Casimir force variability in one-dimensional QED systems

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(Received 10 December 2018; published 10 June 2019)

The Casimir force between two extended charge sources, embedded in a background of one-dimensional massive Dirac fermions, is explored by means of original contour integration techniques. For identical sources with the same (positive) charge, we find that in the nonperturbative region the Casimir interaction between them can reach sufficiently large negative values and simultaneously reveal the features of a long-range force in spite of nonzero fermion mass. For large distances s between sources we recover that their mutual interaction is governed primarily by the structure of the discrete spectrum of a single source, through which it can be tuned to give an attractive, a repulsive, or an almost compensated Casimir force with various rates of the exponential decay, quite different from the standard $\exp(-2ms)$ law. A quite different behavior of the Casimir force is found for the system of two extended sources with the opposite charge. In particular, in this case, there is no possibility for a long-range interaction between sources. The asymptotics of the Casimir force follows the standard $\exp(-2ms)$ law. Moreover, for small separations between sources the Casimir force, being calculated completely nonperturbatively, for symmetric and antisymmetric cases turns out to be of different sign and also opposite to the classic electrostatic force for such Coulomb sources. By means of the same (dubbed ln [Wronskian]) techniques, the case of two pointlike charge sources is also considered in a self-consistent manner with similar results for the variability of the Casimir force.

DOI: 10.1103/PhysRevA.99.062504

I. INTRODUCTION

There is a great deal of interest in essentially nonperturbative vacuum polarization (Casimir) effects in quasi-one-dimensional QED systems caused by charged impurities. Actually, one-dimensional QED systems with impurities appear nowadays in many situations, ranging from relativistic H-like atoms in a strong homogenous magnetic field [1–4] to charged impurities in low-dimensional nanostructures like semiconductor quantum wires, carbon nanotubes, conducting polymers, etc. [5], fermionic atoms in ultracold gases [6–8], and defects in one-dimensional fermionic quantum liquids [9–11].

Impurities have a profound effect on the physical properties of these low-dimensional systems. In certain exceptionally clean systems, impurities can be created and controlled along with the Casimir forces between them mediated by fermions. The general literature on the Casimir effect is vast and the reader may consult Ref. [12] for some basic experimental results and Refs. [13–17] for reviews and background work. More recent studies, including, in particular, such nontrivial effects as spin-dependent photon-photon interactions in arrays of microcavities, Abelian and non-Abelian Josephson effects, and a one-dimensional lattice of Rydberg atoms coupled to an optical cavity, where the dipole interaction competes with the atom-light coupling leading to a rich phase diagram, are considered in Refs. [18–20].

The Casimir interaction mediated by fermions has been intensively studied from different points of view and in different geometries; see, e.g., Ref. [21]. The main result is that for Dirac fermions there is a Casimir force whose strength and sign can be tuned by the impurity separation and their internal structure. This provides a physical situation where the Casimir interaction could be continuously tunable from attractive through almost completely compensated to the repulsive one by variation of an internal control parameter, realizing the known bounds for the one-dimensional Casimir interaction as two limiting cases. In the light of proofs showing the absence of repulsive Casimir interactions for the photonic field in vacuum, this is quite a remarkable situation. Moreover, in Ref. [22] it was shown that the electronic Casimir force between two impurities on a one-dimensional semiconductor quantum wire can be of a very long range, despite the nonzero effective mass of the mediator.

Of special interest in the fermionic Casimir effect is the situation when for some reason the impurities should be modeled as δ-like sources, since the Dirac equation (DE) is inconsistent with direct insertion of external δ-like potentials. This problem was explored in Ref. [23] in terms of the energy density and interaction between two Dirac spikes as a function of a single spike parameter and the distance between them. In this model each spike is represented by a square barrier, which enters the fermion dynamics as an additional mass term, and the limit of δ-like potentials is considered via a transfer matrix, which in this limit allows for a self-consistent treatment. In Ref. [24] the Casimir interaction between two square potential barriers (scatterers), mediated by the massless

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fermions, was considered. The Casimir force between the scatterers was found for both the case of finite width and strength of the barriers and in the limit of δ-like potentials. The result of both works is that for identical δ-like scatterers, separated by a large distance d, the interaction force between them reveals the conventional attractive asymptotics $\sim 1/d^2$. At the same time, for a more general case of inequivalent scatterers, the magnitude and sign of the force depend on their relative spinor polarizations [24].

In this paper, within the framework of a general quasi-one-dimensional QED system, we consider the Casimir interaction of two short-range Coulomb sources, either extended or δ-like, which enter the fermion dynamics as localized electrostatic potentials. In the case of scalar coupling, considered in Refs. [23,24], the scatterers affect equally the positive- and negative-frequency fermionic modes. In the case of vector coupling, the behavior of electronic and positronic components is principally different and leads to a variety of effects, the most significant of which is the discrete levels diving into the lower continuum and related nonperturbative effects of vacuum reconstruction, when the positively charged sources attain the overcritical region. The main question of interest is how these nonperturbative effects, including the effects of supercriticality, manifest themselves in Casimir forces between such sources. For identical positively charged sources, by means of the original ln [Wronskian] contour integration techniques, we find that the interaction energy between sources can exceed sufficiently large negative values and simultaneously reveal the features of a long-range force in spite of a nonzero fermion mass, which could significantly influence the properties of such quasi-one-dimensional QED systems. Moreover, the Casimir force shows a highly nontrivial behavior with increasing distance between sources, which includes separate vertical jumps at finite distances, caused by the effects of discrete level creation and annihilation at the lower threshold, as well as different exponent rates and signs in the asymptotics. The case of two δ-like sources is also considered in detail. In contrast, the antisymmetric source-antisource system reveals quite different features. In particular, in this case there is no possibility for the long-range interaction between sources. The asymptotics of the Casimir force follows the standard $\exp(-2ms)$ law. Moreover, for small separations between sources the Casimir force, being calculated completely nonperturbatively, in the symmetric case is attractive, while in the antisymmetric one it turns into sufficiently strong repulsion. Remarkably enough, the classic electrostatic force for such Coulomb sources should be of opposite sign. There is no evident explanation for this effect. However, the set of parameters used is quite wide to consider this effect as a general one. These results may be relevant for indirect interactions between charged defects and adsorbed species in the quasi-one-dimensional QED systems mentioned above.

The single short-range Coulomb source is described as a potential square well or barrier of width $2a$ and depth or height V_0,

$$V(x) = -V_0\theta(a - |x|), \tag{1}$$

where $V_0 > 0$ for a well and $V_0 < 0$ for a barrier. Actually, the potential (1) could be interpreted as created by the charged impurity considered as a spherical shell of radius R_0 and charge Z, strongly screened for $|x| > R_0$. For this case

$$V_0 = Z\alpha/R_0. \tag{2}$$

In this work we consider the system of two such sources, separated by the distance s. The component of the vacuum polarization energy $\mathcal{E}_{\rm vac}$, responsible for their interaction, is defined as

$$\mathcal{E}_{\rm vac}^{\rm int}(s) = \mathcal{E}_{\rm vac,2}(s) - \mathcal{E}_{\rm vac,2}(s \to \infty), \tag{3}$$

where $\mathcal{E}_{\rm vac,2}(s)$ is the total vacuum polarization energy for the system containing two potentials like (1), located at the distance s from each other. In the case of two sources of the same charge $\mathcal{E}_{\rm vac,2}(s \to \infty) = 2\mathcal{E}_{\rm vac,1}$, with the latter being the vacuum energy of a single source, while in the case of opposite charges $\mathcal{E}_{\rm vac,2}(s \to \infty)$ requires further attention (see Sec. V).

As in other works on vacuum polarization in strong Coulomb fields [25–28], the radiative corrections from virtual photons are neglected. Henceforth, if it is not stipulated otherwise, the units $\hbar = m_e = c = 1$ are used. Thence the coupling constant $\alpha = e^2$ is also dimensionless and the numerical calculations, illustrating the general picture, are performed for $\alpha = 1/137.036$. However, it would be worthwhile to note that for the effective electron-hole vacuum in quasi-one-dimensional systems like nanotubes and wires, as in graphene, the actual value of the finite structure constant, and hence the magnitude of the Casimir effects, could be quite different from the one in pure QED.

II. CALCULATING THE CASIMIR ENERGY AND FORCE VIA ln[WRONSKIAN] CONTOUR INTEGRATION TECHNIQUES

Here we consider the general approach for calculation of the Casimir energy and force via ln[Wronskian] contour integration techniques, exploring for concreteness the case of a single positively charged source with $V_0 > 0$. The starting point for the evaluation of the vacuum energy in QED systems is the Schwinger vacuum average [25–27,29]

$$\mathcal{E}_{\rm vac} = \frac{1}{2}\left(\sum_{\epsilon_n<\epsilon_F}\epsilon_n - \sum_{\epsilon_n\geqslant\epsilon_F}\epsilon_n\right)_V - \frac{1}{2}\left(\sum_{\epsilon_n<\epsilon_F}\epsilon_n - \sum_{\epsilon_n\geqslant\epsilon_F}\epsilon_n\right)_0, \tag{4}$$

where ϵ_n are the eigenvalues of the corresponding DE

$$[\alpha p + \beta + V(x)]\psi(x) = \epsilon\psi(x), \tag{5}$$

while for the positively charged sources ϵ_F should be chosen at the lower threshold, i.e., $\epsilon_F = -1$. The label V indicates the presence of the external potential, while the label 0 corresponds to the free case. Throughout the paper, when solving the DE, the representation $\alpha = \sigma_2$ and $\beta = \sigma_3$ is used.

For the subsequent analysis it is convenient to separate in (4) the contributions from the discrete spectrum and both continua and apply to the latter the well-known tool, which replaces it with the scattering phase $\delta(\epsilon)$ integration (see, e.g., Refs. [23,30,31] and references therein). After a number of almost evident steps one obtains [3]

$$\mathcal{E}_{\mathrm{vac}} = \frac{1}{2\pi}\int_1^{+\infty}\delta_{\mathrm{tot}}(\epsilon)d\epsilon + \frac{1}{2}\sum_{-1\leqslant\epsilon_n<1}(1-\epsilon_n), \tag{6}$$

where $\delta_{\mathrm{tot}}(\epsilon)$ is the total phase shift for the given $|\epsilon|$, including the contributions from scattering states in both continua, while in the discrete spectrum the (effective) electron rest mass is subtracted from each level in order to retain in $\mathcal{E}_{\mathrm{vac}}$ the interaction effects only.

Such an approach to the calculation of $\mathcal{E}_{\mathrm{vac}}$ turns out to be quite effective, since the total phase shift $\delta_{\mathrm{tot}}(\epsilon)$ behaves in both (IR and UV) limits much better than each of the elastic phases separately and is automatically an even function of the external potential. More concretely, in the Coulomb potentials with nonvanishing source size, $\delta_{\mathrm{tot}}(\epsilon)$ is finite for $\epsilon \to 1$ and decreases $\sim 1/\epsilon^3$ for $\epsilon \to \infty$, which provides the convergence of the phase integral in (6) [2–4,23,28,32]. The sum over bound energies $1-\epsilon_n$ of discrete levels in the case of short-range sources like (1) is finite from the very beginning, since such potentials allow for a finite number of discrete levels. As a result, the expression (6) turns out to be finite without any additional renormalization.

However, the convergence of $\mathcal{E}_{\mathrm{vac}}$ in the form (6) is completely caused by the specifics of 1+1 dimensions and in no way means there is no need for a renormalization. Renormalization via a fermionic loop is required on account of the analysis of the vacuum charge density $\rho_{\mathrm{vac}}(x)$, from which it follows that without such UV renormalization the integral induced charge will not acquire the value that follows from quite obvious physical arguments [2–4,26–29]. Such an analysis is performed in Ref. [33] for the cases of the singlet or doublet of positively charged sources.

Another obvious requirement is that for $V_0 \to 0$ the value of $\mathcal{E}_{\mathrm{vac}}$ should coincide with $\mathcal{E}^{(1)}_{\mathrm{vac}}$, obtained in the first order of the QED perturbation theory (PT). The latter is found to be quite similar to the unscreened case considered in Refs. [2–4] and for a single source like (1) is equal to

$$\mathcal{E}^{(1)}_{\mathrm{vac},1} = \frac{2V_0^2}{\pi^2}\int_0^{+\infty} dq\,\frac{\sin^2 qa}{q^2}\left(1-2\frac{\operatorname{arcsinh}(q/2)}{q\sqrt{1+(q/2)^2}}\right), \tag{7}$$

while for the configuration containing a doublet of such identical sources, separated by the distance d, it is given by the expression

$$\mathcal{E}^{(1)}_{\mathrm{vac},2} = \frac{2V_0^2}{\pi^2}\int_0^{+\infty} dq\,\frac{\{\sin[q(a+d)]-\sin[qd]\}^2}{q^2} \times\left(1-2\frac{\operatorname{arcsinh}(q/2)}{q\sqrt{1+(q/2)^2}}\right). \tag{8}$$

It is easy to verify that the nonrenormalized vacuum energy (6) does not satisfy this requirement, since the renormalization coefficient (10) introduced below, which provides also the correspondence between $\mathcal{E}^R_{\mathrm{vac}}$ and $\mathcal{E}^{(1)}_{\mathrm{vac}}$ for $V_0 \to 0$, in the general case turns out to be nonzero with the only exception being for certain parameter sets.

For these reasons, in complete analogy with the renormalization of the charge density, considered in Refs. [2–4,26–32], we should pass from $\mathcal{E}_{\mathrm{vac}}$ to the renormalized vacuum energy

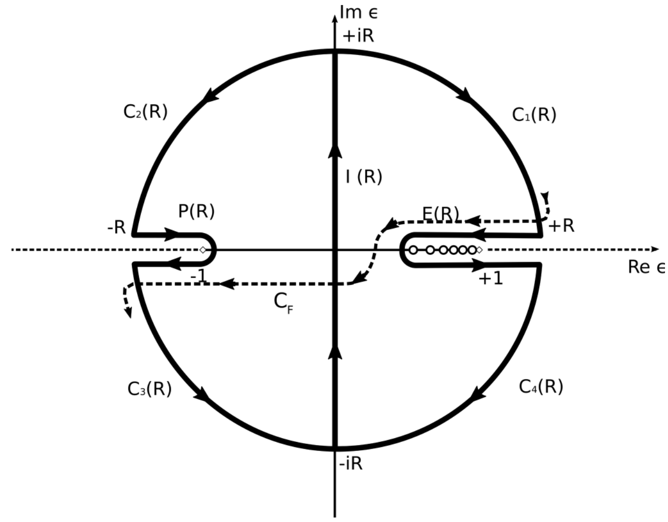


FIG. 1. The WK contours in the complex energy plane, used for the representation of the vacuum charge density and vacuum energy via contour integrals.

$\mathcal{E}^R_{\mathrm{vac}}$. In the practically useful form $\mathcal{E}^R_{\mathrm{vac}}$ should be represented as [2–4,28,32]

$$\mathcal{E}^R_{\mathrm{vac}} = \mathcal{E}_{\mathrm{vac}} + \lambda V_0^2, \tag{9}$$

where the renormalization coefficient

$$\lambda = \lim_{V_0\to 0}\frac{\mathcal{E}^{(1)}_{\mathrm{vac}}(V_0) - \mathcal{E}_{\mathrm{vac}}(V_0)}{V_0^2} \tag{10}$$

depends solely on the shape of the external potential and in the general (1+1)-dimensional case is a function with zeros [2–4,33].

The evaluation of $\mathcal{E}^R_{\mathrm{vac}}$ via the sum of the phase integral and discrete levels is considered in detail in Refs. [2–4,28,32] for the unscreened or partially screened extended Coulomb source and in the present case will differ only by certain technical details. However, for our purposes of a detailed study of Casimir interaction between the localized Coulomb-like external sources, an alternative approach for evaluation of the nonrenormalized $\mathcal{E}_{\mathrm{vac}}$ turns out to be more efficient. This approach is quite similar to the calculation of the vacuum density $\rho_{\mathrm{vac}}(x)$ via integration of a specially constructed function along the Wichmann-Kroll (WK) contours [25,26,29], which are shown in Fig. 1. Namely, it is easy to see that the function

$$F(\epsilon, V_0) = \frac{\epsilon[dJ(\epsilon)/d\epsilon]}{J(\epsilon)}, \tag{11}$$

where $J(\epsilon)$ is the Wronskian for the spectral problem (5), reveals all the pole properties, which are required for the representation of the expression (4) via integrals along the WK contours, since actually $J(\epsilon)$ is nothing but the Jost function of the spectral problem (5) [3]. The real zeros of $J(\epsilon)$ lie in the interval $-1 \leqslant \epsilon_n < 1$ and coincide with discrete energy levels, while the complex ones reside on the second sheet of the Riemann energy surface with the negative imaginary part of the wave number $k=\sqrt{\epsilon^2-1}$ and for $\mathrm{Re}\,k>0$ correspond to the elastic resonances. Moreover, both $J(\epsilon)$

and TrG as functions of the wave number k reveal the same reflection symmetry $f^*(k) = f(-k^*)$ of the Jost function.

To represent $\mathcal{E}_{\rm vac}$ via integration along the WK contours, it suffices to pass to the difference

$$\mathcal{H}(\epsilon, V_0) = F(\epsilon, V_0) - F(\epsilon, 0), \tag{12}$$

with respect to the free case. As a result, the nonrenormalized induced vacuum energy can be represented as

$$\mathcal{E}_{\rm vac} = -\frac{1}{4\pi i}\lim_{R\to\infty}\left(\int_{P(R)} d\epsilon\, \mathcal{H}(\epsilon, V_0) + \int_{E(R)} d\epsilon\, \mathcal{H}(\epsilon, V_0)\right). \tag{13}$$

In the next step one finds by means of the analysis of the asymptotics of the function $\mathcal{H}(\epsilon, V_0)$ on the large circle in Fig. 1 that the initial integration along the contours $P(R)$ and $E(R)$ for the singlet or doublet of external potentials like (1) can be reduced to integration along the imaginary axis [33]. Upon taking into account the (possible) existence of negative discrete levels and proceeding further in complete analogy to the corresponding treatment of the vacuum density performed in Refs. [2–4,26], one finds the final expression for $\mathcal{E}_{\rm vac}$ in the form

$$\mathcal{E}_{\rm vac} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dy\, \mathcal{H}(iy, V_0) - \sum_{-1\leqslant \epsilon_n<0} \epsilon_n. \tag{14}$$

For the single source (1) the integrand in Eq. (14) takes the form

$$\mathcal{H}(iy, V_0) = \frac{iV_0 y\{V_0[V_0+2iy]\gamma(iy) + 2aj^2(iy, V_0)\gamma^2(iy)\sin[2aj(iy, V_0)]\}}{j^2(iy, V_0)\gamma^3(iy)\{j(iy, V_0)\gamma(iy)\cos[2aj(iy, V_0)] + (1 - iV_0 y + y^2)\sin[2aj(iy, V_0)]\}} - \frac{2iaV_0 y j(iy, V_0)\gamma^3(iy)\cos[2aj(iy, V_0)]}{j^2(iy, V_0)\gamma^3(iy)\{j(iy, V_0)\gamma(iy)\cos[2aj(iy, V_0)] + (1 - iV_0 y + y^2)\sin[2aj(iy, V_0)]\}}, \tag{15}$$

where

$$j(\epsilon, V_0) = \sqrt{(V_0+\epsilon)^2 - 1}, \quad \gamma(\epsilon) = \sqrt{1-\epsilon^2}. \tag{16}$$

The corresponding expression for the doublet configuration will be presented below. The direct numerical calculation shows that both approaches to evaluation of the vacuum energy (6) and (14) lead, with high precision, to the same results.

Besides $\mathcal{E}_{\rm vac}$, in 1+1 dimensions the renormalization term λV_0^2 in the expression (9) turns out to be quite important, especially in the nonperturbative region. For a single source (1) the dependence of the renormalization term on the source parameters is determined primarily by the coefficient $\lambda(a)$, which can be represented as

$$\lambda(a) = \lambda_1(a) - \lambda_2(a), \tag{17}$$

where λ_1 emerges from the PT contribution $\mathcal{E}^{(1)}_{\rm vac}$ to the vacuum energy

$$\lambda_1(a) = \frac{a}{\pi} - I_1(a),$$
$$I_1(a) = \frac{4}{\pi^2}\int_0^\infty dq\, \frac{\sin^2(qa)}{q^3}\frac{\mathrm{arcsinh}(q/2)}{\sqrt{1+(q/2)^2}}, \tag{18}$$

while λ_2 corresponds to the first (quadratic in V_0) term in $\mathcal{E}_{\rm vac}$, which is found from the Born series (see Refs. [2–4,26,28,32])

$$\lambda_2(a) = \frac{a}{\pi} - \frac{1}{16} + I_2(a),$$
$$I_2(a) = \frac{1}{4\pi}\int_0^\infty dy\, \frac{e^{-4a\sqrt{1+y^2}}}{(1+y^2)^2}. \tag{19}$$

By means of the relation

$$\lambda_1(a) + \lambda_2(a) = a/\pi, \tag{20}$$

whose derivation requires some additional considerations and so is presented separately in the Appendix, one obtains

$$\lambda(a) = \frac{a}{\pi} - 2\lambda_2(a) = \frac{1}{8} - \frac{a}{\pi} - 2I_2(a). \tag{21}$$

The asymptotics of $\lambda(a)$ for $a \ll 1$ and $a \gg 1$, which are important for further analysis of the Casimir interaction between separate sources, are considered in detail in Ref. [33]. So here we present only the required results. Namely, the asymptotics of $\lambda(a)$ for $a \ll 1$ reads

$$\lambda(a\to 0) = \frac{a}{\pi} - 2a^2 + O(a^3), \tag{22}$$

while for large a neglecting the exponentially small corrections it is given by

$$\lambda(a\to\infty) = \frac{1}{8} - \frac{a}{\pi}. \tag{23}$$

As a result, for small a the coefficient $\lambda(a)$ grows linearly, while for large a it decreases with the same modulus slope $1/\pi$. Hence, the renormalization coefficient $\lambda(a)$ should vanish at certain $a = a_{cr}$. In the case of the single well (1) it has a unique zero when $a_{cr} \simeq 0.297$. More details concerning the behavior of $\lambda(a)$ are given in Ref. [33].

III. CASIMIR FORCES BETWEEN TWO IDENTICAL POSITIVELY CHARGED SHORT-RANGE COULOMB SOURCES

Now, having dealt with the general formulation for calculation of $\mathcal{E}_{\rm vac}$ this way, let us consider the configuration of two such identical positively charged short-range Coulomb sources, described by the potential

$$V_2(x) = -V_0\,\theta(|x| - d)\,\theta(d + a - |x|), \quad V_0 > 0. \tag{24}$$

Let us note that the separate sources have the total width a, which provides the restoration of the initial potential well (1) with the width $2a$ for $d \to 0$.

A detailed analysis of this system is given in Ref. [33]. However, it would be instructive to repeat the main steps of calculation, since in the subsequent sections we will continuously refer to them.

As in the case of one potential well, the calculation of $\mathcal{E}^R_{\text{vac},2}$ requires renormalization in the second order with respect to the external potential

$$\mathcal{E}^R_{\text{vac},2} = \mathcal{E}_{\text{vac},2} + \Lambda(a,d)V_0^2, \tag{25}$$

where

$$\Lambda(a,d) = \Lambda_1(a,d) - \Lambda_2(a,d). \tag{26}$$

The components of the renormalization coefficient $\Lambda_i(a,d)$, $i = 1, 2$, are expressed in terms of the corresponding coefficients $\lambda_i(a)$ for the single source as

$$\Lambda_i(a,d) = \lambda_i(a+d) + \lambda_i(d) + 2\lambda_i(a/2) - 2\lambda_i(d+a/2). \tag{27}$$

From (27) by means of (20) one finds that $\Lambda_i(a,d)$ is subject to the same relation

$$\Lambda_1(a,d) + \Lambda_2(a,d) = a/\pi. \tag{28}$$

As a result, the renormalization coefficient for the two-well problem (24) can be represented as

$$\begin{aligned}\Lambda(a,d) &= a/\pi - 2\Lambda_2(a,d)\\ &= a/\pi - 2\lambda_2(a+d) - 2\lambda_2(d)\\ &\quad - 4\lambda_2(a/2) + 4\lambda_2(d+a/2).\end{aligned} \tag{29}$$

The Casimir interaction energy $\mathcal{E}^{\text{int}}_{\text{vac}}(d)$ for the system of two identical short-range Coulomb sources (24) is determined through the relation (3) with subsequent renormalization. In what follows we will use the parametrization of the source separation in terms of d instead of s as the most pertinent one.

Upon subtraction of $2\mathcal{E}^R_{\text{vac}}(V_0, a/2)$ from the expression (25) the general structure of $\mathcal{E}^{\text{int}}_{\text{vac}}(d)$ takes the form

$$\mathcal{E}^{\text{int}}_{\text{vac}}(d) = I_{\text{int}}(d) - S_{\text{int}}(d) + \Lambda_{\text{int}}(d)V_0^2, \tag{30}$$

with the contributions $I_{\text{int}}(d)$ from the integral term, $S_{\text{int}}(d)$ from the negative discrete levels, and $\Lambda_{\text{int}}(d)V_0^2$ from the renormalization term. It would be pertinent to start with the renormalization term, the asymptotics of which for large d can be explored most simply and in the general form. By means of (21) and (29) the renormalization coefficient $\Lambda_{\text{int}}(d) = \Lambda(a,d) - 2\lambda(a/2)$ can be represented as

$$\begin{aligned}\Lambda_{\text{int}}(d) &= a/\pi - 2\lambda_2(a+d) - 2\lambda_2(d) - 4\lambda_2(a/2)\\ &\quad + 4\lambda_2(d+a/2) - 2[a/2\pi - 2\lambda_2(a/2)]\\ &= 4\lambda_2(d+a/2) - 2\lambda_2(a+d) - 2\lambda_2(d).\end{aligned} \tag{31}$$

In the next step, inserting the definition of λ_2 [Eq. (19)] into (31), one finds

$$\begin{aligned}\Lambda_{\text{int}}(d) &= 4I_2(d+a/2) - 2I_2(a+d) - 2I_2(d)\\ &= -\frac{1}{2\pi}\int_0^\infty \frac{(1-e^{-2a\sqrt{1+y^2}})^2 e^{-4d\sqrt{1+y^2}}}{(1+y^2)^2}dy \leqslant 0.\end{aligned} \tag{32}$$

So the contribution to the interaction energy $\mathcal{E}^{\text{int}}_{\text{vac}}(d)$ from the renormalization term turns out to be strictly negative and exponentially decreasing for $d \gg 1$. The exact form of the asymptotics of $\Lambda_{\text{int}}(d \gg 1)$ can be found from the expression (31) via triple integration of the MacDonald function asymptotics in a way quite similar to the evaluation of the asymptotics of $\lambda(a \to \infty)$, considered in Ref. [33], and takes the form

$$\begin{aligned}\Lambda_{\text{int}}(d \gg 1) = -\frac{e^{-4d}}{\sqrt{2\pi d}}e^{-2a}\bigg[\sinh^2 a\\ + \frac{\sinh a(8ae^{-a} - 13\sinh a)}{32d} + O\left(\frac{1}{d^2}\right)\bigg].\end{aligned} \tag{33}$$

Now let us consider the behavior of the integral term in Eq. (30) for $d \gg 1$, at first without subtracting the contribution from infinitely separated wells. Upon integration by parts it can be written as

$$I(d) = -\frac{1}{\pi}\int_0^\infty dy\,\text{Re}\{\ln[J_{\text{red}}(d, iy)]\}, \tag{34}$$

where the reduced Wronskian

$$J_{\text{red}}(d,\epsilon) = J(d,\epsilon)/J_0(\epsilon) \tag{35}$$

contains in the nominator the Wronskian $J(d,\epsilon)$ for the double-well potential (24),

$$J(d,\epsilon) = \frac{2e^{-2a\sqrt{1-\epsilon^2}}}{\sqrt{1-\epsilon^2}}\left[f_1^2(\epsilon) - e^{-4d\sqrt{1-\epsilon^2}}f_2^2(\epsilon)\right], \tag{36}$$

in which

$$\begin{aligned}f_1(\epsilon) &= \sqrt{1-\epsilon^2}\cos[a\sqrt{(V_0+\epsilon)^2-1}] - (\epsilon^2 - 1 + \epsilon V_0)\\ &\quad \times \sin[a\sqrt{(V_0+\epsilon)^2-1}]/\sqrt{(V_0+\epsilon)^2-1},\\ f_2(\epsilon) &= V_0\sin[a\sqrt{(V_0+\epsilon)^2-1}]/\sqrt{(V_0+\epsilon)^2-1},\end{aligned} \tag{37}$$

while in the denominator the Wronskian $J_0(\epsilon) = 2\sqrt{1-\epsilon^2}$, corresponding to the free case $V_0 = 0$.

The behavior of the integral (34) for large d is found via the expansion of the integrand

$$\begin{aligned}\ln[J_{\text{red}}(d, iy)] = \ln\left(f_1^2(iy)\frac{e^{-2a\sqrt{1+y^2}}}{1+y^2}\right)\\ - e^{-4d\sqrt{1+y^2}}\left(\frac{f_2(iy)}{f_1(iy)}\right)^2 + O(e^{-8d\sqrt{1+y^2}}).\end{aligned} \tag{38}$$

Upon substituting the expansion (38) into the integral (34) one obtains two leading-order terms in the asymptotics of $I(d)$ for $d \gg 1$,

$$\begin{aligned}I(d) \simeq -\frac{1}{\pi}\int_0^\infty dy\,\text{Re}\left[\ln\left(f_1^2(iy)\frac{e^{-2a\sqrt{1+y^2}}}{1+y^2}\right)\right]\\ + \frac{1}{\pi}\int_0^\infty dy\,\text{Re}\left[e^{-4d\sqrt{1+y^2}}\left(\frac{f_2(iy)}{f_1(iy)}\right)^2\right].\end{aligned} \tag{39}$$

Since the first term in Eq. (39) does not depend on d, the leading term in the asymptotics of the integral term in $\mathcal{E}^{\text{int}}_{\text{vac}}(d)$ for

$d \gg 1$ takes the form

$$I_{\rm int}(d) = I(d) - I(d \to \infty)$$
$$= -\frac{1}{\pi}\int_0^\infty dy\,{\rm Re}\left[\ln\left((1+y^2)\frac{e^{2a\sqrt{1+y^2}}}{f_1^2(iy)}J_{\rm red}(d,iy)\right)\right]$$
$$\simeq \frac{1}{\pi}\int_0^\infty dy\,{\rm Re}\left[e^{-4d\sqrt{1+y^2}}\left(\frac{f_2(iy)}{f_1(iy)}\right)^2\right]. \quad (40)$$

For large d the integrand in (40) decreases rapidly with growing y; hence the main contribution to the integral is provided by small y. Therefore, it turns out to be efficient to rewrite the expression (40) in the form

$$I_{\rm int}(d) \simeq \frac{e^{-4d}}{\pi}\int_0^\infty dy\,{\rm Re}\left[e^{-4d(\sqrt{1+y^2}-1-y^2/2)}\left(\frac{f_2(iy)}{f_1(iy)}\right)^2\right] \times e^{-2dy^2} \quad (41)$$

and thereafter to expand the term in square brackets in the integrand in the power series in y. All the integrals emerging this way can be calculated analytically. The final expansion of $I_{\rm int}(d)$ for $d \gg 1$ reads

$$I_{\rm int}(d) = V_0^2\frac{e^{-4d}}{\sqrt{2\pi d}}\left[\frac{A^2}{2} + \frac{1}{8d}\left(\frac{3A^2}{8} + B\right) + O\left(\frac{1}{d^2}\right)\right], \quad (42)$$

where

$$z_0 = \sqrt{V_0^2 - 1}, \quad A = \frac{1}{1 + z_0\cot(az_0)},$$
$$B = A^3\left[-3V_0^2\left(1 - \frac{\cot(az_0)}{z_0} + \frac{a}{\sin^2(az_0)}\right)^2 A - 2 - \frac{(1+z_0^4)}{z_0^3}\cot(az_0) + \frac{a}{z_0^2\sin^2(az_0)}\{1 - 2V_0^2[1 - az_0\cot(az_0)]\}\right]. \quad (43)$$

It should be particularly noted that the formulas (43) work equally well for both $V_0 > 1$ and $V_0 < 1$. For $V_0 = 1$, upon taking in Eq. (43) the limit $z_0 \to 0$, the expressions for A and B are replaced by

$$A = \frac{a}{1+a},$$
$$B = -\frac{a^2}{45(1+a)^4}(45 + 135a + 255a^2 + 210a^3 + 68a^4 + 8a^5). \quad (44)$$

So the asymptotics of the integral term in $\mathcal{E}^{\rm int}_{\rm vac}(d)$ for $d \gg 1$ turns out to be $\sim e^{-4d}/\sqrt{d}$, which is quite similar to the behavior of the renormalization term (33). It should be mentioned that the expansion (42) can be used also for finite d in the case when each next term in the expansion (38) is much less than the preceding one. At the same time, an alternative situation might occur, similar to the case of $a = 1$ and $V_0 = 8$ considered below, when the coefficients A and B turn out to be quite large. The reason is that the zero denominator in A is nothing but the condition for existence of the level with $\epsilon_0 = 0$ in the single well. For $a = 1$ and $V_0 = 8$ the lowest level is $\epsilon_0 \simeq 0.020\,85$ and so by sufficiently small variation of the well parameters this level can be made strictly zero. It follows that in the general case the expansion given above does not hold for the case when there exists in the well a level close to $\epsilon_0 = 0$, since in this case the expansion coefficients A and B become large.

In the latter case it should be taken into account by expanding the term in square brackets in Eq. (41) in the power series in y that the expansion of the function $f_1(iy)$ starts now from the linear in y term, since the first term of the series $\cos(az_0) + \sin(az_0)/z_0$ vanishes. As a result, for the case $\epsilon_0 = 0$ one obtains

$$I_{\rm int}(d) = -\frac{V_0^2 - 1}{(1+a)V_0^2}e^{-2d} + O(e^{-4d}), \quad (45)$$

whence it follows that for this special case the rate of decrease of the integral term in Eq. (30) for $d \gg 1$ becomes sufficiently less. It should be mentioned in addition that the multiplier before the leading exponent in Eq. (45) is strictly negative, since the zero level might appear in the single well only for $V_0 > 1$.

Now let us consider the (possible) contribution to (30) from negative discrete levels for $d \gg 1$. In the general case, the discrete levels are determined by the corresponding zeros of the Wronskian $J(d,\epsilon)$ and satisfy the equation

$$f_1^2(\epsilon) - e^{-4d\sqrt{1-\epsilon^2}}f_2^2(\epsilon) = 0. \quad (46)$$

For $d \to \infty$ Eq. (46) transforms into $f_1(\epsilon) = 0$, which is obviously the equation for degenerate by parity levels in the system with two infinitely separated wells or equivalently for the levels of the single well. Let us consider one of the levels ϵ_0 in the single well for which $f_1(\epsilon_0) = 0$. In the limit $d \to \infty$ the value ϵ_0 serves as the zero approximation for corresponding even and odd levels in the double-well potential (24). To find the splitting of ϵ_0 into the even and odd components for finite $d \gg 1$, let us seek the solution of (46) in the form $\epsilon = \epsilon_0 + \delta\epsilon$, where $\delta\epsilon$ is a small correction to ϵ_0. Inserting this expansion into (46) and decomposing the left-hand side in $\delta\epsilon$ including the third order with account of $f_1(\epsilon_0) = 0$, one obtains a cubic equation

$$-A_1e^{-4d\sqrt{1-\epsilon_0^2}} + B_1e^{-4d\sqrt{1-\epsilon_0^2}}\delta\epsilon + C_1\delta\epsilon^2 + D_1\delta\epsilon^3 = 0, \quad (47)$$

where

$$A_1 = f_2^2(\epsilon_0),$$
$$B_1 = -\frac{2f_2(\epsilon_0)}{\sqrt{1-\epsilon_0^2}}\left[2d\epsilon_0f_2(\epsilon_0) + \sqrt{1-\epsilon_0^2}f_2'(\epsilon_0)\right], \quad (48)$$
$$C_1 = [f_1'(\epsilon_0)]^2, \quad D_1 = f_1'(\epsilon_0)f_1''(\epsilon_0).$$

Solving further Eq. (47) by means of successive iterations, one finds the splitting of the unperturbed level ϵ_0,

$$\delta\epsilon = \pm|K_1(a)|e^{-2d\sqrt{1-\epsilon_0^2}} + K_2(a,d)e^{-4d\sqrt{1-\epsilon_0^2}} + O\big(e^{-6d\sqrt{1-\epsilon_0^2}}\big), \quad (49)$$

where

$$q_0 = V_0 + \epsilon_0, \quad K_1(a) = \frac{(1-\epsilon_0^2)(1-q_0^2)}{V_0\left(\epsilon_0 + q_0 + a q_0\sqrt{1-\epsilon_0^2}\right)}, \tag{50}$$

$$K_2(a,d) = \frac{(1-\epsilon_0^2)^{3/2}(q_0^2-1)^2}{2V_0^2\left(\epsilon_0+q_0+aq_0\sqrt{1-\epsilon_0^2}\right)^2} \times \left[4d\epsilon_0 + \frac{2a^2q_0^2(1-\epsilon_0^2)(\epsilon_0q_0-1) + (2-\epsilon_0^2-q_0^2)(\epsilon_0q_0+1) + a\sqrt{1-\epsilon_0^2}[2\epsilon_0q_0(q_0^2-1)+(\epsilon_0^2-1)(2q_0^2-1)]}{\sqrt{1-\epsilon_0^2}(q_0^2-1)\left(\epsilon_0+q_0+aq_0\sqrt{1-\epsilon_0^2}\right)}\right], \tag{51}$$

where the upper sign in (49) corresponds to the odd level and the lower to the even one. Here is worth noting that for discrete levels in the single well like (1) the relation $q_0 > 1$ always holds (for details see Ref. [34]). So both $K_{1,2}$ are always well defined, since their denominators are strictly positive.

In the case of $\epsilon_0 < 0$ for sufficiently large d both levels ϵ_{odd} and ϵ_{even} become also negative; therefore their total contribution to $\mathcal{E}_{\text{vac}}^{\text{int}}(d)$ is equal to

$$\epsilon_{\text{odd}} + \epsilon_{\text{even}} = 2\epsilon_0 + 2K_2(a,d)e^{-4d\sqrt{1-\epsilon_0^2}} + O\left(e^{-6d\sqrt{1-\epsilon_0^2}}\right). \tag{52}$$

So in this case the contribution to $\mathcal{E}_{\text{vac}}^{\text{int}}(d)$ for large d, caused by the negative discrete level $\epsilon_0 < 0$ in the single well, takes the form

$$\begin{aligned} S_{\text{int}}(d) &= \epsilon_{\text{odd}} + \epsilon_{\text{even}} - 2\epsilon_0 \\ &= 2K_2(a,d)e^{-4d\sqrt{1-\epsilon_0^2}} + O\left(e^{-6d\sqrt{1-\epsilon_0^2}}\right). \end{aligned} \tag{53}$$

At the same time, the zero level $\epsilon_0 = 0$ splits for finite d into a pair, where only the even one is negative, which gives the following term in $\mathcal{E}_{\text{vac}}^{\text{int}}$:

$$S_{\text{int}}(d) = \epsilon_{\text{even}} = -\frac{V_0^2-1}{(1+a)V_0^2}e^{-2d} + O(e^{-4d}). \tag{54}$$

It should be mentioned that the analysis performed above for the discrete level contribution to the interaction energy has the correct status only subject to the condition $d\sqrt{1-\epsilon_0^2} \gg 1$. This means that whenever the single-well parameters are such that the level ϵ_0 lies arbitrarily close to the lower threshold, the expressions (53) and (54) could be valid only for such separations which provide the fulfillment of this condition.

So the resulting behavior of $\mathcal{E}_{\text{vac}}^{\text{int}}(d)$ for $d \gg 1$ to a high degree turns out to be subject to the single-well configuration. If there are only positive levels in the single well, the asymptotics of $\mathcal{E}_{\text{vac}}^{\text{int}}(d)$ should be $O(e^{-4d}/\sqrt{d})$ due to the integral and renormalization terms. The strictly zero level $\epsilon_0 = 0$ yields the contributions to $I_{\text{int}}(d)$ and $S_{\text{int}}(d)$ with half the exponent rates (45) and (54), but in $\mathcal{E}_{\text{vac}}^{\text{int}}(d)$ these terms exactly cancel each other, and hence there remains the same exponential law of decrease $\sim e^{-4d}$.

In the presence of negative levels in the spectrum of the single well the leading term in the asymptotics of $\mathcal{E}_{\text{vac}}^{\text{int}}(d)$ becomes different, namely, the main contribution to the asymptotics of $\mathcal{E}_{\text{vac}}^{\text{int}}(d)$ will be given by the lowest ϵ_0,

$$\mathcal{E}_{\text{vac}}^{\text{int}}(d) = -2K_2(a,d)e^{-4d\sqrt{1-\epsilon_0^2}} + O\left(e^{-6d\sqrt{1-\epsilon_0^2}}\right). \tag{55}$$

It should be mentioned that ϵ_0 can be arbitrarily close to $\epsilon_F = -1$, hence $\sqrt{1-\epsilon_0^2}$ arbitrarily small (but nonzero). In this case the exponential decay of $\mathcal{E}_{\text{vac}}^{\text{int}}(d)$ takes place only at extremely large d subject to the condition $d\sqrt{1-\epsilon_0^2} \gg 1$ and so the Casimir interaction between such wells acquires the features of a long-range force. It is noteworthy that this effect arises due to the lowest discrete level, rather than the replacement of the exponential asymptotics by a powerlike behavior, which could happen only for a massless mediator similar to that considered in Refs. [23,24].

The same effect was found in Ref. [22], where it was shown that the electronic Casimir force between two impurities on a one-dimensional semiconductor quantum wire can be of a very long range, despite the nonzero effective mass of the mediator. It should be emphasized that in that work the electronic Casimir-Polder effect is interpreted in terms of the radiation reaction field, where one of the two sources creates a virtual cloud of the field around itself, and the interaction of this field with the other atom induces the Casimir-Polder force. So in contrast to our approach based on the QED vacuum polarization, there is no need to utilize the idea of vacuum fluctuations of the field as a cause of the electronic Casimir-Polder effect. Although these two interpretations look qualitatively different, Milonni [35,36] and Compagno *et al.* [37] revealed that they are closely related. Moreover, in the present case the analogy between these two approaches can be illustrated by means of the similarity in the answers for the origin of the long-distance behavior of Casimir force. In our case it is the negative discrete level in the single well, which lies close to the lower threshold, while in Ref. [22] it is the single-impurity ground-state energy, which could be very small as one of the striking features of the Van Hove singularity, which causes the appearance of the bound state just below the band edge regardless of the bare impurity energy [38]. Moreover, in both cases we deal with the effect, which cannot be described by means of the perturbative methods.

The concrete type of interaction between the wells can be quite different subject to the single-well parameters V_0 and a, both in the asymptotics and for finite distances between the

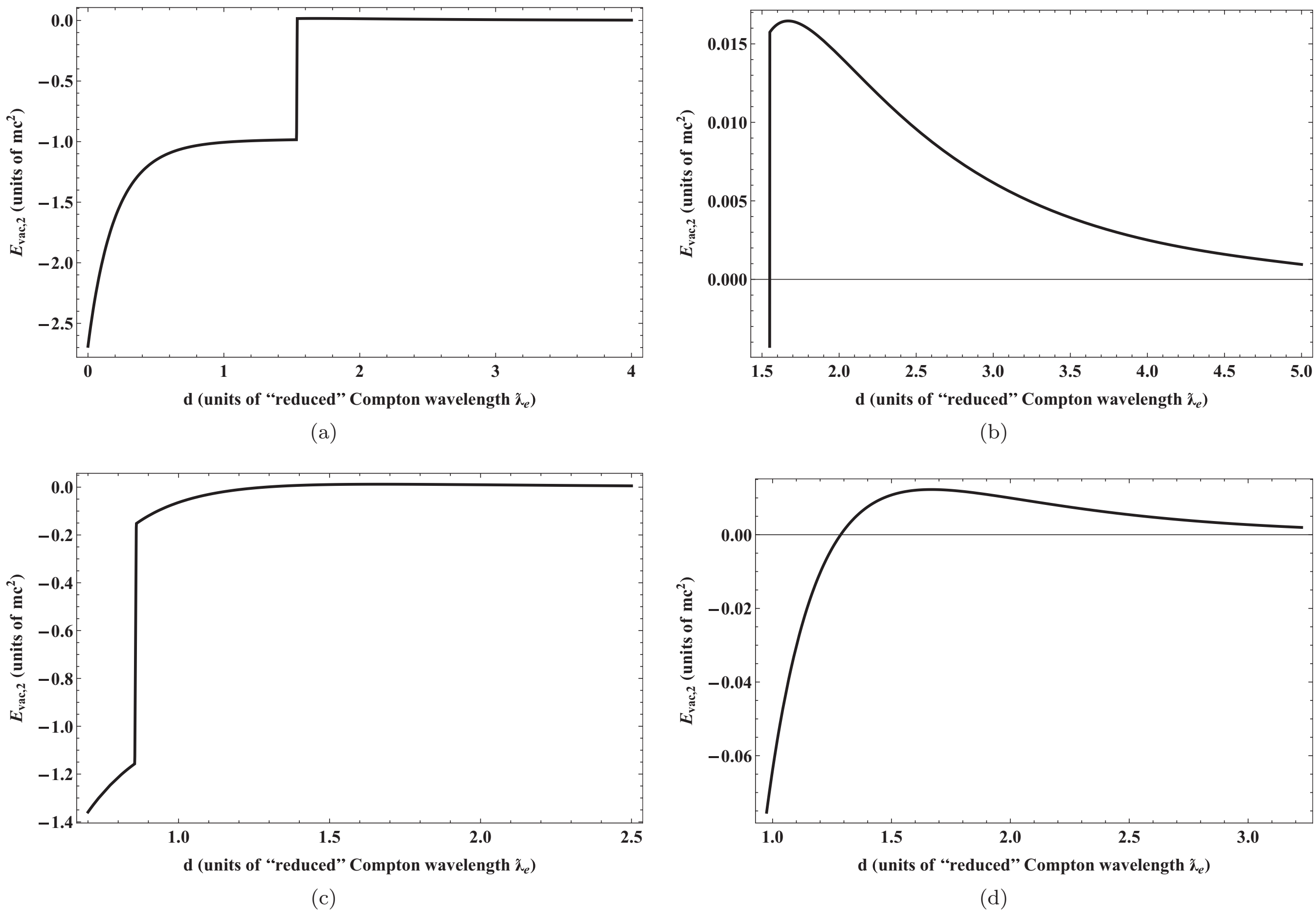


FIG. 2. Dependence of the Casimir interaction energy between two wells on the distance d between them for $a = 1$: (a) and (b) for $V_0 = 4.08$; (c) and (d) for $V_0 = 10$.

wells. In Figs. 2 and 3 $\mathcal{E}_{\text{vac}}^{\text{int}}(d)$ is presented for $a = 1$ and $V_0 = 4.08$, 7.4, 8, and 10. It follows that for $d \gg 1$ and $V_0 = 4.08$, 8, and 10 the interaction energy is positive (reflecting wells), whereas for $V_0 = 7.4$ the energy at large distances becomes negative (attracting wells).

Such behavior can be easily understood by means of the analysis presented above. Actually, for $V_0 = 4.08$ and 10 (Fig. 2) in the corresponding single well the lowest discrete level is negative (-0.9648 and $-0.908\,11$, respectively). As a result, for growing d in $\mathcal{E}_{\text{vac}}^{\text{int}}(d)$ first a jump by $+1$ takes place, provided by the emergence of this discrete level from the lower continuum by passing through the corresponding d_{cr} [quite similar to the picture shown in Fig. 14(b) of Ref. [33]]. For $d \gg 1$ the behavior of $\mathcal{E}_{\text{vac}}^{\text{int}}(d)$ is defined primarily by the contribution from the discrete spectrum, which in this case has the form

$$\begin{aligned}\mathcal{E}_{\text{vac}}^{\text{int}}(d) &\simeq -2K_2(a,d)e^{-4d\sqrt{1-\epsilon_0^2}} \\ &\to -4d\epsilon_0 \frac{\left(1-\epsilon_0^2\right)^{3/2}\left(q_0^2-1\right)^2}{V_0^2\left(\epsilon_0+q_0+aq_0\sqrt{1-\epsilon_0^2}\right)^2} \\ &\quad\times e^{-4d\sqrt{1-\epsilon_0^2}} > 0,\end{aligned} \tag{56}$$

since in the coefficient $K_2(a,d)$ under the condition $d\sqrt{1-\epsilon_0^2} \gg 1$ the main term in the square brackets in (51) will be $4d\epsilon_0$. So in presence of a negative level $\epsilon_0 < 0$ in the initial single well the interaction energy becomes positive for sufficiently large distances between wells.

For $V_0 = 7.4$ and 8 (Fig. 3) the negative levels in the single well are absent; therefore the behavior of $\mathcal{E}_{\text{vac}}^{\text{int}}(d)$ for $d \gg 1$ is defined by the expression

$$\begin{aligned}&\mathcal{E}_{\text{vac}}^{\text{int}}(d \gg 1) \\ &\quad = I_{\text{int}}(d) + V_0^2\Lambda_{\text{int}}(d) \\ &\quad \simeq V_0^2 \frac{e^{-4d}}{\sqrt{2\pi d}}\left[\frac{1}{2}\left(\frac{1}{1+z_0\cot(az_0)}\right)^2 - e^{-2a}\sinh^2(a)\right].\end{aligned} \tag{57}$$

The sign of $\mathcal{E}_{\text{vac}}^{\text{int}}(d \gg 1)$ depends on the sign of the term in square brackets in Eq. (57). For $V_0 = 7.4$ this term in Eq. (57) is negative, and hence, for $d \gg 1$, the wells attract each other [Fig. 3(b)]. For $V_0 = 8$ it is positive, since for these values of (V_0, a) the expression $1 + z_0\cot(az_0)$ is close to zero, as mentioned above, and so the asymptotics of the Casimir force is repulsive, but at the same time takes place for sufficiently larger d [see Fig. 3(d)].

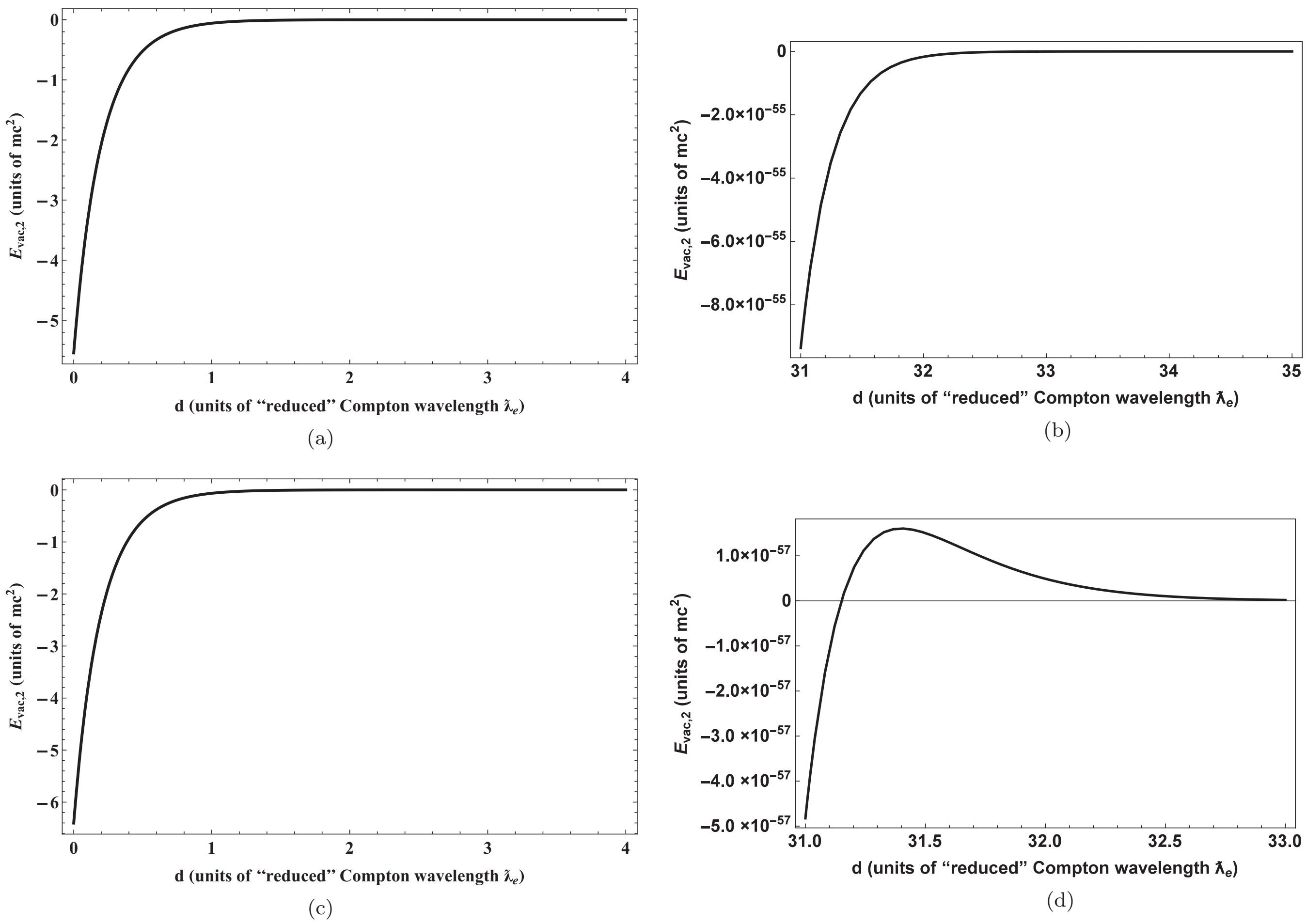


FIG. 3. Dependence of the Casimir interaction energy between two wells on the distance d between them for $a = 1$: (a) and (b) for $V_0 = 7.4$; (c) and (d) for $V_0 = 8$.

IV. CASIMIR FORCES BETWEEN TWO δ-LIKE WELLS

Now let us explore separately the Casimir interaction between two δ-like wells, for which the width and depth are related via $a = C/V_0$ with $V_0 \to \infty$ and $a \to 0$ for $C > 0$ being some constant, proportional to the charge of the source. It is well known that the direct insertion of δ-like potentials into the DE leads to contradictions, since the DE is first order (see, e.g., Ref. [23]). More concretely, the terms involving a delta function are only well defined if ψ is continuous at the points, where the δ-like peaks are located. However, Eq. (5) implies a jump in the lower component of the Dirac wave function ψ_2 for a continuous upper one ψ_1, while the second requires a jump in ψ_1 for continuous ψ_2. Thus the equations are not consistent. In Ref. [23] this problem was solved in terms of the transfer matrix, which in the limit of δ-like potentials remains well defined. Here we present another approach for dealing with δ-like potentials, based on the ln [Wronskian] contour integration, described in the previous sections.

First we consider the case of a single δ-like well, where in order to keep the correspondence with the case of finite wells considered above, it is implied that this δ-like well is twice as wide. Direct evaluation of the corresponding limits for separate components in Eq. (14) yields the following contributions to the renormalized vacuum energy of a single δ-like well. The integral term in Eq. (14) gives

$$I \to -\frac{1}{\pi}\int_0^\infty dy\,\mathrm{Re}\left[\ln\left(\cos(2C) - \frac{iy}{\sqrt{1+y^2}}\sin(2C)\right)\right] = \frac{1-|\cos(2C)|}{2}. \tag{58}$$

The equation for the discrete spectrum takes the form

$$\cos(2C) - \frac{\epsilon}{\sqrt{1-\epsilon^2}}\sin(2C) = 0, \tag{59}$$

which possesses a single root

$$\epsilon_0 = \mathrm{sgn}[\sin(4C)]|\cos(2C)|. \tag{60}$$

Depending on the sign of $\sin(4C)$, this root can be either positive or negative, and hence does not contribute or contribute to the vacuum energy of the single δ-like well. So in the general case the nonrenormalized vacuum energy of a single δ-like well can be represented as

$$\mathcal{E}_{\mathrm{vac}} = I - S = I - \theta(-\epsilon_0)\epsilon_0 = \frac{1-\mathrm{sgn}[\sin(4C)]|\cos(2C)|}{2}. \tag{61}$$

Proceeding further, on account of the asymptotics for the renormalization coefficients $\lambda_1(a)$ and $\lambda_2(a)$ for infinitely small width of the well, which can be easily derived from formulas (17)–(22), one finds

$$V_0^2\lambda_1(a) \to \frac{V_0 C}{\pi} - C^2 \to \infty, \quad V_0^2\lambda_2(a) \to C^2. \tag{62}$$

So in contrast to all the others terms, the PT contribution to the renormalization term does not possess any finite δ-like potentials limit, and hence $\mathcal{E}^R_{\text{vac}}$ for the single δ-like well is divergent:

$$\mathcal{E}^R_{\text{vac}} = I - S + \lambda V_0^2 \to \frac{1 - \text{sgn}[\sin(4C)]|\cos(2C)|}{2} - 2C^2 + \frac{V_0 C}{\pi} \to \infty. \tag{63}$$

Actually, this result should be expected from general considerations, since for discontinuous potentials the Fourier transform $\tilde{A}_0(q)$ of the external potential $A_0^{ext}(x)$ decreases in the momentum space too slowly and so the one-loop perturbative energy diverges. The same in essence effect appears also in more spatial dimensions by screening of the Coulomb asymptotics through the simple vertical cutoff, and it is necessary to introduce additional smoothing in order to maintain the convergence of the perturbative contribution to the energy [32]. It should be clear that in the considered case of a δ-like well such smoothing would also lead to a finite answer.

However, the Casimir interaction energy between two δ-like wells turns out to be a well-defined quantity without any additional smoothing, since the divergent parts do not depend on the distance between wells. Namely, the integral component in (30) will give in this case the following contribution to $\mathcal{E}^{\text{int}}_{\text{vac}}(d)$:

$$I_{\text{int}}(d) = I(d) - I(d \to \infty) \to -\frac{1}{2\pi}\int_0^\infty dy \ln\left[1 + (\sin C)^4 e^{-4d\sqrt{1+y^2}} \times \frac{e^{-4d\sqrt{1+y^2}} - 2[(1+y^2)(\cot C)^2 - y^2]}{[(\cos C)^2 + y^2]^2}\right]. \tag{64}$$

Here it should be mentioned that in this case each δ-like well should be half as wide as the single δ-like well considered in (58)–(63), which implies that $C \to C/2$ in all the subsequent expressions, defining separate components in Eq. (14) for the two-δ-like well configuration.

In particular, the equation for the discrete spectrum (46) splits now into two equations for two levels $\epsilon_\pm$,

$$\cot C\sqrt{1-\epsilon_\pm^2} - \epsilon_\pm = \mp e^{-2d\sqrt{1-\epsilon_\pm^2}}, \tag{65}$$

whence the next contribution to $\mathcal{E}^{\text{int}}_{\text{vac}}(d)$ from the negative part of the discrete spectrum follows

$$S_{\text{int}} \to \theta(-\epsilon_+)\epsilon_+ + \theta(-\epsilon_-)\epsilon_- - 2\theta(-\epsilon_0)\epsilon_0,$$

with ϵ_0 now the single level of a separated δ-like well, which differs from (60) by $C \to C/2$, namely,

$$\epsilon_0 = \text{sgn}[\sin(2C)]|\cos(C)|. \tag{66}$$

Proceeding further, from (32) one finds the limit for the renormalization coefficient in $\mathcal{E}^{\text{int}}_{\text{vac}}(d)$,

$$\Lambda_{\text{int}}(d)V_0^2 \to -\frac{2C^2}{\pi}\int_0^\infty dy\,\frac{e^{-4d\sqrt{1+y^2}}}{1+y^2}. \tag{67}$$

As a result, within the ln [Wronskian] contour integration the renormalized Casimir interaction energy between two δ-like wells turns out to be a well-defined quantity.

Compared to the case of finite wells, the Casimir interaction between two δ-like sources turns out to be no less rich in the variability of the Casimir force both at finite distances and in asymptotic behavior. Namely, for $d \gg 1$ the components of $\mathcal{E}^{\text{int}}_{\text{vac}}(d)$ behave as follows. The integral part (64) turns out to be

$$I_{\text{int}} \simeq e^{-4d}\frac{\tan^2 C}{2\sqrt{2\pi d}} \times \left[1 + \frac{1}{4}\left(\frac{19}{8} - \frac{3}{\cos^2 C}\right)\frac{1}{d} + O\left(\frac{1}{d^2}\right)\right], \tag{68}$$

the renormalization term (67) is equal to

$$\Lambda_{\text{int}}(d)V_0^2 \simeq -e^{-4d}\frac{C^2}{\sqrt{2\pi d}}\left[1 - \frac{5}{32}\frac{1}{d} + O\left(\frac{1}{d^2}\right)\right], \tag{69}$$

and the asymptotics of discrete levels is given by

$$\epsilon_\pm \simeq \epsilon_0 \pm e^{-2d\sqrt{1-\epsilon_0^2}}\left(1-\epsilon_0^2\right) - e^{-4d\sqrt{1-\epsilon_0^2}}\epsilon_0\left(1-\epsilon_0^2\right) \times \left(1 - 4d\sqrt{1-\epsilon_0^2}\right)/2 + O\left(e^{-6d\sqrt{1-\epsilon_0^2}}\right), \tag{70}$$

approaching the level in the single δ-like well (66) from above and from below, respectively.

If $\epsilon_0 < 0$, the contribution from the discrete spectrum for $d\sqrt{1-\epsilon_0^2} \gg 1$ is equal to

$$-S_{\text{int}} = -(\epsilon_+ + \epsilon_- - 2\epsilon_0) \simeq e^{-4d\sqrt{1-\epsilon_0^2}}\epsilon_0\left(1-\epsilon_0^2\right)\left(1 - 4d\sqrt{1-\epsilon_0^2}\right) > 0 \tag{71}$$

and due to the exponent $e^{-4d\sqrt{1-\epsilon_0^2}}$ turns out to be the leading term in $\mathcal{E}^{\text{int}}_{\text{vac}}(d)$, implying for ϵ_0 close to ϵ_F the existence of long-range forces between such δ-like wells quite similar to the case of finite wells. In turn, this is the reason for the behavior of the interaction energy between wells for $C = 3$ and $C = 5$ for large separation [see Figs. 4(d) and 4(f) below]. At the same time, if $\epsilon_0 > 0$, then $S_{\text{int}} = 0$ and the interaction energy $\mathcal{E}^{\text{int}}_{\text{vac}}(d) = I_{\text{int}}(d) + \Lambda_{\text{int}}(d)V_0^2$ decreases with growing d much faster, namely, as $O(e^{-4d})$.

If $\epsilon_0 = 0$, i.e., for $C = \pi/2 + \pi n$, the expression (68) is not valid, since an essential circumstance here is that $\cos C$ entering the denominators in Eqs. (64) and (68) should be nonzero. In this case the integral term transforms into

$$I_{\text{int}} \simeq -e^{-2d} + O(e^{-4d}), \tag{72}$$

while the contribution from the discrete spectrum contains now the level $\epsilon_- < 0$ only and gives

$$-S_{\text{int}} \simeq e^{-2d} + O(e^{-6d}). \tag{73}$$

Therefore, for $\epsilon_0 = 0$ the interaction energy between two δ-like wells decreases also as $O(e^{-4d})$.

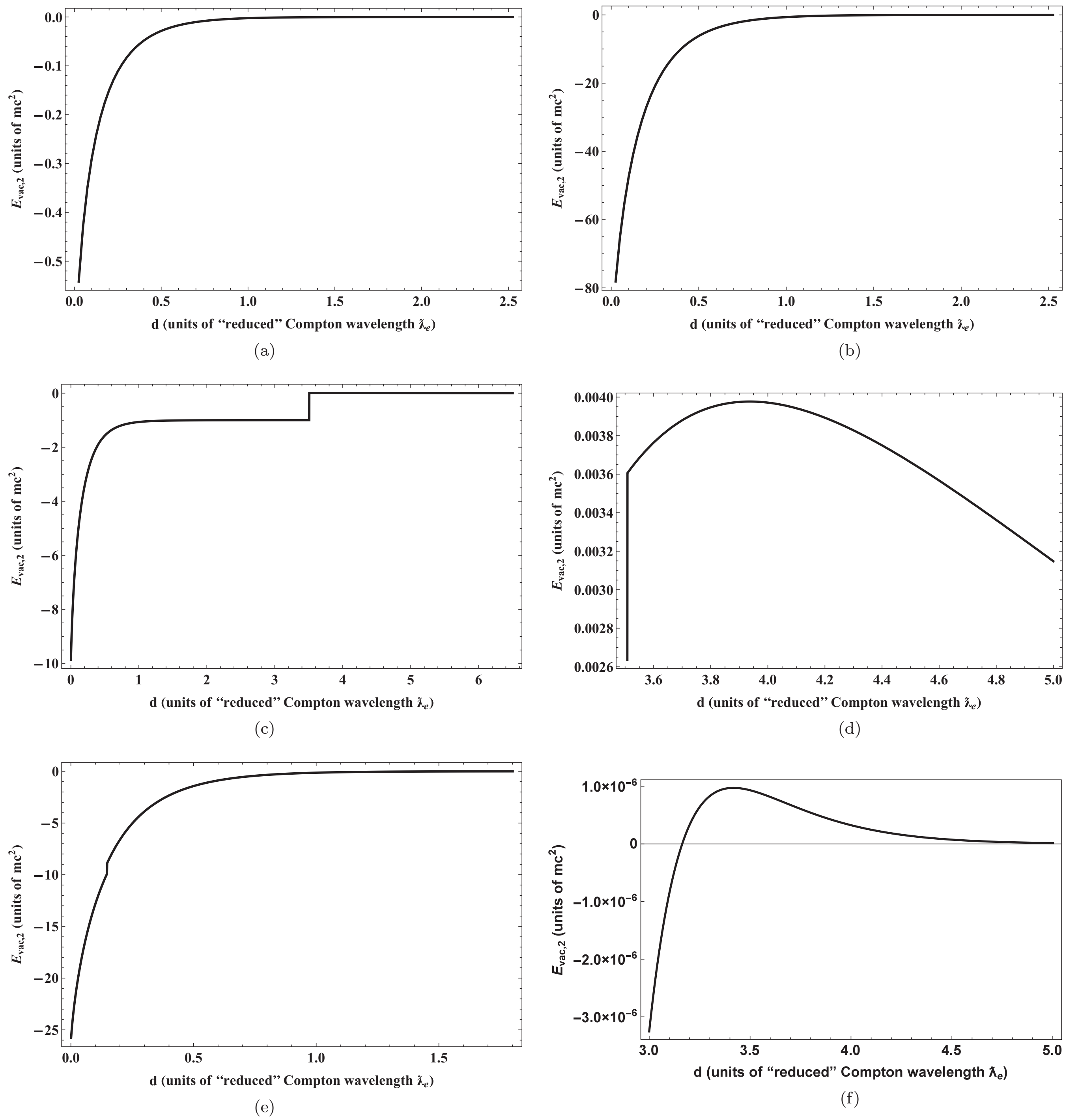


FIG. 4. Different types of the Casimir interaction energy between two δ wells as a function of the distance d between them for (a) $C = 1$, (b) $C = 10$, (c) and (d) $C = 3$, (e) and (f) $C = 5$.

Figure 4 shows the dependence of the interaction energy between two δ-like wells on the distance d between them for a set of different values of the parameter C. As it follows from Figs. 4(c)–4(f), depending on the concrete value of C, the nature of the Casimir force between wells may change from attraction to repulsion with growing d. In the present case this effect takes place for $C = 3$ and $C = 5$. For other values of C shown in Fig. 4, the interaction energy is strictly negative and grows with increasing d, so the wells attract each other.

The jumplike behavior of energy at $d = 3.5076$ for $C = 3$ [Fig. 4(c)] and at $d = 0.1479$ for $C = 5$ [Fig. 4(e)] is caused by the emergence of a new level at the lower threshold, provided the condition

$$d = -\cot(C)/2 > 0, \tag{74}$$

which follows from (65) in the limit $\epsilon_- \to -1$, is fulfilled. Another way to achieve this condition is to use the equation for critical charges

$$\sqrt{(V_0-1)^2-1}\cos[a\sqrt{(V_0-1)^2-1}] + 2dV_0\sin[a\sqrt{(V_0-1)^2-1}]=0 \tag{75}$$

in the δ-like potentials limit (the derivation and status of this equation are considered in detail in Ref. [33]).

With further removal of the wells from each other this level goes up, approaching from below the unique level ϵ_0 in the single δ-like well (66) (for $C=3$ and $C=5$ the latter is negative). Meanwhile the second level goes down, approaching the value ϵ_0 from above. For $C=1$ and $C=10$ there are no negative ϵ_0, and so starting from sufficiently large d the contribution from the discrete spectrum to $\mathcal{E}^{\text{int}}_{\text{vac}}(d)$ disappears.

V. CASIMIR FORCES IN THE SOURCE-ANTISOURCE SYSTEM

There exists only one exception when the effect of the long-range Casimir force in a quasi-one-dimensional QED system with short-range Coulomb sources of the type considered above is not in principle possible. It is the antisymmetric configuration of the source-antisource type, where one of the wells is replaced by a barrier with the same width and height. For our purposes it would be pertinent to consider an even more general situation, described by the external potential of the form

$$W_2(x)=-[V_1\,\theta(x-d)+V_2\,\theta(-x-d)]\,\theta(d+a-|x|), \tag{76}$$

although in what follows we will be interested primarily in the antisymmetric case with $V_1=-V_2=V_0>0$.

In the first step, for such a configuration of external short-range Coulomb sources, the calculation of the corresponding vacuum charge density will be useful. For this purpose one needs to consider the trace of the Green's function

$$\mathrm{Tr}G(x,x;\epsilon)=\frac{1}{J(\epsilon)}\psi_L^T(x)\psi_R(x), \tag{77}$$

with $\psi_{R,L}(x)$ the solutions of the DE (5) with $V(x)$ replaced by $W_2(x)$, which are regular at $\pm\infty$, respectively, while $J(\epsilon)$ is their Wronskian. As in Eq. (11), we use here the denotation

$$[f,g]_a=f_2(a)g_1(a)-g_2(a)f_1(a) \tag{78}$$

for the Wronskian of functions $f(x)$ and $g(x)$, calculated at the point $x=a$. In terms of the latter definition the Wronskian $J(\epsilon)$ in Eq. (77) is equal to

$$J(\epsilon)=[\psi_L,\psi_R]. \tag{79}$$

For the external potential $W_2(x)$ the pertinent solutions of the DE are represented in the form

$$\psi_L(x)=\begin{cases}\Phi(x), & x\leqslant -d-a\\ A_L u(V_2,x)+B_L v(V_2,x), & -d-a\leqslant x\leqslant -d\\ C_L\Phi(x)+D_L\Psi(x), & |x|\leqslant d\\ E_L u(V_1,x)+F_L v(V_1,x), & d\leqslant x\leqslant d+a\\ G_L\Phi(x)+H_L\Psi(x), & a+d\leqslant x,\end{cases} \tag{80}$$

$$\psi_R(x)=\begin{cases}G_R\Psi(x)+H_R\Phi(x), & x\leqslant -d-a\\ E_R u(V_2,x)-F_R v(V_2,x), & -d-a\leqslant x\leqslant -d\\ C_R\Psi(x)+D_R\Phi(x), & |x|\leqslant d\\ A_R u(V_1,x)-B_R v(V_1,x), & d\leqslant x\leqslant d+a\\ \Psi(x), & a+d\leqslant x,\end{cases} \tag{81}$$

with the coefficients $A_{R,L}$, $B_{R,L}$, $C_{R,L}$, $D_{R,L}$, $E_{R,L}$, $F_{R,L}$, $G_{R,L}$, and $H_{R,L}$ obtained via the requirement of continuity of solutions $\psi_{R,L}(x)$ at the points $x=\pm d$, $\pm(d+a)$, while $\Phi(x)$, $\Psi(x)$, $u(V_i,x)$, and $v(V_i,x)$, $i=1,2$, are the linearly independent solutions of the DE in the corresponding regions of constant potential $W_2(x)$,

$$\Phi(x)=\begin{pmatrix}\sqrt{1+\epsilon}\,e^{x\sqrt{1-\epsilon^2}}\\ \sqrt{1-\epsilon}\,e^{x\sqrt{1-\epsilon^2}}\end{pmatrix},\quad \Psi(x)=\begin{pmatrix}\sqrt{1+\epsilon}\,e^{-x\sqrt{1-\epsilon^2}}\\ -\sqrt{1-\epsilon}\,e^{-x\sqrt{1-\epsilon^2}}\end{pmatrix}, \tag{82}$$

$$u(V_i,x)=\begin{pmatrix}\sqrt{\epsilon+V_i+1}\cos[x\sqrt{(\epsilon+V_i)^2-1}]\\ -\sqrt{\epsilon+V_i-1}\sin[x\sqrt{(\epsilon+V_i)^2-1}]\end{pmatrix},$$

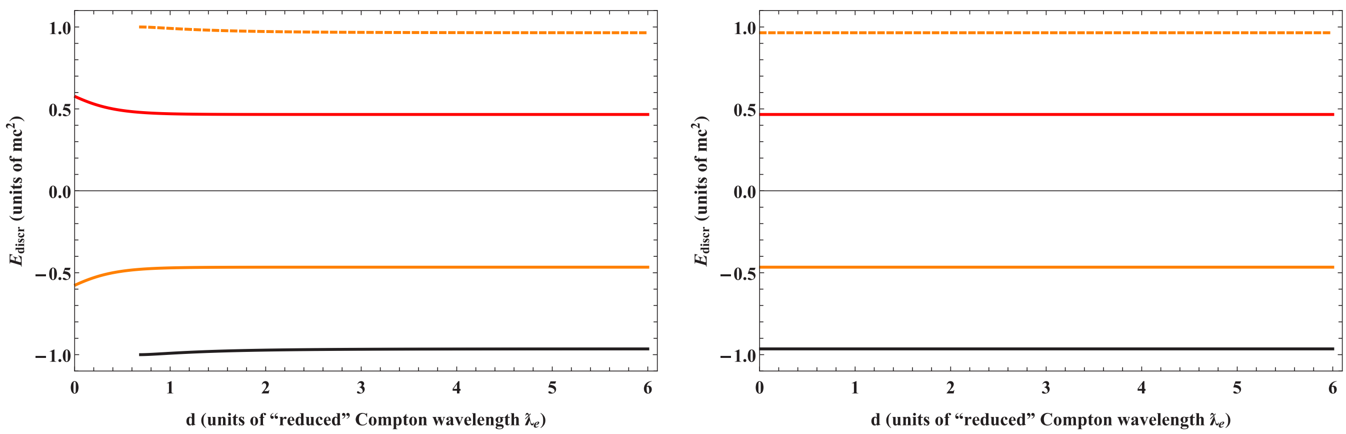
$$v(V_i,x)=\begin{pmatrix}\sqrt{\epsilon+V_i+1}\sin[x\sqrt{(\epsilon+V_i)^2-1}]\\ \sqrt{\epsilon+V_i-1}\cos[x\sqrt{(\epsilon+V_i)^2-1}]\end{pmatrix}. \tag{83}$$

The cross-linking coefficients with the label R take the form

$$\begin{aligned}A_R&=\frac{[\Psi,v(V_1)]_{a+d}}{[u(V_1),v(V_1)]_{a+d}},\\ B_R&=\frac{[\Psi,u(V_1)]_{a+d}}{[u(V_1),v(V_1)]_{a+d}},\\ D_R&=\frac{A_R[u(V_1),\Psi]_d-B_R[v(V_1),\Psi]_d}{[\Phi,\Psi]_d},\\ C_R&=\frac{A_R[\Phi,u(V_1)]_d-B_R[\Phi,v(V_1)]_d}{[\Phi,\Psi]_d},\\ F_R&=\frac{C_R[\Phi,u(V_2)]_d+D_R[\Psi,u(V_2)]_d}{[v(V_2),u(V_2)]_d},\\ E_R&=\frac{C_R[v(V_2),\Phi]_d+D_R[v(V_2),\Psi]_d}{[v(V_2),u(V_2)]_d},\\ H_R&=\frac{E_R[u(V_2),\Phi]_{a+d}+F_R[v(V_2),\Phi]_{a+d}}{[\Psi,\Phi]_{a+d}},\\ G_R&=\frac{E_R[\Psi,u(V_2)]_{a+d}+F_R[\Psi,v(V_2)]_{a+d}}{[\Psi,\Phi]_{a+d}}.\end{aligned} \tag{84}$$

The corresponding coefficients with the label L are obtained from (84) by means of the replacement $R\to L$ and $V_1\leftrightarrow V_2$. By means of (80)–(84) for the explicit form of $J(\epsilon)$ one finds

$$J(d,\epsilon)=2\frac{e^{-2a\sqrt{1-\epsilon^2}}}{\sqrt{1-\epsilon^2}}[f_1(V_1,\epsilon)f_1(V_2,\epsilon) - e^{-4d\sqrt{1-\epsilon^2}}f_2(V_1,\epsilon)f_2(V_2,\epsilon)], \tag{85}$$



(a) (b)

FIG. 5. Energy levels in the case $a = 1$ and $V_0 = 4.08$ for the (a) antisymmetric configuration, (b) single-well (black and red lines), and barrier (orange solid and dashed lines) configurations.

where

$$f_1(V_i,\epsilon) = \sqrt{1-\epsilon^2}\cos[a\sqrt{(V_i+\epsilon)^2-1}] - (\epsilon^2 - 1 + V_i\epsilon) \times \sin[a\sqrt{(V_i+\epsilon)^2-1}]/\sqrt{(V_i+\epsilon)^2-1},$$
$$f_2(V_i,\epsilon) = V_i\sin[a\sqrt{(V_i+\epsilon)^2-1}]/\sqrt{(V_i+\epsilon)^2-1}. \quad (86)$$

Let us consider now more thoroughly the antisymmetric case of the barrier-well configuration, when $V_1 = -V_2 = V_0 > 0$. As it follows from the expressions (85) and (86), in this case $J(d,\epsilon)$ turns out to be an even function of the energy

$$J(d,\epsilon) = J(d,-\epsilon). \quad (87)$$

Therefore, the discrete spectrum of the problem should be sign symmetric, i.e., the levels appear only in pairs with $\pm\epsilon$. Actually, the latter circumstance is the general feature of the source-antisource system, including both the discrete spectrum and continua. Namely, all the energy eigenstates in such a system are related via (up to a phase factor)

$$\psi_{-\epsilon}(x) = \alpha\,\psi_\epsilon(-x). \quad (88)$$

The typical behavior of levels for the antisymmetric case is shown in Fig. 5(a) in relation to the distance d between sources for $a = 1$ and $V_0 = 4.08$. The set (V_0, a) is taken with the same values as for the symmetric case containing two wells, considered in Sec. III. The symmetry of levels relative to the zero-energy line is apparent. Note also that the highest and lowest levels appear only starting from certain $d > 0$. With increasing d all the levels tend to constant values, coinciding with those of the single well and barrier of the same width a and depth or height V_0. Such behavior follows directly from the equation $J(d,\epsilon) = 0$. For $d \gg 1$ the second term in the expression (85) can be neglected; hence the resulting equation for the levels transforms into

$$f_1(V_0,\epsilon) f_1(-V_0,\epsilon) = 0. \quad (89)$$

In turn, the latter splits into two independent equations for the levels in the single well and barrier, namely, $f_1(V_0,\epsilon) = 0$ for the well and $f_1(-V_0,\epsilon) = 0$ for the barrier, which are related by reflection $\epsilon \to -\epsilon$. So the discrete spectra of the well and barrier differ only by the sign, as expected. For more clarity, Fig. 5(b) shows the levels in the single well and barrier with the same parameters $a = 1$ and $V_0 = 4.08$. There exist two levels with values $\epsilon_1 = -0.965$ and $\epsilon_2 = 0.466$ in the well, while for the barrier one finds two levels with opposite signs.

The sign symmetry of the energy spectrum in the source-antisource systems leads to significant changes in the definition and properties of vacuum polarization density and energy. The most important point here is that due to the sign symmetry of the levels the whole spectrum splits into two nonintersecting parts with positive and negative energies, respectively, since the levels cannot intersect, and hence cannot cross the zero line (see Fig. 6). Therefore, in this case the Fermi level, dividing the electronic and positronic (electron-hole) eigenstates in the initial expressions for the vacuum averages similar to (4), should be chosen equal to zero, i.e., $\epsilon_F = 0$.

So the starting expression for the induced density should be written as

$$\rho_{\rm vac}(x) = -\frac{|e|}{2}\left(\sum_{\epsilon_n<0}\psi_n(x)^\dagger\psi_n(x) - \sum_{\epsilon_n>0}\psi_n(x)^\dagger\psi_n(x)\right), \quad (90)$$

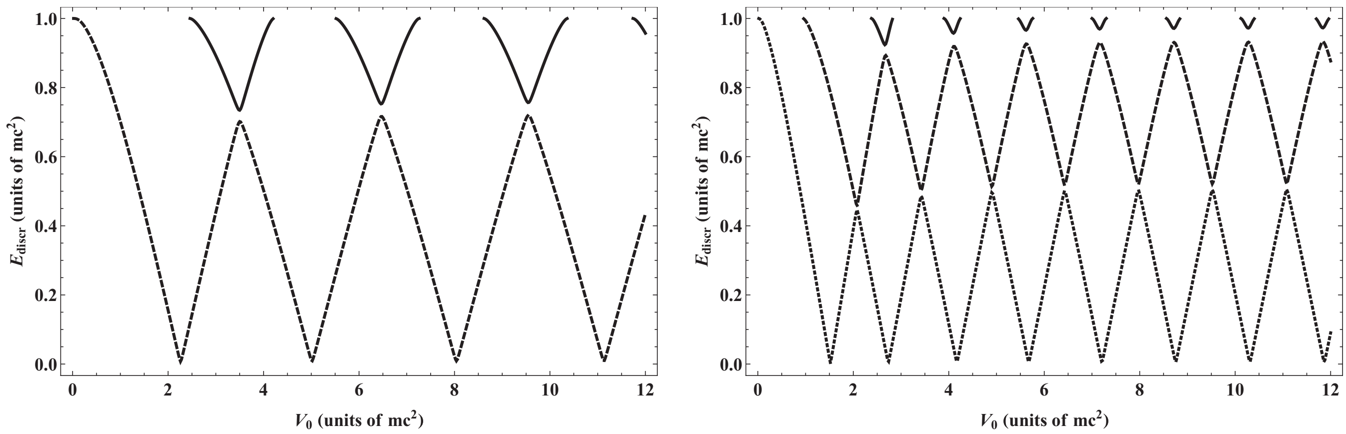
where ϵ_n and $\psi_n(x)$ are the eigenvalues and the eigenfunctions of the corresponding DE for the antisymmetric case. Proceeding further, one finds that due to the sign symmetry of the spectrum, the WK contour, shown in Fig. 1, transforms now into the symmetric one with respect to reflection $\epsilon \to -\epsilon^*$, while its separate parts $P(R)$ and $E(R)$ lie in their respective half planes Re$\epsilon < 0$ for $P(R)$ and Re$\epsilon > 0$ for $E(R)$ and do not intersect with the imaginary axis.

As it should be expected on general grounds, from Eq. (90) combined with the relation (88) it follows that the vacuum density is an odd function

$$\rho_{\rm vac}(x) = -\rho_{\rm vac}(-x), \quad (91)$$

reproducing in this way the similar property of the external potential (76) in the antisymmetric case. Applying further the same technique as in Refs. [2–4,25–27] for the expression of the induced density in terms of TrG, one finds

$$\rho_{\rm vac}(x) = \frac{|e|}{2\pi}\int_{-\infty}^{\infty} dy\,{\rm Tr}G(x,x;iy). \quad (92)$$



(a) (b)

FIG. 6. Behavior of positive energy levels in the antisymmetric case of the barrier-well type in dependence on V_0 for (a) $d = 2$ and $a = 1$; (b) $d = 2$ and $a = 2$.

Note that in the expression (92) there is no separate contribution from negative discrete levels, since the latter appear only in the case when the part $E(R)$ of the WK contour captures a piece of the negative real axis containing these discrete levels.

Since $\rho_{\rm vac}(x)$ is odd from the very beginning, in contrast to the symmetric case [33] and all the more to the one-dimensional QED systems with long-range external Coulomb sources considered in Refs. [2–4], the total induced charge vanishes now without any additional renormalization

$$Q_{\rm vac} = \int_{-\infty}^{\infty} dx\, \rho_{\rm vac}(x) = 0. \tag{93}$$

Nevertheless, a finite renormalization is needed due to the condition that in the perturbative region $V_0 \to 0$ the renormalized vacuum density $\rho^R_{\rm vac}(x)$ should reproduce the perturbative density $\rho^{(1)}_{\rm vac}(x)$, calculated within the standard PT to the leading (one-loop) order [2–4,33]. Actually, this procedure is equivalent to a finite renormalization and normalization conditions as known from perturbative QED (see, e.g., Ref. [39]). The explicit expression for $\rho^{(1)}_{\rm vac}(x)$ reads

$$\begin{aligned}\rho^{(1)}_{\rm vac}(x) = -\frac{|e|}{\pi^2}\int_0^{\infty}\frac{dq}{q}\left(1 - 2\frac{\operatorname{arcsinh}(q/2)}{q\sqrt{1+(q/2)^2}}\right)\\ \times (V_1\{\sin[q(d-x)] - \sin[q(a+d-x)]\}\\ + V_2\{\sin[q(d+x)] - \sin[q(a+d+x)]\}).\end{aligned} \tag{94}$$

It should be quite clear without any additional comments that in the antisymmetric case the perturbative density is an odd function by construction; hence, in this case, the total induced charge $Q^{(1)}_{\rm vac}$, calculated to the leading order of PT by means of $\rho^{(1)}_{\rm vac}(x)$, vanishes (actually this statement holds also for the nonsymmetric case; for details see, e.g., Ref. [28]).

Thus, by means of the standard renormalization procedure for the vacuum density considered in Refs. [2–5,25–27], one obtains

$$\rho^R_{\rm vac}(x) = \rho^{(1)}_{\rm vac}(x) + \rho^{(3+)}_{\rm vac}(x), \tag{95}$$

where

$$\rho^{(3+)}_{\rm vac}(x) = \frac{|e|}{2\pi}\int_{-\infty}^{\infty} dy[\mathrm{Tr}G(x,x;iy) - \mathrm{Tr}G^{(1)}(x,x;iy)]. \tag{96}$$

In the expression (96) the function $\mathrm{Tr}G^{(1)}(x,x;\epsilon)$ is the first-order term in the expansion of the Green's function in the Born series in powers of V_0 (for the antisymmetric case). Figure 7 shows the renormalized vacuum charge density for the following sets of system parameters: $d = 2$, $a = 1$, and $V_0 = 8$ [Fig. 7(a)] and $d = 2$, $a = 2$, and $V_0 = 2$ [Fig. 7(b)]. In general, the behavior of the density is quite similar to those achieved for the case of two wells in Ref. [33], with the main exception that now the density is odd.

After these preliminary considerations let us turn to the calculation of the Casimir energy for the antisymmetric configuration. Repeating the procedure of passing from the initial definition of the vacuum energy by means of the Schwinger average (4) to the integration over the imaginary axis in (34), considered in detail for the symmetric case, for the nonrenormalized vacuum energy one obtains

$$\mathcal{E}_{\rm vac}(d) = -\frac{1}{\pi}\int_0^{\infty} dy\,[\ln J_{\rm red}(d,iy)], \tag{97}$$

with the same definition of the reduced Wronskian $J_{\rm red}(d,iy)$ as in Eq. (35). From the latter for the antisymmetric case one obtains

$$\begin{aligned}J_{\rm red}(d,iy) = \frac{e^{-2a\sqrt{1+y^2}}}{1+y^2}[|f_1(V_0,iy)|^2\\ + e^{-4d\sqrt{1+y^2}}|f_2(V_0,iy)|^2],\end{aligned} \tag{98}$$

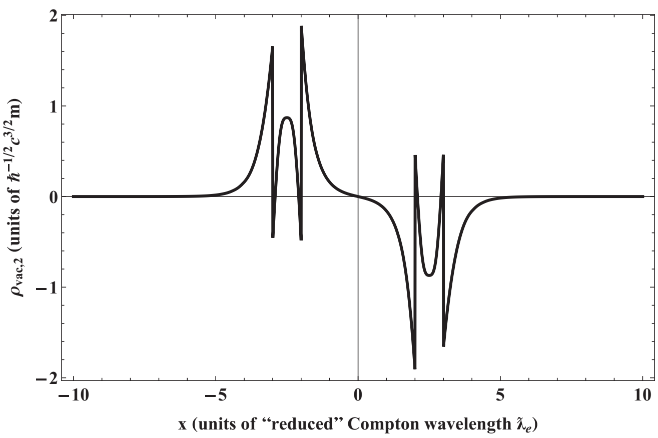
with $f_i(V_0,iy)$ defined in Eq. (86). Note also that in contrast to (14), for the same reasons as in Eq. (92), in the expression (97) there is no separate contribution from the negative discrete levels.

The renormalized vacuum energy is represented as

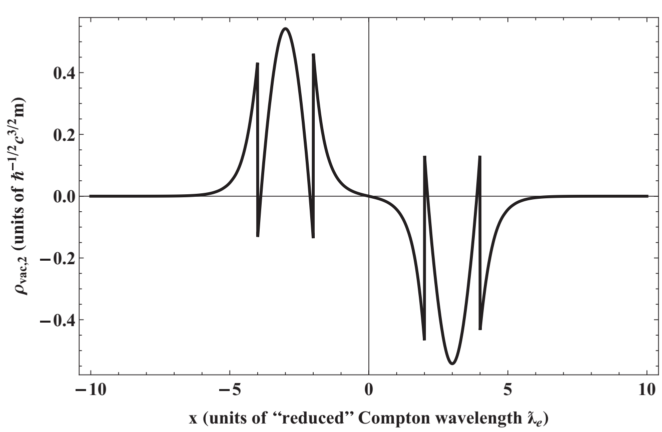
$$\mathcal{E}^R_{\rm vac}(d) = \mathcal{E}_{\rm vac}(d) + \Lambda(d)V_0^2, \tag{99}$$

where the renormalization coefficient

$$\Lambda(d) = \Lambda_1(d) - \Lambda_2(d)$$



(a)



(b)

FIG. 7. Renormalized vacuum charge density in the antisymmetric case for the following sets of the system parameters: (a) $d = 2$, $a = 1$, and $V_0 = 8$; (b) $d = 2$, $a = 2$, and $V_0 = 2$.

contains two terms of the form

$$\Lambda_1(d) = \lim_{V_0\to 0} \mathcal{E}^{(1)}_{\rm vac}(d)/V_0^2,$$

$$\Lambda_2(d) = \lim_{V_0\to 0} \mathcal{E}_{\rm vac}(d)/V_0^2 = \frac{a}{\pi} - \frac{1}{8} + \frac{1}{\pi}\int_0^\infty dy \frac{1 - 2e^{-4d\sqrt{1+y^2}}\sinh^2(a\sqrt{1+y^2})}{2(1+y^2)^2} \times e^{-2a\sqrt{1+y^2}}, \tag{100}$$

where the first-order perturbative vacuum energy $\mathcal{E}^{(1)}_{\rm vac}(d)$ is given by the following expression, calculated within PT in the one-loop approximation for the antisymmetric case:

$$\mathcal{E}^{(1)}_{\rm vac}(d) = \frac{2V_0^2}{\pi^2}\int_0^{+\infty} dq \frac{\{\cos[q(a+d)] - \cos(qd)\}^2}{q^2} \times \left(1 - 2\frac{\operatorname{arcsinh}(q/2)}{q\sqrt{1+(q/2)^2}}\right). \tag{101}$$

In the antisymmetric case a relation similar to (20) and (28) holds (see the Appendix), which allows one to represent the renormalization coefficient in a more convenient form, namely,

$$\Lambda(d) = \frac{a}{\pi} - 2\Lambda_2(d). \tag{102}$$

To explore the Casimir force in the source-antisource system let us start with the behavior of nonrenormalized vacuum energy $\mathcal{E}_{\rm vac}(d)$ for large $d \gg 1$. In this case the expression (97) simplifies up to

$$\mathcal{E}_{\rm vac}(d \gg 1) = -\frac{1}{\pi}\int_0^\infty dy \ln\left[\frac{e^{-2a\sqrt{1+y^2}}}{1+y^2}|f_1(V_0, iy)|^2\right] \tag{103}$$

and coincides with the nonrenormalized total energy of the system, containing an infinitely separated barrier and well with the same width a and depth or height V_0, but preserving the antisymmetry property (88) of the whole configuration. Otherwise, considering the limiting configuration as a direct sum of the single barrier and single well without the antisymmetry property, we should deal with their contributions according to (14) for the well and to a similar expression for the barrier, where the additional sum includes now positive discrete levels and enters with opposite sign. As a result, in this case the limiting vacuum energy will contain twice the sum over discrete levels entering the expression (14). However, such a configuration cannot be considered as a physically correct limit for $\mathcal{E}_{\rm vac}(d \gg 1)$, since the antisymmetry property is lost.

So the nonrenormalized interaction energy in the coupled barrier-well system is equal to

$$\mathcal{E}_{\rm int}(d) = \mathcal{E}_{\rm vac}(d) - \mathcal{E}_{\rm vac}(d \to \infty) = -\frac{1}{\pi}\int_0^\infty dy \ln\left[J_{\rm red}(d, iy)(1+y^2)\frac{e^{2a\sqrt{1+y^2}}}{|f_1(V_0, iy)|^2}\right]. \tag{104}$$

Expanding the integrand on the right-hand side of (104) for $d \gg 1$ up to $O(e^{-8d\sqrt{1+y^2}})$, one obtains

$$\mathcal{E}_{\rm int}(d) \simeq -\frac{1}{\pi}\int_0^\infty dy\, e^{-4d\sqrt{1+y^2}}\left|\frac{f_2(V_0, iy)}{f_1(V_0, iy)}\right|^2. \tag{105}$$

Further expansion of the expression (105) for large d proceeds quite similarly to the symmetric case and leads to the answer

$$\mathcal{E}_{\rm int}(d) \simeq -V_0^2\frac{e^{-4d}}{\sqrt{2\pi d}}\left[\frac{A^2}{2} + \frac{1}{8d}\left(\frac{3A^2}{8} + B\right)\right] + O\left(\frac{1}{d^2}\right), \tag{106}$$

where z_0 and A are defined as in Eq. (43) and

$$B = A^3\left[-V_0^2\left(1 - \frac{\cot(az_0)}{z_0} + \frac{a}{\sin^2(az_0)}\right)^2 A - 2 - \frac{(1+z_0^4)}{z_0^3}\cot(az_0) + \frac{a}{z_0^2\sin^2(az_0)}\{1 - 2V_0^2[1 - az_0\cot(az_0)]\}\right]. \tag{107}$$

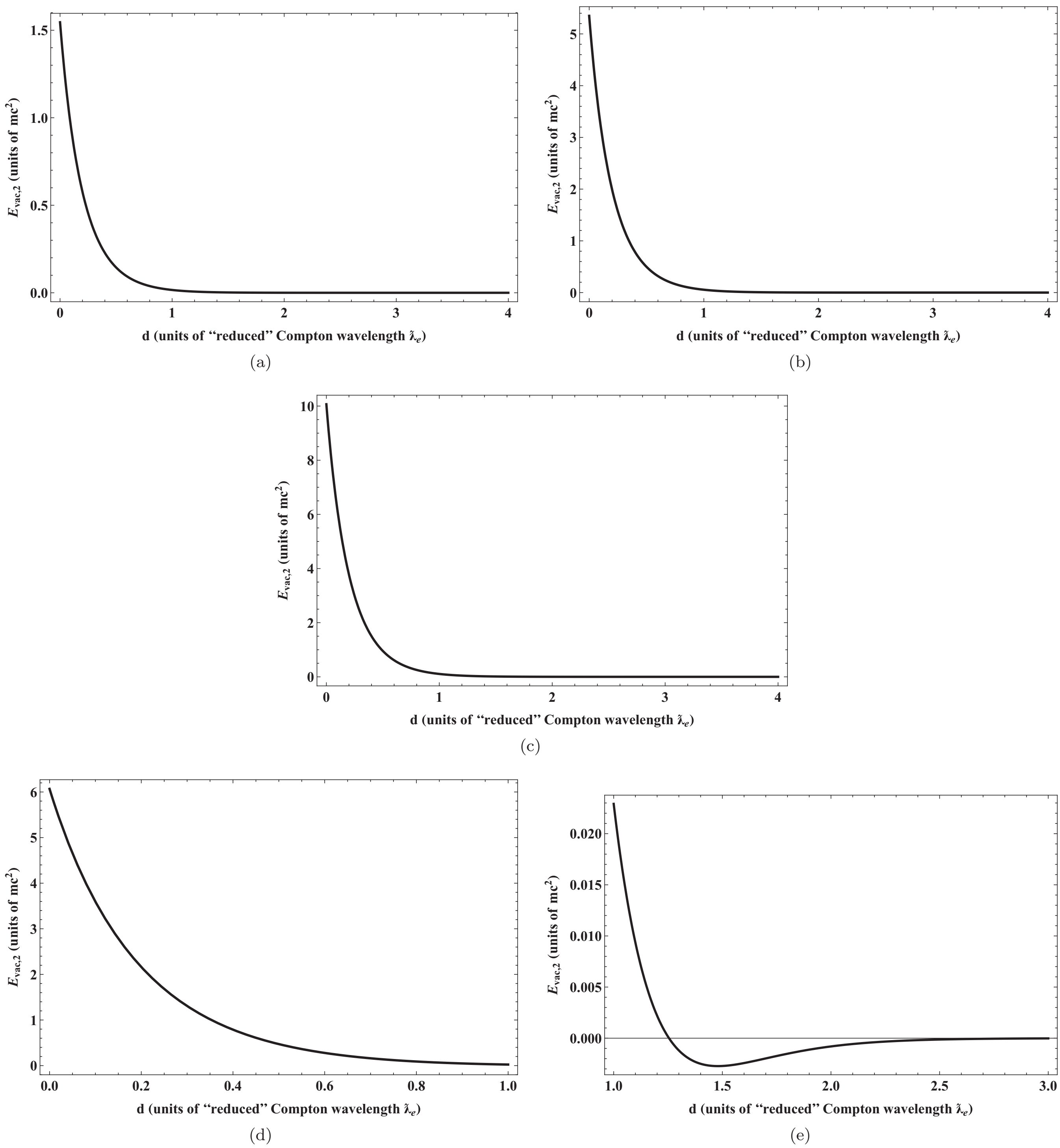


FIG. 8. Behavior of $\mathcal{E}^R_{\text{int}}(d)$ in the antisymmetric case for $a = 1$ and (a) $V_0 = 4.08$, (b) $V_0 = 7.4$, (c) $V_0 = 10$, (d) and (e) $V_0 = 8$.

As in the symmetric case, these expressions are valid for both $V_0 < 1$ and $V_0 > 1$, while for $V_0 = 1$, when $z_0 = 0$, they should be replaced by

$$A = \frac{a}{1+a}, \tag{108}$$

$$B = -a^2 \frac{45 + 135a + 165a^2 + 90a^3 + 28a^4 + 8a^5}{45(1+a)^4}. \tag{109}$$

Moreover, the expansion of $\mathcal{E}_{\text{int}}(d)$ for large d, presented above, becomes invalid when in the single well (or barrier) there exists the level with zero energy, since in this case the denominator in A vanishes, i.e., $\sin(az_0) + z_0 \cos(az_0) = 0$. Therefore, this case requires a separate analysis, similar to that considered for two wells in Sec. III.

The renormalization coefficient for $\mathcal{E}^R_{\text{int}}(d)$ coincides with the corresponding one in the two-well configuration up to the

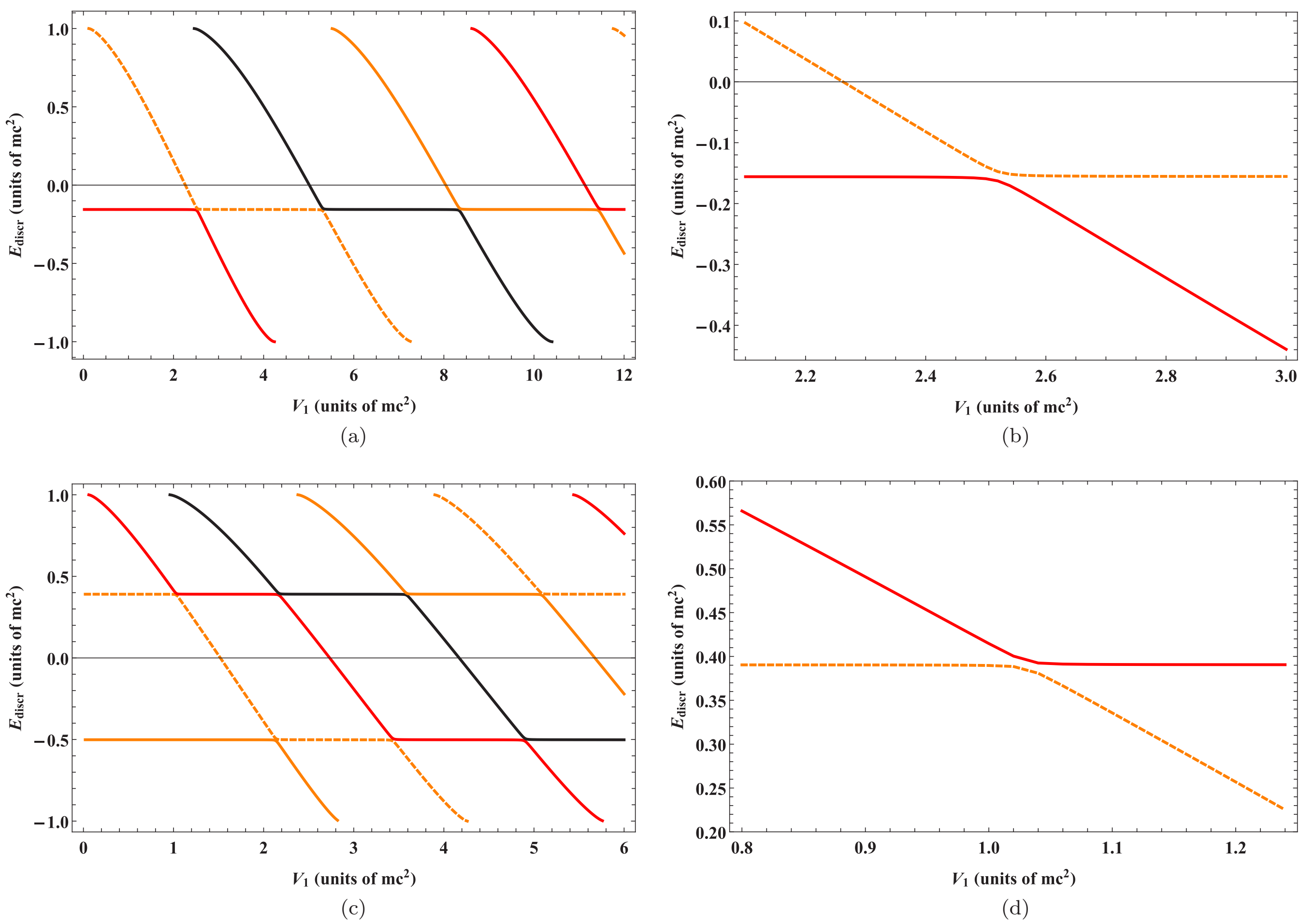


FIG. 9. Behavior of the levels in the barrier-well system without antisymmetry of the potential dependent on the well depth V_1 with fixed height of the barrier $V_2 = -2$ for (a) and (b): $d = 2$ and $a = 1$; (c) and (d): $d = 2$ and $a = 2$.

sign, namely,

$$\Lambda_{\text{int}}(d) = \frac{1}{2\pi}\int_0^\infty dy\, e^{-4d\sqrt{1+y^2}}\frac{(1 - e^{-2a\sqrt{1+y^2}})^2}{(1+y^2)^2} \geqslant 0, \tag{110}$$

and so for large d reveals the same asymptotics as in (33) with different sign. As a result, the leading term in the renormalized Casimir energy $\mathcal{E}^R_{\text{int}}(d)$ for the antisymmetric case turns out to be

$$\mathcal{E}^R_{\text{int}}(d) = \mathcal{E}_{\text{int}}(d) + \Lambda_{\text{int}}(d)V_0^2 \simeq V_0^2\frac{e^{-4d}}{\sqrt{2\pi d}}\left[e^{-2a}\sinh^2 a - \frac{A^2}{2}\right]. \tag{111}$$

In Eq. (111) the multiplier in square brackets changes sign depending on the single source parameters (V_0, a). In particular, for the set $a = 1$ and $V_0 = 4.08$, 7.4, and 10, considered in Sec. III, this multiplier is positive; hence the sources repel each other at large separations. In contrast, for $a = 1$ and $V_0 = 8$ it is negative and so the sources attract. Note that in the last case the Casimir force changes from repulsion to attraction by increasing d. The behavior of $\mathcal{E}^R_{\text{int}}(d)$ starting from sufficiently small separations up to large-d asymptotics is shown in Fig. 8.

Apart from these peculiar features, the general answer for the Casimir force in the antisymmetric case is substantially different from the symmetric one, since now the asymptotics of the Casimir force for large separations between sources is subject to the standard $\exp(-2ms)$ law. Moreover, it is the unique specifics of the source-antisource system, since it is the only case, when the symmetry between the positive and negative energy eigenstates according to (88) takes place. The direct consequence of this symmetry is that the separate contributions from negative discrete levels are absent in the final expressions (92) and (97) for $\rho_{\text{vac}}(x)$ and $\mathcal{E}_{\text{vac}}(d)$. Indeed, this circumstance underlies the standard $\exp(-2ms)$ decay of the Casimir force for large separations between sources, since in the symmetric case the breakdown of the latter is caused by the contribution from the negative discrete levels. Namely, the main contribution to the asymptotics of $\mathcal{E}^{\text{int}}_{\text{vac}}(d)$ will be given by the lowest $\epsilon_0 < 0$ according to Eq. (55). As soon as the strict antisymmetry of the external potential (76) is broken, the spectrum immediately transforms into the standard nonsymmetric form, where the levels are able to approach the threshold of the lower continuum, for instance, with growing depth of the well. This circumstance is reminiscent of the well-known quantum-mechanical effect, when in the one-dimensional potential well with arbitrarily small

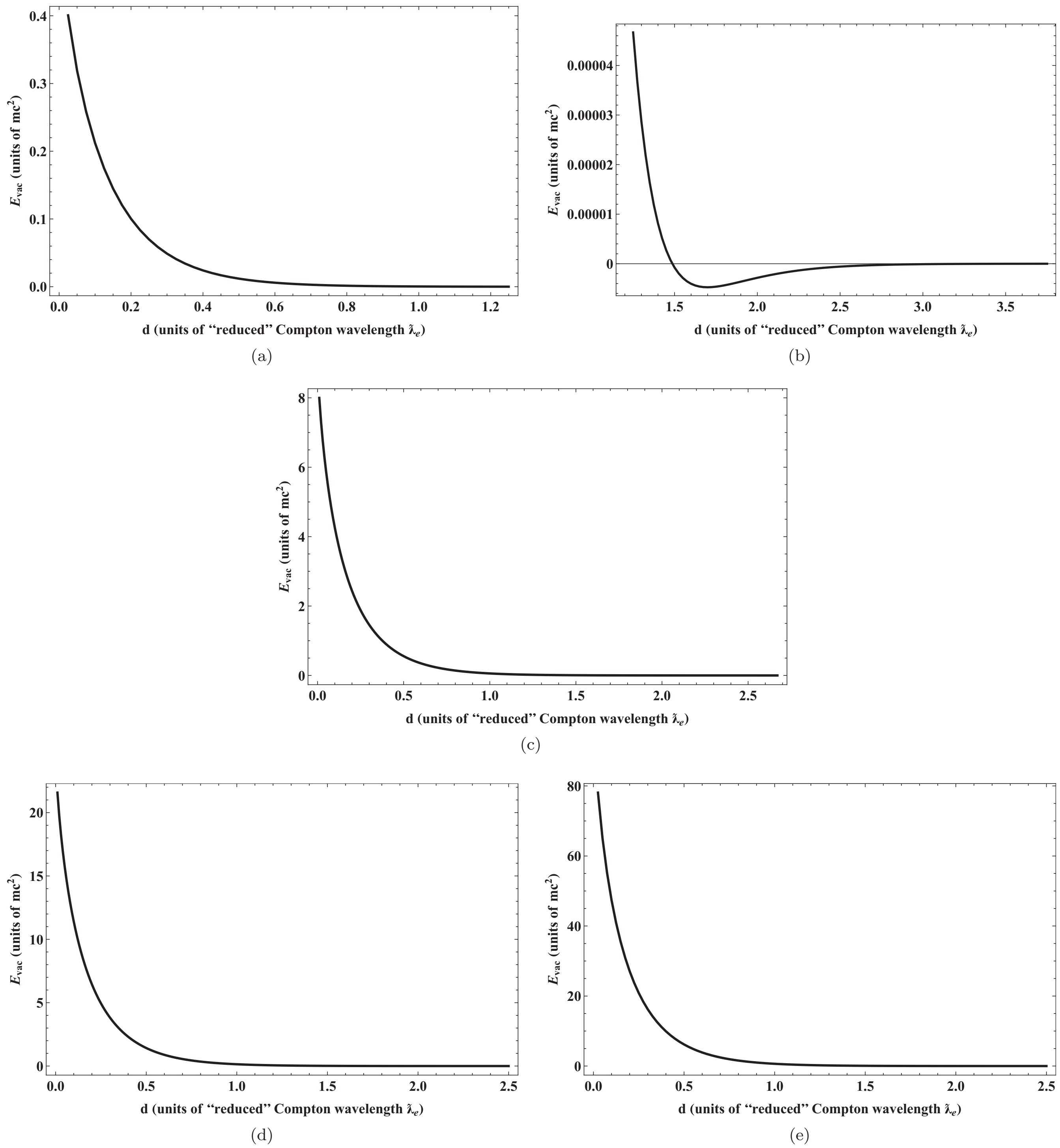


FIG. 10. Different types of the Casimir interaction energy in the antisymmetric case between the δ barrier and δ well as a function of the distance d between them for (a) and (b) $C = 1$, (c) $C = 3$, (d) $C = 5$, and (e) $C = 10$.

depth and size, but with equal height of both walls, there exists always at least one discrete level, which can be very shallow, but disappears as soon as the height of the walls becomes different. As an illustration of this property of the antisymmetric case Fig. 9 shows the behavior of the levels in the barrier-well system without antisymmetry of the potential $W_2(x)$ dependent on the well depth V_1 with fixed height of the barrier V_2.

Actually, one finds the same picture in the δ limit. With the same definitions $a = C/V_0$ with $V_0 \to \infty$ and $a \to 0$ and $C > 0$ some constant for the renormalized interaction energy $\mathcal{E}^R_{\rm int}(d) = \mathcal{E}_{\rm int}(d) + V_0^2 \Lambda_{\rm int}(d)$ between δ barrier and δ well one finds

$$V_0^2 \Lambda_{\rm int}(d) \to \frac{2C^2}{\pi} \int_0^\infty dy \, \frac{e^{-4d\sqrt{1+y^2}}}{1+y^2}, \tag{112}$$

while

$$\mathcal{E}_{\text{int}}(d) \to -\frac{1}{\pi}\int_0^\infty dy \ln\left[1 + e^{-4d\sqrt{1+y^2}}\frac{\sin^2 C}{\cos^2 C + y^2}\right], \tag{113}$$

and so $\mathcal{E}^R_{\text{int}}(d)$ is finite.

It is worth noting that in the antisymmetric case the δ limit reveals the same singularities as for a single-well or -barrier and two-δ-well configurations, namely, $V_0^2\Lambda_1(d) \to aV_0^2/\pi = CV_0/\pi \to \infty$, while $V_0^2\Lambda_2(d)$ remains finite. However, these singularities do not depend on d and so do not influence the answer for the interaction energy, which is always defined via subtraction (104).

Figure 10 shows the dependence of the interaction energy between the δ barrier and δ well on the distance d between them for a set of different values of the parameter C. For $C = 3$, 5, and 10 the corresponding interaction energy is positive and so decreases with growing distance between δ sources, leading to a repulsive Casimir force. For $C = 1$ the repulsion at large distances is replaced by attraction. Note also that at small distances d with the same values of the constant C the interaction between two δ wells is always attractive, while in the antisymmetric case it is strictly repulsive.

VI. CONCLUSION

In this work, by means of the ln [Wronskian] contour integration techniques for calculating the Casimir effect, we have shown the magnitude of the Casimir force variability for two short-range Coulomb sources, embedded in the background of one-dimensional massive Dirac fermions. The main result is that essentially nonperturbative vacuum QED effects, including the effects of supercriticality, are able to add a set of properties to Casimir forces between such sources, which turn out to be more diverse than the case of scattering potentials with scalar coupling to fermions, considered in Refs. [23,24]. In particular, we have shown that the interaction energy between two identical positively charged short-range Coulomb sources can exceed sufficiently large negative values and simultaneously reveal some features similar to a long-range force, like the electronic Casimir force between two impurities on a one-dimensional semiconductor quantum wire despite the nonzero effective mass of the mediator [22], which could significantly alter the properties of such quasi-one-dimensional QED systems.

The most intriguing circumstance here is that in the symmetric case their mutual interaction is governed primarily by the structure of the discrete spectrum of the single source, through which it can be tuned to give an attractive, a repulsive, or an almost compensated Casimir force with various rates of the exponential decay, quite different from the standard $\exp(-2ms)$ law. Let us mention once more that the essence of the long-range interaction between sources, which appears whenever the single well contains a level ϵ_0 close to the lower threshold, is that under these conditions the exponential decay starts at extremely large distances $d \gg (1-\epsilon_0^2)^{-1/2}$ between sources, rather than by replacement of the exponential asymptotics by a powerlike behavior, which could happen only for a massless mediator. No less interesting is the pattern of Casimir interaction observed in the δ limit with sources of negligible width, which can also be explored in detail within the presented ln [Wronskian] contour integration approach. The latter circumstance could be quite important, since in some reasonable cases the best description for impurities is achieved indeed in the δ limit.

Particular attention should be paid to the antisymmetric source-antisource system, which reveals quite different features. In particular, in this case there is no possibility for the long-range interaction between sources. The asymptotics of the Casimir force follows the standard $\exp(-2ms)$ law. Moreover, being calculated completely nonperturbatively, the symmetric and antisymmetric cases are substantially different for small separations between sources. Namely, there follows from Figs. 2–4, 8, and 10, which are calculated for the same sets of single-source parameters up to the replacement well with the barrier, that in the symmetric case the Casimir interaction between sources is attractive, while in the antisymmetric one it turns into sufficiently strong repulsion. At the same time, in the perturbative region $|V_0| \ll 1$ both cases correspond to a strictly positive vacuum energy in accordance with PT one-loop calculations (8) and (101), which vanishes for large distances $d \gg 1$, and hence describes repulsion. Remarkably enough, the classic electrostatic force for such Coulomb sources should be of opposite sign. There is no evident explanation for this effect. However, the set of parameters used is quite wide to consider this effect as a general one. These results may be relevant for indirect interactions between charged defects and adsorbed species in the quasi-one-dimensional QED systems mentioned above.

ACKNOWLEDGMENTS

The authors are indebted to Prof. P. K. Silaev and Dr. O. V. Pavlovsky from Department of Physics, MSU for interest and helpful discussions. K.S. is especially grateful to A. Kondakova, Solid State Division, Department of Physics, MSU for information on the current situation in the research of quantum wires. This work was supported in part by the RF Ministry of Science and Education Scientific Research Program, Projects No. 01-2014-63889 and No. A16-116021760047-5 and by RFBR Grant No. 14-02-01261.

APPENDIX

Here we obtain the relation (20), which plays an important role in the calculation presented above. For this purpose we will show that for all $a > 0$ the relation

$$I_1(a) - I_2(a) = \frac{a}{\pi} - \frac{1}{16} \tag{A1}$$

holds, with $I_1(a)$ (further I_1) and $I_2(a)$ (further I_2) defined in Eqs. (18) and (19), respectively.

In the first step it would be useful to remove the irrationalities, which for the integration variables are replaced by $y = t - 1/t$ in I_1 and by $y = (z - 1/z)/2$ in I_2, whereupon the integrals take the form

$$I_1 = \frac{8}{\pi^2}\int_1^\infty \frac{\sin^2(a(t-1/t))}{(t^2-1)^3} t^2 \ln(t) dt,$$

$$I_2 = \frac{2}{\pi}\int_1^\infty \frac{e^{-2a(z+1/z)}}{(z^2+1)^3} z^2 dz. \tag{A2}$$

Proceeding further, I_1 is expanded in the sum of three terms

$$I_1 = J_1 + J_2 + J_3, \tag{A3}$$

where

$$J_1 = \frac{4}{\pi^2}\int_1^{\infty} \frac{t^2 \ln(t)}{(t^2-1)^3} dt,$$
$$J_2 = -\frac{2}{\pi^2}\int_1^{\infty} \frac{e^{2ia(t-1/t)} t^2 \ln(t)}{(t^2-1)^3} dt, \tag{A4}$$
$$J_3 = -\frac{2}{\pi^2}\int_1^{\infty} \frac{e^{-2ia(t-1/t)} t^2 \ln(t)}{(t^2-1)^3} dt.$$

Each of the integrals (A4), when considered separately, diverges at the lower limit. We regularize them by transforming the integration in J_1, J_2, and J_3 to the imaginary axis

$$J_1 = -\frac{4i}{\pi^2}\int_i^{+i\infty} \frac{z^2}{(z^2+1)^3}\left(\ln(z) - \frac{i\pi}{2}\right) dz,$$
$$J_2 = -\frac{2i}{\pi^2}\int_{-i}^{-i\infty} \frac{e^{-2a(z+1/z)} z^2}{(z^2+1)^3}\left(\ln(z) + \frac{i\pi}{2}\right) dz, \tag{A5}$$
$$J_3 = \frac{2i}{\pi^2}\int_i^{+i\infty} \frac{e^{-2a(z+1/z)} z^2}{(z^2+1)^3}\left(\ln(z) - \frac{i\pi}{2}\right) dz$$

and making use of auxiliary contours in the complex z plane, shown in Fig. 11, namely, the contour [Fig. 11(a)] for J_1 and J_3 and the contour [Fig. 11(b)] for J_2.

Each of the integrals (A5) can be expressed via the integrals along the real axis and corresponding integral along the arc of the first quadrant of the circle with radius $r = 1 + \varepsilon$, $\varepsilon \to +0$ (see contours C_r and D_r in Fig. 11), namely,

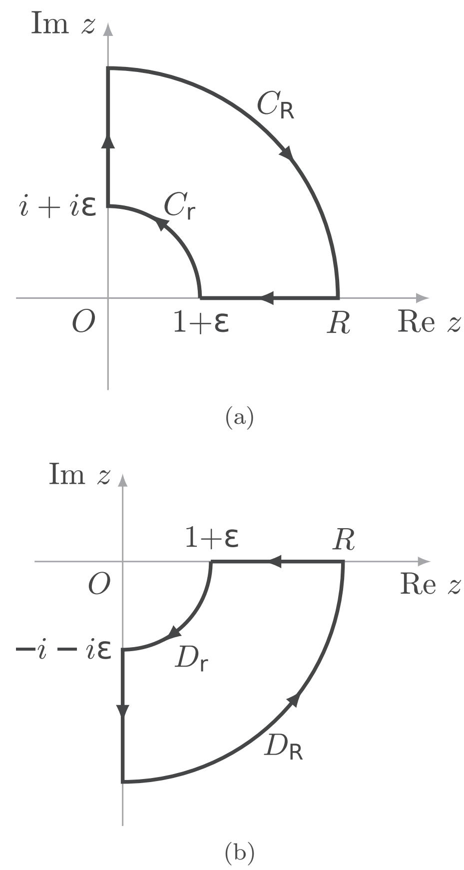


FIG. 11. Auxiliary contours for dealing with the integrals (a) J_1 and J_3 and (b) J_2.

$$J_1 = -\frac{4i}{\pi^2}\int_1^{\infty} \frac{z^2}{(z^2+1)^3}\left(\ln(z) - \frac{i\pi}{2}\right) dz - \frac{4(1+\varepsilon)^3}{\pi^2}\int_0^{\pi/2} \frac{e^{3i\phi}}{[(1+\varepsilon)^2 e^{2i\phi}+1]^3}\left(i\phi - \frac{i\pi}{2} + \ln(1+\varepsilon)\right) d\phi,$$
$$J_2 = -\frac{2i}{\pi^2}\int_1^{\infty} \frac{e^{-2a(z+1/z)} z^2}{(1+z^2)^3}\left(\ln(z) + \frac{i\pi}{2}\right) dz$$
$$+ \frac{2(1+\varepsilon)^3}{\pi^2}\int_0^{\pi/2} \frac{\exp\{-2a[(1+\varepsilon)e^{-i\phi} + e^{i\phi}/(1+\varepsilon)]\} e^{-3i\phi}}{[(1+\varepsilon)^2 e^{-2i\phi}+1]^3}\left(-i\phi + \frac{i\pi}{2} + \ln(1+\varepsilon)\right) d\phi,$$
$$J_3 = \frac{2i}{\pi^2}\int_1^{\infty} \frac{e^{-2a(z+1/z)} z^2}{(1+z^2)^3}\left(\ln(z) - \frac{i\pi}{2}\right) dz$$
$$+ \frac{2(1+\varepsilon)^3}{\pi^2}\int_0^{\pi/2} \frac{\exp\{-2a[(1+\varepsilon)e^{i\phi} + e^{-i\phi}/(1+\varepsilon)]\} e^{3i\phi}}{[(1+\varepsilon)^2 e^{2i\phi}+1]^3}\left(i\phi - \frac{i\pi}{2} + \ln(1+\varepsilon)\right) d\phi. \tag{A6}$$

To achieve the integral $I_1 = J_1 + J_2 + J_3$ let us summarize the right-hand side of Eqs. (A6). The sum of the real integrals in J_1 and J_2 is equal to I_2, whence it follows that

$$I_1 = I_2 + \mathcal{I}_1 + \mathcal{I}_2, \tag{A7}$$

where

$$\mathcal{I}_1 = -\frac{4i}{\pi^2}\int_1^{\infty} \frac{z^2}{(z^2+1)^3}\left(\ln(z) - \frac{i\pi}{2}\right) dz = -\frac{1}{16} + \frac{i}{4\pi^2}(1-2G), \tag{A8}$$

$$\mathcal{I}_2 = \lim_{\varepsilon\to+0}\left\{\frac{(1+\varepsilon)^3}{\pi^2}\int_0^{\pi/2} e^{3i\phi}\left(\frac{2i[\pi-2\phi+2i\ln(1+\varepsilon)]}{[1+e^{2i\phi}(1+\varepsilon)^2]^3}+\frac{\exp\{-2a[e^{-i\phi}/(1+\varepsilon)+e^{i\phi}(1+\varepsilon)]\}[-i(\pi-2\phi)+2\ln(1+\varepsilon)]}{(1+e^{2i\phi}(1+\varepsilon)^2)^3}\right.\right.$$
$$\left.\left.+\frac{\exp\{-2a[e^{i\phi}/(1+\varepsilon)+e^{-i\phi}(1+\varepsilon)]\}[i(\pi-2\phi)+2\ln(1+\varepsilon)]}{(e^{2i\phi}+(1+\varepsilon)^2)^3}\right)d\phi\right\}, \quad \text{(A9)}$$

where $G = 0.9159\ldots$ is the Catalan constant.

To calculate $\mathcal{I}_2$ we note that it is convergent; hence the integrand can be represented by a series in powers of a, which in turn can be exchanged with the limit $\varepsilon \to +0$. Afterward, for each of the terms of the emerging series the integral over ϕ can be calculated explicitly. It remains to note that after the limit $\varepsilon \to +0$ only the linear in a terms survive. As a result,

$$\mathcal{I}_2 = -\frac{i}{4\pi^2}(1-2G) + \frac{a}{\pi}. \quad \text{(A10)}$$

Inserting further the answers for $\mathcal{I}_1$ and $\mathcal{I}_2$, found in this way, into (A7), one obtains

$$I_1 = I_2 + \frac{a}{\pi} - \frac{1}{16}, \quad \text{(A11)}$$

whence the relation (A1) follows and hence the relations (20) and (28). ■

For the antisymmetric case the relation $\Lambda_1(d) + \Lambda_2(d) = a/\pi$ turns out to be the direct consequence of the previous analysis. It is easy to verify that $\Lambda_1(d)$ can be expressed in the form

$$\Lambda_1(d) = a/\pi - 2I_1(a/2) + I_1(d) + I_1(a+d) - 2I_1(a/2+d), \quad \text{(A12)}$$

where $I_1(a)$ is defined as previously via (18). Proceeding further, from (A12) by means of (A11) one obtains the required relation

$$\Lambda_1(d) = \frac{1}{8} - \frac{1}{\pi}\int_0^\infty dy\, \frac{e^{-2a\sqrt{1+y^2}}[1-2e^{-4d\sqrt{1+y^2}}\sinh^2(a\sqrt{1+y^2})]}{2(1+y^2)^2} = \frac{a}{\pi} - \Lambda_2(d). \quad \text{(A13)}$$

It is worth noting that the property similar to (20) between the renormalization coefficients of the polarization problems in one-dimensional QED is not the exceptional feature of potentials containing square barriers and wells similar to that considered above. Actually, it also holds for the wells with Coulomb-like asymptotics [2–4], where the parameter a plays the role of the size of the central charged sphere or just of the smoothing of the Coulomb point singularity.

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