Quantum computation from fermionic anyons on a one-dimensional lattice

Allan D. C. Tosta, Daniel J. Brod, and Ernesto F. Galvão

Instituto de Física, Universidade Federal Fluminense, Av. Gal. Milton Tavares de Souza s/n, Niterói, RJ, 24210-340, Brazil

Fermionic linear optics corresponds to the dynamics of free fermions and is known to be efficiently simulable classically. We define fermionic anyon models by deforming the fermionic algebra of creation and annihilation operators, and consider the dynamics of number-preserving, quadratic Hamiltonians on these operators. We show that any such deformation results in an anyonic linear-optical model which allows for universal quantum computation.

I. INTRODUCTION

There are good reasons to study computational models using particles with some kind of fractional statistics. Experimentally, there has been a growing interest in finding such exotic particles in condensed-matter systems such as the one exhibiting the fractional quantum Hall effect (FQHE) [1], as they can simplify the description of a multitude of complex condensed-matter phenomena. They can also be used as a platform for models of quantum computation that are intrinsically robust to decoherence under specific conditions [2–5]. Theoretically, understanding the computational power of models based on such particles can give us more knowledge about the transition from classical to quantum computation and what resources make this transition possible.

In this paper we generalize the model of quantum computation based on the dynamics of noninteracting fermions (known as fermionic linear optics, FLO [6–8]) to one of anyons on a one-dimensional (1D) lattice [9,10]. This anyon system is defined via a deformation of the fermionic anticommutation relations that introduces nontrivial exchange phases (also called fractional exchange phases).

A system of identical particles with Abelian fractional exchange statistics is one where the multiparticle wave function gains an arbitrary complex phase factor under an operation that exchanges particle positions [11,12]. A non-Abelian system is very similar but, in this case, each particle has an internal Hilbert space, and because of it the exchange phases are replaced by arbitrary unitary matrices. Particles with these properties are called Abelian anyons in the first case and non-Abelian anyons in the second [13]. Such statistics have a topological origin related to the dimensionality of the physical space and can be observed in two-dimensional many-body systems subject to a process know as transmutation of statistics [13]. This involves many-body interactions described by nondynamical effective vector potentials, whose only roles are to associate fictitious magnetic fluxes to charged particles, generating the exchange phase factor via the Aharonov-Bohm effect. These fictitious fields are described by topological quantum field theories (TQFT), such as the Chern-Simons theory [14], and are used to describe many classes of states of FQHE systems [15,16].

In contrast to this topological description, known to be valid only in two dimensions, a system is said to have fractional exclusion statistics if the maximum number of particles allowed per mode is a finite integer not equal to 1 [17]. This is a dimension-independent generalization of the Pauli exclusion principle, which also applies to the description of some subsets of FQH states that possess exchange statistics. This generalization is unrelated to statistical transmutation and is, in fact, independent of it. For such a definition, spinless fermions and spinless hard-core bosons are both seen as fermions, since both obey the exclusion principle, even though their commutation relations differ if particles are in different states.

The type of anyonic system we will be concerned with here is defined by modified commutation relations, referred to in the literature as deformed commutation relations [9,10,18–20], obtainable from the conventional fermionic and bosonic ones via specific mappings [21–23]. These mappings, which have a similar form to the Jordan-Wigner transformation [24], relate fermionic and bosonic operators on a lattice to corresponding anyonic operators on the same lattice.

Systems defined in terms of deformed commutation relations allow for the study of anyonic gases and other analogs of one-dimensional condensed-matter systems using conventional many-body quantum mechanics and renormalization group methods [9,25–27]. Besides effects in thermodynamics and the study of phase transitions [28–30], these systems are interesting also due to practical proposals for implementations in systems such as in optical lattices [31–33], and even a proposal for simulation using classical optics [34].

We define a computational model similar to the boson sampling model [35], where a finite number of photons are input into a circuit made of optical devices such as beam splitters and phase shifters, followed by photon-counting detectors. The anyonic model used in this work interpolates between one-dimensional fermions and one-dimensional hard-core bosons. We remark that our proposal is different from the one made in topological quantum computing (TQC), which deals with the computational power of braiding non-Abelian anyons and is intrinsically fault-tolerant [3,36–40].

This paper is organized as follows. In Sec. II we define and review the properties of several computational models based on quadratic number-preserving particle dynamics, including

FLO, hard-core bosons, and the anyonic model we investigate [41,42]. In Sec. III we solve the general two-particle dynamics and define equivalent "linear-optical" devices, highlighting the differences with respect to its bosonic and fermionic counterparts. In Sec. IV we construct a universal set of gates inspired by the one obtained in [42], showing that our model is universal for quantum computation (as could be expected from the literature on extensions of FLO and related models [43,44]). Finally, in Sec. V we offer some concluding remarks.

II. REVIEW

In this section we review how to describe noninteracting bosons and fermions, whose dynamics are respectively known as bosonic (Sec. II A) and fermionic linear optics (Sec. II B). In Sec. II C we review how qubits can be understood as a system of bosons under a hard-core interaction. In Sec. II D we define the anyon model we use and how it can be constructed from deformed fermionic commutation relations, although we defer the discussion of their dynamics to a subsequent section.

A. Linear optics

Photons are described by their degrees of freedom (which we call *modes*, for short). We consider a system with m modes described by creation and annihilation operators $b_i^\dagger$ and b_i with $i = 1, \ldots, m$, which respectively create and destroy a single photon in mode i. These operators obey the canonical bosonic commutation relations

$$b_i b_j^\dagger - b_j^\dagger b_i = \delta_{ij}, \tag{1a}$$

$$b_i b_j - b_j b_i = 0, \tag{1b}$$

$$b_i^\dagger b_j^\dagger - b_j^\dagger b_i^\dagger = 0, \tag{1c}$$

for all modes i, j. The basis vectors for this system's Hilbert space can be chosen to be

$$\left| n_1^B, \ldots, n_m^B \right\rangle = \frac{(b_1^\dagger)^{n_1^B} \ldots (b_m^\dagger)^{n_m^B}}{\sqrt{n_1^B! \ldots n_m^B!}} |0_B\rangle, \tag{2}$$

where n_i^B is the eigenvalue of the number operator $N_i^B = b_i^\dagger b_i$. This basis is called the *Fock basis* or the *occupation number basis*, and we use this basis throughout this work.

As we will shortly see, any free-particle bosonic dynamics can be expressed in terms only of one- and two-mode passive linear-optical elements (as described, e.g., in [45]). A *phase shifter* is a single-mode passive linear device described as the time-evolution operator $PS_i(\theta) = \exp i\theta H_i^{PS}$ of the Hamiltonian $H_i^{PS} = N_i^B$. Its effect on Fock states results in an occupation-number-dependent phase:

$$PS_i(\theta) \left| n_1^B, \ldots, n_m^B \right\rangle = e^{i\theta n_i^B} \left| n_1^B, \ldots, n_m^B \right\rangle. \tag{3}$$

A phase shifter acts on creation operators as

$$PS_i(\theta) b_j^\dagger PS_i(-\theta) = e^{i\theta \delta_{ij}} b_j^\dagger. \tag{4}$$

A *beam splitter* is a two-mode passive linear device described by the time-evolution operator $BS_{ij}(\theta) = \exp i\theta H_{ij}^{BS}$, corresponding to the Hamiltonian $H_{ij}^{BS} = b_i^\dagger b_j + b_j^\dagger b_i$. Its ef-

fect in creation operators is given by the matrix equation

$$BS_{ij}(\theta) \begin{bmatrix} b_i^\dagger \\ b_j^\dagger \end{bmatrix} BS_{ij}(-\theta) = \begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} b_i^\dagger \\ b_j^\dagger \end{bmatrix}. \tag{5}$$

Another way to understand how these devices act is to consider a system with a single photon which can be in any of the m modes. The basis states of this system in the occupation number representation are the m states $|1, 0, 0, \ldots, 0\rangle, |0, 1, 0, \ldots, 0\rangle, \ldots, |0, 0, 0, \ldots, 1\rangle$. In terms of these states, the actions of beam splitters and phase shifters are SU(2) matrices in the subspaces on which they act. In fact it was proven that any SU(m) matrix can be constructed in this single-photon system using only successive applications of phase shifters and beam splitters [46]. The computational model where the computational basis states are the Fock states, the unitaries are arbitrary circuits of phase shifters and beam splitters between two arbitrary modes, and measurements are made in the Fock basis is called the boson sampling model [35]. This model is not known to be universal for quantum computational, but there is evidence that it is hard to simulate on a classical computer given some complexity-theoretic assumptions.

B. Fermionic linear optics

We now turn to the FLO model as described, e.g., in [6,7,47], and specialize it to our needs. Consider a system of abstract fermionic modes described by creation and annihilation operators $f_i^\dagger$ and f_i on m modes, satisfying the *canonical fermionic anticommutation relations*

$$f_i f_j^\dagger + f_j^\dagger f_i = \delta_{ij}, \tag{6a}$$

$$f_i f_j + f_j f_i = 0, \tag{6b}$$

$$f_i^\dagger f_j^\dagger + f_j^\dagger f_i^\dagger = 0, \tag{6c}$$

for all modes i, j. The occupation number operator is $N_i^F = f_i^\dagger f_i$, and the vacuum state is $|0_F\rangle$. The Fock basis for fermions comprises the basis states

$$\left| n_1^F, \ldots, n_m^F \right\rangle = (f_1^\dagger)^{n_1^F} \ldots (f_m^\dagger)^{n_m^F} |0_F\rangle, \tag{7}$$

where the n_j^F are eigenvalues of the corresponding number operators which, due to the commutation relations, can only be 0 or 1.

We call *passive fermionic linear-optical elements* unitaries of the form

$$PS_i(\theta) = \exp i\theta H_i^{PS}, \tag{8a}$$

$$BS_{ij}(\theta) = \exp i\theta H_{ij}^{BS}, \tag{8b}$$

with Hamiltonians given by

$$H_i^{PS} = N_i^F, \tag{8c}$$

$$H_{ij}^{BS} = f_i^\dagger f_j + f_j^\dagger f_i. \tag{8d}$$

The action of these elements over creation operators is very similar to the bosonic case and is given by the equations

$$PS_i(\theta) f_j^\dagger PS_i(-\theta) = e^{i\theta \delta_{ij}} f_j^\dagger, \tag{9a}$$

$$BS_{ij}(\theta) \begin{bmatrix} f_i^\dagger \\ f_j^\dagger \end{bmatrix} BS_{ij}(-\theta) = \begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} f_i^\dagger \\ f_j^\dagger \end{bmatrix}. \tag{9b}$$

As in the bosonic case, one can construct any SU(m) operator over the Hilbert space of a single fermion in m modes using only compositions of beam splitters and phase shifters.

By taking as computational basis states the occupation number basis and as logic gates fermionic linear-optical elements between arbitrary modes and measurements in the occupation number basis, we define the computational model called fermionic linear optics. This model of computation is proven to be exactly efficiently simulable by a classical computer in [7], which means that there is an efficient classical algorithm to evaluate $|\langle y|U|x\rangle|^2$ for arbitrary computational states $|x\rangle$ and $|y\rangle$ and arbitrary unitaries generated by the application of passive optical elements (and any desired marginal probabilities).

C. Qubits as hard-core bosons

To make a point of comparison, we now give an alternate description of the usual circuit model of computation on qubits. We will call this model *qubit linear optics*. Consider a system of m qubits, with the Pauli matrices acting on qubit i denoted by X_i, Y_i, Z_i. Following [41] we can write the usual computational basis states as a Fock space basis given by the equations

$$\left| n_1^Q, \ldots, n_m^Q \right\rangle = (\pi_1^\dagger)^{n_1^Q} \ldots (\pi_m^\dagger)^{n_m^Q} |0_Q\rangle, \tag{10}$$

with the creation and annihilation operators given by $\pi_i^\dagger = \frac{1}{2}(X_i + iY_i)$ and $\pi_i = \frac{1}{2}(X_i - iY_i)$ acting on state $|0_Q\rangle = |0, \ldots, 0\rangle$, and the number operator $N_i^Q = \pi_i^\dagger \pi_i = \frac{1}{2}(\mathbf{1}_i + Z_i)$, whose eigenvalues n_i^Q can be only 0 or 1. These operators must obey the algebra

$$\pi_i \pi_j^\dagger - \pi_j^\dagger \pi_i = 0, \tag{11a}$$

$$\pi_i \pi_j - \pi_j \pi_i = 0, \tag{11b}$$

$$\pi_i^\dagger \pi_j^\dagger - \pi_j^\dagger \pi_i^\dagger = 0, \tag{11c}$$

for all modes i, j with $i \neq j$, [compare with Eqs. (1)] and

$$\pi_i \pi_i^\dagger + \pi_i^\dagger \pi_i = 1, \tag{11d}$$

$$(\pi_i^\dagger)^2 = (\pi_i)^2 = 0, \tag{11e}$$

for each mode i [compare with Eqs. (6)]. This algebra forbids more than one boson in the same mode, which is why this system can be understood as bosons with a hard-core interaction.

Similarly to bosonic and fermionic linear optics, qubit passive linear-optical elements are defined as unitary operators of the form

$$PS_i(\theta) = \exp i\theta H_i^{PS}, \tag{12a}$$

$$BS_{ij}(\theta) = \exp i\theta H_{ij}^{BS}, \tag{12b}$$

with Hamiltonians given by

$$H_i^{PS} = N_i^Q, \tag{12c}$$

$$H_{ij}^{BS} = \pi_i^\dagger \pi_j + \pi_j^\dagger \pi_i. \tag{12d}$$

Using a bit of algebra we can write the beam-splitter Hamiltonian in terms of Pauli operators as $H_{ij}^{BS} = X_i X_j + Y_i Y_j$. In [42], Kempe and Whaley showed that this interaction acting between nearest and next-nearest neighbors on a 1D chain, along with Pauli Z rotations, can perform universal quantum computation in an encoded subspace of the Fock space (next-to-nearest neighbor interactions are essential for the protocol to work). This result suggests that qubit linear optics is computationally more powerful than fermionic linear optics, even though the only differences are the signs of commutators of operators in different sites.

D. The definition of fermionic anyons

The model we consider in this paper has been studied previously in the theory of interacting bosonic one-dimensional gases [9,23], interacting fermionic one-dimensional gases [26], and their simulations in several physical systems [28,31–33]. In this model the creation and annihilation operators $a_i^\dagger$ and a_i satisfy the deformed anticommutation relations

$$a_i a_j^\dagger + e^{-i\varphi\epsilon_{ij}} a_j^\dagger a_i = \delta_{i,j}, \tag{13a}$$

$$a_i a_j + e^{i\varphi\epsilon_{ij}} a_j a_i = 0, \tag{13b}$$

where the symbol ϵ_{ij} is given by

$$\epsilon_{ij} = \begin{cases} 1, & \text{if } i < j \\ 0, & \text{if } i = j \\ -1, & \text{if } i > j \end{cases}. \tag{13c}$$

The dependence of the anticommutation relations on ϵ_{ij} defines an order over the lattice, coming from the way the deformation is defined [see Eqs. (14)]. When $\varphi = 0$ or π, this order is irrelevant and we reobtain a fermionic system [see Eqs. (1)] and hard-core bosons [or qubits, see Eqs. (11d) and (11e)], respectively. For all $0 < \varphi < \pi$ we have a nontrivial anyonic model. The deformed anticommutation relations come from the generalized Jordan-Wigner transformation below:

$$a_i^\dagger = JW_i^{(\varphi)\dagger} f_i^\dagger, \tag{14a}$$

$$a_i = JW_i^{(\varphi)} f_i, \tag{14b}$$

$$JW_i^{(\varphi)} = \exp i\varphi \sum_{k=1}^{i-1} f_k^\dagger f_k. \tag{14c}$$

The Jordan-Wigner operator $JW_i^{(\varphi)}$ transmutes statistics in a way similar to the Chern-Simons field [48].

Number operators for this system are as in the previous cases, and we represent them by $N_i^A = a_i^\dagger a_i$, with eigenvalues n_i^A either 0 or 1. The Fock basis states for anyons are, therefore,

$$\left| n_1^A, \ldots, n_m^A \right\rangle = (a_1^\dagger)^{n_1^A} \ldots (a_m^\dagger)^{n_m^A} |0_A\rangle. \tag{15}$$

Anyonic passive linear-optical elements are the unitaries

$$PS_i(\theta) = \exp i\theta H_i^{PS}, \tag{16a}$$

$$BS_{ij}(\theta) = \exp i\theta H_{ij}^{BS}, \tag{16b}$$

with Hamiltonians given by

$$H_i^{PS} = N_i^A, \tag{16c}$$

$$H_{ij}^{BS} = a_i^\dagger a_j + a_j^\dagger a_i. \tag{16d}$$

The computational model is as before: the computational basis states are the Fock states, the unitaries are all possible combinations of phase-shifters and beam-splitters for arbitrary pairs of modes (this arbitrariness is essential for our result), and measurements are allowed in the Fock basis only.

III. LINEAR OPTICS OF 1D FERMIONIC ANYONS

In this section we define the dynamics corresponding to anyonic linear optics. In Sec. III A, we solve the equations of motion for creation operators acted on by the linear-optical Hamiltonians exactly and explain the difference in relation to the fermionic case. Next, in Sec. III B we show that when beam splitters act on non-neighboring modes, the presence of particles in between gives rise to an Aharonov-Bohm effect, and we argue that this is responsible for the differences in the action of beam splitters with respect to free fermions.

A. Anyonic optical elements and equations of motion

In the anyonic case, the evolution of creation operators is more involved than for the bosonic and fermionic cases and it is what we discuss now. For phase shifters we have that

$$PS_i(\theta) a_j^\dagger PS_i(-\theta) = e^{i\theta \delta_{ij}} a_j^\dagger, \tag{17}$$

just like before. The action of beamsplitters, on the other hand, requires more attention. We will solve the dynamical problem defined by the H_{ij}^{BS} Hamiltonian. The Heisenberg equations of motion are

$$i\frac{da_i^\dagger}{d\theta} = \left[H_{ij}^{BS}, a_i^\dagger\right], \tag{18a}$$

$$i\frac{da_j^\dagger}{d\theta} = \left[H_{ij}^{BS}, a_j^\dagger\right], \tag{18b}$$

where the commutators are computed using the algebra of anyonic operators [Eqs. (13)]:

$$\left[H_{ij}^{BS}, a_i^\dagger\right] = a_j^\dagger\left\{1 - (1 - e^{i\varphi})N_i^A\right\}, \tag{19a}$$

$$\left[H_{ij}^{BS}, a_j^\dagger\right] = a_i^\dagger\left\{1 - (1 - e^{-i\varphi})N_j^A\right\}. \tag{19b}$$

Let us start by rewriting the equation for mode i in a more suggestive form:

$$i\frac{da_i^\dagger}{d\theta} = a_j^\dagger W_i^{(\varphi)}, \tag{20}$$

where we have introduced a short-hand notation for the nonlinear term $W_i^{(\varphi)} \equiv 1 - (1 - e^{i\varphi})N_i^A$. By computing the commutator $[H_{ij}^{BS}, a_j^\dagger W_i^{(\varphi)}]$ we find that the equation of motion for this operator is given by

$$i\frac{d\left(a_j^\dagger W_i^{(\varphi)}\right)}{d\theta} = a_i^\dagger. \tag{21}$$

Therefore this is a coupled linear system of equations for the operators $a_i^\dagger$ and $a_j^\dagger W_i^{(\varphi)}$, which is exactly solvable. The

equation of motion for mode j has a similar property, giving the system

$$i\frac{da_j^\dagger}{d\theta} = a_i^\dagger W^{\dagger(\varphi)}_j, \tag{22a}$$

$$i\frac{d\left(a_i^\dagger W^{\dagger(\varphi)}_j\right)}{d\theta} = a_j^\dagger. \tag{22b}$$

By linearity, the solutions of the equations for $a_i^\dagger$ and $a_j^\dagger$ must be

$$BS_{ij}(\theta) a_i^\dagger BS_{ij}(-\theta) = \cos\theta a_i^\dagger + i\sin\theta a_j^\dagger W_i^{(\varphi)}, \tag{23a}$$

$$BS_{ij}(\theta) a_j^\dagger BS_{ij}(-\theta) = \cos\theta a_j^\dagger + i\sin\theta a_i^\dagger W^{\dagger(\varphi)}_j. \tag{23b}$$

Notice that these solutions are very similar to the corresponding dynamics of fermions and bosons [Eqs. (5) and (9b)].

Equations (23a) and (23b) represent the exact dynamics for arbitrary pairs of modes for these systems. Previous treatments have focused on many-body anyonic Hamiltonians (for either the bosonic and fermionic anyon varieties) using methods such as the Bethe-Ansatz, as in [9].

B. Aharonov-Bohm effect in anyonic beam splitters

To complete our description of a general beam splitter, we need to discuss what happens with the modes that are between the ones appearing in the Hamiltonian. In the fermionic and bosonic cases these modes commute with H^{BS}, but in the anyonic case creation operators $a_k^\dagger$ with $i < k < j$ do not commute with H_{ij}^{BS}. In fact, they satisfy the relation

$$\exp i\theta(a_i^\dagger a_j + a_j^\dagger a_i) a_k^\dagger = a_k^\dagger \exp i\theta(e^{i2\varphi} a_i^\dagger a_j + e^{-i2\varphi} a_j^\dagger a_i), \tag{24}$$

which we can treat as an effective beam splitter that acts on states by introducing a phase correction dependent on the number of modes occupied between i and j. Or putting it in a more compact notation,

$$BS_{ij}(\theta) a_k^\dagger = a_k^\dagger BS_{ij}^{(2\varphi)}(\theta), \tag{25a}$$

with the new effective beam-splitter unitary defined by

$$BS_{ij}^{(\alpha)}(\theta) = \exp i\theta(e^{i\alpha} a_i^\dagger a_j + e^{-i\alpha} a_j^\dagger a_i), \tag{25b}$$

and the solution to the equations of motion for modes i and j is

$$a^{(\alpha)\dagger}_i(\theta) = \cos\theta a_i^\dagger + ie^{i\alpha}\sin\theta a_j^\dagger W_i^{(\varphi)}, \tag{25c}$$

$$a^{(\alpha)\dagger}_j(\theta) = \cos\theta a_j^\dagger - ie^{-i\alpha}\sin\theta a_i^\dagger W^{\dagger(\varphi)}_j. \tag{25d}$$

Example. To illustrate this phase correction, consider a system with three modes, and choose $i = 1$ and $j = 3$. Let us act with a balanced beam splitter ($\theta = \frac{\pi}{4}$) on state $|0, 1, 1\rangle = a_2^\dagger a_3^\dagger |0_A\rangle$. The first step of the calculation is to use Eq. (25a):

$$BS_{13}\left(\frac{\pi}{4}\right) a_2^\dagger a_3^\dagger |0_A\rangle = a_2^\dagger \left[BS_{13}^{(2\varphi)}\left(\frac{\pi}{4}\right)\right] a_3^\dagger |0_A\rangle. \tag{26}$$

Now we need to analyze the dynamics of $a_3^\dagger |0_A\rangle$ under the action of $BS_{13}^{(2\varphi)}(\frac{\pi}{4})$. Using Eqs. (25c) and (25d), and the fact

that $W_i^{(\varphi)}|0_A\rangle = |0_A\rangle$, for every mode i and angle φ we obtain

$$a_2^\dagger \left[BS_{13}^{(2\varphi)}\left(\frac{\pi}{4}\right) \right] a_3^\dagger |0_A\rangle = \frac{1}{\sqrt{2}} a_2^\dagger (ie^{i2\varphi} a_1^\dagger + a_3^\dagger)|0_A\rangle. \quad (27)$$

We now use the commutation relation $a_2^\dagger a_1^\dagger = -e^{-i\varphi} a_1^\dagger a_2^\dagger$ to write the final equation in normally ordered form:

$$\frac{1}{\sqrt{2}} a_2^\dagger (ie^{i2\varphi} a_1^\dagger + a_3^\dagger)|0_A\rangle = \frac{1}{\sqrt{2}}(-ie^{i\varphi} a_1^\dagger a_2^\dagger + a_2^\dagger a_3^\dagger)|0_A\rangle. \quad (28)$$

If we compare this result with the same Hamiltonian applied on state $a_1^\dagger a_2^\dagger |0_A\rangle$, we see that the effect of the phase-corrected Hamiltonian is to guarantee unitarity of the time-evolution operator. To see this use the solution of the equations of motion to obtain

$$BS_{13}\left(\frac{\pi}{4}\right) a_1^\dagger a_2^\dagger |0_A\rangle = \frac{1}{\sqrt{2}}(a_1^\dagger + ia_3^\dagger) a_2^\dagger |0_A\rangle. \quad (29)$$

Finally, using $a_3^\dagger a_2^\dagger = -e^{-i\varphi} a_2^\dagger a_3^\dagger$ we get

$$\frac{1}{\sqrt{2}}(a_1^\dagger + ia_3^\dagger) a_2^\dagger |0_A\rangle = \frac{1}{\sqrt{2}}(a_1^\dagger a_2^\dagger - ie^{-i\varphi} a_2^\dagger a_3^\dagger)|0_A\rangle. \quad (30)$$

Therefore the total effect on the Fock basis is given by

$$BS_{13}\left(\frac{\pi}{4}\right)|1, 1, 0\rangle = \frac{1}{\sqrt{2}}(|1, 1, 0\rangle - ie^{-i\varphi}|0, 1, 1\rangle), \quad (31a)$$

$$BS_{13}\left(\frac{\pi}{4}\right)|0, 1, 1\rangle = \frac{1}{\sqrt{2}}(-ie^{i\varphi}|1, 1, 0\rangle + |0, 1, 1\rangle), \quad (31b)$$

which is manifestly unitary.

If the anyon in mode 1 tunnels to mode 3 when another anyon occupies mode 2, a relative phase factor between the modes appears. It is easy to see that this phase would be absent if mode 2 was empty. This dynamics can be understood as an one-dimensional analog of the Aharonov-Bohm effect [49], where the magnetic flux carried by the particle is given by $\pi - \varphi$, and the π factor accounts for the -1 fermionic phase appearing when $\varphi = 0$.

This phenomenon, due to particle statistics, was seen previously in [50], where it affects the phase difference of a driven Bose-Einstein condensate (of bosonic anyons) in ring-shaped optical lattices threaded by a magnetic flux. These phase factors, as we will see in the next section, are crucial in determining the computational power of the different models.

IV. THE COMPUTATIONAL POWER OF ANYONS

In [42], it was shown how to construct an encoded, entangling two-qubit gate using only nearest and next-to-nearest neighbor Hamiltonians (XY interaction) between physical qubits to achieve universal quantum computation. In this section we will generalize this construction to show that the quadratic dynamics of fermionic anyons is also capable of universal quantum computation. We begin by defining a qubit encoding that is preserved by dynamics of either free fermions, hard-core bosons, or free fermionic anyons. We will then describe linear-optical circuits on fermionic anyons and its action on encoded states. We prove that the two-qubit logical gate implemented by this circuit is deterministic
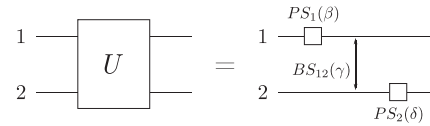

FIG. 1. Single-qubit unitary decomposed in optical elements. A single-qubit unitary needs four parameters ($\alpha, \beta, \gamma, \delta$). The first one is a global phase, and the others are realized by the optical elements in the figure.

and entangling for any value of the statistical parameter $\varphi \neq 0$. This, together with single-qubit unitaries, shows that linear optics on fermionic anyons is universal for quantum computation.

A. Encoding

We use $2n$ modes to encode n qubits such that each logical qubit corresponds to a pair of neighboring modes as in the equations

$$|0_L\rangle = |1, 0\rangle, \quad (32a)$$

$$|1_L\rangle = |0, 1\rangle. \quad (32b)$$

So, for example, a two-qubit system needs four modes and the logical states are given by

$$|00\rangle_L = |1, 0, 1, 0\rangle, \quad (33a)$$

$$|01\rangle_L = |1, 0, 0, 1\rangle, \quad (33b)$$

$$|10\rangle_L = |0, 1, 1, 0\rangle, \quad (33c)$$

$$|11\rangle_L = |0, 1, 0, 1\rangle, \quad (33d)$$

where the right-hand sides of these equations are Fock states. This encoding is independent of the parameter φ, since all of these particles obey the Pauli exclusion principle, which allows the direct comparison of logical gates between models.

B. Encoded one- and two-qubit gates

With this encoding it is possible to do any logical one-qubit gate using only phase shifters and beam splitters on the two corresponding modes. To prove this, consider a single qubit encoded in modes 1 and 2, and notice that a phase shifter in mode 2 acts in the logical basis states as

$$PS_2(\theta)|1, 0\rangle = |1, 0\rangle, \quad (34a)$$

$$PS_2(\theta)|0, 1\rangle = e^{i\theta}|0, 1\rangle, \quad (34b)$$

which is equivalent to a logical Z rotation in the Bloch sphere by θ degrees. Note also that a beam splitter between modes 1 and 2 acts in the logical basis states as

$$BS_{12}(\theta)|1, 0\rangle = \cos\theta|1, 0\rangle + i\sin\theta|0, 1\rangle, \quad (35a)$$

$$BS_{12}(\theta)|0, 1\rangle = i\sin\theta|1, 0\rangle + \cos\theta|0, 1\rangle, \quad (35b)$$

which is equivalent to a logical X rotation in the Bloch sphere by an angle θ. With arbitrary rotations around two axes in the Bloch sphere, we can perform an arbitrary single-qubit gate, as in Fig. 1.
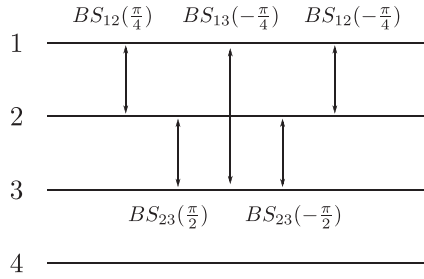
FIG. 2. Two-qubit gate, sequence of beam splitters that generate our entangling gate.

To implement an encoded two-qubit gate we generalize the *XY* interaction protocol found in [42] (see Fig. 2), adapting the construction to our anyonic model.

When the particles considered are hard-core bosons (or spins), we recover the original result of [42] and the circuit executes the logical gate $\sqrt{-ZZ}$, which is a maximally entangling gate.

In Appendix we calculate the effect of this circuit on fermionic anyons characterized by the deformed fermionic algebra of Eqs. (13), for any value of the deformation parameter φ. Its action on the encoded qubits is the gate

$$C(\varphi) = R_{\hat{z}}\left(\frac{\pi}{2}\right) \otimes |0\rangle\langle 0| + R_{\hat{n}}\left(\frac{\pi}{2}\right) \otimes |1\rangle\langle 1|, \quad (36)$$

where $R_{\hat{z}}(\frac{\pi}{2})$ is a rotation of $\frac{\pi}{2}$ around the Z axis in the Bloch sphere, and $R_{\hat{n}}(\frac{\pi}{2})$ is a $\frac{\pi}{2}$ rotation around the axis $\hat{n} = (-\sin\varphi, 0, \cos\varphi)$ in the Bloch sphere. So this is a controlled rotation gate whose action depends continuously on the parameter φ characterizing our anyonic model. We have, for the special cases of fermions and hard-core bosons,

$$C(\varphi) = \begin{cases} Z \otimes \mathbf{1}, & \text{if } \varphi = 0 \\ \sqrt{-ZZ}, & \text{if } \varphi = \pi \end{cases}, \quad (37)$$

which in the fermionic case is a local gate and for hard-core bosons is as described above. Therefore this gate interpolates between the corresponding ones for the other two models.

C. The entangling power of $C(\varphi)$

We claim that gate $C(\varphi)$ [Eq. (36)], together with arbitrary single-qubit gates, form a set that is universal for quantum computing whenever $\varphi \neq 0$. To prove this it is sufficient to show that $C(\varphi)$ has a nonzero entangling power (as defined by [51]).

The entangling power $e_p(U)$ of a unitary gate U is defined as the average entanglement of formation generated by the action of U on product states $|\psi_1\rangle \otimes |\psi_2\rangle$:

$$e_p(U) = \overline{E(U|\psi_1\rangle \otimes |\psi_2\rangle)}^{(\psi_1, \psi_2)}, \quad (38)$$

where the bar denotes an average with respect to some probability distribution $p(\psi_1, \psi_2)$. It can be shown that if the average is taken over the uniform distribution, the entangling power is both local invariant and SWAP invariant (that is, it remains the same if U is conjugated by SWAP or by single-qubit gates). In fact, this invariant can be easily calculated in terms of simpler invariants, which was done in [52]. Two-

qubit gates have two local invariant quantities given by

$$G_1 = \frac{\text{Tr}^2 U_B^T U_B}{16 \det(U)} \quad (39)$$

and

$$G_2 = \frac{\text{Tr}^2 U_B^T U_B - \text{Tr}(U_B^T U_B)^2}{4 \det(U)}, \quad (40)$$

where U_B is the matrix representation of the gate U written in the *Bell basis*. With these invariants, the (normalized) entangling power $e_p(U)$ of a two-qubit gate U over the uniform distribution is simply given by [52]

$$e_p(U) = 1 - |G_1|. \quad (41)$$

With this in hand, the entangling power of $C(\varphi)$ [Eq. (36)] is

$$e_p(C(\varphi)) = 1 - \cos^4\frac{\varphi}{2}. \quad (42)$$

This shows that any $\varphi \neq 0$ results in a fermionic anyon model that allows for universal quantum computation. In fact, since this gate generates entanglement, it can be used to construct encoded controlled-NOT gates using the argument given in [53], with the number of required $C(\varphi)$ gates depending on the value of $e_p(C(\varphi))$.

V. CONCLUSION

We generalized the model of passive fermionic linear optics by studying anyonic systems defined by deformations of fermionic anticommutation relations. We have taken quadratic, number-preserving Hamiltonians as the analog of fermionic linear-optical dynamics and solved the generated dynamics exactly. We showed how the difference to fermionic dynamics is due to a one-dimensional analog of the Aharonov-Bohm effect, which is intimately related to the computational power of the model we consider and also gives a simple interpretation to a previously known effect [50].

We adapted a scheme for quantum computing with nearest- and next-nearest-neighbor spin-$\frac{1}{2}$ interactions to systems of identical particles, showing that the analogous interactions in our model allow for universal quantum computation when the particles have fractional exchange statistics. This happens for any value of the statistical deformation parameter $\varphi \neq 0$, that is, as long as our anyons differ from usual fermions. Given that free fermions can be simulated efficiently, this means the transition in computational power is abrupt, as is common to extensions of FLO [43,44], going from classically simulable to quantum universal for any $\varphi \neq 0$.

This raises the question of whether such interactions arising from statistics alone can lead to a computational advantage in other settings. For example, it would be interesting to study deformed bosonic commutation relations. This is likely to be a harder problem than the fermionic one due to the absence of an exclusion principle, which simplifies the dynamical equations. Deformed bosonic relations would result in a generalization of the boson sampling model. Boson sampling is known to be hard to simulate classically (given some complexity-theoretic assumptions), but is not known to be universal for quantum computation. We leave the exact

dynamics of two-body hopping Hamiltonians for anyonic bosons as an interesting open problem.

The fact that our fermionic anyon system allows for universal quantum computation is a statement about the complexity of the model's dynamics. As we have proven, the model is universal for quantum computation—no efficient classical simulation scheme is believed to exist for it. Also, no quantum computational device which is not universal will be able to efficiently simulate fermionic anyons. Our results show that studying the computational capacity of general anyonic models may yield insights on each model's complexity.

APPENDIX: EXPLICIT EXPRESSION OF $C(\varphi)$

In this Appendix we calculate the two-qubit unitary implemented by the anyonic linear-optical circuit of Fig. 2. For this, we need to calculate the matrix elements of the various beam splitters in the two-particle basis and see if our protocol preserves the encoding of qubits. Given the four modes of Fig. 2, the two-particle basis is given by the states

$$|1, 1, 0, 0\rangle, |1, 0, 1, 0\rangle, |1, 0, 0, 1\rangle, \quad (A1a)$$

$$|0, 1, 1, 0\rangle, |0, 1, 0, 1\rangle, |0, 0, 1, 1\rangle. \quad (A1b)$$

There are three kinds of beam splitters in the circuit, BS_{12}, BS_{23}, and BS_{13}, where the first two appear more then once, with two different angles (defining the beam-splitting ratios). First let us calculate the matrix representation of the general beam splitter $BS_{12}(\theta)$ in our two-particle basis. The idea is to use the representation in terms of creation operators and use the solutions to the equations of motion to find the matrix elements. For brevity, we will do the calculation for the more involved basis states, as the others will be similar in form and easier to calculate.

The first case will be $a_1^\dagger a_2^\dagger |0_A\rangle$,

$$BS_{12}(\theta)a_1^\dagger a_2^\dagger |0_A\rangle = [BS_{12}(\theta)a_1^\dagger BS_{12}(-\theta)]BS_{12}(\theta)a_2^\dagger |0_A\rangle, \quad (A2)$$

where $BS_{12}(\theta)|0_A\rangle = |0_A\rangle$. We can now use the solutions to the equations of motion to obtain

$$[BS_{12}(\theta)a_1^\dagger BS_{12}(-\theta)]BS_{12}(\theta)a_2^\dagger |0_A\rangle$$
$$= \left(\cos\theta a_1^\dagger + i\sin\theta a_2^\dagger W_1^{(\varphi)}\right)\left(\cos\theta a_2^\dagger + i\sin\theta a_1^\dagger W_2^{\dagger(\varphi)}\right)|0_A\rangle. \quad (A3)$$

Using that $[W_1^{(\varphi)}, a_2^\dagger] = 0$ we can rewrite this as

$$\cos^2\theta a_1^\dagger a_2^\dagger |0_A\rangle - \sin^2\theta a_2^\dagger W_1^{(\varphi)} a_1^\dagger |0_A\rangle. \quad (A4)$$

In the next step, we use the identity $N_1^A a_1^\dagger = a_1^\dagger$ to obtain

$$\cos^2\theta a_1^\dagger a_2^\dagger |0_A\rangle - \sin^2\theta a_2^\dagger [1 - (1 - e^{i\varphi})N_1^A]a_1^\dagger |0_A\rangle$$
$$= \cos^2\theta a_1^\dagger a_2^\dagger |0_A\rangle - e^{i\varphi}\sin^2\theta a_2^\dagger a_1^\dagger |0_A\rangle. \quad (A5)$$

Finally, using the commutation relation $a_1^\dagger a_2^\dagger = -e^{i\varphi}a_2^\dagger a_1^\dagger$, we conclude that

$$BS_{12}(\theta)a_1^\dagger a_2^\dagger |0_A\rangle$$
$$= \cos^2\theta a_1^\dagger a_2^\dagger |0_A\rangle - e^{i\varphi}\sin^2\theta a_2^\dagger a_1^\dagger |0_A\rangle$$
$$= (\cos^2\theta + \sin^2\theta)a_1^\dagger a_2^\dagger |0_A\rangle = a_1^\dagger a_2^\dagger |0_A\rangle. \quad (A6)$$

So the matrix element $\langle 1, 1, 0, 0|BS_{12}(\theta)|1, 1, 0, 0\rangle$ is 1. Similarly, $\langle 0, 0, 1, 1|BS_{12}(\theta)|0, 0, 1, 1\rangle$ is also 1, because the beam splitter has no action on these modes. Now we illustrate one more case and then give the expression for $BS_{12}(\theta)$. Consider the state $a_1^\dagger a_3^\dagger |0_A\rangle$. We can proceed in pretty much the same way we did before and obtain

$$BS_{12}(\theta)a_1^\dagger a_3^\dagger |0_A\rangle$$
$$= [BS_{12}(\theta)a_1^\dagger BS_{12}(-\theta)]BS_{12}(\theta)a_3^\dagger |0_A\rangle$$
$$= [BS_{12}(\theta)a_1^\dagger BS_{12}(-\theta)]a_3^\dagger |0_A\rangle$$
$$= \left(\cos\theta a_1^\dagger + i\sin\theta a_2^\dagger W_1^{(\varphi)}\right)a_3^\dagger |0_A\rangle$$
$$= \cos\theta a_1^\dagger a_3^\dagger |0_A\rangle + i\sin\theta a_2^\dagger a_3^\dagger |0_A\rangle, \quad (A7)$$

which tells us that $\langle 1, 0, 1, 0|BS_{12}(\theta)|1, 0, 1, 0\rangle = \cos\theta$ and $\langle 0, 1, 1, 0|BS_{12}(\theta)|1, 0, 1, 0\rangle = i\sin\theta$. Doing the calculation of the other matrix elements, we obtain

$$[BS_{12}(\varphi)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & i\sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & 0 & i\sin\theta & 0 \\ 0 & i\sin\theta & 0 & \cos\theta & 0 & 0 \\ 0 & 0 & i\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (A8)$$

The matrix for $BS_{23}(\theta)$ is very similar, since most of the calculations of matrix are elements repeated with different indices. The matrix is

$$[BS_{23}(\theta)] = \begin{bmatrix} \cos\theta & i\sin\theta & 0 & 0 & 0 & 0 \\ i\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta & i\sin\theta \\ 0 & 0 & 0 & 0 & i\sin\theta & \cos\theta \end{bmatrix}. \quad (A9)$$

The remaining matrix is the one which marks a departure between fermions and anyons, as the 1D analog of the Aharonov-Bohm phase appears explicitly. To calculate it we must use the result of example in Sec. III B for the matrix elements $\langle 1100|BS_{13}(\theta)|1100\rangle$, $\langle 0110|BS_{13}(\theta)|1100\rangle$, $\langle 1100|BS_{13}(\theta)|0110\rangle$, and $\langle 0110|BS_{13}(\theta)|0110\rangle$. The other

matrix elements are trivial. The result of this calculation is

$$[BS_{13}(\theta)] = \begin{bmatrix} \cos\theta & 0 & 0 & i\sin\theta & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & 0 & 0 & -ie^{-i\varphi}\sin\theta \\ i\sin\theta & 0 & 0 & \cos\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -ie^{i\varphi}\sin\theta & 0 & 0 & \cos\theta \end{bmatrix}. \tag{A10}$$

Now, by combining all of these results to evaluate the matrix products indicated by the circuit (Fig. 2) we obtain the matrix

$$[C(\varphi)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-i\frac{\pi}{4}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-i\cos\varphi}{\sqrt{2}} & 0 & \frac{i\sin\varphi}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & e^{i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & \frac{i\sin\varphi}{\sqrt{2}} & 0 & \frac{1+i\cos\varphi}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{A11}$$

To stay in the encoding defined in Sec. IV, the image of the encoded states under the unitary must stay in the encoded

subspace, and to guarantee this we only need to show that $|1, 1, 0, 0\rangle$ and $|0, 0, 1, 1\rangle$ are eigenstates of $C(\varphi)$, which is easily seen in the matrix above. In fact, $C(\varphi)$ in the encoded basis is

$$\begin{bmatrix} e^{-i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & \frac{1-i\cos\varphi}{\sqrt{2}} & 0 & \frac{i\sin\varphi}{\sqrt{2}} \\ 0 & 0 & e^{i\frac{\pi}{4}} & 0 \\ 0 & \frac{i\sin\varphi}{\sqrt{2}} & 0 & \frac{1+i\cos\varphi}{\sqrt{2}} \end{bmatrix}, \tag{A12}$$

showing that $C(\varphi) = R_{\hat{z}}(\frac{\pi}{2}) \otimes |0\rangle\langle0| + R_{\hat{n}}(\frac{\pi}{2}) \otimes |1\rangle\langle1|$, with $\hat{n} = (-\sin\varphi, 0, \cos\varphi)$ as claimed.

[1] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).

[2] C. Nayak and F. Wilczek, Nucl. Phys. B **479**, 529 (1996).

[3] A. Kitaev, Ann. Phys. **303**, 2 (2003).

[4] A. Kitaev, Ann. Phys. **321**, 2 (2006).

[5] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Rev. Mod. Phys. **80**, 1083 (2008).

[6] L. G. Valiant, SIAM J. Comput. **31**, 1229 (2002).

[7] B. M. Terhal and D. P. DiVincenzo, Phys. Rev. A **65**, 032325 (2002).

[8] S. Bravyi, Quantum Inf. Comput. **5**, 216 (2005).

[9] L. Amico, A. Osterloh, and U. Eckern, Phys. Rev. B **58**, R1703 (1998).

[10] C. D. Batista and G. Ortiz, Adv. Phys. **53**, 1 (2004).

[11] R. Mirman, Nuovo Cimento B **18**, 110 (1973).

[12] J. M. Leinaas and J. Myrheim, Nuovo Cimento B **37**, 1 (1977).

[13] F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982).

[14] E. Witten, Commun. Math. Phys. **117**, 353 (1988).

[15] J. Fröhlich and A. Zee, Nucl. Phys. B **364**, 517 (1991).

[16] J. Fröhlich and T. Kerler, Nucl. Phys. B **354**, 369 (1991).

[17] F. D. M. Haldane, Phys. Rev. Lett. **67**, 937 (1991).

[18] O. W. Greenberg, Phys. Rev. Lett. **64**, 705 (1990).

[19] S. Meljanac and A. Perica, J. Phys. A **27**, 4737 (1994).

[20] V. Bardek, M. Dorešić, and S. Meljanac, Int. J. Mod. A **09**, 4185 (1994).

[21] S. Meljanac, M. Milekovic, and A. Perica, Europhys. Lett. **28**, 79 (1994).

[22] M. El Baz and Y. Hassouni, Int. J. Mod. Phys. A **18**, 3015 (2003).

[23] M. D. Girardeau, Phys. Rev. Lett. **97**, 100402 (2006).

[24] P. Jordan and E. Wigner, Z. Phys. **47**, 631 (1928).

[25] Y. Hao, Y. Zhang, and S. Chen, Phys. Rev. A **78**, 023631 (2008).

[26] Y. Hao and S. Chen, Phys. Rev. A **86**, 043631 (2012).

[27] G. Tang, S. Eggert, and A. Pelster, New J. Phys. **17**, 123016 (2015).

[28] T. Keilmann, S. Lanzmich, I. McCulloch, and M. Roncaglia, Nat. Commun. **2**, 361 (2011).

[29] F. Liu, J. R. Garrison, D. L. Deng, Z. X. Gong, and A. V. Gorshkov, Phys. Rev. Lett. **121**, 250404 (2018).

[30] L. Wang, L. Wang, and Y. Zhang, Phys. Rev. A **90**, 063618 (2014).

[31] S. Greschner and L. Santos, Phys. Rev. Lett. **115**, 053002 (2015).

[32] L. Cardarelli, S. Greschner, and L. Santos, Phys. Rev. A **94**, 023615 (2016).

[33] C. Sträter, S. C. L. Srivastava, and A. Eckardt, Phys. Rev. Lett. **117**, 205303 (2016).

[34] S. Longhi and G. Della Valle, Opt. Lett. **37**, 2160 (2012).

[35] S. Aaronson and A. Arkhipov, Theory Comput. **9**, 143 (2013).

[36] M. H. Freedman, A. Kitaev, and Z. Wang, Commun. Math. Phys. **227**, 587 (2002).

[37] S. Bravyi, Phys. Rev. A **73**, 042313 (2006).

[38] S. X. Cui, S. M. Hong, and Z. Wang, Quantum Inf. Proc. **14**, 2687 (2015).

[39] S. X. Cui and Z. Wang, J. Math. Phys. **56**, 032202 (2015).

[40] C. Levaillant, B. Bauer, M. Freedman, Z. Wang, and P. Bonderson, Phys. Rev. A **92**, 012301 (2015).

[41] L. A. Wu and D. A. Lidar, J. Math. Phys. **43**, 4506 (2002).

[42] J. Kempe and K. B. Whaley, Phys. Rev. A **65**, 052330 (2002).

[43] D. J. Brod and E. F. Galvão, Phys. Rev. A **84**, 022310 (2011).

[44] M. Oszmaniec and Z. Zimborás, Phys. Rev. Lett. **119**, 220502 (2017).

[45] S. Prasad, M. O. Scully, and W. Matienssen, Opt. Commun. **62**, 139 (1987).

[46] M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, Phys. Rev. Lett. **73**, 58 (1994).

[47] S. B. Bravyi and A. Y. Kitaev, Ann. Phys. **298**, 210 (2002).

[48] A. Lerda and S. Sciuto, Nucl. Phys. B **401**, 613 (1993).

[49] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).

[50] T. Haug, H. Heimonen, R. Dumke, L.-C. Kwek, and L. Amico, arXiv:1706.05180.

[51] P. Zanardi, C. Zalka, and L. Faoro, Phys. Rev. A **62**, 030301(R) (2000).

[52] S. Balakrishnan and R. Sankaranarayanan, Phys. Rev. A **82**, 034301 (2010).

[53] M. J. Bremner, C. M. Dawson, J. L. Dodd, A. Gilchrist, A. W. Harrow, D. Mortimer, M. A. Nielsen, and T. J. Osborne, Phys. Rev. Lett. **89**, 247902 (2002).