Relativistic $\mathcal{PT}$-symmetric fermionic theories in 1 + 1 and 3 + 1 dimensions

Alireza Beygi* and S. P. Klevansky†
Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 12, 69120 Heidelberg, Germany

Carl M. Bender‡
Department of Physics, Washington University, St. Louis, Missouri 63130, USA

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Relativistic $\mathcal{PT}$-symmetric fermionic interacting systems are studied in 1 + 1 and 3 + 1 dimensions. The noninteracting Dirac equation is separately $\mathcal{P}$ and $\mathcal{T}$ invariant. The objective here is to include non-Hermitian $\mathcal{PT}$-symmetric interaction terms that give rise to *real* spectra. Such interacting systems could be physically realistic and could describe new physics. The simplest such non-Hermitian Lagrangian density is $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\rm int} = \bar\psi(i\partial\!\!\!/ - m)\psi - g\bar\psi\gamma^5\psi$. The associated relativistic Dirac equation is $\mathcal{PT}$ invariant in 1 + 1 dimensions and the associated Hamiltonian commutes with $\mathcal{PT}$. However, the dispersion relation $p^2 = m^2 - g^2$ shows that the $\mathcal{PT}$ symmetry is broken (the eigenvalues become complex) in the chiral limit $m \to 0$. For field-theoretic interactions of the form $\mathcal{L}_{\rm int} = -g(\bar\psi\gamma^5\psi)^N$ with $N = 2,\ 3$, which we can only solve approximately, we also find that if the associated (approximate) Dirac equation is $\mathcal{PT}$ invariant, the dispersion relation always gives rise to complex energies in the chiral limit $m \to 0$. Other models are studied in which x-dependent $\mathcal{PT}$-symmetric potentials such as ix^3, $-x^4$, $i\kappa/x$, Hulthén, or periodic potentials are coupled to the fermionic field ψ using vector or scalar coupling schemes or combinations of both. For each of these models the classical trajectories in the complex-x plane are examined. Some combinations of these potentials can be solved numerically, and it is shown explicitly that a real spectrum can be obtained. In 3 + 1 dimensions, while the simplest system $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\rm int} = \bar\psi(i\partial\!\!\!/ - m)\psi - g\bar\psi\gamma^5\psi$ resembles the 1 + 1-dimensional case, the Dirac equation is *not* $\mathcal{PT}$ invariant because $\mathcal{T}^2 = -\mathbb{1}$. This explains the appearance of complex eigenvalues as $m \to 0$. Other Lorentz-invariant two-point and four-point interactions are considered that give non-Hermitian $\mathcal{PT}$-symmetric terms in the Dirac equation. Only the axial vector and tensor Lagrangian interactions $\mathcal{L}_{\rm int} = -i\bar\psi\tilde B_\mu\gamma^5\gamma^\mu\psi$ and $\mathcal{L}_{\rm int} = -i\bar\psi T_{\mu\nu}\sigma^{\mu\nu}\psi$ fulfill both requirements of $\mathcal{PT}$ invariance of the associated Dirac equation and non-Hermiticity. The dispersion relations show that both interactions lead to complex spectra in the chiral limit $m \to 0$. The effect on the spectrum of the additional constraint of self-adjointness of the Hamiltonian with respect to the $\mathcal{PT}$ inner product is investigated.

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I. INTRODUCTION

A non-Hermitian quantum-mechanical Hamiltonian H that is invariant under combined parity (space reflection) $\mathcal{P}$ and time reversal $\mathcal{T}$ can have real eigenvalues [1,2]. If the spectrum is entirely real, we say that H has an *unbroken* $\mathcal{PT}$ symmetry. However, if H has complex eigenvalues, we say that H has a *broken* $\mathcal{PT}$ symmetry. Numerous theoretical studies of classical and quantum-mechanical $\mathcal{PT}$-symmetric systems have been done and many experiments on such systems have been performed. The remarkable features of $\mathcal{PT}$-symmetric systems include $\mathcal{PT}$ symmetry breaking in coupled wave guides, unidirectional invisibility, enhanced sensing at exceptional points, level bifurcation in superconducting wires, and robust wireless power transfer [3–10].

In quantum mechanics $x \to -x$ under parity $\mathcal{P}$ and $i \to -i$ under time reversal $\mathcal{T}$. Thus, the quantum-mechanical Hamiltonian $H = p^2 + x^2(ix)^\varepsilon$ (ε real) is $\mathcal{PT}$ invariant; H has a *real positive discrete* spectrum when $\varepsilon \geqslant 0$ [1]. This quantum theory generalizes to relativistic quantum field theory if the operator $x(t)$ is replaced by the pseudoscalar field $\phi(t, \mathbf{x})$ so that $\phi(t, \mathbf{x}) \to -\phi(t, -\mathbf{x})$ under $\mathcal{P}$ and $\phi(t, \mathbf{x}) \to \phi^*(-t, \mathbf{x})$ under $\mathcal{T}$. The analogous bosonic field-theoretic Hamiltonian density $(\partial\phi)^2 + \phi^2(i\phi)^\varepsilon$ also appears to have a real spectrum; this was shown to first order in ε for $0 \leqslant D < 2$ [11].

While $\mathcal{PT}$-symmetric bosonic systems have been studied heavily (there are over 4000 papers on such systems), only a few papers have been written on $\mathcal{PT}$-symmetric fermionic systems. Early work on matrix models of fermionic systems can be found in Refs. [12–15]. The Lagrangian density for a free relativistic fermionic field with mass m was extended by including a non-Hermitian axial mass term $\mathcal{L}_{\rm int} = -g\bar\psi\gamma^5\psi$, where g is a real mass parameter [16]. Further developments were made in Ref. [17] in which quantum electrodynamics was extended to include such a term and the restoration of gauge symmetry was investigated. In Ref. [18] the relationship between conserved currents and invariances of the Lagrangian in the framework of non-Hermitian field theories

*beygi@thphys.uni-heidelberg.de
†spk@physik.uni-heidelberg.de
‡cmb@wustl.edu

was examined. An application of $\mathcal{PT}$-symmetric fermionic field theory to neutrino species oscillation was proposed in Ref. [19] in which an eight-dimensional Dirac equation was analyzed. Neutrino oscillations in the context of $\mathcal{PT}$ symmetry were studied further in Ref. [20].

$\mathcal{PT}$-symmetric fermionic field theories in $1+1$ dimensions share the property with quantum-mechanical and bosonic field theories that $\mathcal{T}^2 = \mathbb{1}$ [16]. However, in Ref. [13] it was noted that $\mathcal{PT}$-symmetric fermionic systems in $3+1$ dimensions have the property that $\mathcal{T}^2 = -\mathbb{1}$. To explain this we first examine what happens in $1+1$ dimensions, where the gamma matrices are [21]

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1}$$

Note that $(\gamma^0)^2 = \mathbb{1}$, $(\gamma^1)^2 = -\mathbb{1}$, and $\gamma^5 = \gamma^0\gamma^1 = -\sigma_3$, where σ_3 is a Pauli matrix. Let us identify the discrete symmetries of the free Dirac equation

$$[i\gamma^0\partial_0 + i\gamma^1\partial_1 - m]\psi(t,x) = 0. \tag{2}$$

(Here $\partial_0 = \partial_t$ and $\partial_1 = \partial_x$). To determine the effect of a space reflection we let $x \to -x$ and then multiply (2) on the left by γ^0 to get

$$[i\gamma^0\partial_0 + i\gamma^1\partial_1 - m]\gamma^0\psi(t,-x) = 0.$$

Because this equation has the *same form* as (2) we identify that the action of parity reflection $\mathcal{P}$ on the spinor $\psi(t,x)$ is given by

$$\mathcal{P}: \ \psi(t,x) \to \mathcal{P}\psi(t,x)\mathcal{P}^{-1} = \gamma^0\psi(t,-x). \tag{3}$$

Next, to determine the effect of time reversal $\mathcal{T}$ we let $t \to -t$ in (2), take the complex conjugate of the resulting equation, and again multiply on the left by γ^0. We get

$$[i\gamma^0\partial_0 + i\gamma^1\partial_1 - m]\gamma^0\psi^*(-t,x) = 0.$$

Again, from form invariance we conclude that time reversal for spinors in $1+1$ dimensions is given by

$$\mathcal{T}: \ \psi(t,x) \to \mathcal{T}\psi(t,x)\mathcal{T}^{-1} = \gamma^0\psi^*(-t,x). \tag{4}$$

Since γ^0 is real we see that applying $\mathcal{P}$ or $\mathcal{T}$ twice leaves $\psi(t,x)$ invariant. Thus, $\mathcal{P}^2 = \mathbb{1}$ and $\mathcal{T}^2 = \mathbb{1}$. (Interestingly, this property of time reversal in $1+1$ dimensions implies that the Dirac electron behaves like a boson [22]).

In $3+1$ dimensions the Dirac representation of the gamma matrices is [23]

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$
$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \tag{5}$$

where σ^i are the Pauli matrices. The actions of parity and time reversal obtained similarly, are now [23]

$$\begin{aligned} \mathcal{P}: \ \psi(t,\mathbf{x}) &\to \mathcal{P}\psi(t,\mathbf{x})\mathcal{P}^{-1} = \gamma^0\psi(t,-\mathbf{x}), \\ \mathcal{T}: \ \psi(t,\mathbf{x}) &\to \mathcal{T}\psi(t,\mathbf{x})\mathcal{T}^{-1} = i\gamma^1\gamma^3\psi^*(-t,\mathbf{x}). \end{aligned} \tag{6}$$

If we apply $\mathcal{T}$ twice, we observe a change of sign: $\mathcal{P}^2 = \mathbb{1}$, but now $\mathcal{T}^2 = -\mathbb{1}$. This underscores the different nature of fermions in $3+1$ dimensions.

The purpose of this paper is to investigate the behavior of $1+1$- and $3+1$-dimensional relativistic $\mathcal{PT}$-invariant fermionic theories. An exploratory study in Ref. [24] examined in part the properties of a $\mathcal{PT}$-symmetric fermionic Lee model. This paper begins by reexamining the results in [16], where it was assumed that for real g the Lagrangian $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m - g\gamma^5)\psi$ is $\mathcal{PT}$ symmetric. We find that including the axial term gives a dispersion relation $p^2 = m^2 - g^2$ that yields a real value for the physical mass only when $m^2 \geqslant g^2$. This implies that the spectrum is not real in the chiral limit $m \to 0$. This result holds for Lagrangians in both $1+1$ and $3+1$ dimensions. We ask, why is this so and under what conditions is it not so? Obtaining spectral reality in the chiral limit is part of the motivation for this paper. One of our long-range goals is to construct a $\mathcal{PT}$-symmetric version of the Nambu-Jona-Lasinio model. The challenge is to identify additional non-Hermitian terms that are both $\mathcal{PT}$ symmetric and chiral and give rise to a real spectrum in the chiral limit [25].

Two ingredients are required for a precise analysis of fermionic systems: (i) Care must be taken in analyzing time reversal, which is nontrivial for fermionic systems. (ii) Care is needed in deciding on the form of $\mathcal{PT}$-adjoint operators. In this paper we focus first on time reversal and then address the constraint of self-adjointness with respect to the $\mathcal{PT}$ inner product for fermions.

For various interactions in $1+1$ and $3+1$ dimensions we use the Euler-Lagrange equations to construct the Dirac equation that results from a Lagrangian density and investigate whether this (quantum-mechanical) Dirac equation is form invariant under the actions of $\mathcal{P}$ and $\mathcal{T}$. This enables us to identify the transformation properties of the interaction term and also to calculate the dispersion relation associated with plane-wave solutions of the Dirac equation. In addition, by rewriting the Dirac equation in the form $i\partial_t\psi = H\psi$, we identify the *effective Hamiltonian* H [23] associated with the interaction. We will see that the form invariance of the Dirac equation under $\mathcal{PT}$ is equivalent to the statement that H commutes with $\mathcal{PT}$.

In analyzing the case of $1+1$ dimensions, we find the surprising result that for complex fermionic fields, the bilinear interaction form $-g\bar{\psi}\gamma^5\psi$ gives a Dirac equation that is *odd* under time reversal and also odd under parity. Thus, the Dirac equation with the interaction term is form invariant under $\mathcal{PT}$. The $\mathcal{PT}$ symmetry can also be verified by determining the Hamiltonian H associated with this Dirac equation $i\partial_t\psi = H\psi$. The 2×2 matrix representation clarifies this result. Comparing with the general result for a 2×2 $\mathcal{PT}$-symmetric fermionic Hamiltonian in $1+1$ dimensions [24], it becomes evident that in $1+1$ dimensions the $\mathcal{PT}$ symmetry is broken when m vanishes. In a second example, due to the similarities in the transformation properties of this interaction with those of $\phi(t,x)$, we surmise that higher integer powers of the interaction Lagrangian density $-g(\bar{\psi}\gamma^5\psi)^N$ might lead to a spectral relation that has real energies; we investigate this for $N=2$ and 3. We find that the $\mathcal{PT}$ symmetry is always broken if we assume that the expectation value $\langle\bar{\psi}\gamma^5\psi\rangle$ is negative imaginary. There are no other matrix potentials in $1+1$ dimensions.

We then turn to further examples for which x-dependent $\mathcal{PT}$-symmetric potentials ix^3, $-x^4$, and $i\kappa/x$ introduced via vector or scalar coupling various combinations, as well as the complex $\mathcal{PT}$-symmetric lattice potentials $i\kappa\cot(x)+i\gamma^0\sin(x)$ and the Hulthén potential are included in the Dirac equation of motion. In order to gain some understanding of these systems, we construct the analogous classical systems for which a classical phase structure can be obtained.

The situation in $3+1$ dimensions is different because $\mathcal{T}^2=-\mathbb{1}$. Studying the algebra in $3+1$ dimensions, we confirm that the interaction term $-g\gamma^5\psi$ in the equation of motion is *even* under time reversal. Since the parity transformation is still odd in $3+1$ dimensions, we conclude that the interaction term in the Dirac equation is not invariant under $\mathcal{PT}$. While the dispersion relation is superficially the same as for the $1+1$-dimensional case, which implies that there is *no* region in which the spectrum is real in the chiral limit, the associated interaction Hamiltonian is anti-$\mathcal{PT}$ symmetric, which is consistent with the complex nature of the spectrum.

In $3+1$ dimensions, we search for other bilinear combinations of fermionic fields with the aim of determining all possible combinations that give a Dirac equation that is form invariant under $\mathcal{PT}$ and that are not Hermitian. We find two types of terms having either an axial vector or a tensor structure. The spectra of both of the non-Hermitian $\mathcal{PT}$-symmetric interactions are analyzed. Here too we find that the $\mathcal{PT}$ symmetry is always broken in the chiral limit. We also look at the consequences of imposing an additional condition that the Hamiltonian be self-adjoint under the $\mathcal{PT}$ inner product for fermions [13,14] and investigate the restrictions that this implies. We demonstrate that the $\mathcal{PT}$ symmetry is always broken in the chiral limit, a feature that prevails in the analysis of the Dirac equation in the dimensions studied.

This paper is organized as follows: In Sec. II we investigate possible $\mathcal{PT}$-symmetric interactions in $1+1$ dimensions. We analyze $\mathcal{L}_{\rm int}=-g\bar\psi\gamma^5\psi$ in Sec. II A and extensions to this as $-g(\bar\psi\gamma^5\psi)^N$ in Sec. II B. We introduce the spatially dependent potentials ix^3, $-x^4$, and $i\kappa/x$, and the lattice and Hulthén potentials in Sec. II C. In Sec. III we analyze $3+1$-dimensional interactions, starting with $\mathcal{L}_{\rm int}=-g\bar\psi\gamma^5\psi$ in Sec. III A and other two-body (four-point) interactions in Sec. III B. Our conclusions and outlook are presented in Sec. IV.

II. NON-HERMITIAN $\mathcal{PT}$-SYMMETRIC FERMIONS IN 1 + 1 DIMENSIONS

A. Axial bilinear fermionic interaction

We start with the Lagrangian density for a conventional Hermitian free fermionic field theory,

$$\mathcal{L}_0=\bar\psi(i\partial\!\!\!/-m)\psi, \tag{7}$$

where $\bar\psi=\psi^\dagger\gamma^0$ and $\psi^\dagger$ is the Hermitian conjugate of ψ. In $1+1$ dimensions the gamma matrices are given in (1). We have shown that the free Dirac equation (2) associated with (7) is form invariant under the operation of $\mathcal{P}$ in (3) and of $\mathcal{T}$ in (4). Note that (2) is also form invariant under the combined operations of $\mathcal{P}$ and $\mathcal{T}$ because the functions $\psi(t,x)$ and $\mathcal{PT}\psi(t,x)=\gamma^0\gamma^0\psi^*(-t,-x)=\psi^*(-t,-x)$ both satisfy (2). A plane-wave solution to (2) gives the dispersion relation $E^2=p^2+m^2$. Finally, we read off the *effective* or *quantum-mechanical Hamiltonian H* from the free Dirac equation $i\partial_t\psi=H\psi$ in (2): $H=-i\gamma^0\gamma^1\partial_1+m\gamma^0$. (This form is often written using the definitions $\alpha=\gamma^0\gamma^1$ and $\beta=\gamma^0$ [23]).

We observe that the form invariance of the Dirac equation under $\mathcal{PT}$ is equivalent to the statement that H commutes with $\mathcal{PT}$: $H(\mathcal{PT}\psi)=\mathcal{PT}(H\psi)$. This is so because the left-hand side is

$$H(\mathcal{PT}\psi)=H(\gamma^0\gamma^0\psi^*)=H\psi^*,$$

and the right-hand side is

$$\mathcal{PT}(H\psi)=\gamma^0\gamma^0[+i\gamma^0\gamma^1(-\partial_1)+m\gamma^0]\psi^*=H\psi^*.$$

Next, we examine what happens if a pseudoscalar bilinear term is included in the Lagrangian density $\mathcal{L}=\mathcal{L}_0+\mathcal{L}_{\rm int}$, where $\mathcal{L}_{\rm int}=-g\bar\psi\gamma^5\psi$ and g is a real parameter. Now the associated quantum-mechanical Dirac equation is altered to read

$$(i\partial\!\!\!/-m-g\gamma^5)\psi=0. \tag{8}$$

Parity transforms this equation into

$$(i\partial\!\!\!/-m+g\gamma^5)\gamma^0\psi(t,-x)=0, \tag{9}$$

and time reversal has the effect

$$(i\partial\!\!\!/-m+g\gamma^5)\gamma^0\psi^*(-t,x)=0. \tag{10}$$

This Dirac equation is not invariant under $\mathcal{P}$ or $\mathcal{T}$ separately but it is invariant under $\mathcal{PT}$ because the axial interaction term changes sign twice; it is odd under both $\mathcal{P}$ and $\mathcal{T}$. So this axial non-Hermitian term is $\mathcal{PT}$ symmetric.

We can formulate this differently: We identify the effective quantum-mechanical Dirac Hamiltonian associated with the Dirac equation as

$$H=H_0+H_{\rm int}=-i\gamma^0\gamma^1\partial_1+m\gamma^0+g\gamma^0\gamma^5,$$

where $H_0=-i\gamma^0\gamma^1\partial_1+m\gamma^0$ and $H_{\rm int}=g\gamma^0\gamma^5$. We have shown that H_0 commutes with $\mathcal{PT}$, and from the effect of $\mathcal{P}$ and $\mathcal{T}$ in $1+1$ dimensions and the reality of $H_{\rm int}$, we see that $H_{\rm int}$ also commutes with $\mathcal{PT}$. Thus, the effective Hamiltonian H reflects the symmetry of the Dirac equation.

For this case the dispersion relation is obtained from a plane-wave solution $\psi(t,x)$, and multiplying (8) by $(p\!\!\!/+m+g\gamma^5)$, where $p\!\!\!/=\gamma^0p_0+\gamma^1p_1$, yields the result [16] $p^2=m^2-g^2$, which is non-negative only when $m^2\geqslant g^2$. Thus, in the chiral limit $m\to0$ the spectrum is complex and the $\mathcal{PT}$ symmetry is broken in this limit.

The matrix representation makes this result clearer. Recall that a general two-dimensional $\mathcal{PT}$-symmetric fermionic Hamiltonian, which is self-adjoint with respect to the $\mathcal{PT}$ inner product for fermions and which commutes with $\mathcal{PT}$, can be written as [24]

$$H^{\mathcal{PT}}=\begin{pmatrix}a & b\\ f & a\end{pmatrix}, \tag{11}$$

where a, b, and f are real numbers. The eigenvalues are $E_\pm=a\pm\sqrt{bf}$. Thus, if b and f have the same sign, the spectrum is real and the $\mathcal{PT}$ symmetry is unbroken.

Now, if the interaction Lagrangian density is $-g\bar{\psi}\gamma^5\psi$, the quantum-mechanical interaction Hamiltonian is

$$H_{\rm int} = g\gamma^0\gamma^5 = g\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Comparing this with (11), we confirm that $H_{\rm int}$ is $\mathcal{PT}$ symmetric and that this symmetry is always broken. Note that $H_{\rm int}$ is non-Hermitian.

If we add the conventional mass term to the interaction, the effective Hamiltonian in matrix form becomes

$$H = m\gamma^0 + g\gamma^0\gamma^5 = \begin{pmatrix} 0 & m+g \\ m-g & 0 \end{pmatrix}.$$

We see immediately that it is $\mathcal{PT}$ symmetric and that the $\mathcal{PT}$ symmetry is unbroken if $g^2 \leqslant m^2$.

Observe that the equation of motion resulting from the Dirac equation with an *imaginary* axial term,

$$(i\partial\!\!\!/ - m - ig\gamma^5)\psi = 0, \tag{12}$$

gives the dispersion relation $p^2 = m^2 + g^2$. So p is real for all g, including the chiral limit $m \to 0$. However, this axial term is not $\mathcal{PT}$ symmetric. In fact, it is *anti*-$\mathcal{PT}$ symmetric, so that (12) is not form invariant under $\mathcal{PT}$.

B. Approximate solution for higher-power field-theoretic interactions

This section explores the effect that higher-power interaction terms have in $1+1$-dimensional systems. Our starting point is the general Lagrangian density

$$\mathcal{L}(N) = \mathcal{L}_0 + \mathcal{L}_{\rm int} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - g(\bar{\psi}\gamma^5\psi)^N.$$

The Euler-Lagrange equations give the corresponding Dirac or single-particle equation of motion as

$$[i\partial\!\!\!/ - m - Ng\gamma^5(\bar{\psi}\gamma^5\psi)^{N-1}]\psi = 0,$$

which is nonlinear if $N > 1$. The case $N = 1$ reduces to that examined in Sec. II A. In the following we examine the cases $N = 2$ and $N = 3$.

1. N = 2

When $N = 2$, $\mathcal{L}(2) = \bar{\psi}(i\partial\!\!\!/ - m)\psi - g(\bar{\psi}\gamma^5\psi)^2$, and the interaction term is $\mathcal{PT}$ symmetric. The associated Euler-Lagrange equation is

$$[i\partial\!\!\!/ - m - 2g\gamma^5(\bar{\psi}\gamma^5\psi)]\psi = 0, \tag{13}$$

from which we deduce that the Hamiltonian H satisfying $i\partial_t\psi = H\psi$ is

$$H = -i\gamma^0\gamma^1\partial_1 + m\gamma^0 + 2g\gamma^0\gamma^5(\bar{\psi}\gamma^5\psi).$$

To solve (13) approximately we replace $\bar{\psi}\gamma^5\psi$ by its average value $\langle\bar{\psi}\gamma^5\psi\rangle = \langle\phi\rangle$. This is suggested by the fact that Wick contracting the fields would lead to this result in a first-level or mean-field approximation. Furthermore, by operating on the approximate version of (13) by $(i\partial\!\!\!/ - m - 2g\gamma^5\langle\phi\rangle)$ we can solve for the spectrum. In the chiral limit $m \to 0$ this is

$$p^2 = -4g^2\langle\phi\rangle^2. \tag{14}$$

Now, noting that the expectation value of a bosonic pseudoscalar field should be negative imaginary [11],

$$\langle\phi\rangle = -iA,$$

where A is a constant, it follows from (14) that $p^2 = 4g^2A^2$ is real. However, with this choice of $\langle\phi\rangle$, $H_{\rm int} = 2g\gamma^0\gamma^5\langle\phi\rangle$ is anti-$\mathcal{PT}$ symmetric, as is the interaction term in (13). Thus, the quantum-mechanical Dirac equation is no longer form invariant under $\mathcal{PT}$; also $\mathcal{PT}$ does not commute with H. Yet we obtain a real spectrum because now $H_{\rm int}$ is Hermitian. The opposite case, namely, when the Dirac equation is $\mathcal{PT}$ symmetric and H commutes with $\mathcal{PT}$, can be simulated by letting $g \to ig$. Then $p^2 < 0$ so, as in Sec. II A, $\mathcal{PT}$ symmetry is again realized in the broken phase.

2. N = 3

When $N = 3$, $\mathcal{L}(3) = \bar{\psi}(i\partial\!\!\!/ - m)\psi - g(\bar{\psi}\gamma^5\psi)^3$. This resembles the case for $N = 1$. The Euler-Lagrange equation now reads

$$[i\partial\!\!\!/ - m - 3g\gamma^5(\bar{\psi}\gamma^5\psi)^2]\psi = 0. \tag{15}$$

It follows that the interaction part of the Hamiltonian is

$$H_{\rm int} = 3g\gamma^0\gamma^5(\bar{\psi}\gamma^5\psi)^2. \tag{16}$$

Again, to find an approximate solution we replace $(\bar{\psi}\gamma^5\psi)^2$ by its average value $\langle(\bar{\psi}\gamma^5\psi)^2\rangle$. Solving (15) we get

$$p^2 = -9g^2\langle(\bar{\psi}\gamma^5\psi)^2\rangle^2,$$

in the chiral limit. We expect $\langle(\bar{\psi}\gamma^5\psi)^2\rangle$ to be real, so $p^2 < 0$ and the $\mathcal{PT}$ symmetry is always broken. We can confirm this explicitly by noting that (16) is simply proportional to γ^1 and thus only has off-diagonal values of opposite sign, see (1). Comparing this with (11), we note that (16) is manifestly $\mathcal{PT}$ symmetric.

We conclude that (i) if we construct a $1+1$-dimensional Lagrangian density containing the axial $\mathcal{PT}$-symmetric interaction $(\bar{\psi}\gamma^5\psi)^N$ (N odd), our approximation scheme shows that we obtain an equation of motion that is form invariant under $\mathcal{PT}$, and correspondingly a $\mathcal{PT}$-symmetric Hamiltonian. The $\mathcal{PT}$ symmetry is broken in the chiral limit. (ii) For even N the equation of motion contains an anti-$\mathcal{PT}$-symmetric term and the associated interaction Hamiltonian is also anti-$\mathcal{PT}$ symmetric but we obtain a dispersion relation that has real masses as a result of Hermiticity. If we modify the interaction by replacing $g \to ig$, we obtain a $\mathcal{PT}$-symmetric system but once again the $\mathcal{PT}$ symmetry is broken.

C. Dirac particle in $\mathcal{PT}$-symmetric potentials

In $1+1$ dimensions there are no other γ-matrix-based interactions. However, in addition to these, we can include $\mathcal{PT}$-symmetric potentials having a spatial dependence such as ix^3, $-x^4$, $i\kappa/x$, or even periodic potentials into the relativistic Dirac equation and study the effects of these. Unlike nonrelativistic potentials, which are scalars and can only be included as such in the Schrödinger equation, in the Dirac equation, such potentials can be incorporated either as the nonvanishing scalar part of the four-vector potential (which we refer to as vector coupling), or as pure scalar interactions, or as combinations thereof. We consider some examples below.

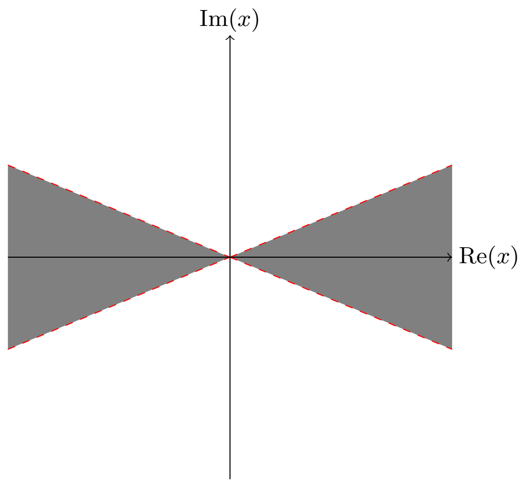

FIG. 1. Stokes sectors in the complex-x plane for ψ_1 with an opening angle of $\pi/4$ for the massless Dirac particle in the vector coupled potential ix^3; ψ_1 vanishes exponentially as $|x| \to \infty$ inside these sectors.

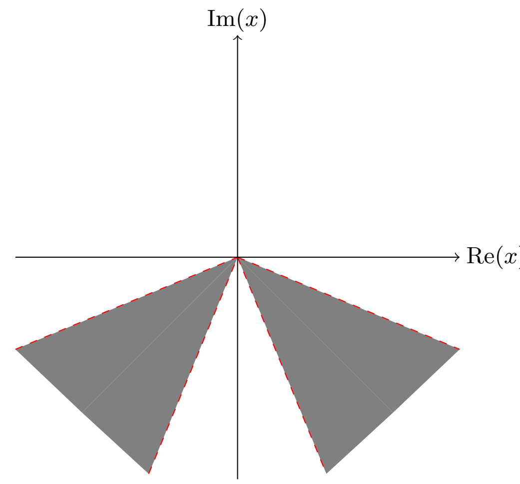

FIG. 2. Stokes sectors in the complex-x plane for ψ_2 with an opening angle of $\pi/4$. In this case, the sectors rotate below the real-x axis; ψ_2 vanishes exponentially as $|x| \to \infty$ inside these sectors.

1. Vector coupling with ix^3

The $1+1$-dimensional Dirac equation that includes the non-Hermitian $\mathcal{PT}$-symmetric vector-coupled potential ix^3 reads

$$(i\partial\!\!\!/ - ix^3\gamma^0)\psi(t,x) = 0. \tag{17}$$

This is form invariant under $\mathcal{PT}$ and the associated relativistic Hamiltonian

$$H = -i\alpha\partial_x + ix^3 \quad (\alpha \equiv \gamma^0\gamma^1),$$

is also $\mathcal{PT}$ invariant. If we look for solutions of the form $\psi(t,x) = e^{-iEt}\psi(x)$, we arrive at the corresponding eigenvalue problem

$$H\psi = (-i\alpha\partial_x + ix^3)\psi = E\psi.$$

The eigenvectors $\psi_1(x)$ and $\psi_2(x)$ that solve this equation have the asymptotic behavior

$$\psi_1(x) \sim \begin{pmatrix} e^{-x^4/4} \\ 0 \end{pmatrix}, \quad \psi_2(x) \sim \begin{pmatrix} 0 \\ e^{x^4/4} \end{pmatrix}.$$

The convergence domain for $\psi_1(x)$ and $\psi_2(x)$ in the complex-x plane are the $\mathcal{PT}$-symmetric Stokes sectors shown in Figs. 1 and 2, respectively. In these sectors $\psi_{1,2}(x)$ vanish exponentially as $|x| \to \infty$.

To obtain the self-energy of the propagating particle we apply $i\partial\!\!\!/$ to (17) and obtain the differential equation

$$\left(E^2 + \partial_x^2\right)\psi = -(x^6 + 3x^2\gamma^1\gamma^0 + 2x^3\partial_x\gamma^1\gamma^0)\psi. \tag{18}$$

Since the matrix $\gamma^1\gamma^0 = \mathrm{diag}(1,-1)$ is diagonal, the two-component equations in (18) decouple. Although they are not Schrödinger-like, each is individually $\mathcal{PT}$ symmetric. We first examine the classical analog of these equations obtained by replacing $-i\partial_x$ by p,

$$\begin{pmatrix} E^2 - p^2 & 0 \\ 0 & E^2 - p^2 \end{pmatrix} = \begin{pmatrix} -x^6 - 2ipx^3 - 3x^2 & 0 \\ 0 & -x^6 + 2ipx^3 + 3x^2 \end{pmatrix}.$$

The classical Hamiltonian associated with ψ_1 is $H_1 = \sqrt{p^2 - 2ix^3p - x^6 - 3x^2}$. The equation of motion of a classical particle described by H_1 is obtained by combining Hamilton's equations $dx/dt = \partial H_1/\partial p$ and $dp/dt = -\partial H_1/\partial x$: $dx/dt = \pm\sqrt{1 + 3x^2/E^2}$. By rescaling both x and t this equation becomes

$$\frac{dx}{dt} = \pm\sqrt{1+x^2}.$$

We find that $x(t)$ forms *open* trajectories in the complex-x plane, as shown in Fig. 3.

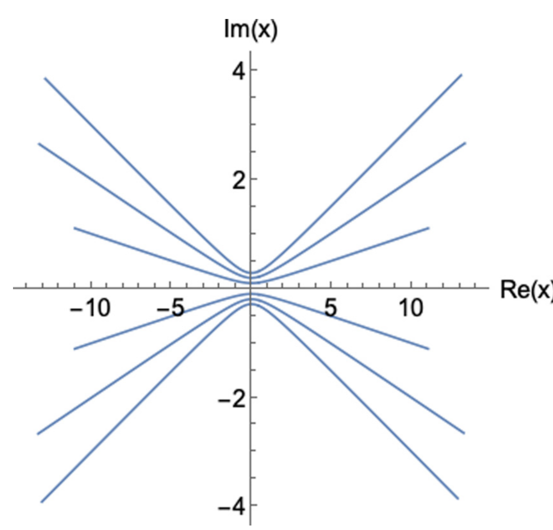

FIG. 3. Classical trajectories in the complex-x plane described by $H_1 = \sqrt{p^2 - 2ix^3p - x^6 - 3x^2}$.

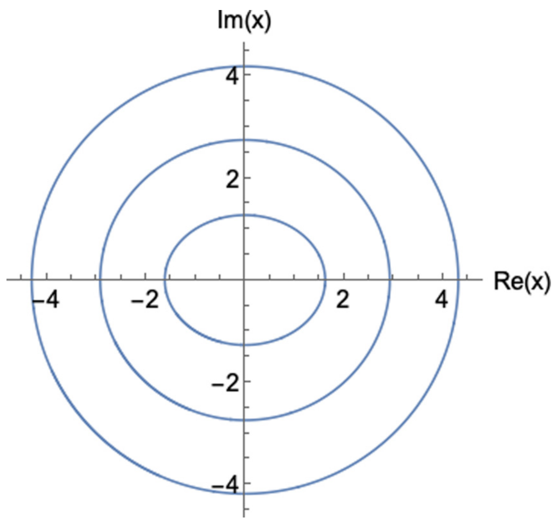

FIG. 4. Classical trajectories in the complex-x plane described by $H_2 = \sqrt{p^2 + 2ix^3 p - x^6 + 3x^2}$.

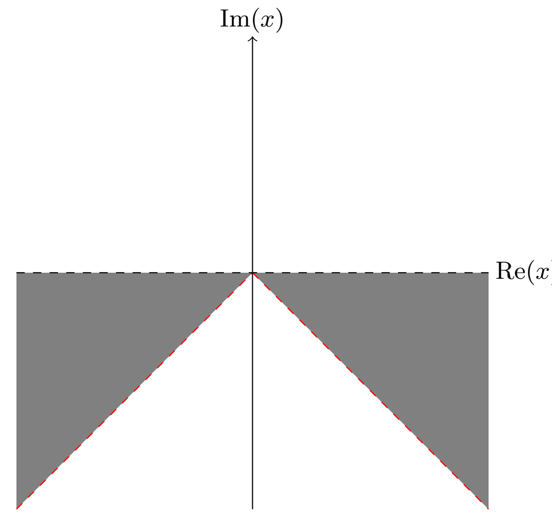

FIG. 5. Stokes sectors in the complex-x plane for ϕ_1 in (22). ϕ_1 vanishes exponentially inside these sectors.

The open classical trajectories of the particle in the complex-x plane reflects the behavior seen in the quantum case: By setting $p = 0$, we observe that the self-energy Σ_1 of the particle corresponding to ψ_1 is given by $\Sigma_1^2 = -x^6 - 3x^2$, which implies that Σ_1 cannot be real [26].

On the other hand, the trajectories of the classical particle in the complex-x plane that are associated with the classical Hamiltonian $H_2 = \sqrt{p^2 + 2ix^3 p - x^6 + 3x^2}$, are closed, as can be seen in Fig. 4. In the quantum system, the self-energy corresponding to ψ_2 is $\Sigma_2^2 = -x^6 + 3x^2$. By parametrizing x as $-i(\sqrt{1+ir} - 1)$ where r is real, ψ_2 vanishes exponentially as $r \to \pm\infty$. We note that the ends of this path lie in the left and right Stokes sectors of Fig. 2 as $|x| \to \infty$. When $-\sqrt[4]{3} < x < \sqrt[4]{3}$, Σ_2^2 is positive. Thus, the self-energy associated with the particle is real.

2. Scalar coupling with ix^3 and vector coupling with $i\kappa/x$

In the previous subsection we treated the $\mathcal{PT}$-symmetric potential ix^3 in a vector-coupling scheme; now we consider it as a *scalar* potential, where, in addition, the Dirac particle is also under the influence of a complex $\mathcal{PT}$-symmetric Coulomb potential. The non-Hermitian $\mathcal{PT}$-symmetric Dirac equation now reads

$$[i\partial\!\!\!/ - (i\kappa/x)\gamma^0 - ix^3]\psi(t,x) = 0, \tag{19}$$

where κ is a real parameter. The associated relativistic quantum-mechanical Hamiltonian is

$$H = -i\alpha\partial_x + i\kappa/x + \beta i x^3 \quad (\beta \equiv \gamma^0).$$

Again, looking for solutions of the form $\psi(t,x) = e^{-iEt}\psi(x)$ leads to an eigenvalue problem

$$H\psi = (-i\alpha\partial_x + i\kappa/x + \beta i x^3)\psi = E\psi.$$

Writing the eigenfunction $\psi(x)$ in terms of its two spinor components, $\psi(x) = (\phi_1(x), \phi_2(x))$ [27] we find two coupled differential equations for the scalar functions $\phi_{1,2}(x)$,

$$i\phi_1' + i\kappa\phi_1/x + ix^3\phi_2 = E\phi_1, \tag{20}$$

$$-i\phi_2' + i\kappa\phi_2/x + ix^3\phi_1 = E\phi_2. \tag{21}$$

We can eliminate the second component ϕ_2 from (20) by exploiting (21), and after rescaling ϕ_1, and choosing κ to be $-3/2$ for convenience, we obtain the simple form

$$-\phi_1'' - x^6\phi_1 = E^2\phi_1, \tag{22}$$

which is a Schrödinger-like equation with a $-x^6$ potential. On the real-x axis this upside-down potential is unstable, but by imposing appropriate $\mathcal{PT}$-symmetric boundary conditions we can obtain a real spectrum. As in the previous subsection, we find that to have a convergent eigenfunction, we must treat the problem in the complex-x plane.

The WKB approximation for the solutions of (22) to leading order is [28]

$$\phi_{\rm WKB}(x) = C_\pm[Q(x)]^{-1/4} e^{\pm i\int^x ds\sqrt{Q(s)}}, \tag{23}$$

where $Q(x) = E^2 + x^6$. For large $|x|$ the exponential component of this asymptotic behavior is

$$\phi_1 \sim e^{\pm ix^4/4}. \tag{24}$$

There are eight Stokes sectors in the complex-x plane, each with an opening angle of $\pi/4$. To have a $\mathcal{PT}$-symmetric pair of Stokes sectors, we choose the minus sign in (24) for the right Stokes sector, which is located just below the positive-real-x axis. For the left Stokes sector we choose the positive sign in (24), which determines a sector located just below the negative-real-x axis. These two Stokes sectors are depicted in Fig. 5.

We can also approximate the eigenenergies of (22). To do so, we first find the two turning points which are determined by $E = -x^6$ and which lie in the Stokes sectors in Fig. 5. These two points are

$$x_1 = \sqrt[6]{E}\, e^{-5i\pi/6}, \quad x_2 = \sqrt[6]{E}\, e^{-i\pi/6}.$$

The WKB quantization condition is

$$\int_{x_1}^{x_2} ds\sqrt{E_n^2 + s^6} = \left(n + \tfrac{1}{2}\right)\pi \quad (n \to \infty).$$

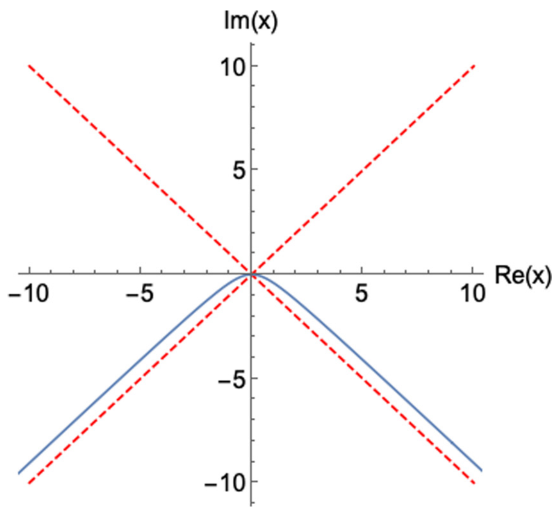

FIG. 6. The contour (solid line) on which the eigenvalue problem in (22) is posed (blue). The dashed lines (red) denote the edges of the sectors.

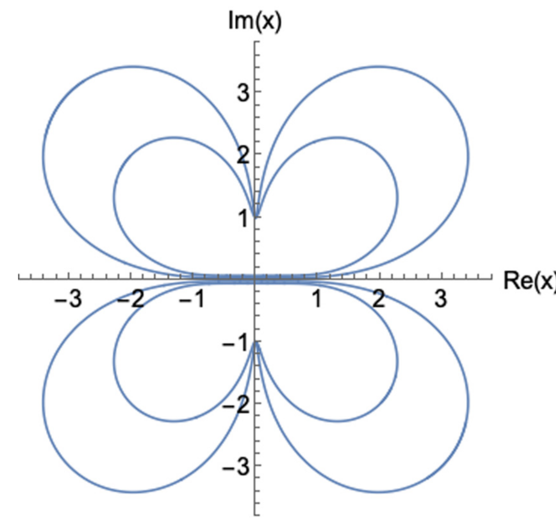

FIG. 7. Classical trajectories in the complex-x plane described by $H=\sqrt{p^2-x^6}$.

Thus,

$$E_n=\pm\left[4\sqrt{\pi/3}\Gamma\left(\tfrac{2}{3}\right)(2n+1)/\Gamma\left(\tfrac{1}{6}\right)\right]^{3/4}\quad(n\to\infty).$$

For $n=0$ or 1, we obtain $E_0=\pm1.0$ and $E_1=\pm2.27$.

An exact calculation of the eigenvalues can be made on parametrizing x as $-i(\sqrt{1+ir}-1)$, where r is a real variable. As depicted in Fig. 6, the ends of this path lie inside the Stokes sectors as $|x|\to\infty$, so we pose the eigenvalue problem for the differential equation in (22) on this contour. We determine the ground-state and first-excited-state energies numerically as

$$E_0=\pm1.16,\quad E_1=\pm2.29,$$

which illustrates the accuracy of the WKB approximation. Thus, the energy spectrum of the Dirac particle in the combined non-Hermitian $\mathcal{PT}$-symmetric potentials ix^3 and $i\kappa/x$ is real and discrete.

The trajectories of a classical particle in the complex-x plane described by the classical Hamiltonian $H=\sqrt{p^2-x^6}$ obtained from (22) are shown in Fig. 7. These trajectories are closed, which reflects the reality and the discreteness of the spectrum at the quantum level.

3. Vector coupling with $-x^4$

Next, we consider a massless Dirac particle under the influence of the upside-down quartic potential $-x^4$. In the vector-coupling scheme, the relativistic Dirac equation is modified to read

$$(i\partial\!\!\!/+x^4\gamma^0)\psi(t,x)=0. \tag{25}$$

As in the previous examples, this equation is form invariant under $\mathcal{PT}$ and the associated Hamiltonian

$$H=-i\alpha\partial_x-x^4,$$

commutes with $\mathcal{PT}$. Looking for solutions of the form $\psi(t,x)=e^{-iEt}\psi(x)$ leads to an eigenvalue equation $H\psi=E\psi$, whose eigenvectors behave asymptotically as

$$\psi_1(x)\sim\begin{pmatrix}e^{-ix^5/5}\\0\end{pmatrix},\quad\psi_2(x)\sim\begin{pmatrix}0\\e^{ix^5/5}\end{pmatrix}.$$

Note that ψ_1 vanishes exponentially in a Stokes sector with opening angle $\pi/5$. This sector contains the negative-imaginary-x axis, so it vanishes exponentially as $x\to-i\infty$. The function ψ_2 also vanishes exponentially in the same Stokes sector, but one that has rotated upward; that is, $\psi_2\to0$ as $x\to i\infty$.

Following the analysis given in Sec. II C 1, we iteratively apply $i\partial\!\!\!/$ to the corresponding Dirac equation and find the decoupled system of equations

$$\begin{pmatrix}E^2-p^2&0\\0&E^2-p^2\end{pmatrix}=\begin{pmatrix}x^8+2px^4-4ix^3&0\\0&x^8-2px^4+4ix^3\end{pmatrix}. \tag{26}$$

The self-energies Σ_1 and Σ_2 of the particle corresponding to ψ_1 and ψ_2 are given by $\Sigma_1^2=x^8-4ix^3$ and $\Sigma_2^2=x^8+4ix^3$. As ψ_1 and ψ_2 converge on $x=-ir$ and $x=ir$, the self-energies become real.

The trajectories of the classical particle described by both of the classical Hamiltonians obtained from (26) are closed in the complex-x plane. In Fig. 8 this is shown for the classical Hamiltonian $H=\sqrt{p^2+2x^4p+x^8-4ix^3}$.

4. Scalar coupling with $-x^4$ and vector coupling with $i\kappa/x$

We now treat the upside-down potential $-x^4$ as a scalar potential and, in addition, we consider the effect of a complex $\mathcal{PT}$-symmetric Coulomb potential on the Dirac particle in a vector-coupling scheme, satisfying the modified Dirac equation

$$[i\partial\!\!\!/-(i\kappa/x)\gamma^0+x^4]\psi(t,x)=0,$$

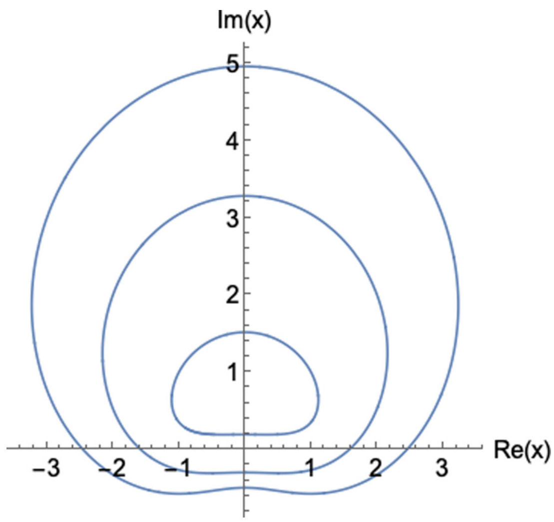

FIG. 8. Classical trajectories in the complex-x plane described by $H = \sqrt{p^2 + 2x^4 p + x^8 - 4ix^3}$.

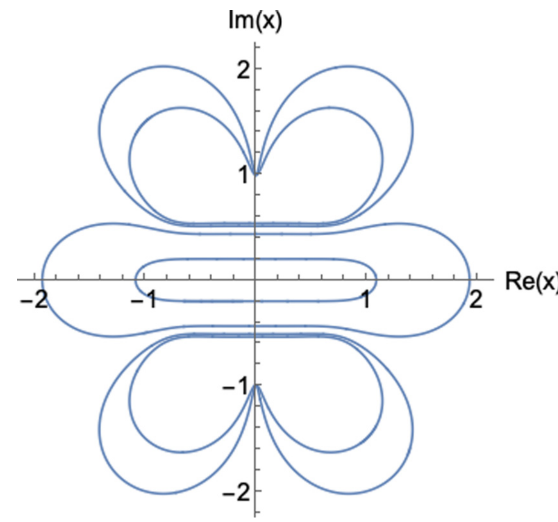

FIG. 9. Classical trajectories in the complex-x plane described by $H = \sqrt{p^2 + x^8}$.

where κ is a real parameter. This equation is form invariant under $\mathcal{PT}$ and the associated Hamiltonian

$$H = -i\alpha\partial_x + i\kappa/x - \beta x^4,$$

commutes with $\mathcal{PT}$. The search for solutions of the form $\psi(t,x) = e^{-iEt}\psi(x)$ requires solutions of the eigenvalue equation

$$H\psi = (-i\alpha\partial_x + i\kappa/x - \beta x^4)\psi = E\psi.$$

As in Sec. II C 2, it is convenient to write $\psi(x)$ in terms of its (scalar) components $\psi = (\phi_1, \phi_2)$ and derive the coupled equations that ϕ_1 and ϕ_2 satisfy. Following the procedure outlined in Sec. II C 2, we eliminate ϕ_2 and arrive at a Schrödinger-like equation for ϕ_1,

$$-\phi_1'' + x^8\phi_1 = E^2\phi_1, \tag{27}$$

where, for convenience, we have set $\kappa = -2$. We have thus found an octic potential with positive sign. Hence, we pose the eigenvalue problem on the real-x axis. As before we use the WKB approximation to obtain the eigenvalues for large n,

$$E_n = \pm\left[\sqrt{\pi}\,\Gamma\left(\tfrac{13}{8}\right)\left(n + \tfrac{1}{2}\right)/\Gamma\left(\tfrac{9}{8}\right)\right]^{4/5} \quad (n \to \infty).$$

From this equation we find that $E_0 = \pm 0.87$ and $E_1 = \pm 2.10$. A direct numerical calculation gives $E_0 = \pm 1.11$ and $E_1 = \pm 2.18$. Thus, once again, we find that the energy spectrum of a Dirac particle in the presence of combined non-Hermitian $\mathcal{PT}$-symmetric vector and scalar potentials $i\kappa/x$ and $-x^4$ is real and discrete.

Here again we see that the reality and discreteness of the spectrum is evident at the classical level with closed trajectories in the complex-x plane. We recognize the classical Hamiltonian of the system from (27) as being $H = \sqrt{p^2 + x^8}$. Figure 9 shows that the classical trajectories described by this Hamiltonian H are closed.

5. Complex $\mathcal{PT}$-symmetric lattice potentials

The methods in the previous subsections are general enough to be applied to a Dirac particle in complex $\mathcal{PT}$-symmetric lattices. The relativistic Dirac equation

$$[i\partial\!\!\!/ - i\kappa\cot(x)\gamma^0 - i\sin(x)]\psi(t,x) = 0, \tag{28}$$

with κ real, has non-Hermitian interaction terms, but is form invariant with respect to $\mathcal{PT}$. The associated Hamiltonian,

$$H = -i\alpha\partial_x + i\kappa\cot(x) + i\beta\sin(x),$$

commutes with $\mathcal{PT}$.

As before, we can search for time-independent solutions of (28). Writing $\psi(t,x) = e^{-iEt}\psi(x)$, we obtain coupled equations for the components of the spinor eigenfunction ϕ_1 and ϕ_2, where $\psi = (\phi_1, \phi_2)$. Eliminating ϕ_2, we find a Schrödinger-like equation for ϕ_1, which after suitably rescaling, is

$$-\phi_1'' - \sin^2(x)\phi_1 = E^2\phi_1,$$

where we have set $\kappa = -1/2$.

The spectrum of the operator $-d^2/dx^2 - \sin^2(x)$ is real and consists of spectral bands separated by infinitely many spectral gaps [29]. The absence of discrete energies and the reality of the band-structure manifest themselves via periodic, open trajectories of the classical particle described by $H = \sqrt{p^2 - \sin^2(x)}$ in the complex-x plane, as depicted in Fig. 10.

Before closing this subsection, we make a side remark: We note that the (quantum-mechanical, nonrelativistic) Hamiltonian $H = p^2 + i\sin(x)$ describes a particle subject to the periodic potential $V(x) = i\sin(x)$ in a $\mathcal{PT}$-symmetric crystal. As was shown in Ref. [30], by examining a discriminant, one can conclude that this Hamiltonian has real energy bands. However, to verify that the band structure is real, one can alternatively show that the *eigenfunctions* are $\mathcal{PT}$ symmetric; that is, that the $\mathcal{PT}$ symmetry of the Hamiltonian is unbroken. To this end we plot the absolute values of the eigenfunctions of the two states of $H = p^2 + i\sin(x)$ in Fig. 11 and observe that both are in fact symmetric. The energy bands are real, and are shown in Fig. 12. We use this technique in the next subsection.

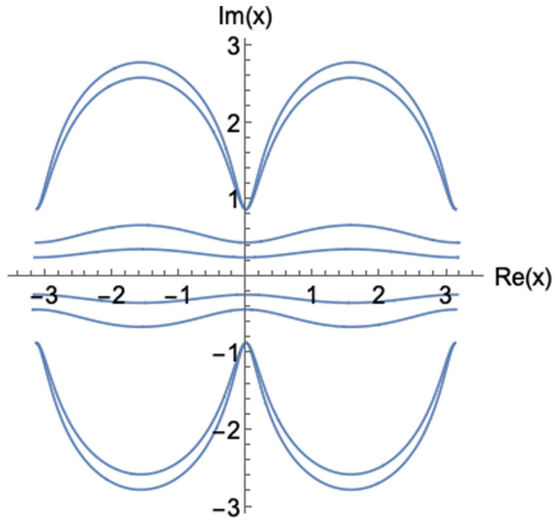

FIG. 10. Classical paths for $H = \sqrt{p^2 - \sin^2(x)}$.

6. Scalar coupling with complex $\mathcal{PT}$-symmetric Hulthén potential

The complex $\mathcal{PT}$-symmetric Hulthén potential is

$$V(x) = \frac{e^{-ix}}{1 - e^{-ix}}.$$

If we regard $V(x)$ as a potential in the *nonrelativistic* time-independent Schrödinger equation, $H\psi = E\psi$, with $H = p^2 + V(x)$, we find that the band structure for the energies is entirely complex, and, as is the case with $\mathcal{PT}$-symmetric potentials in the broken-symmetry phase, the eigenvalues occur in complex-conjugate pairs. We illustrate this by plotting the absolute values of the eigenfunctions of the two states of the Hamiltonian that correspond to the complex-conjugate pairs of the band-edge energies $E = 0.75 \pm 0.59i$, see Fig. 13. Note that the eigenfunctions display no symmetry, which implies the complex nature of the band structure.

We now consider the relativistic Dirac equation that includes the $\mathcal{PT}$-symmetric Hulthén potential in a scalar-coupling scheme, together with an additional $\mathcal{PT}$-symmetric vector potential:

$$\left(i\partial\!\!\!/ - \kappa\frac{1}{1 - e^{-ix}}\gamma^0 - \frac{e^{-ix}}{1 - e^{-ix}}\right)\psi(t, x) = 0, \tag{29}$$

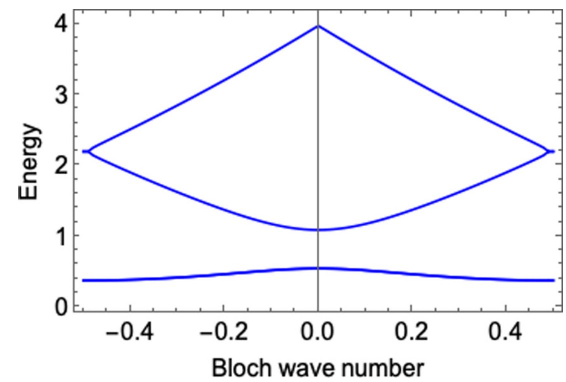

FIG. 12. The energy bands associated with the potential $i\sin(x)$ in the first Brillouin zone.

with κ being a real parameter. This equation has been constructed so as to be form invariant with respect to $\mathcal{PT}$ and the associated Hamiltonian

$$H = -i\alpha\partial_x + \kappa\frac{1}{1 - e^{-ix}} + \beta\frac{e^{-ix}}{1 - e^{-ix}},$$

once again commutes with $\mathcal{PT}$. Following the same procedure as in the previous subsections, we search for time-independent solutions of the Dirac equation, and find the equations for the components of $\psi = (\phi_1, \phi_2)$. On eliminating ϕ_2, we obtain a Schrödinger-like equation for the first component of the two-component spinor eigenfunction as

$$-\phi_1'' + \frac{1}{(1 - e^{ix})^2}\phi_1 = E^2\phi_1, \tag{30}$$

where we have set $\kappa = -1/2$ for convenience.

By using spectral methods, we determine numerically that the band structure in (30) is entirely real; that is, the symmetry is unbroken. We have shown the absolute values of the first two eigenfunctions in Fig. 14, which are clearly symmetric as expected. The (real) energy bands corresponding to this potential are shown in Fig. 15.

The classical Hamiltonian associated with this system is $H = \sqrt{p^2 + 1/(1 - e^{ix})^2}$. The trajectories of the classical particle, as shown in Fig. 16, are periodic and open. This appears to correspond to the fact that the quantum Hamiltonian has real energy bands but no discrete eigenvalues.

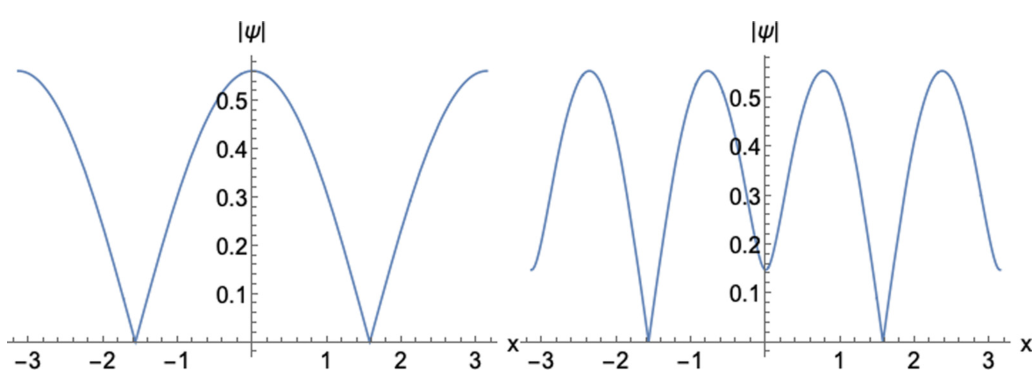

FIG. 11. Absolute values of the eigenfunctions corresponding to the band-edge energies of 1.08 (left panel) and 3.97 (right panel) of $H = p^2 + i\sin(x)$.

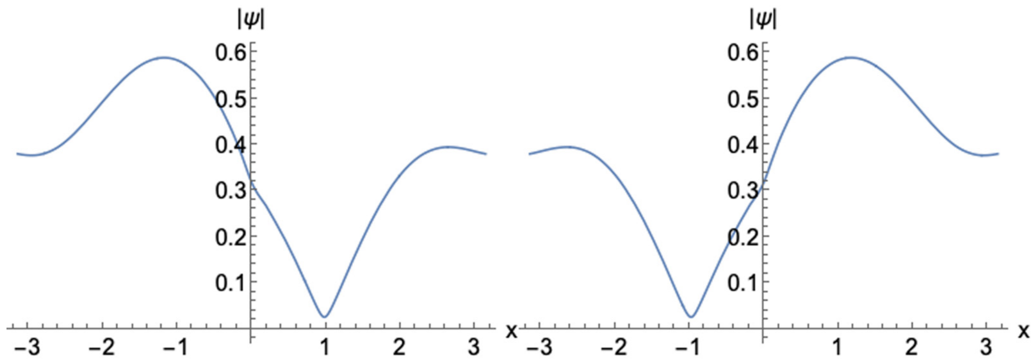

FIG. 13. Absolute values of the eigenfunctions corresponding to the band-edge energies $0.75 + 0.59i$ (left panel) and $0.75 - 0.59i$ (right panel) for $H = p^2 + e^{-ix}/(1 - e^{-ix})$.

III. NON-HERMITIAN $\mathcal{PT}$-SYMMETRIC FERMIONS IN 3 + 1 DIMENSIONS

A. Axial bilinear fermionic interaction

In $3 + 1$ dimensions, we again start with the free fermionic Lagrangian $\mathcal{L}_0 = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ of (7) and the Dirac equation of motion,

$$(i\partial\!\!\!/ - m)\psi(t, \mathbf{x}) = 0, \tag{31}$$

and recall that the actions of $\mathcal{P}$ and $\mathcal{T}$ are given in (6), where the gamma matrices are given in (5). Equation (31) is form invariant under the combined operations $\mathcal{P}$ and $\mathcal{T}$ because the functions $\psi(t, \mathbf{x})$ and $\mathcal{PT}\psi(t, \mathbf{x}) = \gamma^0(i\gamma^1\gamma^3)\psi^*(-t, -\mathbf{x})$ satisfy the same equation. For the free Dirac equation, this is true for $\mathcal{P}\psi = \gamma^0\psi(t, -\mathbf{x})$ and $\mathcal{T}\psi = i\gamma^1\gamma^3\psi^*(-t, \mathbf{x})$ individually. By setting $\mathbf{x} \to -\mathbf{x}$ in (31), it becomes

$$(i\gamma^0\partial_0 - i\gamma^i\partial_i - m)\psi(t, -\mathbf{x}) = 0,$$

where $i = 1, 2, 3$ denote the spatial components. Multiplying this result from the left with γ^0 and using the anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, with $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ results in

$$(i\partial\!\!\!/ - m)\gamma^0\psi(t, -\mathbf{x}) = 0.$$

On the other hand, taking the complex conjugate of (31) and replacing $t \to -t$ gives

$$[-i(-\gamma^0\partial_0 + \gamma^1\partial_1 - \gamma^2\partial_2 + \gamma^3\partial_3) - m]\psi^*(-t, \mathbf{x}) = 0,$$

because $(\gamma^2)^* = -\gamma^2$. Multiplying this equation from the left by $i\gamma^1\gamma^3$ and using the anticommutation relations for the gamma matrices then gives

$$(i\partial\!\!\!/ - m)i\gamma^1\gamma^3\psi^*(-t, \mathbf{x}) = 0.$$

The form invariance of the equation satisfied by $\mathcal{PT}\psi(t, \mathbf{x})$ then follows.

Next we include an axial non-Hermitian bilinear term into the Lagrangian density, $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$, where $\mathcal{L}_{\text{int}} = -g\bar{\psi}\gamma^5\psi$. The corresponding Euler-Lagrange equation

$$(i\partial\!\!\!/ - m - g\gamma^5)\psi(t, \mathbf{x}) = 0 \tag{32}$$

superficially resembles the $1 + 1$-dimensional case. However, here, while parity transforms this equation into

$$(i\partial\!\!\!/ - m + g\gamma^5)\gamma^0\psi(t, -\mathbf{x}) = 0, \tag{33}$$

time reversal transforms it into

$$(i\partial\!\!\!/ - m - g\gamma^5)i\gamma^1\gamma^3\psi^*(-t, \mathbf{x}) = 0. \tag{34}$$

Note the minus sign before the last term in (34): While parity flips the sign of the axial term, time reversal in $3 + 1$ dimensions does not. Parity is odd, but time reversal is *even* in $3 + 1$ dimensions. So the combination of $\mathcal{PT}$ does not lead to a form-invariant Dirac equation. The axial term by itself is anti-$\mathcal{PT}$ symmetric. This differs from the $1 + 1$-dimensional case [see (9) and (10)].

The dispersion relation that one obtains from (32) is formally the same as in the $1 + 1$-dimensional case; assuming

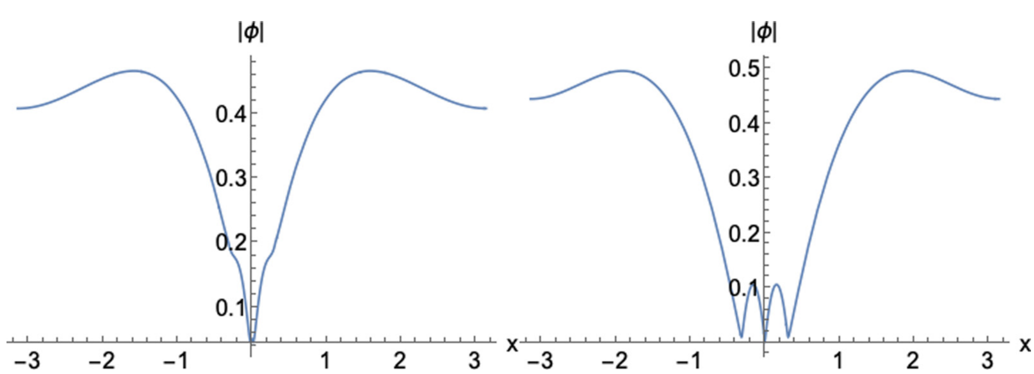

FIG. 14. Absolute values of the eigenfunctions corresponding to the band-edge energies of $E = 0.65$ (left panel) and $E = 0.98$ (right panel), obtained from (30). The symmetry of the eigenfunctions implies the reality of the energy band.

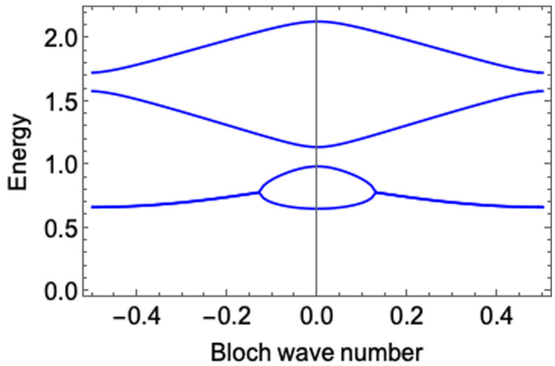

FIG. 15. The energy bands for the potential $(1-e^{ix})^{-2}$ in the first Brillouin zone.

plane-wave solutions of the form $\psi = e^{-ip^\mu x_\mu}$ and multiplying (32) by $(\not{p}+m+g\gamma^5)$, we arrive at the same spectral relation as in 1 + 1 dimensions,

$$p^2 = m^2 - g^2,$$

which is non-negative only when $m^2 \geqslant g^2$ and is complex in the chiral limit $m \to 0$.

As before, the form invariance of the Dirac equation under $\mathcal{PT}$ implies that $H(\mathcal{PT}\psi) = \mathcal{PT}(H\psi)$, where H is the Dirac Hamiltonian identified through $i\partial_t\psi = H\psi$. Thus, we can ascertain the properties of various interaction terms by testing them with this commutation relation. For (32) the associated Hamiltonian is

$$H = \alpha(-i\nabla) + \beta m + \beta g\gamma^5.$$

Let us check the symmetry of the axial interaction term $H_{\rm int} = g\gamma^0\gamma^5$ under $\mathcal{P}$ and $\mathcal{T}$. Using (6), we evaluate $\mathcal{PT}\psi(t,\mathbf{x}) = \gamma^0 i\gamma^1\gamma^3\psi^*(-t,-\mathbf{x})$ and apply $H_{\rm int}$:

$$\begin{aligned} H_{\rm int}(\mathcal{PT}\psi) &= g\gamma^0\gamma^5\gamma^0 i\gamma^1\gamma^3\psi^* = -\gamma^0 i\gamma^1\gamma^3 g\gamma^0\gamma^5\psi^* \\ &= -\mathcal{P}T H_{\rm int}\psi^* = -\mathcal{PT}H^*_{\rm int}\psi \\ &= -\mathcal{PT}(H_{\rm int}\psi). \end{aligned} \tag{35}$$

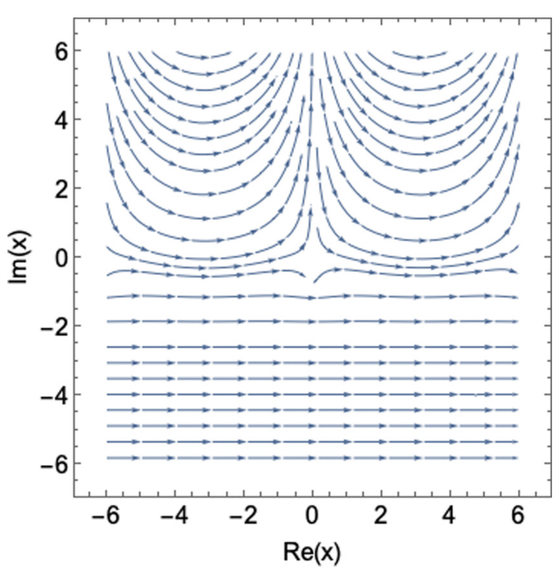

FIG. 16. Classical trajectories in the complex-x plane described by $H = \sqrt{p^2 + 1/(1-e^{ix})^2}$.

$H_{\rm int}$ *anticommutes* with $\mathcal{PT}$, confirming that this term is not $\mathcal{PT}$ symmetric. It thus explains the complex nature of the dispersion relation in the chiral limit. By contrast, if $H_{\rm int}$ is imaginary, that is $H_{\rm int} = ig\gamma^0\gamma^5$, we have a $\mathcal{PT}$-symmetric Hamiltonian, which is also Hermitian, and does have a real spectrum for all g, $p^2 = g^2$ in the chiral limit.

Once again, to clarify this point, we turn to an explicit matrix representation. Then $H_{\rm int}$ becomes

$$H_{\rm int} = g\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{36}$$

which is not Hermitian. By comparison, a general four-dimensional $\mathcal{PT}$-symmetric fermionic Hamiltonian that is invariant under $\mathcal{PT}$ and also self-adjoint under the $\mathcal{PT}$ inner product has a matrix form [13,14,31]

$$H = \begin{pmatrix} a_0 & 0 & -C_- & -B_- \\ 0 & a_0 & -B_+ & C_+ \\ C_+ & B_- & -a_0 & 0 \\ B_+ & -C_- & 0 & -a_0 \end{pmatrix}, \tag{37}$$

where $B_\pm = b_1 \pm ib_2$ and $C_\pm = b_3 \pm ib_0$. The parameters a_0, b_0, b_1, b_2, and b_3 are real. This matrix has twofold degenerate real eigenvalues

$$E_\pm = \pm\sqrt{a_0^2 - b_0^2 - b_1^2 - b_2^2 - b_3^2}, \tag{38}$$

for $a_0^2 \geqslant \sum_{i=0}^3 b_i^2$ [32]. Equation (36) is not a special case of (37), so it does not represent a $\mathcal{PT}$-symmetric fermionic Hamiltonian.

Evidently, the symmetry properties of the axial term $-g\gamma^5\psi$ in the Dirac equation in 1 + 1 dimensions differ from those in 3 + 1 dimensions. The Dirac equation is form invariant in 1 + 1 dimensions under $\mathcal{PT}$, but not in 3 + 1 dimensions. This corresponds to a relativistic $\mathcal{PT}$-symmetric quantum-mechanical Hamiltonian in 1 + 1 dimensions, but not in 3 + 1 dimensions. This difference is caused by the different effect of time reversal in 1 + 1 and 3 + 1 dimensions. The spectrum obtained in both cases is formally the same, so we conclude that the $\mathcal{PT}$ symmetry is always broken in 1 + 1 dimensions when $m \to 0$. However, in 3 + 1 dimensions the Hamiltonian is anti-$\mathcal{PT}$ symmetric in the chiral limit, which explains the complex nature of the spectrum when $m \to 0$.

Interestingly, if we include the conventional mass term $m\gamma^0$, (36) becomes

$$H_{\rm int} = \begin{pmatrix} m & 0 & g & 0 \\ 0 & m & 0 & g \\ -g & 0 & -m & 0 \\ 0 & -g & 0 & -m \end{pmatrix}, \tag{39}$$

which is neither Hermitian nor $\mathcal{PT}$ symmetric. However, $H_{\rm int}$ is *pseudo*-Hermitian in the sense of [33] because $H^\dagger_{\rm int} = \gamma^0 H_{\rm int}(\gamma^0)^{-1}$. Hence, this Hamiltonian can be used to describe pseudo-Hermitian fermions.

We can construct fermionic creation and annihilation operators which are quadratically nilpotent, and investigate their anticommutation relations. First, we note that the eigenvalues of (39) are

$$E_{\pm} = \pm\omega = \pm\sqrt{m^2 - g^2},$$

with corresponding eigenvectors

$$|E_-^{(1)}\rangle = \frac{1}{\sqrt{2w}}\begin{pmatrix} 0 \\ -\sqrt{m+w}(m-w)/g \\ 0 \\ \sqrt{m+w} \end{pmatrix}, \quad |E_-^{(2)}\rangle = \frac{1}{\sqrt{2w}}\begin{pmatrix} -\sqrt{m+w}(m-w)/g \\ 0 \\ \sqrt{m+w} \\ 0 \end{pmatrix},$$

$$|E_+^{(1)}\rangle = \frac{1}{\sqrt{2w}}\begin{pmatrix} 0 \\ -\sqrt{m-w}(m+w)/g \\ 0 \\ \sqrt{m-w} \end{pmatrix}, \quad |E_+^{(2)}\rangle = \frac{1}{\sqrt{2w}}\begin{pmatrix} -\sqrt{m-w}(m+w)/g \\ 0 \\ \sqrt{m-w} \\ 0 \end{pmatrix}.$$

The spectrum is twofold degenerate and is real if $g^2 \leqslant m^2$. This degeneracy is the analog of the phenomenon of Kramer's theorem in conventional Hermitian quantum mechanics, where the Hamiltonian is invariant under odd time reversal, as is the case with (39).

We introduce the annihilation operator for the Hamiltonian (39) as

$$\eta = \frac{1}{2w}\begin{pmatrix} g & 0 & m-w & 0 \\ 0 & g & 0 & m-w \\ -m-w & 0 & -g & 0 \\ 0 & -m-w & 0 & -g \end{pmatrix},$$

which is nilpotent ($\eta^2 = 0$) as required. We verify that

$$\eta|E_-^{(1)}\rangle = \eta|E_-^{(2)}\rangle = 0,$$

$$\eta|E_+^{(1)}\rangle = |E_-^{(1)}\rangle, \quad \eta|E_+^{(2)}\rangle = |E_-^{(2)}\rangle.$$

The creation operator reads

$$\eta' = \frac{1}{2w}\begin{pmatrix} g & 0 & m+w & 0 \\ 0 & g & 0 & m+w \\ -m+w & 0 & -g & 0 \\ 0 & -m+w & 0 & -g \end{pmatrix}.$$

One can now establish the anticommutation relations

$$\{N, \eta\} = -\eta, \quad \{N, \eta'\} = -\eta',$$

where N is the number operator, $N = \eta'\eta$, as well as the peculiar anticommutation relation $\eta\eta' + \eta'\eta = -\mathbb{1}$. The minus sign indicates that the number operator gives the negative of the state occupation number. For further illustrations of this in the context of $\mathcal{PT}$ symmetry see Refs. [15,24].

Finally, we comment that in terms of the number operator N, we can write the four-dimensional pseudo-Hermitian fermionic Hamiltonian in (39) in the form of a free (bosonic) harmonic oscillator as

$$H = \Delta\omega(-N) + \omega_-\mathbb{1},$$

where $\Delta\omega = \omega_+ - \omega_-$ and $\mathbb{1}$ is the four-dimensional identity matrix.

B. Other matrix-type two-body (four-point) $\mathcal{PT}$- and anti-$\mathcal{PT}$-symmetric interactions and the resulting $\mathcal{PT}$-symmetric Hamiltonians

Having determined that an axial non-Hermitian interaction Lagrangian density of the form $-g\bar{\psi}\gamma^5\psi$ in 3 + 1 dimensions does not give rise to a Dirac equation that is form invariant with respect to $\mathcal{PT}$, we seek other types of interactions that are $\mathcal{PT}$ symmetric but non-Hermitian. Usually, the standard method of analyzing two-body (four-point) interactions involves constructing the 16 independent bilinears from the 16 4×4 independent matrices and considering the Lagrangian density associated with each of these. The standard Hermitian combinations are (1) $\bar{\psi}\psi$, (2) $\bar{\psi}\gamma^\mu\psi$, (3) $\bar{\psi}\sigma^{\mu\nu}\psi$, (4) $\bar{\psi}\gamma^5\gamma^\mu\psi$, and (5) $i\bar{\psi}\gamma^5\psi$. This Lagrangian-density approach is suitable for a discussion of symmetries that lead to conserved currents through Noether's theorem, but the analysis of $\mathcal{PT}$ symmetry is most simply done by examining the form invariance of the appropriate Dirac-like equation that can be derived using the Euler-Lagrange equations. Since this in turn translates into a commutation relation of the Hamiltonian with $\mathcal{PT}$, in a form of *reverse engineering*, we only need to identify possible $\mathcal{PT}$-symmetric forms of the interaction Hamiltonians. Thus, we consider the five interaction Hamiltonians below and show that these combinations are all $\mathcal{PT}$ symmetric:

$$H_{\text{int},1} = g\gamma^0, \quad H_{\text{int},2} = B_\mu\gamma^0\gamma^\mu,$$
$$H_{\text{int},3} = iT_{\mu\nu}\gamma^0\sigma^{\mu\nu}, \quad H_{\text{int},4} = i\tilde{B}_\mu\gamma^0\gamma^5\gamma^\mu,$$
$$H_{\text{int},5} = ig_A\gamma^0\gamma^5,$$

where g, B_μ, $T_{\mu\nu}$, $\tilde{B}_\mu$, and g_A are taken to be real.

Using the procedure in (35) in which $H_{\text{int},i}$ is applied to $\mathcal{PT}\psi$, we evaluate the commutator of $H_{\text{int},i}$ and $\mathcal{PT}$ using

(6), and where necessary, make use of the relation $\gamma^\mu i\gamma^1\gamma^3 = i\gamma^1\gamma^3\gamma^*_\mu$. Then

$$H_{\text{int},1}(\mathcal{PT}\psi) = g\gamma^0\gamma^0 i\gamma^1\gamma^3\psi^* = \gamma^0 i\gamma^1\gamma^3 g\gamma^0\psi^* = PTH_{\text{int},1}\psi^* = \mathcal{PT}(H_{\text{int},1}\psi), \tag{40}$$

$$H_{\text{int},2}(\mathcal{PT}\psi) = B_\mu\gamma^0\gamma^\mu\gamma^0 i\gamma^1\gamma^3\psi^* = \gamma^0 i\gamma^1\gamma^3 B_\mu\gamma^0\gamma^{\mu*}\psi^* = PTH^*_{\text{int},2}\psi^* = \mathcal{PT}(H_{\text{int},2}\psi), \tag{41}$$

$$H_{\text{int},3}(\mathcal{PT}\psi) = iT_{\mu\nu}\gamma^0\sigma^{\mu\nu}\gamma^0 i\gamma^1\gamma^3\psi^* = -\gamma^0 i\gamma^1\gamma^3 iT_{\mu\nu}\gamma^0\sigma^{\mu\nu*}\psi^* = -PTi\gamma^0\sigma^{\mu\nu*}T_{\mu\nu}\psi^* = \mathcal{PT}(H_{\text{int},3}\psi), \tag{42}$$

$$H_{\text{int},4}(\mathcal{PT}\psi) = i\tilde{B}_\mu\gamma^0\gamma^5\gamma^\mu\gamma^0 i\gamma^1\gamma^3\psi^* = \gamma^0 i\gamma^1\gamma^3(-i)\tilde{B}_\mu\gamma^0\gamma^5\gamma^{\mu*}\psi^* = PTH^*_{\text{int},4}\psi^* = \mathcal{PT}(H_{\text{int},4}\psi), \tag{43}$$

$$H_{\text{int},5}(\mathcal{PT}\psi) = ig_A\gamma^0\gamma^5\gamma^0 i\gamma^1\gamma^3\psi^* = \gamma^0 i\gamma^1\gamma^3(-i)g_A\gamma^0\gamma^5\psi^* = PTH^*_{\text{int},5}\psi^* = \mathcal{PT}(H_{\text{int},5}\psi). \tag{44}$$

We conclude that

$$[\mathcal{PT}, H_{\text{int},i}] = 0 \quad (i = 1, \dots, 5).$$

Thus, the general form of a relativistic quantum-mechanical Dirac equation, which is form invariant under $\mathcal{PT}$ transformations, reads

$$(i\not{\partial} - g - B_\mu\gamma^\mu - iT_{\mu\nu}\sigma^{\mu\nu} - i\tilde{B}_\mu\gamma^5\gamma^\mu - ig_A\gamma^5)\psi(t, \mathbf{x}) = 0.$$

A brief analysis shows that $H_{\text{int},3}$ and $H_{\text{int},4}$ are anti-Hermitian, while $H_{\text{int},1}$, $H_{\text{int},2}$, and $H_{\text{int},5}$ are Hermitian. So we have identified two types of terms that give rise to non-Hermitian but $\mathcal{PT}$-symmetric Hamiltonians. We consider each of these in turn.

1. $H_{\text{int},3} = iT_{\mu\nu}\gamma^0\sigma^{\mu\nu}$

To understand the structure of $H_{\text{int},3}$ we write it in matrix form:

$$H_{\text{int},3} = \begin{pmatrix} iq_4 & -q_5 + iq_6 & -q_3 & -q_1 + iq_2 \\ q_5 + iq_6 & -iq_4 & -q_1 - iq_2 & q_3 \\ q_3 & q_1 - iq_2 & -iq_4 & q_5 - iq_6 \\ q_1 + iq_2 & -q_3 & -q_5 - iq_6 & iq_4 \end{pmatrix}, \tag{45}$$

where the coefficients q_i, $i = 1, \dots, 6$, are abbreviations for combinations of the $T_{\mu\nu}$,

$$q_1 = T_{01} - T_{10}, \quad q_2 = T_{02} - T_{20}, \quad q_3 = T_{03} - T_{30},$$
$$q_4 = T_{12} - T_{21}, \quad q_5 = T_{13} - T_{31}, \quad q_6 = T_{23} - T_{32}.$$

The eigenvalues of (45) are

$$\pm\big\{-Q^2 \pm 2\big[(q_1^2 + q_2^2)q_4^2 + (q_1^2 + q_3^2)q_5^2 + (q_2^2 + q_3^2)q_6^2 + 2q_2q_3q_4q_5 + 2q_1q_2q_5q_6 - 2q_1q_3q_4q_6\big]^{1/2}\big\}^{1/2},$$

where $Q^2 = \sum_{i=1}^6 q_i^2$. Thus, the eigenvalues are complex and the $\mathcal{PT}$ symmetry is broken. Including a finite mass term $m\gamma^0$ in general does not change this result. The eigenvalues of $H_{\text{int},3} + m\gamma^0$ are modified to read

$$\pm\big\{m^2 - Q^2 \pm 2\big[(q_1^2 + q_2^2 - m^2)q_4^2 + (q_1^2 + q_3^2 - m^2)q_5^2 + (q_2^2 + q_3^2 - m^2)q_6^2 + 2q_2q_3q_4q_5 + 2q_1q_2q_5q_6 - 2q_1q_3q_4q_6\big]^{1/2}\big\}^{1/2}.$$

As we have already argued, only if the spectrum is twofold degenerate, can the eigenvalues be real [32].

If we compare (45) with (37), we see that both have a quaternionic structure. However, in addition to being $\mathcal{PT}$ symmetric, (37) fulfills the additional condition that this Hamiltonian is self-adjoint with regard to the $\mathcal{PT}$ inner product according to [13]. This means that, in addition, $H_{\text{int},3}$ should fulfill the condition $H^{\mathcal{PT}}_{\text{int},3} = PH^\dagger_{\text{int},3}P = H_{\text{int},3}$. If we construct $H^{\mathcal{PT}}_{\text{int},3}$, we find that

$$q_4 = q_5 = q_6 = 0,$$

for this condition to hold. The eigenvalues are twofold degenerate and if a mass term is included, they are

$$E_\pm = \pm\sqrt{m^2 - q_1^2 - q_2^2 - q_3^2},$$

which is real provided that $m^2 \geqslant q_1^2 + q_2^2 + q_3^2$. Thus, $\mathcal{PT}$ symmetry is broken in the chiral limit. The regions of unbroken $\mathcal{PT}$ symmetry for the Hamiltonian $H_{\text{int},3} + m\gamma^0$ for some specific parameters are shown in Fig. 17.

2. $H_{\text{int},4} = i\tilde{B}_\mu\gamma^0\gamma^5\gamma^\mu$

We now consider the equation of motion resulting from the non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian $H_{\text{int},4}$ (as well as its corresponding Lagrangian $\mathcal{L}_{\text{int},4}$),

$$(i\not{\partial} - i\gamma^5\not{\tilde{B}})\psi = 0.$$

The spectrum associated with this equation can be obtained by calculating the poles of the associated Green function in momentum space, which satisfies

$$(\not{p} - i\gamma^5\not{\tilde{B}})S(p) = 1.$$

Rationalizing this expression for $S(p)$, we identify the dispersion relation as

$$(p^2 - \tilde{B}^2)^2 + 4(p\cdot\tilde{B})^2 = 0.$$

This has no real solutions for all p_0. Thus, again we find that the $\mathcal{PT}$ symmetry of the Hamiltonian is broken. We also notice that an anti-$\mathcal{PT}$-symmetric but Hermitian Hamiltonian would give a real spectrum with dispersion relation $(p^2 - \tilde{B}^2)^2 - 4(p\tilde{B})^2 = 0$.

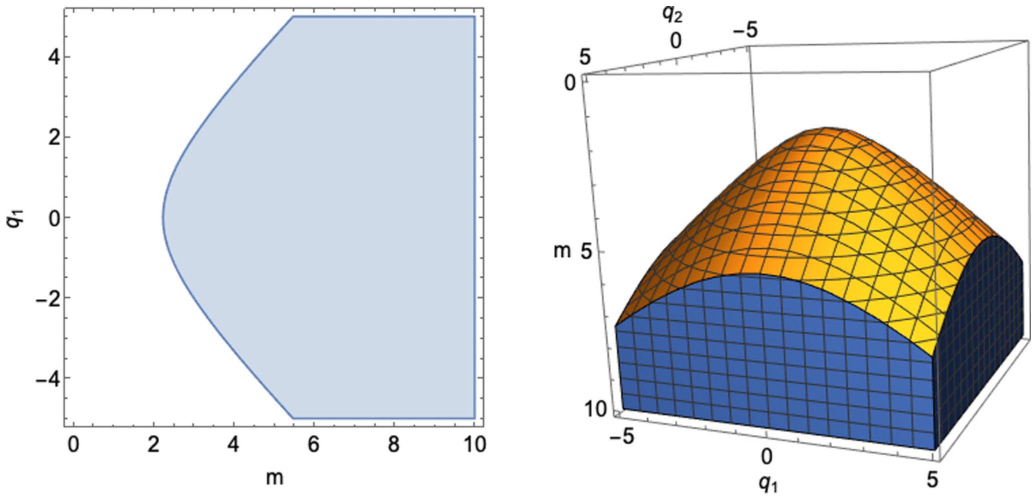

FIG. 17. Parametric regions of unbroken $\mathcal{PT}$ symmetry (shaded regions) for the Hamiltonian $H_{\text{int},3} + m\gamma^0$, where $q_4 = q_5 = q_6 = 0$. Left panel: In the (m, q_1) plane, $q_2 = 1$ and $q_3 = 2$. Right panel: $q_3 = 2$.

Note that the matrix form of the Hamiltonian $H_{\text{int},4}$, with components $\tilde{B}_\mu = (\tilde{B}_0, \tilde{B}_1, \tilde{B}_2, \tilde{B}_3)$ is

$$H_{\text{int},4} = \begin{pmatrix} -i\tilde{B}_3 & -i\tilde{B}_1 - \tilde{B}_2 & -i\tilde{B}_0 & 0 \\ \tilde{B}_2 - i\tilde{B}_1 & i\tilde{B}_3 & 0 & -i\tilde{B}_0 \\ -i\tilde{B}_0 & 0 & -i\tilde{B}_3 & -i\tilde{B}_1 - \tilde{B}_2 \\ 0 & -i\tilde{B}_0 & \tilde{B}_2 - i\tilde{B}_1 & i\tilde{B}_3 \end{pmatrix},$$

which has complex eigenvalues for all $\tilde{B}_\mu$ real,

$$E_{1,2} = i\tilde{B}_0 \pm i\sqrt{\tilde{B}_1^2 + \tilde{B}_2^2 + \tilde{B}_3^2},$$
$$E_{3,4} = -i\tilde{B}_0 \pm i\sqrt{\tilde{B}_1^2 + \tilde{B}_2^2 + \tilde{B}_3^2}.$$

If, as in Sec. III B 1, we demand that the Hamiltonian $H_{\text{int},4}$ satisfies the self-adjointness condition according to [13,14,24], that is, $H^{\mathcal{PT}}_{\text{int},4} = PH^\dagger_{\text{int},4}P = H_{\text{int},4}$, we calculate that $\tilde{B}_0 \neq 0$ and $\tilde{B}_1 = \tilde{B}_2 = \tilde{B}_3 = 0$. The resulting twofold degenerate energies are

$$E_\pm = \pm\sqrt{m^2 - \tilde{B}_0^2}, \tag{46}$$

where we have included a mass term. This implies a real spectrum for $m^2 \geqslant \tilde{B}_0^2$. Once again, in the chiral limit the $\mathcal{PT}$ symmetry is broken.

IV. MAIN CONCLUSIONS AND OUTLOOK

Our focus in this paper has been on investigating non-Hermitian $\mathcal{PT}$-symmetric extensions to fermionic systems in $1+1$ and $3+1$ dimensions. The main findings are the following:

(a) Usually we explore the symmetries of a field theory by examining the Lagrangian density. However, the properties associated with $\mathcal{PT}$ symmetry are more easily found by forming the Euler-Lagrange equations and demanding form invariance of the relativistic equation of motion with respect to $\mathcal{PT}$. This is equivalent to constructing the quantum-mechanical relativistic Hamiltonian and investigating its commutation relation with $\mathcal{PT}$.

(b) For a pure axial interaction the symmetry properties in $1+1$ dimensions differ from those in $3+1$ dimensions even though the formal structure of the energy relation is unchanged. This can be traced back to the different transformation properties of time reversal in $1+1$ and $3+1$ dimensions and is ultimately due to the fact that $\mathcal{T}^2 = -\mathbb{1}$ in $3+1$ dimensions.

(c) In $1+1$ dimensions including a complex $\mathcal{PT}$-symmetric position-dependent potential in both scalar- and vector-coupling schemes and combinations thereof can result in real and discrete eigenvalues, when searching for plane wave solutions. For appropriately chosen combinations of scalar and vector couplings, a Schrödinger-like equation can be found and the spectrum can be determined numerically. The analogous classical systems give information about the nature of the spectrum. They display closed contours when the eigenvalues are real and discrete and they are periodic and open if there is a real band structure. If the eigenvalues are complex, the paths are open and nonperiodic.

(d) In $3+1$ dimensions only two possible Lorentz-invariant two-body combinations are $\mathcal{PT}$ symmetric and not Hermitian. These, however, give rise to a complex spectrum in the chiral limit. Including a mass term can result in a real spectrum. In addition, further constraints are placed on the parameters if the condition of self-adjointness with respect to the $\mathcal{PT}$ inner product is placed on the Hamiltonian. This does not change the conclusion.

It remains an open question as to whether including non-Hermitian $\mathcal{PT}$-symmetric terms can play a role in physical fermionic systems, for example, affecting chiral symmetry restoration in the Nambu-Jona-Lasinio model. Here, in the simplest case, the internal SU(2) symmetry initially coexists with the global $\mathcal{PT}$ symmetry of the Lagrangian. In the standard formulation, when the internal SU(2) symmetry is broken, the global $\mathcal{PT}$ symmetry is broken concurrently; the two appear to go hand in hand. However, it is not clear what the effect of including further $\mathcal{PT}$-symmetric interaction terms may do. Additional interesting fermionic models that can be considered are the Thirring models or fermionic models of the weak interactions.

We conclude by observing that the occurrence of features associated with complex spectra, such as the nonconservation of a probability current and associated nonvanishing cross sections occurs in many other formalisms, for example (and in particular) in quantum many-body theories in which the Hilbert space is partially eliminated by projection onto a limited subspace. Such effects can occur and may not be physical if the procedures involved are not *a priori* symmetry preserving. Methods of deriving $\mathcal{PT}$-symmetry-respecting projection techniques could alleviate this problem.

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[1] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).
[2] P. E. Dorey, C. Dunning, and R. Tateo, J. Phys. A: Math. Gen. **34**, 5679 (2001).
[3] C. E. Rüter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, Nat. Phys. **6**, 192 (2010).
[4] Z. Lin, H. Ramezani, T. Eichelkraut, T. Kottos, H. Cai, and D. N. Christodoulides, Phys. Rev. Lett. **106**, 213901 (2011).
[5] Y. D. Chong, L. Ge, and A. D. Stone, Phys. Rev. Lett. **106**, 093902 (2011).
[6] J. Doppler, A. A. Mailybaev, J. Böhm, U. Kuhl, A. Girschik, F. Libisch, T. J. Milburn, P. Rabl, N. Moiseyev, and S. Rotter, Nature (London) **537**, 76 (2016).
[7] H. Xu, D. Mason, L. Jiang, and J. G. E. Harris, Nature (London) **537**, 80 (2016).
[8] W. Chen, S. K. Özdemir, G. Zhao, J. Wiersig, and L. Yang, Nature (London) **548**, 192 (2017).
[9] J. Rubinstein, P. Sternberg, and Q. Ma, Phys. Rev. Lett. **99**, 167003 (2007).
[10] S. Assawaworrarit, X. Yu, and S. Fan, Nature (London) **546**, 387 (2017).
[11] C. M. Bender, N. Hassanpour, S. P. Klevansky, and S. Sarkar, Phys. Rev. D **98**, 125003 (2018).
[12] A. Mostafazadeh, J. Phys. A **37**, 10193 (2004).
[13] K. Jones-Smith and H. Mathur, Phys. Rev. A **82**, 042101 (2010).
[14] K. Jones-Smith, Ph.D. thesis, Case Western Reserve University, 2010.
[15] C. M. Bender and S. P. Klevansky, Phys. Rev. A **84**, 024102 (2011).
[16] C. M. Bender, H. F. Jones, and R. J. Rivers, Phys. Lett. B **625**, 333 (2005).
[17] J. Alexandre, C. M. Bender, and P. Millington, J. High Energy Phys. 11 (2015) 111.
[18] J. Alexandre, P. Millington, and D. Seynaeve, Phys. Rev. D **96**, 065027 (2017).
[19] K. Jones-Smith and H. Mathur, Phys. Rev. D **89**, 125014 (2014).
[20] T. Ohlsson, Europhys. Lett. **113**, 61001 (2016).
[21] E. Abdalla, M. G. B. Abdalla, and K. D. Rothe, *Non-Perturbative Methods in Two-Dimensional Quantum Field Theory* (World Scientific, New York, 1991).
[22] B. Thaller, *The Dirac Equation* (Springer, Berlin, 1992).
[23] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1965).
[24] A. Beygi, S. P. Klevansky, and C. M. Bender, Phys. Rev. A **97**, 032128 (2018).
[25] In this paper most of the Hamiltonians that are considered are bilinear in the fermionic field. One result of this paper is that it is not possible to construct a $\mathcal{PT}$-symmetric bilinear fermionic Hamiltonian that has a real spectrum in the chiral limit. However, we firmly believe that a Hamiltonian having higher-power interactions will achieve this goal. A simple bosonic model that illustrates this point is the Hamiltonian $H = p^2 + (m^2 - \mu^2)x^2 - gx^4$. In the quadratic case $g = 0$ we see that the spectrum becomes complex in the $m \to 0$ limit. However, if $g \neq 0$ it is shown in C. M. Bender, D. C. Brody, J.-H. Chen, H. F. Jones, K. A. Milton, and M. C. Ogilvie, Phys. Rev. D **74**, 025016 (2006) that H is equivalent to the Hermitian Hamiltonian $\tilde{H} = p^2 - 2\hbar x\sqrt{g} + 4g[x^2 - (m^2 - \mu^2)/(4g)]^2$, which has a spectrum that is real and bounded below for *all* values of m and μ.
[26] In C. M. Bender, D. C. Brody, and D. W. Hook, J. Phys. A: Math. Theor. **41**, 352003 (2008) it is shown that closed classical orbits correspond to real quantum-mechanical energies and open classical orbits correspond to complex quantum-mechanical energies.
[27] We reserve the notation ψ_1, ψ_2 for two different spinor solutions of the same equation.
[28] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978).
[29] M. Eastham, *The Spectral Theory of Periodic Differential Equations* (Scottish Academic, Edinburgh, 1973).
[30] C. M. Bender, G. V. Dunne, and P. N. Meisinger, Phys. Lett. A **252**, 272 (1999).
[31] A. Beygi and S. P. Klevansky, Phys. Rev. A **98**, 022105 (2018).
[32] When H and $\mathcal{PT}$ commute, in general, in order to have a real spectrum, we have two possibilities. One option is ψ to be $\mathcal{PT}$ symmetric, i.e., it has the symmetry of the Hamiltonian. The other option is that ψ and $\mathcal{PT}\psi$ be degenerate eigenvectors of H. For $\mathcal{T}^2 = -\mathbb{1}$ the first option is ruled out since, if we require $\mathcal{PT}\psi = \psi$, then ψ must vanish; So the degeneracy of the Hamiltonian is a necessary condition for the spectrum to be real.
[33] G. W. Mackey, *Commutative Banach Algebras* (Instituto de Matematica pura e Aplicada do Conselho Nacional de Pesquisa, Rio De Janeiro, 1959).