\}^T[\mathcal{U}_{21}(t)\mathcal{U}_{22}(t)^T]^{-1}\left\{\binom{\zeta^*}{\zeta}-\mathcal{U}_{22}(t)\left[\binom{\bar{\zeta}^*}{\bar{\zeta}}+\int_0^t dt_1[\mathcal{U}_{21}(t_1)^T d_1-\mathcal{U}_{11}(t_1)^T d_2]\right]\right\}},\end{aligned} \tag{11}$$

where $\begin{pmatrix}\mathcal{U}_{11}(t) & \mathcal{U}_{12}(t)\\ \mathcal{U}_{21}(t) & \mathcal{U}_{22}(t)\end{pmatrix}=e^{\begin{pmatrix}\mathcal{B} & -\mathcal{A}\\ \mathcal{C} & -\mathcal{B}^T\end{pmatrix}t}$. Using Eq. (10) along with Eq. (11) in $\mathcal{Z}[\chi;t]=\mathcal{G}_\chi[0,0;t]$ and performing $\bar{\zeta}$ integrals, we get

$$\begin{aligned}\mathcal{Z}[\chi;t]=&\frac{1}{\sqrt{\det[\mathcal{U}_{22}(t)+\mathcal{U}_{21}(t)D]}}\\&\times e^{\int_0^t dt_1\int_{t_1}^t dt_2[\mathcal{U}_{21}(t_1)^T d_1-\mathcal{U}_{11}(t_1)^T d_2]^T D[\mathcal{U}_{21}(t)D+\mathcal{U}_{22}(t)]^{-1}\mathcal{U}_{21}(t)[\mathcal{U}_{22}(t_2)^T d_1-\mathcal{U}_{12}(t_2)^T d_2]}\\&\times e^{-\int_0^t dt_1\int_0^{t_1} dt_2[\mathcal{U}_{21}(t_1)^T d_1-\mathcal{U}_{11}(t_1)^T d_2]^T\left[\mathbb{I}_{2\times 2}-D[\mathcal{U}_{21}(t)D+\mathcal{U}_{22}(t)]^{-1}\mathcal{U}_{21}(t)\right][\mathcal{U}_{22}(t_2)^T d_1-\mathcal{U}_{12}(t_2)^T d_2]}\\&\times e^{\frac{1}{2}\int_0^t dt_1\int_0^t dt_2[\mathcal{U}_{22}(t_1)^T d_1-\mathcal{U}_{12}(t_1)^T d_2]^T[\mathcal{U}_{21}(t)D+\mathcal{U}_{22}(t)]^{-1}\mathcal{U}_{21}(t)[\mathcal{U}_{22}(t_2)^T d_1-\mathcal{U}_{12}(t_2)^T d_2]}\\&\times e^{-\frac{1}{2}\int_0^t dt_1\int_0^t dt_2[\mathcal{U}_{21}(t_1)^T d_1-\mathcal{U}_{11}(t_1)^T d_2]^T D[\mathcal{U}_{21}(t)D+\mathcal{U}_{22}(t)]^{-1}\mathcal{U}_{22}(t)[\mathcal{U}_{21}(t_2)^T d_1-\mathcal{U}_{11}(t_2)^T d_2]}.\end{aligned} \tag{12}$$

The final expression for the moment generating function for work can be obtained by using explicit expressions for $\mathcal{U}_{xy}(t)$. This gives

$$\mathcal{Z}[\chi,\phi;t] = \frac{e^{\frac{|z_1|^2\left(\frac{\sinh(|\frac{z_2}{2}|t)}{|\frac{z_2}{2}|}\right)^2\left[(1+f)\left(e^{i\chi}-1\right)+f\left(e^{-i\chi}-1\right)\right]\left\{\cosh(|z_2|t)+\cos(\phi)\sinh(|z_2|t)\left[(1+f)e^{i\chi}-fe^{-i\chi}\right]\right\}}{1-\sinh^2(|z_2|t)\left[(1+f)^2\left(e^{i2\chi}-1\right)+f^2\left(e^{-i2\chi}-1\right)\right]}}}{\sqrt{1-\sinh^2(|z_2|t)[(1+f)^2(e^{i2\chi}-1)+f^2(e^{-i2\chi}-1)]}}, \tag{13}$$

where $\phi = 2\text{Arg}(z_1) - \text{Arg}(z_2)$.

It is clear that

$$\mathcal{Z}[\chi,\phi;t] \neq \mathcal{Z}[-\chi + i\beta\epsilon,\phi;t], \tag{14}$$

which is related to the broken time-reversal symmetry of the Hamiltonian [2,3], i.e, $\hat{H} \neq \mathcal{T}\hat{H}\mathcal{T}^{-1}$ ($\mathcal{T}$ is the time-reversal operator). To recover the Jarzynski-Crooks-Bochkov-Kuzovlev fluctuation theorem [2,3,83,84], work distributions for time-forward and time-backward trajectories need to be compared. The backward evolution is governed by $\mathcal{T}\hat{H}\mathcal{T}^{-1} = -i\hbar[z_1^* b^\dagger - z_1 b] - \frac{i\hbar}{2}[z_2^* b^\dagger b^\dagger - z_2 bb] \neq \hat{H}$. In order to recover the time-reversal symmetry, we also need to change ϕ to $\pi - \phi$. This leads to the Gallavotti-Cohen symmetry for the work moment generating function:

$$\mathcal{Z}[\chi,\phi;t] = \mathcal{Z}[-\chi + i\beta\epsilon,\pi-\phi;t]. \tag{15}$$

This, in turn, leads to the transient work fluctuation theorem:

$$\frac{P[n,\phi;t]}{P[-n,\pi-\phi;t]} = e^{\beta\epsilon n} \Rightarrow \langle e^{-\beta\epsilon n}\rangle = 1. \tag{16}$$

However, for cases where only one of the pumps is present (i.e., $z_1 = 0$ or $z_2 = 0$), $\mathcal{Z}[\chi,\phi;t]$ becomes independent of ϕ. This is because, for the case when only one of the pumps is present, the phase of the electric field can be gauged out (for the initial thermal state) and does not appear in the expression for $\mathcal{Z}[\chi,\phi;t]$. When both the pumps are present, the phases of both the pumps cannot be gauged out. Hence work statistics is unaffected by the phase of the classical fields for the case $z_1 = 0$ or $z_2 = 0$, which is similar to the result noted in Ref. [9] where the phase of the classical field is shown to have no influence on the work statistics during the process of coherent displacement of a harmonic oscillator from a thermal state. For the cases when only one of the pumps is present, the moment generating function and hence the probability distribution function for the work for time-forward and time-backward processes are the same.

In the next section, we discuss the cases $z_1 = 0$, $z_2 = 0$, and $z_1 \neq 0 \neq z_2$ separately and present work fluctuations and work distribution functions for each case. For the cases $z_1 = 0$ and $z_2 = 0$, we denote $\mathcal{Z}[\chi,\phi;t]$ and $P[n,\phi;t]$ as $\mathcal{Z}[\chi;t]$ and $P[n;t]$, suppressing the ϕ dependence.

III. STATISTICS OF THE WORK DONE

It is convenient to represent $P[n,\phi;t]$, defined in Eqs. (3) and (13) as a contour integral around the unit circle in the complex plane [85], as

$$P[n,\phi;t] = \oint_{|\xi|=1}\frac{d\xi}{2\pi i\xi^{n+1}}\sqrt{\frac{\left(1-\frac{1}{\xi_+(0)^2}\right)(1-\xi_-(0)^2)}{\left(1-\frac{\xi^2}{\xi_+(0)^2}\right)\left(1-\frac{\xi_-(0)^2}{\xi^2}\right)}}\, e^{\alpha(\phi,t)(\xi-1)\frac{\left(1+\frac{\xi_-(\phi)}{\xi}\right)\left(1+\frac{\xi}{\xi_+(\phi)}\right)\left(1+\frac{\xi_-(\phi)\xi_+(\phi)}{\xi}\right)}{(1+\xi_-(\phi))\left(1+\frac{1}{\xi_+(\phi)}\right)(1+\xi_-(\phi)\xi_+(\phi))}\frac{\left(1-\frac{1}{\xi_+(0)^2}\right)(1-\xi_-(0)^2)}{\left(1-\frac{\xi^2}{\xi_+(0)^2}\right)\left(1-\frac{\xi_-(0)^2}{\xi^2}\right)}}, \tag{17}$$

where

$$\alpha(\phi,t) = |z_1|^2\left(\frac{\sinh\left(\left|\frac{z_2}{2}\right|t\right)}{\left|\frac{z_2}{2}\right|}\right)^2[\cosh(|z_2|t)+\cos(\phi)\sinh(|z_2|t)], \tag{18}$$

and

$$\xi_\pm(\phi) = \frac{1 \pm \sqrt{1+4f(1+f)\cos^2(\phi)\tanh^2(|z_2|t)}}{2(1+f)\cos(\phi)\tanh(|z_2|t)}. \tag{19}$$

For the case $z_1 \neq 0 \neq z_2$, we were not able to invert the moment generating function analytically to get the probability function for work. Below we present analytical results for $z_1 = 0$ and $z_2 = 0$ cases and then discuss numerical results for the general case.

A. $z_2 = 0$ case

Taking the $z_2 \to 0$ limit of Eq.(13), the moment generating function of work is

$$\mathcal{Z}[\chi;t] = e^{|z_1|^2t^2[(1+f)(e^{i\chi}-1)+f(e^{-i\chi}-1)]}. \quad (20)$$

This moment generating function corresponds to a bi-Poissonian stochastic process. The above expression for $\mathcal{Z}[\chi;t]$ is a special case (resonant drive) of a more general expression for the work generating function derived in Ref. [9] for the general displacement drive. The cumulant generating function for work, $\ln \mathcal{Z}[\chi;t]$, clearly scales quadratically with time (t) and hence all cumulants, obtained using $(-i)^n(\frac{d}{d\chi})^n \ln \mathcal{Z}[\chi;t]|_{\chi=0}$, scale as t^2. The first two cumulants of work are $\langle n\rangle = |z_1|^2t^2$ and $\langle(n-\langle n\rangle)^2\rangle = |z_1|^2t^2(1+2f)$. The probability function of work can be obtained analytically by converting Fourier inversion to an integral over the unit circle in the complex plane [Eq. (17)] as

$$P[n;t] = \oint_{|\xi|=1} \frac{d\xi}{2\pi i}\frac{1}{\xi^{n+1}} e^{|z_1|^2t^2\left[(1+f)(\xi-1)+f\left(\frac{1}{\xi}-1\right)\right]}. \quad (21)$$

Observing that the function $e^{|z_1|^2t^2[(1+f)(\xi-1)+f(\frac{1}{\xi}-1)]}$ is an analytic function of complex variable ξ in the entire complex plane except at $\xi = 0, \infty$ where it has essential singularity (hence it is analytic in the center punctured unit disk), the above integral gives Laurent series (around $\xi = 0$) coefficients of $e^{|z_1|^2t^2[(1+f)(\xi-1)+f(\frac{1}{\xi}-1)]}$. This gives

$$P[n;t] = \left(\frac{1+f}{f}\right)^{\frac{n}{2}} \frac{|z_1|^{2|n|}t^{2|n|}[f(1+f)]^{|n|/2}}{\Gamma[1+|n|]} e^{-|z_1|^2t^2(1+2f)}$$

$$\mathbb{F}_1^0[1+|n|; f(1+f)|z_1|^4t^4], \quad (22)$$

where the generalized hypergeometric function of variable x is defined as $\mathbb{F}_n^m[a_1,\ldots,a_m,b_1,\ldots,b_n;x] = \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_m)_k}{(b_1)_k\cdots(b_n)_k}\frac{x^k}{\Gamma[k+1]}$ (here the Pocchammer symbol is $(c)_r = \frac{\Gamma[c+r]}{\Gamma[c]}$) [85–87]. It is important to note that the probability function for work takes the same form for both the forward and backward processes. Using this explicit expression for $P[n;t]$, the detailed work fluctuation theorem and hence the integral fluctuation theorem can be verified.

The probability that no work is performed on the system ($P[0;t]$) is given as

$$P[0;t] = e^{-|z_1|^2t^2(1+2f)} I_0[2\sqrt{f(1+f)}|z_1|^2t^2] \quad (23)$$

where $I_0[z]$ is the modified Bessel function of the first kind of order zero of variable z. In the zero-temperature limit, $\beta\epsilon \to \infty \Rightarrow f \to 0$, $P[0;t] = e^{-|z_1|^2t^2}$, which means, if the oscillator's initial state is the ground state, the number of microscopic realizations where no net work is performed by the displacement or linear drive on the oscillator decays with time as a Gaussian function. Further, in the zero-temperature limit

$$P[n;t] = \frac{|z_1|^{2n}t^{2n}}{\Gamma[1+n]} e^{-|z_1|^2t^2}\Theta[n], \quad (24)$$

where $\Theta[n] = 1$ iff $n \geqslant 0$ else $\Theta[n] = 0$. This means, if the oscillator's initial state is the ground state, there are no microscopic realizations where work is extracted by the classical drive from the oscillator. Further, the term t^{2n} competes with the exponential decay ($e^{-|z_1|^2t^2}$), shifting the value of most probable work (n) to higher values for larger times.

B. $z_1 = 0$ case

For the $z_1 = 0$ case, the moment generating function for work can be obtained by taking the $z_1 \to 0$ limit of Eq. (13). This gives

$$\mathcal{Z}[\chi;t] = \frac{1}{\sqrt{1-\sinh^2(|z_2|t)[(1+f)^2(e^{i2\chi}-1)+f^2(e^{-i2\chi}-1)]}}. \quad (25)$$

This leads to an average work given by $\langle n\rangle = (1+2f)\sinh^2(|z_2|t) \geqslant 0$. Thus, on average, work is done on the quantum oscillator and grows exponentially in time. This is to be contrasted with the quadratic time dependence for the $z_2 = 0$ case discussed above. The second cumulant of work distribution is $\langle(n-\langle n\rangle)^2\rangle = [1+(1+2f)^2\cosh(2|z_2|t)]\sinh^2(|z_2|t) \geqslant 0$. This indicates that the distribution function becomes broader exponentially with time. Similar to the previous section, the probability function for the work is expressed as the complex contour integral [Eq. (17)] as

$$P[n;t] = \oint_{|\xi|=1}\frac{d\xi}{2\pi i}\frac{1}{\xi^{n+1}}\sqrt{\frac{\left(1-\frac{1}{\xi_+(0)^2}\right)(1-\xi_-(0)^2)}{\left(1-\frac{\xi^2}{\xi_+(0)^2}\right)\left(1-\frac{\xi_-(0)^2}{\xi^2}\right)}}. \quad (26)$$

Noticing $|\xi_-(0)|<1<|\xi_+(0)|$ (for $0<|z_2|t<\infty$ and $0<f<\infty$), we observe that the function with square root in the above integrand is a multivalued complex function with four branch points at $\pm\xi_-(0)$ and $\pm\xi_+(0)$. A single valued branch can be chosen for this function by defining the branch cut as the union of two straight lines joining $-\xi_-(0)$ with $+\xi_-(0)$ and $-\xi_+(0)$ with $+\xi_+(0)$ (through ∞). With this choice, we get a single valued function which is analytic in the strip $|\xi_-(0)| < |\xi| < |\xi_+(0)|$. Hence, the above complex integral just gives the Laurent expansion coefficients of $\sqrt{\frac{\left(1-\frac{1}{\xi_+(0)^2}\right)(1-\xi_-(0)^2)}{\left(1-\frac{\xi^2}{\xi_+(0)^2}\right)\left(1-\frac{\xi_-(0)^2}{\xi^2}\right)}}$ expanded around $\xi = 0$. The final expression for the distribution is obtained as

$$P[2n;t] = \left(\frac{1+f}{f}\right)^n \sqrt{\frac{\left(1-\frac{1}{\xi_+(0)^2}\right)(1-\xi_-(0)^2)}{\pi}}\left(-\frac{\xi_-(0)}{\xi_+(0)}\right)^{|n|}\frac{\Gamma\left[\frac{1}{2}+|n|\right]}{\Gamma[1+|n|]}\mathbb{F}_1^2\left[\frac{1}{2},\frac{1}{2}+|n|,1+|n|;\left(\frac{\xi_-(0)}{\xi_+(0)}\right)^2\right], \quad (27)$$

and

$$P[2n+1;t] = 0. \tag{28}$$

It is important to observe here that there are no microscopic realizations where the classical drive does odd quanta of work on the optical oscillator. Further, similar to the $z_2 = 0$ case, the probability function for work takes the same form for both the forward and backward processes. The above explicit expression for $P[n;t]$ satisfies the detailed work fluctuation theorem and hence the integral fluctuation theorem.

We find that the work distribution is always maximum at $n = 0$. [10,13]. The probability that no work is performed on the optical oscillator is given by

$$P[0;t] = \frac{2}{\pi}\sqrt{\left(1-\frac{1}{\xi_+(0)^2}\right)(1-\xi_-(0)^2)}K\left[\left(\frac{\xi_-(0)}{\xi_+(0)}\right)^2\right], \tag{29}$$

where $K[x]$ is the complete elliptic integral of the first kind [85,86]. For the zero-temperature case ($\beta\epsilon \to \infty \Rightarrow f \to 0$), $\xi_-(0) = 0$ and $\xi_+(0) = \coth(|z_2|t)$, which makes, $P[0;t] = \text{sech}(|z_2|t)$, indicating that, for long time, the number of microscopic realizations where no work is done on the oscillator is exponentially suppressed. Further, in the limit of zero temperature,

$$P[2n;t] = \frac{1}{\sqrt{\pi}}\frac{\Gamma\left[\frac{1}{2}+n\right]}{\Gamma[1+n]}\text{sech}(|z_2|t)\tanh^{2n}(|z_2|t)\Theta[n], \tag{30}$$

indicating that there are no microscopic realizations where work is extracted from the system; this is intuitive, since for the zero-temperature case oscillators initial state is the ground state and so it is not possible to extract any work from it. For large time $|z_2|t \to \infty$, $P[2n \geqslant 0;t]$ decays as an exponential in time. This behavior is different from the $z_2 = 0$ case discussed previously, where $P[n \geqslant 0;t]$ decays as a Gaussian with time.

For large fluctuations (large n) probability can be approximated by

$$\bar{P}[2n;t] \approx \frac{1}{\sqrt{\pi}}\sqrt{\frac{\left(1-\frac{1}{\xi_+(0)^2}\right)(1-\xi_-(0)^2)}{\left(1-\frac{\xi_-(0)^2}{\xi_+(0)^2}\right)}} \times \frac{e^{-2n[\Theta[n]\ln(|\xi_+(0)|)+\Theta[-n]\ln(|\xi_-(0)|)]}}{\sqrt{|n|}}. \tag{31}$$

Thus, the distribution function falls off exponentially in tails with different rates determined by $|\xi_+(0)|$ and $|\xi_-(0)|$. This

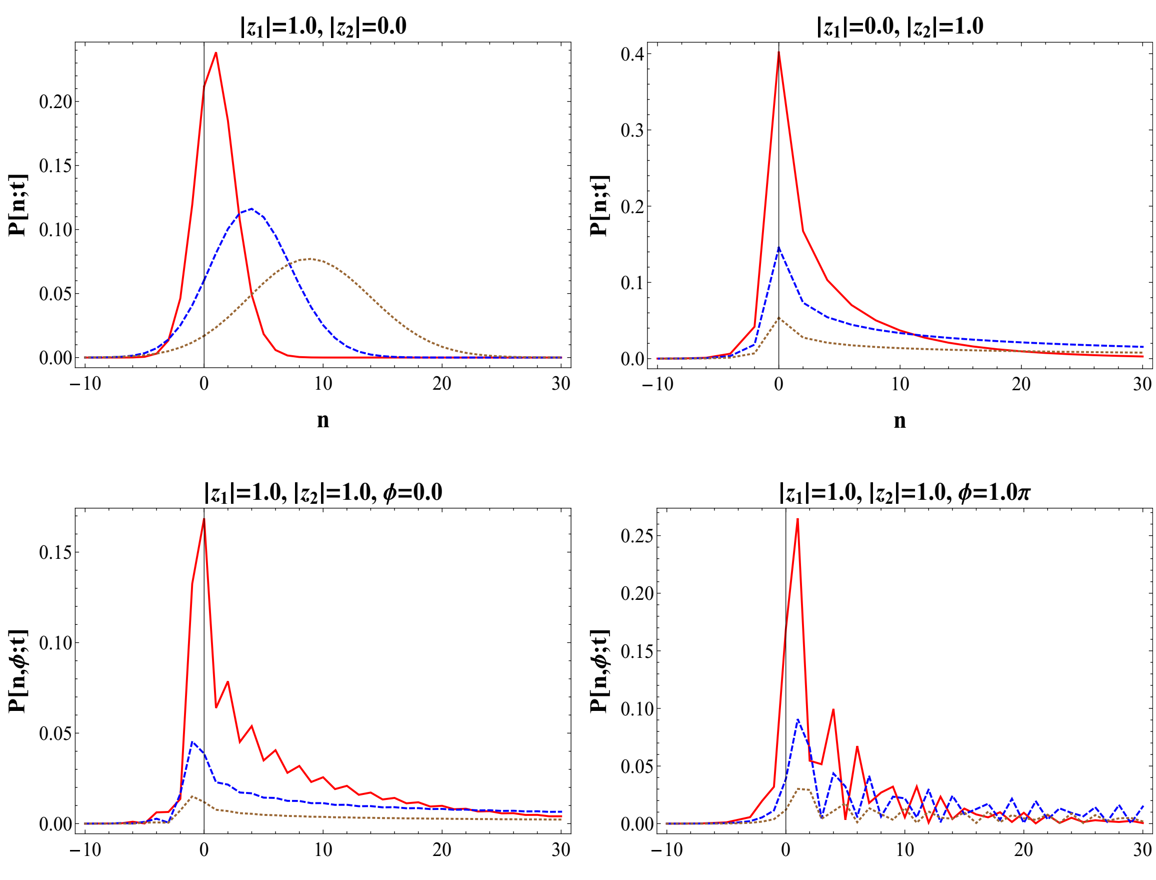

FIG. 2. Probability distribution function ($P[n,\phi;t]$) of work done by both classical drives on the quantum optical oscillator for different measurement times $t = 1.0$ (red), $t = 2.0$ (dashed blue), and $t = 3.0$ (dotted brown) with initial (thermal) average photon number $f = 1.0$. The strength of the classical drives $|z_1|$ and $|z_2|$ and phase difference between them (ϕ) are indicated in the plots. For $z_1 = 0.0$ and $z_2 = 1.0$ cases, only $P[n;t]$ for even n is shown as $P[n;t] = 0$ for odd n.

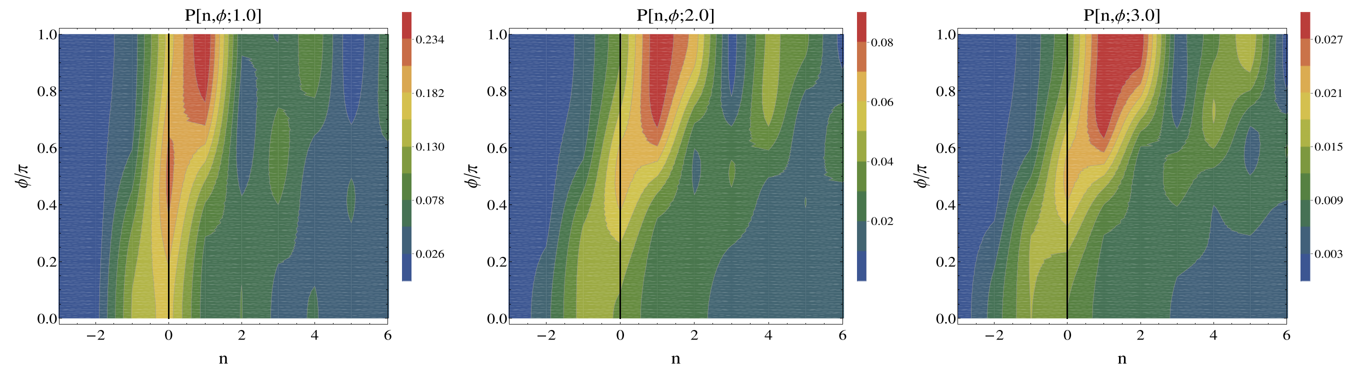

FIG. 3. Probability distribution function of work done ($P[n, \phi; t]$) for different measurement times $t = 1.0$ (left), $t = 2.0$ (center), and $t = 3.0$ (right) as a function of phase difference between the drives (ϕ) with $|z_1| = |z_2| = 1.0$ and $f = 1.0$.

shows that the large fluctuations in n (or work) are suppressed exponentially.

The probability weight for smaller values of n falls quickly as time ($|z_2|t$) increases; however, since $\xi_+(0)$ approaches to unity, the weight for larger values of n increases. Further, for large times ($|z_2|t \to \infty$), $\xi_+(0) \to 1$ [$\xi_-(0)^2\xi_+(0)^2 = (\frac{f}{1+f})^2$ for any time], making the distribution function flatter for positive n with time, but for negative n tails decay with finite rate even for long time.

C. $z_1 \neq 0 \neq z_2$ case

For the general case, the moment generating function of work is given in Eq. (13); the first two cumulants of work are given as

$$\langle n \rangle = (1 + 2f)\sinh^2(|z_2|t) + \alpha(\phi, t) \tag{32}$$

and

$$\langle (n - \langle n \rangle)^2 \rangle = [1 + (1 + 2f)^2\cosh(2|z_2|t)]\sinh^2(|z_2|t) + (1 + 2f)\alpha(\phi, t)\frac{\cosh(3|z_2|t) + \cos(\phi)\sinh(3|z_2|t)}{\cosh(|z_2|t) + \cos(\phi)\sinh(|z_2|t)}. \tag{33}$$

It is interesting to note that the ϕ independent terms of both the cumulants are the same as that of the $z_1 = 0$ case. This is because the cumulant generating function is the sum of the cumulant generating functions for the $z_1 = 0$ case and another ϕ dependent term. Note that the two contributions to both the cumulants are positive, but for large times $|z_2|t \to \infty$ the second ϕ dependent terms grow exponentially (the first term always grows exponentially) with time except for the case $\phi = \pi$ where they saturate to a finite value or decay to zero, respectively.

We numerically invert the generating function for the general case to obtain the probability distribution. The probability distribution function for work done $P[n, \phi; t]$ obtained analytically for cases $z_1 = 0$ and $z_2 = 0$ and numerically for the general case for different values of ϕ at different times for fixed values of $|z_1|$, $|z_2|$, and f is shown in Fig. 2.

For $z_2 = 0$ (case A), the distribution is more or less symmetric and the average work roughly corresponds to the peak position which increases quadratically with time. However the distribution function behaves very differently for $z_1 = 0$ (case B), where the peak of the distribution is always fixed at zero work ($n = 0$) while the average work increases exponentially with time, leading to the asymmetric distribution. For the general case (case C), the two drives compete and we find that work distribution becomes more noisy as the value of ϕ is increased from zero to π. Further, for certain values of ϕ around zero, the maximum of the distribution function shifts to a negative value of n with time as can be seen from Fig. 3. This is clearly an interference effect between the two drives, for such a behavior is not possible when only one drive is present, as is clear from the analytical results obtained above.

IV. CONCLUSION

We have computed the statistics of work done by two classical drives, one-photon and two-photon resonant pumps, on the quantum optical oscillator (a variant of the degenerate parametric amplification process). This simple model allows us to obtain an exact analytic expression for the moment generating function for work. When only one of the drives is present, the probability function for work is analytically obtained. Our results show very different behavior of the work distribution when only individual drivings are present. We found that for the case when only one of the pumps is present work statistics is not influenced by the phase of the drive. When both drives are present, the relative phase between the drives influences the work statistics. For recovering the Jarzynski-Crooks fluctuation theorem, the phase has to be reflected around π (i.e., $\phi \to \pi - \phi$), which is related to the broken time-reversal symmetry of the Hamiltonian. Furthermore, work distribution functions exhibit an interesting behavior, where for small values of the relative phase difference between the drives the most probable work shifts to negative values with time.

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APPENDIX: SOLUTION OF EQ. (7)

Here we present a sketch of the method used for solving the parabolic partial differential equation [Eq. (7)] [88] with the initial condition $\mathcal{G}_\chi[\zeta,\zeta^*;t]|_{t=0} = \mathcal{G}[\zeta,\zeta^*;0]$. If $\mathcal{A} = \mathbf{0}$, the above equation is equivalent to the standard Ornstein-Uhlenbeck equation, the solution of which can be obtained by the method of characteristics [85]. For general $\mathcal{A}$, the transformation $\mathcal{G}_\chi[\zeta,\zeta^*;t] = e^{-\frac{1}{2}\binom{\zeta^*}{\zeta}^T \mathcal{U}_{12}(t)\mathcal{U}_{22}(t)^{-1}\binom{\zeta^*}{\zeta}} \tilde{\mathcal{G}}_\chi[\zeta,\zeta^*;t]$ can be used to eliminate the quadratic term. Here $\mathcal{U}_{xy}(t)$ are square blocks of the following 2×2 partitioned square matrix:

$$\begin{pmatrix} \mathcal{U}_{11}(t) & \mathcal{U}_{12}(t) \\ \mathcal{U}_{21}(t) & \mathcal{U}_{22}(t) \end{pmatrix} = e^{\begin{pmatrix} \mathcal{B} & -\mathcal{A} \\ \mathcal{C} & -\mathcal{B}^T \end{pmatrix} t}. \tag{A1}$$

With this, $\tilde{\mathcal{G}}_\chi[\zeta,\zeta^*;t]$ satisfies the following partial differential equation:

$$\frac{\partial}{\partial t}\tilde{\mathcal{G}}_\chi[\zeta,\zeta^*;t] = \left[\frac{1}{2}\begin{pmatrix} \zeta^* \\ \zeta \\ \frac{\partial}{\partial \zeta^*} \\ \frac{\partial}{\partial \zeta} \end{pmatrix}^T \begin{pmatrix} \mathbf{0} & [\mathcal{B} - \mathcal{U}_{12}(t)\mathcal{U}_{22}(t)^{-1}\mathcal{C}] \\ [\mathcal{B} - \mathcal{U}_{12}(t)\mathcal{U}_{22}(t)^{-1}\mathcal{C}]^T & \mathcal{C} \end{pmatrix} \begin{pmatrix} \zeta^* \\ \zeta \\ \frac{\partial}{\partial \zeta^*} \\ \frac{\partial}{\partial \zeta} \end{pmatrix} + \begin{pmatrix} d_1 - \mathcal{U}_{12}(t)\mathcal{U}_{22}(t)^{-1} d_2 \\ d_2 \end{pmatrix}^T \begin{pmatrix} \zeta^* \\ \zeta \\ \frac{\partial}{\partial \zeta^*} \\ \frac{\partial}{\partial \zeta} \end{pmatrix}\right] \tilde{\mathcal{G}}_\chi[\zeta,\zeta^*;t]. \tag{A2}$$

This partial differential equation can be solved by the method of characteristics to get $\tilde{\mathcal{G}}_\chi[\zeta,\zeta^*;t]$. Using this, $\mathcal{G}_\chi[\zeta,\zeta^*;t]$ is given as

$$\mathcal{G}_\chi[\zeta,\zeta^*;t] = \int_{\bar{\zeta}\in\mathbb{C}} d^2\bar{\zeta}\, \mathbf{G}_\chi[\zeta,\zeta^*;t|\bar{\zeta},\bar{\zeta}^*;0]\mathcal{G}[\bar{\zeta},\bar{\zeta}^*;0], \tag{A3}$$

with $\mathbf{G}_\chi[\zeta,\zeta^*;t|\bar{\zeta},\bar{\zeta}^*;0]$ given in Eq. (11).

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