Family of bound entangled states on the boundary of the Peres set

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Bound entangled (BE) states are strange in nature: a nonzero amount of free entanglement is required to create them but no free entanglement can be distilled from them under local operations and classical communication (LOCC). Even though the usefulness of such states has been shown in several information processing tasks, there exists no simple method to characterize them for an arbitrary composite quantum system. Here we present a $(d-3)/2$-parameter family of BE states each with positive partial transpose (PPT). This family of PPT-BE states is introduced by constructing an unextendible product basis (UPB) in $\mathbb{C}^d \otimes \mathbb{C}^d$ with d odd and $d \geqslant 5$. The range of each such PPT-BE state is contained in a $2(d-1)$-dimensional entangled subspace, whereas the associated UPB subspace is of dimension $(d-1)^2+1$. We further show that each of these PPT-BE states can be written as a convex combination of $(d-1)/2$ rank-4 PPT-BE states. Moreover, we prove that these rank-4 PPT-BE states are extreme points of the convex compact set $\mathcal{P}$ of all PPT states in $\mathbb{C}^d \otimes \mathbb{C}^d$, namely, the Peres set. An interesting geometric implication of our result is that the convex hull of these rank-4 PPT-BE extreme points—the $(d-3)/2$ simplex—is sitting on the boundary between the set $\mathcal{P}$ and the set of non-PPT states. We also discuss consequences of our construction in the context of quantum state discrimination by LOCC.

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I. INTRODUCTION

Entanglement is one of the fundamental features of multipartite quantum systems. Though this very concept was recognized in the early days of quantum theory [1], its physical meaning has remained elusive to date. The advent of quantum information theory identifies quantum entanglement as a useful resource for several information processing tasks (see Refs. [2,3] and references therein). Thus, characterization, detection, and quantification of quantum entanglement are of practical relevance and it is one of the main objectives to pursue research in quantum information theory.

It has been shown that the *quantum separability problem*, i.e., to verify whether an arbitrary density matrix of a given bipartite quantum system is entangled or separable, is an NP-hard problem [4]. But we have some sufficient criteria to detect entanglement of a given state. One such useful criterion is negative partial transposition (NPT): given a bipartite quantum state, if its partial transposition is negative then the state is entangled [5]. However, positive partial transposition (PPT) does not always guarantee separability; indeed there exist PPT-entangled states [6].

Quantifying entanglement is another challenging aspect in entanglement theory, and different operational as well as geometric measures have been introduced so far for this purpose. One such operationally motivated measure is *entanglement of distillation* [7–9]. It is defined as the optimal rate of obtaining a pure entangled state (singlet state) given many (asymptotically large) copies of the noisy (impure) entangled state under a sequence of local operations and classical communication (LOCC). However, this measure is not faithful[1] as it has been shown that PPT entangled states are undistillable and hence are also called bound entangled (BE) states [6]. These PPT-BE states may not be the only type of BE states as it has been conjectured that there may exist NPT-BE states [10,11].

During the past few years, the usefulness of PPT-BE states has been extensively studied in different contexts, e.g., entanglement activation [12,13], probabilistic interconvertibility among multipartite pure states [14,15], universal usefulness [16], secure key distillation [17–19], and quantum metrology [20]. Their connections to quantum steering [21] as well as to quantum nonlocality [22] have also been established. However, due to the hardness of the *quantum separability problem* there is no simple scheme to decide whether a given state is PPT-BE or not. We only know some examples and special constructions of such states [6,23–32]. One elegant construction comes from the structure of unextendible product bases (UPBs). It was shown that the normalized projector onto the subspace orthogonal to the UPB subspace (subspace spanned by a UPB) is a PPT-BE state [23,26].

In order to explore the geometry of the set of PPT-BE states, researchers have considered following sets of density

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[1]A *faithful* entanglement measure is strictly positive if and only if the state is entangled.

matrices: (i) the set $\mathcal{S}$ of all separable density matrices and (ii) the set $\mathcal{P}$ of all PPT density matrices. Both sets are convex and compact. These sets are identical for any two-qubit system as well as for any qubit-qutrit system while for other systems $\mathcal{P}$ strictly contains $\mathcal{S}$. Clearly, $\mathcal{S}$ contains only the rank-1 separable density matrices (pure product states) as its extreme points. However, except for the two-qubit and the qubit-qutrit systems, $\mathcal{P}$ contains not only the extreme points of $\mathcal{S}$ but also some additional extreme points. These new extreme points are nothing but PPT-BE states, and identifying such PPT-BE states is a troublesome task. Moreover, there exist some PPT-BE states, called *edge states*, living on the boundary of the set $\mathcal{P}$ and that of NPT states. To understand the extremely complicated structure of $\mathcal{P}$ it is important to identify the extremal PPT-BE states as well as the *edge states*. In the past few years a considerable effort has been given to addressing this question. A series of interesting results can be found in Refs. [33–44]. In Ref. [33], entanglement witness operators were constructed for edge states. In Ref. [34], it was shown that for any bipartite system the ratio between the probabilities of finding a PPT state in the interior of the set of PPT mixed quantum states and at its boundary is equal to 2. Later in Refs. [35–44] different methods were proposed to identify the extremal PPT-BE states of the set $\mathcal{P}$. In particular, Chen and Đoković have shown that all rank-4 two-qutrit PPT-BE states can be constructed from unextendible product bases and all such PPT-BE states are extreme points of $\mathcal{P}$ [40]. For a higher-dimensional system, the problem of identifying the extremal PPT-BE states as well as the edge states becomes more complicated; indeed very little is known so far. The main goal of the present work is to understand the set $\mathcal{P}$ for higher-dimensional systems by exploring new classes of extreme points and edge states. In the following, we summarize the main findings of this paper:

(i) We construct a UPB in $\mathbb{C}^d \otimes \mathbb{C}^d$, where $d \geqslant 5$ and d is odd. We also provide the tile structures corresponding to the UPB. The cardinality of such a UPB is $(d-1)^2+1$ and the corresponding entangled subspace is of dimension $2(d-1)$. The present construction is a generalization of the Tiles UPB in $\mathbb{C}^3 \otimes \mathbb{C}^3$, given in Ref. [23].

(ii) We show that the PPT-BE state proportional to the full rank projector onto the entangled subspace allows a convex decomposition in terms of $(d-1)/2$ rank-4 PPT-BE states and hence is not an extreme point of the set $\mathcal{P}$. Interestingly, it turns out that the rank-4 PPT-BE states appearing in the above decomposition are extreme points of the set $\mathcal{P}$.

(iii) We further show that any convex mixture of the aforesaid extreme points are edge states. Geometrically, a $(d-3)/2$ simplex is formed by the aforesaid $(d-1)/2$ rank-4 extreme points and the simplex resides on the boundary between the set $\mathcal{P}$ and the set of NPT states. At this point the result of Ref. [40] is worth mention. It turns out that the entangled subspace corresponding to a two-qutrit UPB contains only one edge state and hence it is also an extreme point of $\mathcal{P}$.

(iv) We study the cardinality of different locally indistinguishable sets (both *completable* and *uncompletable*) of orthogonal product states. We also discuss the merits of our construction in the context of orthogonal mixed-state discrimination by LOCC.

The paper is organized in the following way. In Sec. II we provide the notations used here and discuss the prerequisite ideas. In Sec. III we provide the main results where we first briefly review the Tiles UPB in $\mathbb{C}^3 \otimes \mathbb{C}^3$ and then present the generalized Tiles UPB in $\mathbb{C}^5 \otimes \mathbb{C}^5$ and $\mathbb{C}^d \otimes \mathbb{C}^d$, respectively. Then we provide the parametric family of PPT-BE edge states. Finally, in Sec. IV we make concluding remarks with some open problems for further research.

II. NOTATIONS AND PRELIMINARIES

A complex Hilbert space of dimension d is denoted by $\mathbb{C}^d$. To represent a non-null (unnormalized) vector, we use the ket notation $|v\rangle \in \mathbb{C}^d$, while $|\tilde{v}\rangle$ denotes the normalized vector parallel to $|v\rangle$. A linear operator maps a vector $|v\rangle \in \mathbb{C}^d$ to another vector $|v'\rangle \in \mathbb{C}^d$. The rank of a linear operator $\mathcal{O} : \mathbb{C}^d \mapsto \mathbb{C}^d$ is the dimension of its range denoted by $\mathcal{R}(\mathcal{O})$. Given a set of vectors, $S = \{|v_1\rangle, \ldots, |v_k\rangle\} \subset \mathbb{C}^d$, their linear span forms a subspace $\mathrm{Span}(S) := \{\sum_{i=1}^{k} \alpha_i |v_i\rangle \mid \alpha_i \in \mathbb{C},\ \forall\, i\}$; sometimes we use the notation $|\sum_{i=1}^{k} \alpha_i v_i\rangle \equiv \sum_{i=1}^{k} \alpha_i |v_i\rangle$.

The tensor product of two Hilbert spaces $\mathbb{C}^{d_A}$ and $\mathbb{C}^{d_B}$ is denoted by $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$. Consider an n-dimensional subspace of $\mathbb{C}^{d_A}$ spanned by a set of orthogonal vectors $\{|u_i\rangle\}_{i=1}^{n}$ and an m-dimensional subspace of $\mathbb{C}^{d_B}$ spanned by $\{|v_i\rangle\}_{i=1}^{m}$. We say that $\{|u_1\rangle_A, \ldots, |u_n\rangle_A\} \otimes \{|v_1\rangle_B, \ldots, |v_m\rangle_B\}$ spans the subspace $\mathbb{C}^n \otimes \mathbb{C}^m$ of $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$, where $\{|u_1\rangle_A, \ldots, |u_n\rangle_A\} \otimes \{|v_1\rangle_B, \ldots, |v_m\rangle_B\} \equiv \{|u_i\rangle_A \otimes |v_j\rangle_B\}_{i,j=1}^{n,m}$.

A convex set $\mathcal{A}$ is a subset of an affine space that is closed under convex combinations; i.e., for any $a_i \in \mathcal{A}$, $\sum_{i=1}^{n} p_i a_i \in \mathcal{A}$, where $p_i \geqslant 0$, $\forall\, i$ and $\sum_{i=1}^{n} p_i = 1$. A point $b \in \mathcal{A}$ is called an extreme point of $\mathcal{A}$ if it cannot be expressed as a convex combination of other points in $\mathcal{A}$. The set of all extreme points of $\mathcal{A}$ is denoted by $\mathcal{E}(\mathcal{A})$. A subset in Euclidean space is called compact if it is closed (contains all limit points) and bounded. According to the Krein-Milman theorem [45], any convex compact set of a finite-dimensional vector space is equal to the convex hull of its extreme points. Thus, this theorem ensures that while maximizing a linear functional over a convex compact set, it is sufficient to scan over only the extreme points instead of the whole convex compact set.

Every quantum system is associated with a Hilbert space. The state of a d-level quantum system is described by a density matrix ρ which is a positive semidefinite, Hermitian, trace-1 operator acting on $\mathbb{C}^d$. The set of all these density matrices $\mathcal{D}(\mathbb{C}^d)$ forms a convex compact subset of a real Euclidean space $\mathbb{R}^{(d^2-1)}$. For the density matrices of rank 1, i.e., $\rho = |\psi\rangle\langle\psi|$, $|\psi\rangle \in \mathbb{C}^d$ constitutes $\mathcal{E}(\mathcal{D})$.

A bipartite quantum system is associated with a tensor-product Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$, where d_k is the dimension of $\mathcal{H}_k$, $k \in \{A, B\}$. A quantum state $\rho_{AB} \in \mathcal{D}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$ is called a separable state (or product state) if it can be written as $\rho_{AB} = \sum_i p_i \rho_A^i \otimes \rho_B^i$, where $\rho_k^i \in \mathcal{D}(\mathbb{C}^{d_k})$, $\forall\, i, k$; $p_i \in \{0, 1\}$ and $\sum_i p_i = 1$. States that cannot be expressed in this form are entangled. The set of all separable states $\mathcal{S}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$ is also a convex compact set which is strictly contained in $\mathcal{D}$, i.e., $\mathcal{S} \subset \mathcal{D}$. Note that $\mathcal{E}(\mathcal{D})$ is constituted by both pure product and entangled states while only the pure product states constitute $\mathcal{E}(\mathcal{S})$.

To certify entanglement of a bipartite state, the partial transpose operation plays a crucial role. Partial transpose of a density matrix ρ_{AB} is denoted by $\rho_{AB}^{T_k}$, where T_k is the transposition operation in a chosen basis with respect to the kth party. If $\rho_{AB}^{T_k} \ngeqslant 0$, ρ_{AB} must be entangled [5]. However, $\rho_{AB}^{T_k} \geqslant 0$ guarantees separability of a given density matrix for the systems $\mathbb{C}^2 \otimes \mathbb{C}^2$, $\mathbb{C}^3 \otimes \mathbb{C}^2$, and $\mathbb{C}^2 \otimes \mathbb{C}^3$. In fact, in higher dimensions, there exist entangled states with positive partial transpose [6]. For a composite Hilbert space, the states having positive partial transpose again form a convex compact set $\mathcal{P}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$ (say), also known as the Peres set [35]. Clearly, $\mathcal{S} \subseteq \mathcal{P} \subset \mathcal{D}$ with set equality holds true for lower dimensions, i.e., for any two-qubit and qubit-qutrit systems. Thus, $\mathcal{E}(\mathcal{P})$ is exactly the same as $\mathcal{E}(\mathcal{S})$ in these dimensions and $\mathcal{E}(\mathcal{P})$ is strictly bigger than $\mathcal{E}(\mathcal{S})$ for all other dimensions. So, the nontriviality of characterizing $\mathcal{E}(\mathcal{P})$ lies in the fact that it contains not only pure product states but also some additional PPT-BE states. One of the goals of the present work is to understand these additional PPT-BE states of $\mathcal{E}(\mathcal{P})$. Here, these PPT-BE states are connected to UPBs, the definition of which is given below.

Definition 1. Consider a bipartite quantum system $\mathcal{H} = \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$. A complete orthogonal product basis (COPB) is a set of orthogonal product states that spans $\mathcal{H}$ while an incomplete orthogonal product basis (ICOPB) is a set of pure orthogonal product states that spans a subspace $\mathcal{H}_S$ of $\mathcal{H}$. An uncompletable product basis (UCPB) is an ICOPB whose complementary subspace $\mathcal{H}_S^{\perp}$ contains fewer pairwise orthogonal pure product states than its dimension. A UPB is a UCPB whose complementary subspace $\mathcal{H}_S^{\perp}$ contains no product states.

Note that $\mathcal{H}_S^{\perp}$, in the case of a UPB, is a fully entangled subspace. Here, we denote such a subspace as $\mathcal{H}_E$. Bennett *et al.* showed that the normalized projector onto the subspace $\mathcal{H}_E$ for a given UPB is a PPT-BE state [23]. These PPT-BE states are known to be edge states as they reside on the boundary between the set of PPT-BE states and that of the NPT states. In the following, we recall the mathematical definition of an edge state [33].

Definition 2. A PPT-BE state δ_{AB} is an edge state if and only if there exists no product state $|\varphi_A\rangle|\varphi_B\rangle$ and $\epsilon > 0$, such that $\delta_{AB} - \epsilon\mathbb{P}(|\varphi_A\rangle|\varphi_B\rangle)$ is positive or does have a PPT, where $\mathbb{P}(\cdot)$ denotes the one-dimensional projection operator.

A necessary and sufficient criterion for an edge state is given in Ref. [33]. We recall that criterion in the following remark.

Remark 1. A PPT-BE state δ_{AB} is an edge state if and only if there exists no $|\varphi_A\rangle|\varphi_B\rangle \in \mathcal{R}(\delta_{AB})$, such that $|\varphi_A\rangle|\varphi_B^{\star}\rangle \in \mathcal{R}(\delta_{AB}^{T_B})$, where $|\varphi_B^{\star}\rangle$ is the complex conjugate of $|\varphi_B\rangle$.

Obviously, any PPT-BE state belonging to $\mathcal{E}(\mathcal{P})$ is an edge state, but the converse is not true in general. Still, identifying the edge states is important to decipher the complicated geometrical structure of the set $\mathcal{P}$. This is another aspect of the present study. In particular, we provide a parametric family of edge states and the number of parameters increases linearly with the dimension of subsystems.

Another important aspect of UPBs is that they exhibit the phenomenon quantum nonlocality without entanglement [46] and hence the orthogonal pure product states within a UPB cannot be perfectly distinguished by LOCC [23,26]. Suppose that no party can perform nontrivial and orthogonality-preserving measurement[2] [47–49] in order to distinguish a set of orthogonal pure product states. Then this guarantees that the states of the given set cannot be distinguished perfectly by LOCC. Indeed, not even a single state from that set can be perfectly identified by such measurements. Again, in the context of orthogonal mixed-state discrimination, UPBs play a crucial role. It has been shown that any state supported in $\mathcal{H}_E$ cannot be *conclusively* distinguished from the mixed state proportional to the projector onto the UPB subspace [50]. Our construction leads to a few interesting observations regarding the state discrimination problem by LOCC.

III. RESULTS

Finding all the extreme points of the set $\mathcal{P}$ of a given system $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ is a highly nontrivial task. This is because of two reasons: first, identifying a PPT-BE state is itself a nontrivial job, and then determining whether such a PPT-BE state is an extreme point of the set $\mathcal{P}$ is the second hurdle. Leinaas and co-authors, for the first time, derived a necessary and sufficient condition for uniquely identifying the extreme points of $\mathcal{P}$ [35]. However, the useful implication of their condition requires an algorithmic search to detect any such point. Subsequently, this method has been studied in several bipartite and multipartite systems [36,37]. Later, these works motivated Chen and Đoković to come up with an analytical approach to explore the nontrivial extreme points of $\mathcal{P}$ [40]. In particular, using the techniques of projective geometry, they proved that any rank-4 PPT-BE state in $\mathbb{C}^3 \otimes \mathbb{C}^3$ is an extreme point of $\mathcal{P}$. As an immediate extension of this result, we give the following lemma.

Lemma 1. Consider a rank-4 PPT-BE state ρ_{AB} of $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ with $d_A, d_B \geqslant 3$. Assume that the range of ρ_{AB} is supported in $\mathcal{H}_A' \otimes \mathcal{H}_B'$, where $\mathcal{H}_A'$ ($\mathcal{H}_B'$) is a three-dimensional subspace of $\mathbb{C}^{d_A}$ ($\mathbb{C}^{d_B}$) and in $\mathcal{H}_A' \otimes \mathcal{H}_B'$, the tensor product is the induced version of "$\otimes$" used for the full Hilbert space $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$. Then the state ρ_{AB} is an extreme point of the set $\mathcal{P}$ of $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$.

Proof. In contradiction to the statement of the above lemma, let us assume that ρ_{AB} is not an extreme point of $\mathcal{P}$. Therefore, ρ_{AB} allows at least one decomposition of the form $\rho_{AB} = p\eta_{AB} + (1-p)\eta_{AB}'$, where $p \in (0, 1)$ and $\rho_{AB} \neq \eta_{AB}$, $\rho_{AB} \neq \eta_{AB}'$, and $\eta_{AB} \neq \eta_{AB}'$. The ranges of η_{AB} and η_{AB}' are fully contained in the range of ρ_{AB}. Consider the projector $\mathbb{P}'$ onto the subspace $\mathcal{H}_A' \otimes \mathcal{H}_B'$. $\mathbb{P}'\sigma\mathbb{P}' = \sigma$, for $\sigma \in \{\rho_{AB}, \eta_{AB}, \eta_{AB}'\}$; therefore, $\mathbb{P}'\rho_{AB}\mathbb{P}' = p\mathbb{P}'\eta_{AB}\mathbb{P}' + (1-p)\mathbb{P}'\eta_{AB}'\mathbb{P}'$, which contradicts the result of Chen *et al.* [40] that any PPT-BE state of rank 4 in $\mathbb{C}^3 \otimes \mathbb{C}^3$ is an extreme point. ■

Next we construct a $(d-3)/2$-parameter family of PPT-BE states in $\mathbb{C}^d \otimes \mathbb{C}^d$ for odd d and corresponding UPBs.

[2]If not all the positive operator-valued measure (POVM) elements describing a measurement are proportional to the identity operator then the measurement is a nontrivial measurement. Moreover, while distinguishing a given set of orthogonal states if the postmeasurement states remain pairwise orthogonal then it is an orthogonality-preserving measurement.

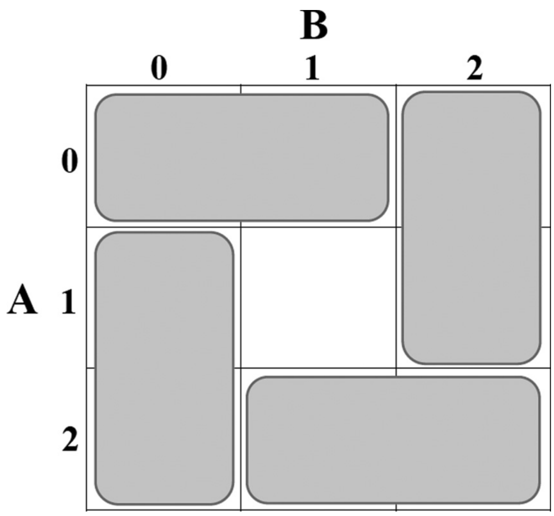

FIG. 1. Tile structure of a UPB in $\mathbb{C}^3 \otimes \mathbb{C}^3$.

Our construction is a generalization of Tiles UPB in $\mathbb{C}^3 \otimes \mathbb{C}^3$ introduced by Bennett *et al.* [8]. So, before presenting our main results we first briefly review different aspects of the Tiles UPB in $\mathbb{C}^3 \otimes \mathbb{C}^3$.

A. Tiles UPB in $\mathbb{C}^3 \otimes \mathbb{C}^3$

The orthogonal pure product states forming the Tiles UPB in $\mathbb{C}^3 \otimes \mathbb{C}^3$ are given below (see Fig. 1):

$$\begin{aligned} |\psi_1\rangle &= |0\rangle|0-1\rangle, \;\; |\psi_2\rangle = |2\rangle|1-2\rangle, \\ |\psi_3\rangle &= |0-1\rangle|2\rangle, \;\; |\psi_4\rangle = |1-2\rangle|0\rangle, \\ |S\rangle &= |0+1+2\rangle|0+1+2\rangle. \end{aligned} \tag{1}$$

The states are written without the normalization coefficients and throughout the paper we follow this convention unless stated otherwise. We say a subspace spanned by the product states forming a UPB is a UPB subspace ($\mathcal{H}_U$). The subspace orthogonal to $\mathcal{H}_U$ contains no product state and hence is an entangled subspace ($\mathcal{H}_E$). The normalized projector on $\mathcal{H}_E$ is given by

$$\rho_3 = \frac{1}{4}\left(\mathbb{I}_3 \otimes \mathbb{I}_3 - \sum_{i=1}^{4} |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| - |\tilde{S}\rangle\langle\tilde{S}|\right), \tag{2}$$

where $\mathbb{I}_n$ denotes an $n \times n$ identity matrix, and $|\tilde{\psi}_i\rangle$'s and $|\tilde{S}\rangle$ are normalized states of that given in Eqs. (1). We use the subscript ρ_3 to indicate that it is a density matrix on $\mathbb{C}^3 \otimes \mathbb{C}^3$. Since $\mathcal{H}_E$ contains no product state, ρ_3 is an entangled state. To prove the positivity of ρ_3 under partial transpose we make use of the following observation taken from Ref. [10].

Observation 1. Under transposition on Alice's side any pure product state $|\alpha_A\rangle\langle\alpha_A| \otimes |\alpha_B\rangle\langle\alpha_B|$ becomes $|\alpha_A^*\rangle\langle\alpha_A^*| \otimes |\alpha_B\rangle\langle\alpha_B|$ and hence a set of orthogonal product states is mapped into another set of orthogonal product states.

This guarantees the positivity of ρ_3 under partial transposition. If we remove the state $|S\rangle$ from Eqs. (1), the remaining four states together with the following five product states form a COPB:

$$\begin{aligned} |\psi_5\rangle &= |0\rangle|0+1\rangle, \;\; |\psi_6\rangle = |2\rangle|1+2\rangle, \\ |\psi_7\rangle &= |0+1\rangle|2\rangle, \;\; |\psi_8\rangle = |1+2\rangle|0\rangle, \\ |\psi_9\rangle &= |1\rangle|1\rangle. \end{aligned} \tag{3}$$

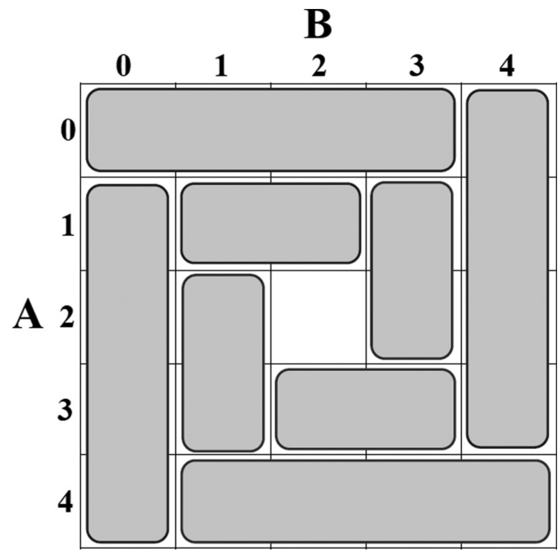

FIG. 2. Tile structure of a UPB in $\mathbb{C}^5 \otimes \mathbb{C}^5$.

When the state $|S\rangle$ is considered with the set of states $\{|\psi_i\rangle\}_{i=1}^4$, it stops inclusion of any the above states from forming a COPB. In other words it plays the role of "stopper" to construct the UPB. Using the states of Eqs. (3) one can construct four pairwise orthogonal entangled states $\{|\phi_i\rangle\}_{i=1}^4$ that span $\mathcal{H}_E$. One such construction is given as follows:

$$\begin{aligned} |\phi_1\rangle &= |\psi_5\rangle + |\psi_6\rangle - |\psi_7\rangle - |\psi_8\rangle, \\ |\phi_2\rangle &= |\psi_5\rangle - |\psi_6\rangle + |\psi_7\rangle - |\psi_8\rangle, \\ |\phi_3\rangle &= |\psi_5\rangle - |\psi_6\rangle - |\psi_7\rangle + |\psi_8\rangle, \\ |\phi_4\rangle &= a_1(|\psi_5\rangle + |\psi_6\rangle + |\psi_7\rangle + |\psi_8\rangle) + a_2|\psi_9\rangle. \end{aligned} \tag{4}$$

Note that the orthogonality of the states $\{|\psi_i\rangle\}_{i=1}^4$ with that of Eqs. (4) is immediate from construction. The orthogonality of $\{|\phi_i\rangle\}_{i=1}^3$ with $|S\rangle$ is also ensured due to the construction but to make $|\phi_4\rangle$ orthogonal to $|S\rangle$ we need to fix the values of a_1, a_2 accordingly. Now the PPT-BE state of (2) can be rewritten as $\rho_3 = 1/4\sum_{i=1}^4 |\tilde{\phi}_i\rangle\langle\tilde{\phi}_i|$. With reference to this context, it is quite worthy to mention that $\rho_3' = \sum_{i=1}^4 p_i|\tilde{\phi}_i\rangle\langle\tilde{\phi}_i|$ is an NPT, where p_i's are probabilities (excluding the case when all p_i's are equal) [40]. Indeed all such states are one-copy distillable [51]. Next we generalize this tile structure to higher-dimensional Hilbert spaces and explore different intriguing aspects of such generalization.

B. Tiles UPB in $\mathbb{C}^5 \otimes \mathbb{C}^5$

In $\mathbb{C}^5 \otimes \mathbb{C}^5$, it is possible to construct a COPB based on the tile structure given in Fig. 2. If we choose a suitable stopper $|S\rangle \in \mathbb{C}^5 \otimes \mathbb{C}^5$ and remove the product states that are not orthogonal to $|S\rangle$ then the remaining states of the COPB along with $|S\rangle$ form a UPB in $\mathbb{C}^5 \otimes \mathbb{C}^5$. Such a UPB is given by

$$
\begin{aligned}
&|\psi_1\rangle = |0\rangle|0-1+2-3\rangle, \quad |\psi_2\rangle = |0\rangle|0+1-2-3\rangle,\\
&|\psi_3\rangle = |0\rangle|0-1-2+3\rangle, \quad |\psi_4\rangle = |4\rangle|1-2+3-4\rangle,\\
&|\psi_5\rangle = |4\rangle|1+2-3-4\rangle, \quad |\psi_6\rangle = |4\rangle|1-2-3+4\rangle,\\
&|\psi_7\rangle = |0-1+2-3\rangle|4\rangle, \quad |\psi_8\rangle = |0+1-2-3\rangle|4\rangle,\\
&|\psi_9\rangle = |0-1-2+3\rangle|4\rangle, \quad |\psi_{10}\rangle = |1-2+3-4\rangle|0\rangle,\\
&|\psi_{11}\rangle = |1+2-3-4\rangle|0\rangle, \quad |\psi_{12}\rangle = |1-2-3+4\rangle|0\rangle,\\
&|\psi_{13}\rangle = |1\rangle|1-2\rangle, \quad |\psi_{14}\rangle = |3\rangle|2-3\rangle,\\
&|\psi_{15}\rangle = |1-2\rangle|3\rangle, \quad |\psi_{16}\rangle = |2-3\rangle|1\rangle,\\
&|S\rangle = |0+1+2+3+4\rangle|0+1+2+3+4\rangle. \qquad (5)
\end{aligned}
$$

Note that the cardinality of a UPB depends on the choice of stopper state. We say the product states that are not orthogonal to the stopper $|S\rangle$ but are orthogonal to all the other states in Eqs. (5) are *missing states*. The missing states $\{|\psi_i\rangle\}_{i=17}^{25}$ are given by

$$
\begin{aligned}
&|\psi_{17}\rangle = |0\rangle|0+1+2+3\rangle, \quad |\psi_{18}\rangle = |4\rangle|1+2+3+4\rangle,\\
&|\psi_{19}\rangle = |0+1+2+3\rangle|4\rangle, \quad |\psi_{20}\rangle = |1+2+3+4\rangle|0\rangle,\\
&|\psi_{21}\rangle = |1\rangle|1+2\rangle, \quad |\psi_{22}\rangle = |3\rangle|2+3\rangle,\\
&|\psi_{23}\rangle = |1+2\rangle|3\rangle, \quad |\psi_{24}\rangle = |2+3\rangle|1\rangle, \quad |\psi_{25}\rangle = |2\rangle|2\rangle. \qquad (6)
\end{aligned}
$$

Note that, in case of the COPB, each tile of the outermost layer corresponds to four pairwise orthogonal product states while each tile of the inner layer corresponds to two pairwise orthogonal product states, and the middle one corresponds to the state $|\psi_{25}\rangle = |2\rangle|2\rangle$. Because of the nonorthogonality with the stopper, one has to remove a pure product state from each tile in order to build the UPB. Hence, there are 17 states in the present UPB. This construction is different from that of Ref. [26].

Clearly, the states $\{|\psi_i\rangle\}_{i=1}^{16}$ of Eqs. (5) and the states $\{|\psi_i\rangle\}_{i=17}^{25}$ of Eqs. (6) together form a COPB in $\mathbb{C}^5\otimes\mathbb{C}^5$. Such a class of COPB in $\mathbb{C}^d\otimes\mathbb{C}^d$ and their local indistinguishability is discussed in Ref. [52]. Notice the structure given in Fig. 2. To include an orthogonal product state in any of the tiles, the new state must be orthogonal to the existing states of that tile and the stopper $|S\rangle$. But it is not possible and hence guarantees the unextendibility. Next, we construct the entangled basis $\{|\phi_i\rangle\}_{i=1}^{8}$ that spans the entangled subspace $\mathcal{H}_E$ of $\mathbb{C}^5\otimes\mathbb{C}^5$:

$$
\begin{aligned}
|\phi_1\rangle &= |\psi_{17}\rangle + |\psi_{18}\rangle - |\psi_{19}\rangle - |\psi_{20}\rangle,\\
|\phi_2\rangle &= |\psi_{17}\rangle - |\psi_{18}\rangle + |\psi_{19}\rangle - |\psi_{20}\rangle,\\
|\phi_3\rangle &= |\psi_{17}\rangle - |\psi_{18}\rangle - |\psi_{19}\rangle + |\psi_{20}\rangle,\\
|\phi_4\rangle &= a_3(|\psi_{17}\rangle + |\psi_{18}\rangle + |\psi_{19}\rangle + |\psi_{20}\rangle)\\
&\quad + a_4(a_2(|\psi_{21}\rangle + |\psi_{22}\rangle + |\psi_{23}\rangle + |\psi_{24}\rangle) - a_1|\psi_{25}\rangle),\\
|\phi_5\rangle &= |\psi_{21}\rangle + |\psi_{22}\rangle - |\psi_{23}\rangle - |\psi_{24}\rangle,\\
|\phi_6\rangle &= |\psi_{21}\rangle - |\psi_{22}\rangle + |\psi_{23}\rangle - |\psi_{24}\rangle,\\
|\phi_7\rangle &= |\psi_{21}\rangle - |\psi_{22}\rangle - |\psi_{23}\rangle + |\psi_{24}\rangle,\\
|\phi_8\rangle &= a_1(|\psi_{21}\rangle + |\psi_{22}\rangle + |\psi_{23}\rangle + |\psi_{24}\rangle) + a_2|\psi_{25}\rangle. \qquad (7)
\end{aligned}
$$

The states $\{|\phi_i\rangle\}_{i=1}^{8}$ are pairwise orthogonal by construction. The pairwise orthogonality also holds when we consider $\{|\psi_i\rangle\}_{i=1}^{16}$ of Eqs. (5) and $\{|\phi_i\rangle\}_{i=1}^{8}$ of Eqs. (7) together. Except $|\phi_4\rangle$ and $|\phi_8\rangle$ the other states of Eqs. (7) are also orthogonal to the stopper $|S\rangle$ by construction. But to make $|\phi_4\rangle$ and $|\phi_8\rangle$ orthogonal to $|S\rangle$ the coefficients a_i are chosen judicially. The PPT-BE state corresponding to the UPB of Eqs. (5) is of rank 8. The explicit form of the state is given by

$$\rho_5 = \frac{1}{8}\left(\mathbb{I}_5\otimes\mathbb{I}_5 - \sum_{i=1}^{16}|\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| - |\tilde{S}\rangle\langle\tilde{S}|\right) \qquad (8)$$

$$= \frac{1}{8}\sum_{i=1}^{8}|\tilde{\phi}_i\rangle\langle\tilde{\phi}_i|. \qquad (9)$$

From Observation 1 it follows that ρ_5 is a PPT state. Furthermore, ρ_5 is supported in an entangled subspace complementary to a UPB subspace. Therefore, according to Remark 1 it is an edge state. Note that the PPT-BE state corresponding to the Tiles UPB of $\mathbb{C}^3\otimes\mathbb{C}^3$ is not only an edge state but also an extreme point of the set $\mathcal{P}$ of $\mathbb{C}^3\otimes\mathbb{C}^3$ [40]. In the following we show that ρ_5 is not an extreme point of the set $\mathcal{P}$ of $\mathbb{C}^5\otimes\mathbb{C}^5$, though it corresponds to a Tiles UPB.

Theorem 1. The PPT-BE state ρ_5 allows a decomposition of the form $\rho_5 = \frac{1}{2}\sigma_1 + \frac{1}{2}\sigma_2$, where both σ_1 and σ_2 are rank-4 PPT-BE extreme points of the set $\mathcal{P}$ of $\mathbb{C}^5\otimes\mathbb{C}^5$.

Proof. Rewriting Eq. (9), we get $\rho_5 = \frac{1}{2}(\frac{1}{4}\sum_{i=1}^{4}|\tilde{\phi}_i\rangle\langle\tilde{\phi}_i|) + \frac{1}{2}(\frac{1}{4}\sum_{i=5}^{8}|\tilde{\phi}_i\rangle\langle\tilde{\phi}_i|)$, where we take $\sigma_1 \equiv \frac{1}{4}\sum_{i=1}^{4}|\tilde{\phi}_i\rangle\langle\tilde{\phi}_i|$ and $\sigma_2 \equiv \frac{1}{4}\sum_{i=5}^{8}|\tilde{\phi}_i\rangle\langle\tilde{\phi}_i|$. Both σ_1 and σ_2 are entangled as their ranges are contained in $\mathcal{H}_E$. In particular, the range of σ_2 is contained in a two-qutrit subspace (of $\mathbb{C}^5\otimes\mathbb{C}^5$) spanned by $\{|1\rangle_A, |2\rangle_A, |3\rangle_A\}\otimes\{|1\rangle_B, |2\rangle_B, |3\rangle_B\}$. In fact, σ_2 can be written as

$$\sigma_2 = \frac{1}{4}\left(\mathbb{I}_3'\otimes\mathbb{I}_3' - \sum_{i=13}^{16}|\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| - |\tilde{S}'\rangle\langle\tilde{S}'|\right), \qquad (10)$$

where $|S'\rangle = |1+2+3\rangle|1+2+3\rangle$ and $\mathbb{I}_3'$ is defined as

$$\mathbb{I}_3' := |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3|. \qquad (11)$$

Observation 1, while employed in Eq. (10), ensures the positivity of σ_2 under partial transpose. Since σ_2 turns out to be a rank-4 PPT-BE state of $\mathbb{C}^5\otimes\mathbb{C}^5$, then according to Lemma 1, σ_2 is an extreme point of the set $\mathcal{P}$ of $\mathbb{C}^5\otimes\mathbb{C}^5$.

To show the positivity of σ_1 under partial transposition, we rewrite it as

$$\sigma_1 = \frac{1}{4}\Bigg(\mathbb{I}_5\otimes\mathbb{I}_5 - \mathbb{I}_3'\otimes\mathbb{I}_3' - \sum_{i=1}^{12}|\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| - |\tilde{S}\rangle\langle\tilde{S}| + |\tilde{S}'\rangle\langle\tilde{S}'|\Bigg). \qquad (12)$$

A similar argument as in the case of σ_2 applies to the above expression for σ_1 and ensures the positivity of σ_1 under partial transposition. Then, applying the mapping $|1\rangle\to|0\rangle$, $|2\rangle\to|1+2+3\rangle$, $|3\rangle\to|4\rangle$ (for both Alice and Bob) to Eq. (10), we get $\sigma_2\to\sigma_1$. This reveals the fact that the range of σ_1 is contained in a two-qutrit subspace (of $\mathbb{C}^5\otimes\mathbb{C}^5$) spanned by

$\{|0\rangle_A, |1+2+3\rangle_A, |4\rangle_A\} \otimes \{|0\rangle_B, |1+2+3\rangle_B, |4\rangle_B\}$. Therefore, according to Lemma 1, σ_1 is an extreme point of the set $\mathcal{P}$ of $\mathbb{C}^5 \otimes \mathbb{C}^5$. This completes the proof. ∎

Please note that the two-qutrit subspaces of $\mathbb{C}^5 \otimes \mathbb{C}^5$ mentioned in the above theorem (in which the ranges of states σ_1 and σ_2 are respectively contained) are not orthogonal; they have exactly one-dimensional overlap. As already discussed in Sec. III A, the entangled subspace of the Tiles UPB in $\mathbb{C}^3 \otimes \mathbb{C}^3$ contains only one PPT-BE state which is also an extreme point of $\mathcal{P}$. Interestingly, for our construction in $\mathbb{C}^5 \otimes \mathbb{C}^5$, there exist three PPT-BE states ρ_5, σ_1, and σ_2. Here, σ_1 and σ_2 are extreme points of $\mathcal{P}$ as shown in Theorem 1 while ρ_5 is an edge state but not an extreme point. Indeed, in the following corollary we give a one-parameter family of PPT-BE states that are edge states. The range of all these states is contained in the same entangled subspace $\mathcal{H}_E$ as that of ρ_5.

Corollary 1. Consider a one-parameter family of states of the form $\sigma(p) := p\sigma_1 + (1-p)\sigma_2$, where $p \in [0, 1]$ and σ_1, σ_2 are same as those used in the proof of Theorem 1. All of these states are PPT-BE edge states.

Proof. Any convex mixture of PPT states is again a PPT state, either a PPT-BE state or a separable state. As $\mathcal{R}(\sigma(p))$ is contained in the fully entangled subspace $\mathcal{H}_E$, $\sigma(p)$ must be a PPT-BE state. Moreover, Remark 1 guarantees it to be an edge state. ∎

Another important point is that if the Tiles UPB of $\mathbb{C}^3 \otimes \mathbb{C}^3$ is trivially extended to a higher-dimensional Hilbert space (by adding suitable product states) then it is always possible to get a new UPB [53]. Such a UPB contains $D-4$ product states, where D is the net dimension of the extended system. Thus, if someone wants to construct a UPB in $\mathbb{C}^5 \otimes \mathbb{C}^5$ preserving the tile structure of $\mathbb{C}^3 \otimes \mathbb{C}^3$, it is then always possible to get a new UPB that contains 21 pure product states. This extension is trivial in the sense that the PPT-BE state corresponding to the higher-dimensional UPB is of the same rank as the old one. We, therefore, can say that our construction of Tiles UPB in $\mathbb{C}^5 \otimes \mathbb{C}^5$ is a "nontrivial extension" of the Tiles UPB in $\mathbb{C}^3 \otimes \mathbb{C}^3$. In the following lemma, it is shown that a nontrivial extension imposes a constraint on the cardinality of such a new UPB.

Lemma 2. Preserving the tile structure of the UPB in $\mathbb{C}^3 \otimes \mathbb{C}^3$, if one constructs a new UPB in $\mathbb{C}^5 \otimes \mathbb{C}^5$ then it is not possible to get a UPB with n pure product states, where $n = 18, 19, 20$.

Proof. We suppose that it is possible to construct a UPB in $\mathbb{C}^5 \otimes \mathbb{C}^5$ with 18 pure product states, i.e., $n = 18$. We also assume that the tile structure of the UPB in $\mathbb{C}^3 \otimes \mathbb{C}^3$ is preserved. These result in a PPT-BE state ρ_5' of rank 7 which can be written as $\rho_5' = \frac{1}{2}\sigma_1' + \frac{1}{2}\sigma_2$, where σ_1' is a PPT-BE state of rank 3 (see Theorem 1). But rank-3 PPT-BE states do not exist [54,55]. A similar argument holds for $n = 19, 20$. ∎

C. Tiles UPB in $\mathbb{C}^d \otimes \mathbb{C}^d$

We now generalize the results of the previous section for a system in $\mathbb{C}^d \otimes \mathbb{C}^d$ with d being odd. For this purpose, we first describe the tile structure given in Fig. 3. In this figure, there are two types of layers: type I and type II. Type I corresponds to the central layer which contains only one tile. We label this central tile by $k = 0$. On the other hand, all other

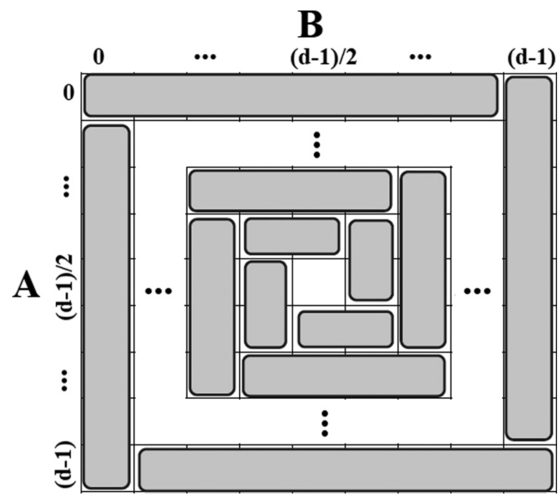

FIG. 3. Tile structure of a UPB in $\mathbb{C}^d \otimes \mathbb{C}^d$.

layers are type-II layers, each of which contains four tiles. We label these type-II tiles by $k = 1, \ldots, (d-1)/2$, where the outermost layer is labeled by $k = (d-1)/2$ and the innermost layer is labeled by $k = 1$. Now to construct the tiles UPB, we accumulate $2k-1$ pairwise orthogonal pure product states from each of the four tiles of the kth layer belonging to type II (for $k = 1, 2, \ldots, (d-1)/2$). Clearly, from the kth layer, we take $4(2k-1)$ states. In this way, we accumulate $(d-1)^2$ states in total. Along with these states, we add a stopper $|S\rangle = |0+1+\cdots+(d-1)\rangle|0+1+\cdots+(d-1)\rangle$. These result in a UPB of cardinality $(d-1)^2+1$. In Table I, we give explicit forms of the states taken from each layer (type II) in order to construct the UPB. We label these states by $\{|\psi_i\rangle\}_{i=1}^{(d-1)^2}$.

The stopper $|S\rangle$ blocks exactly one state from each of the four tiles of the kth layer of type II for $k = 1, 2, \ldots, (d-1)/2$, and also blocks the state of the central layer. So,

TABLE I. Explicit forms of the states taken from each layer (type II) in order to construct the UPB.

Value of k	States ($\omega = e^{\frac{\pi i}{k}}$, $i = \sqrt{-1}$, $k' = 1, \ldots, 2k-1$)
$(d-1)/2$	$|0\rangle|\sum_{j=0}^{d-2}\omega^{jk'}(j)\rangle$, $|d-1\rangle|\sum_{j=1}^{d-1}\omega^{jk'}(j)\rangle$ $|\sum_{j=0}^{d-2}\omega^{jk'}(j)\rangle|d-1\rangle$, $|\sum_{j=1}^{d-1}\omega^{jk'}(j)\rangle|0\rangle$
$(d-3)/2$	$|1\rangle|\sum_{j=1}^{d-3}\omega^{jk'}(j)\rangle$, $|d-3\rangle|\sum_{j=2}^{d-2}\omega^{jk'}(j)\rangle$ $|\sum_{j=1}^{d-3}\omega^{jk'}(j)\rangle|d-3\rangle$, $|\sum_{j=2}^{d-3}\omega^{jk'}(j)\rangle|1\rangle$
⋮	⋮
1	$|(d-3)/2\rangle|(d-3)/2-(d-1)/2\rangle$ $|(d+1)/2\rangle|(d-1)/2-(d+1)/2\rangle$ $|(d-3)/2-(d-1)/2\rangle|(d+1)/2\rangle$ $|(d-1)/2-(d+1)/2\rangle|(d-3)/2\rangle$

TABLE II. Missing states used to construct a set of orthogonal entangled states that spans $\mathcal{H}_E$.

k	Missing states ($t = (d-1)^2$)
$\frac{(d-1)}{2}$	$\lvert\psi_{t+1}\rangle = \lvert 0\rangle\lvert 0+1+\cdots+(d-2)\rangle$
	$\lvert\psi_{t+2}\rangle = \lvert d-1\rangle\lvert 1+2+\cdots+(d-1)\rangle$
	$\lvert\psi_{t+3}\rangle = \lvert 0+1+\cdots+(d-2)\rangle\lvert d-1\rangle$
	$\lvert\psi_{t+4}\rangle = \lvert 1+2+\cdots+(d-1)\rangle\lvert 0\rangle$
$\frac{(d-3)}{2}$	$\lvert\psi_{t+5}\rangle = \lvert 1\rangle\lvert 1+2+\cdots+(d-3)\rangle$
	$\lvert\psi_{t+6}\rangle = \lvert d-2\rangle\lvert 2+3+\cdots+(d-2)\rangle$
	$\lvert\psi_{t+7}\rangle = \lvert 1+2+\cdots+(d-3)\rangle\lvert d-2\rangle$
	$\lvert\psi_{t+8}\rangle = \lvert 2+3+\cdots+(d-2)\rangle\lvert 1\rangle$
⋮	⋮
1	$\lvert\psi_{d^2-4}\rangle = \lvert (d-3)/2\rangle\lvert (d-3)/2+(d-1)/2\rangle$
	$\lvert\psi_{d^2-3}\rangle = \lvert (d+1)/2\rangle\lvert (d-1)/2+(d+1)/2\rangle$
	$\lvert\psi_{d^2-2}\rangle = \lvert (d-3)/2+(d-1)/2\rangle\lvert (d+1)/2\rangle$
	$\lvert\psi_{d^2-1}\rangle = \lvert (d-1)/2+(d+1)/2\rangle\lvert (d-3)/2\rangle$
0	$\lvert\psi_{d^2}\rangle = \lvert (d-1)/2\rangle\lvert (d-1)/2\rangle$

to form a COPB, these $(2d-1)$ missing states along with the aforementioned $(d-1)^2$ orthogonal states are required. Here, the entangled subspace $\mathcal{H}_E$ is of dimension $2(d-1)$. Obviously, the density matrix ρ_d, proportional to the full rank projector onto $\mathcal{H}_E$, is of rank $2(d-1)$. However, to construct a set of orthogonal entangled states that spans $\mathcal{H}_E$, we use the missing states as shown earlier. The missing states are listed in Table II.

Next, we construct the set of entangled states that spans $\mathcal{H}_E$. These states are given in Table III. For the last states of each row, the coefficients are chosen in a way to make the state orthogonal to the stopper. Observe that the layer with $k = 1$ and that with $k = 0$ represent five missing states.

TABLE III. The set of entangled states that spans $\mathcal{H}_E$.

Entangled states ($l = 2d-2$)
$\lvert\phi_{l-3}\rangle = \lvert\psi_{d^2-4}\rangle + \lvert\psi_{d^2-3}\rangle - \lvert\psi_{d^2-2}\rangle - \lvert\psi_{d^2-1}\rangle$
$\lvert\phi_{l-2}\rangle = \lvert\psi_{d^2-4}\rangle - \lvert\psi_{d^2-3}\rangle + \lvert\psi_{d^2-2}\rangle - \lvert\psi_{d^2-1}\rangle$
$\lvert\phi_{l-1}\rangle = \lvert\psi_{d^2-4}\rangle - \lvert\psi_{d^2-3}\rangle - \lvert\psi_{d^2-2}\rangle + \lvert\psi_{d^2-1}\rangle$
$\lvert\phi_{l}\rangle = a_1(\lvert\psi_{d^2-4}\rangle + \lvert\psi_{d^2-3}\rangle + \lvert\psi_{d^2-2}\rangle + \lvert\psi_{d^2-1}\rangle) + a_2\lvert\psi_{d^2}\rangle$
$\lvert\phi_{l-7}\rangle = \lvert\psi_{d^2-8}\rangle + \lvert\psi_{d^2-7}\rangle - \lvert\psi_{d^2-6}\rangle - \lvert\psi_{d^2-5}\rangle$
$\lvert\phi_{l-6}\rangle = \lvert\psi_{d^2-8}\rangle - \lvert\psi_{d^2-7}\rangle + \lvert\psi_{d^2-6}\rangle - \lvert\psi_{d^2-5}\rangle$
$\lvert\phi_{l-5}\rangle = \lvert\psi_{d^2-8}\rangle - \lvert\psi_{d^2-7}\rangle - \lvert\psi_{d^2-6}\rangle + \lvert\psi_{d^2-5}\rangle$
$\lvert\phi_{l-4}\rangle = a_3(\lvert\psi_{d^2-8}\rangle + \lvert\psi_{d^2-7}\rangle + \lvert\psi_{d^2-6}\rangle + \lvert\psi_{d^2-5}\rangle) + a_4\lvert\phi_l^\perp\rangle$
⋮
$\lvert\phi_1\rangle = \lvert\psi_{t+1}\rangle + \lvert\psi_{t+2}\rangle - \lvert\psi_{t+3}\rangle - \lvert\psi_{t+4}\rangle$
$\lvert\phi_2\rangle = \lvert\psi_{t+1}\rangle - \lvert\psi_{t+2}\rangle + \lvert\psi_{t+3}\rangle - \lvert\psi_{t+4}\rangle$
$\lvert\phi_3\rangle = \lvert\psi_{t+1}\rangle - \lvert\psi_{t+2}\rangle - \lvert\psi_{t+3}\rangle + \lvert\psi_{t+4}\rangle$
$\lvert\phi_4\rangle = a_{d-2}(\lvert\psi_{t+1}\rangle + \lvert\psi_{t+2}\rangle + \lvert\psi_{t+3}\rangle + \lvert\psi_{t+4}\rangle) + a_{d-1}\lvert\phi_8^\perp\rangle$

Using these five missing states we construct four orthogonal entangled states given in the first row of Table III. Taking an equal mixture of these four states we construct a mixed state $\sigma_{(d-1)/2}$. Clearly, $\sigma_{(d-1)/2}$ is an entangled state as its range is contained in $\mathcal{H}_E$. Moreover, following the same procedure as in the case of Theorem 1, it can be shown that $\sigma_{(d-1)/2}$ is indeed a PPT-BE state.

Thereafter, we consider the missing states of the layer with $k = 2$. Again, using these states we construct three entangled states given in the second row (first three states) of Table III. The last state of the same row is formed by taking a linear combination of two states: the first state is chosen making it orthogonal to the other three states of this row and the second state is chosen making it orthogonal to the last state of the previous row. The coefficients a_3 and a_4 (in the second row) are chosen in a way that the state becomes orthogonal to the stopper $\lvert S\rangle$. Using these entangled states we again construct another PPT-BT state $\sigma_{(d-3)/2}$. In this way this process is repeated up to the outermost layer ($k = (d-1)/2$) and up to PPT-BE state σ_1.

We are now ready to present Theorem 2, which is a generalized version of Theorem 1. The proof of Theorem 2 is straightforward from the above discussion.

Theorem 2. The PPT-BE state ρ_d allows a decomposition of the form $\rho_d = \frac{2}{(d-1)}\sum_{k=1}^{(d-1)/2}\sigma_k$, where σ_k's are rank-4 PPT-BE extreme points of the set $\mathcal{P}$ of $\mathbb{C}^d\otimes\mathbb{C}^d$.

Note that the range of any σ_k is contained in two-qutrit subspaces of $\mathbb{C}^d\otimes\mathbb{C}^d$. For example, the two-qutrit subspace corresponding to $\sigma_{(d-1)/2}$ is spanned by $\{\lvert (d-3)/2\rangle_A, \lvert (d-1)/2\rangle_A, \lvert (d+1)/2\rangle_A\}\otimes\{\lvert (d-3)/2\rangle_B, \lvert (d-1)/2\rangle_B, \lvert (d+1)/2\rangle_B\}$; the two-qutrit subspace corresponding to $\sigma_{(d-3)/2}$ is spanned by $\{\lvert (d-5)/2\rangle_A, \lvert (d-3)/2+(d-1)/2+(d+1)/2\rangle_A, \lvert (d+3)/2\rangle_A\}\otimes\{\lvert (d-5)/2\rangle_B, \lvert (d-3)/2+(d-1)/2+(d+1)/2\rangle_B, \lvert (d+3)/2\rangle_B\}$, and so on. A corollary to the above theorem is stated as the following.

Corollary 2. Consider a $(d-3)/2$-parameter family of states of the form $\sigma(\vec{p}) := \sum_{k=1}^{(d-1)/2}p_k\sigma_k$, where $\vec{p}$ is a probability vector of dimension $(d-1)/2$ and σ_k's are the same as in Theorem 2. All these states are PPT-BE edge states.

In Ref. [36], the authors conjectured that the rank of an extremal PPT-BE state in $\mathbb{C}^d\otimes\mathbb{C}^d$ with *full local ranks* is always greater than or equal to $2(d-1)$. In the above corollary, if all p_k's are nonzero then the edge states of rank $2(d-1)$ have full local ranks. Furthermore, it is an open problem whether $2(d-1)$ is also the minimum rank for any edge state to possess the property that it has full local ranks.

D. State discrimination by LOCC

The study of UPB results in another interesting aspect called "quantum nonlocality without entanglement," where a set of bipartite product states allows local preparation but the set cannot be perfectly distinguished by LOCC [23,26,46].[3]

[3]Very recently, a nontrivial multipartite generalization of this phenomenon is reported, where a set of multipartite product states allow local preparation but for perfect discrimination all the parties must come together or an entangled resource across every bipartite cut is required [56–58].

As already mentioned, a trivially extended UPB from $\mathbb{C}^3 \otimes \mathbb{C}^3$ to $\mathbb{C}^5 \otimes \mathbb{C}^5$ contains 21 pairwise orthogonal pure product states [53]. But, our nontrivial construction in $\mathbb{C}^5 \otimes \mathbb{C}^5$ contains 17 pairwise orthogonal pure product states. Both the trivial and the nontrivial constructions exhibit the phenomenon called *quantum nonlocality without entanglement*. Thus, it is impossible to perfectly distinguish all the states within a UPB by LOCC only. However, our nontrivial construction attributes a notable property compared to the trivial one. In the case of trivial extension it is always possible to distinguish few states perfectly from the UPB by orthogonality-preserving LOCC. But in our case, not even a single state can be perfectly distinguished by such LOCC. This clearly indicates a stronger notion of local indistinguishability. Such a notion is also captured by all the higher-dimensional UPBs constructed in this paper. Note that the analogous notion of local indistinguishability has also been studied for multipartite systems [59]. Now, one may raise the question whether, to exhibit this notion, it is necessary to consider all 17 states of the present UPB in $\mathbb{C}^5 \otimes \mathbb{C}^5$. Interestingly, the answer is negative as it is possible to choose 9 states among these 17 states that can exhibit the aforesaid stronger notion. One possible choice of such 9 states is $\{|\psi_2\rangle, |\psi_5\rangle, |\psi_8\rangle, |\psi_{11}\rangle, |\psi_{13}\rangle, |\psi_{14}\rangle, |\psi_{15}\rangle, |\psi_{16}\rangle, |S\rangle\}$ given in Eqs. (5). In order to distinguish this set by LOCC, at least one party has to start with a nontrivial and orthogonality-preserving measurement. However, it can be shown that such a measurement does not exist for the present set. The proof follows from the argument given in Refs. [47,48]. Notice that from each tile (except the central tile) of Fig. 2, we take only one state and then we add the stopper and this results in a UCPB with cardinality 9 in $\mathbb{C}^5 \otimes \mathbb{C}^5$. Another interesting construction is the following: It is possible to construct a set of 14 states that also exhibit the above-mentioned notion but this set can be extended to a COPB. One such set of 14 states can be formed by excluding the states $|S\rangle$, $|\psi_6\rangle$, and $|\psi_{12}\rangle$ from the UPB of Eqs. (5). A similar construction of a completable set of locally indistinguishable states also follows from Refs. [60,61]. However, the cardinality there will be higher than 14. Both the UCPB and the set that is extendible to a COPB can be realized in $\mathbb{C}^d \otimes \mathbb{C}^d$.

Our construction also leads to an interesting observation in the context of orthogonal mixed-state discrimination. It is known that any state supported in the entangled subspace $\mathcal{H}_E$ cannot be distinguished unambiguously from the normalized projector onto the UPB subspace [50]. In $\mathbb{C}^3 \otimes \mathbb{C}^3$ there exists only one such state which is again PPT [40]. However, our construction ensures that there exists more than one PPT-BE state in $\mathbb{C}^d \otimes \mathbb{C}^d$ (with $d \geqslant 5$ and d odd) that cannot be distinguished unambiguously from the normalized projector onto the UPB subspace. In particular, the states $\sigma(\vec{p})$ defined in Corollary 2 possess the above feature.

IV. DISCUSSIONS

Our construction of PPT-BE states is based on Tiles UPBs. Theses UPBs can be thought of as a generalization of Tiles UPB in $\mathbb{C}^3 \otimes \mathbb{C}^3$. However, for higher dimensions the structure is more intriguing than the simplest case. For example, the entangled subspace in $\mathbb{C}^3 \otimes \mathbb{C}^3$ contains only one PPT-BE state—the equal mixture of any orthonormal basis of that entangled subspace—in other words, the state proportional to the projector onto the entangled subspace is that PPT-BE state. All other states corresponding to unequal mixtures of the projectors of pure states forming an orthonormal basis of the entangled subspace are NPT [40]; indeed they are distillable [51]. But our construction shows that for a higher-dimensional system, the entangled subspace contains a parametric family of PPT-BE states. Here, one may raise the following question: Consider any set of orthonormal states spanning the entangled subspace for a higher-dimensional system. Other than the given parametric family of states, consider an arbitrary state which is a mixture of those entangled states. Will that mixed state be an NPT state? To answer to this question, further analysis is required in this context. Another interesting research direction may be to generalize the present construction for even-dimensional quantum systems and explore different properties of the corresponding PPT entangled states. It is also important to reduce the cardinality of the present UPBs so that it is possible to find new edge states with higher rank. Again, by reducing the cardinality it may be possible to find new PPT entangled states (other than rank 4) that are extremal points of the set $\mathcal{P}$. We have also discussed consequences of our construction in the context of quantum state discrimination by LOCC. We have introduced a stronger notion of local indistinguishability. Quantification of this stronger notion is another aspect for further research. It will also be interesting to find the usefulness of these PPT-BE states in quantum information processing tasks.

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