Figure of merit for single-photon generation based on cavity quantum electrodynamics

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We investigate a tradeoff relation between the internal generation efficiency and the escape efficiency for single-photon generation based on cavity quantum electrodynamics, where cavity internal loss is treated explicitly. Consequently, we analytically derive an upper bound on the overall efficiency. The bound is expressed only with an *internal cooperativity*, introduced here as the cooperativity parameter with respect to the cavity internal loss rate. This result means that the internal cooperativity is a figure of merit for single-photon generation based on cavity QED. The bound is derived by optimizing the cavity external loss rate, which can be experimentally controlled by designing or tuning the transmissivity of the output coupler. The model here is general enough to treat various cavity-QED effects, such as the Purcell effect, on-resonant or off-resonant cavity-enhanced Raman scattering, and vacuum-stimulated Raman adiabatic passage. For typical optical systems, we additionally take into account a "reexcitation" process, where the atom is reexcited after its decay to the initial ground state.

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I. INTRODUCTION

Single-photon sources are a key component for photonic quantum information processing and quantum networking [1]. Single-photon sources based on cavity quantum electrodynamics [2–10] are particularly promising, because they enable deterministic emission into a single mode, which is desirable for low-loss and scalable implementations. Many single-photon generation schemes have been proposed and studied using various cavity-QED effects, such as the Purcell effect [2–4], on-resonant [4–6] or off-resonant [7,8] cavity-enhanced Raman scattering, and vacuum-stimulated Raman adiabatic passage (vSTIRAP) [2–4,6,8–10].

The overall efficiency of single-photon generation based on cavity QED is composed of two factors: the internal generation efficiency $\eta_{\rm in}$ (probability that a photon is generated inside the cavity) and the escape efficiency $\eta_{\rm esc}$ (probability that a generated photon is extracted to the desired external mode). The upper bounds on $\eta_{\rm in}$ have been derived for some of the above schemes [3–6], where the upper bound is expressed with the cooperativity parameter C [3]. C is inversely proportional to the total cavity loss rate, $\kappa = \kappa_{\rm ex} + \kappa_{\rm in}$ ($\kappa_{\rm ex}$ and $\kappa_{\rm in}$ are the external and internal loss rates, respectively [11]). Note that $\kappa_{\rm ex}$ can be experimentally controlled by designing or tuning the transmissivity of the output coupler [12]. Thus, $\eta_{\rm in}$ is maximized by setting $\kappa_{\rm ex}$ to a small value so that $\kappa \approx \kappa_{\rm in}$. However, a low $\kappa_{\rm ex}$ results in a low escape efficiency $\eta_{\rm esc} = \kappa_{\rm ex}/\kappa$, which limits the channelling of the generated photons into the desired mode. There is therefore a *tradeoff* relation between $\eta_{\rm in}$ and $\eta_{\rm esc}$ with respect to $\kappa_{\rm ex}$, and $\kappa_{\rm ex}$ should be optimized to maximize the overall efficiency. This tradeoff relation has not been examined in previous studies, where the internal loss rate $\kappa_{\rm in}$ has not been treated explicitly.

By treating the cavity internal loss explicitly for the above tradeoff relation and using a general cavity-QED model shown in Fig. 1, here we analytically derive the following lower bound on the failure probability, P_F, of single-photon generation based on cavity QED:

$$P_F \geqslant \frac{2}{1+\sqrt{1+2C_{\rm in}}} \approx \sqrt{\frac{2}{C_{\rm in}}}, \tag{1}$$

where we have introduced an *internal cooperativity*, $C_{\rm in} = g^2/(2\kappa_{\rm in}\gamma)$, as the cooperativity parameter with respect to $\kappa_{\rm in}$ instead of κ for the standard definition, $C = g^2/(2\kappa\gamma)$ [3]. The approximation in Eq. (1) holds when $C_{\rm in} \gg 1$. This result suggests that $C_{\rm in}$, instead of C, is a figure of merit for single-photon generation based on cavity QED. Here it is notable that similar lower bounds on failure probabilities, inversely proportional to $\sqrt{C_{\rm in}}$, have been derived for quantum gate operations based on cavity QED [13–15]. This fact suggests that $C_{\rm in}$ may be a figure of merit for quantum applications of cavity-QED systems in a more general sense.

The lower bound on P_F in Eq. (1) is obtained when $\kappa_{\rm ex}$ is set to its optimal value,

$$\kappa_{\rm ex}^{\rm opt} \equiv \kappa_{\rm in}\sqrt{1+2C_{\rm in}}, \tag{2}$$

and is simply expressed as $2\kappa_{\rm in}/\kappa^{\rm opt}$, where $\kappa^{\rm opt} \equiv \kappa_{\rm in} + \kappa_{\rm ex}^{\rm opt}$. Remarkably, this optimal value of $\kappa_{\rm ex}$ is exactly the same as that for a quantum gate operation in Ref. [14].

Note that the experimental values of $(g, \gamma, \kappa_{\rm in})$ determine which regime the system should be in: the Purcell regime ($\kappa \gg g^2/\kappa \gg \gamma$), the strong-coupling regime [$g \gg (\kappa, \gamma)$], or the intermediate regime ($\kappa \approx g^2/\kappa \gg \gamma$).

The remainder of this paper is organized as follows. In Sec. II, we show that the present model is applicable to various cavity-QED single-photon generation schemes. In Sec. III, we provide the basic equations for the present analysis. Using these equations, we analytically derive an upper bound on

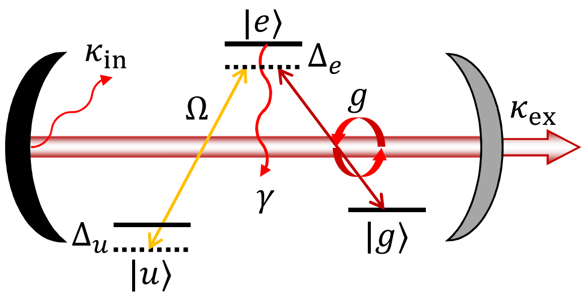

FIG. 1. Cavity-QED system for single-photon generation. The atom is initially prepared in $|u\rangle$. κ_{in} and κ_{ex}: cavity internal and external loss rates, respectively. g: atom-cavity coupling rate via the $|g\rangle - |e\rangle$ transition. Ω: Rabi frequency of the external field for the $|u\rangle - |e\rangle$ transition. Δ_e and Δ_u: one-photon and two-photon detunings, respectively. γ: atomic decay rate due to spontaneous emission.

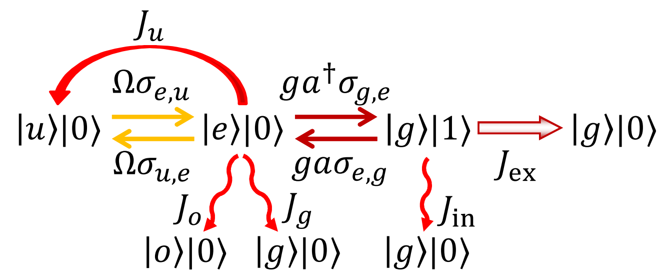

FIG. 2. Transitions in Eqs. (3) and (4).

the success probability, $P_S = 1 - P_F$, of single-photon generation in Sec. IV. Using the bound, we optimize κ_{ex} and derive Eq. (1). In Sec. V, we briefly discuss the condition for typical optical cavity-QED systems, where the effect of a "reexcitation" process is also discussed. Finally, the conclusion and outlook are presented in Sec. VI.

II. MODEL

As shown in Fig. 1, we consider a cavity-QED system with a Λ-type three-level atom in a one-sided cavity. The atom is initially prepared in $|u\rangle$. The $|u\rangle - |e\rangle$ transition is driven with an external classical field, while the $|g\rangle - |e\rangle$ transition is coupled to the cavity. This system is general enough to describe most of the cavity-QED single-photon generation schemes.

For instance, by first exciting the atom to $|e\rangle$ with a resonant π pulse (with time-dependent Ω), or fast adiabatic passage (with time-dependent Δ_u), the atom is able to decay to $|g\rangle$ with a decay rate enhanced by the Purcell effect [16], generating a single photon. Here, the Purcell regime is assumed [2–4].

Another example is where the atom is weakly excited with small Ω and a cavity photon is generated by cavity-enhanced Raman scattering. Here, $\kappa \gg g$ is assumed in the on-resonant case ($\Delta_e = \Delta_u = 0$) [4–6], while $\Delta_e \gg g$ is assumed in the off-resonant case ($\Delta_u = 0$) [7,8].

A third example is based on vSTIRAP [2–4,6,8–10], where Ω is gradually increased, and where the strong-coupling regime [$g \gg (\kappa, \gamma)$] and small detunings ($|\Delta_e|, |\Delta_u| \ll g$) are assumed.

III. BASIC EQUATIONS

The starting point of our study is the following master equation describing the cavity-QED system (here we use the natural units, $c = \hbar = 1$):

$$\dot{\rho} = \mathcal{L}\rho, \mathcal{L} = \mathcal{L}_{\mathcal{H}} + \mathcal{J}_u + \mathcal{J}_g + \mathcal{J}_o + \mathcal{J}_{\text{in}}, \tag{3}$$

$$\mathcal{L}_{\mathcal{H}}\rho = -i(\mathcal{H}\rho - \rho\mathcal{H}^\dagger), \mathcal{H} = H - i(\gamma\sigma_{e,e} + \kappa_{\text{in}}a^\dagger a),$$

$$H = \Delta_e\sigma_{e,e} + \Delta_u\sigma_{u,u} + \int kb^\dagger(k)b(k)dk$$
$$+ i\Omega(\sigma_{e,u} - \sigma_{u,e}) + ig(a\sigma_{e,g} - a^\dagger\sigma_{g,e})$$
$$+ i\sqrt{\frac{\kappa_{\text{ex}}}{\pi}}\int_{-\infty}^{\infty}[b^\dagger(k)a - a^\dagger b(k)]dk,$$
$$\mathcal{J}_u\rho = 2\gamma r_u\sigma_{u,e}\rho\sigma_{e,u}, \mathcal{J}_g\rho = 2\gamma r_g\sigma_{g,e}\rho\sigma_{e,g},$$
$$\mathcal{J}_o\rho = 2\gamma r_o\sigma_{o,e}\rho\sigma_{e,o}, \mathcal{J}_{\text{in}}\rho = 2\kappa_{\text{in}}a\rho a^\dagger, \tag{4}$$

where ρ is the density operator describing the state of the system; the dot denotes differentiation with respect to time t; H is the Hamiltonian for the cavity-QED system including the terms for the output mode; a and $a^\dagger$ are respectively the annihilation and creation operators for cavity photons; $b(k)$ and $b^\dagger(k)$ are respectively the annihilation and creation operators for output-mode photons with wave number, or frequency, of k; $|o\rangle$ is, if it exists, a ground state other than $|u\rangle$ and $|g\rangle$; r_u, r_g, and $r_o = 1 - r_u - r_g$ are respectively the branching ratios for spontaneous emission from $|e\rangle$ to $|u\rangle$, $|g\rangle$, and $|o\rangle$; and $\sigma_{j,l} = |j\rangle\langle l|$ ($j, l = u, g, e, o$) are atomic operators. In the present work, we assume no pure dephasing [17].

The transitions corresponding to the terms in Eqs. (3) and (4) are depicted in Fig. 2, where the second and third ket vectors denote cavity photon number states and output-mode states, respectively, and $|k\rangle = b^\dagger(k)|0\rangle$. Once the state of the system becomes $|g\rangle|0\rangle_c|0\rangle$ or $|o\rangle|0\rangle_c|0\rangle$ by a quantum jump, $\mathcal{J}_g$, $\mathcal{J}_o$, or $\mathcal{J}_{\text{in}}$, the time evolution stops, and the single-photon generation ends up in failure. On the other hand, the quantum jump $\mathcal{J}_u$ initializes the state to $|u\rangle$, and the single-photon generation restarts. However, a photon generated by the "reexcitation" process will have a different envelope from that of a photon generated without $\mathcal{J}_u$, and therefore such photons may be not useful for some applications. Thus, we consider that the single-photon generation ends up in failure if the quantum jump $\mathcal{J}_u$ occurs. That is, the success probability P_S for the single-photon generation is given by the probability that all the quantum jumps do not occur. (We will discuss the reexcitation process later.)

Under the condition of no quantum jumps, the time evolution is given by the non-Hermitian Schrödinger equation: $i|\dot{\psi}\rangle = \mathcal{H}|\psi\rangle$ [18,19]. Expressing $|\psi\rangle$ as

$$|\psi\rangle = \alpha_u|u\rangle|0\rangle_c|0\rangle + \alpha_e|e\rangle|0\rangle_c|0\rangle + \alpha_g|g\rangle|1\rangle_c|0\rangle$$
$$+ \int_{-\infty}^{\infty}\alpha_k|g\rangle|0\rangle_c|k\rangle dk, \tag{5}$$

the Schrödinger equation becomes

$$\dot{\alpha_u} = -i\Delta_u\alpha_u - \Omega\alpha_e, \tag{6}$$

$$\dot{\alpha_e} = -(\gamma + i\Delta_e)\alpha_e + \Omega\alpha_u + g\alpha_g, \tag{7}$$

$$\dot{\alpha_g} = -\kappa\alpha_g - g\alpha_e, \tag{8}$$

$$\alpha_k(t) = \sqrt{\frac{\kappa_{\rm ex}}{\pi}}\int_0^t \alpha_g(t')e^{-ik(t-t')}dt'. \tag{9}$$

The norm of $|\psi\rangle$ decreases from unity. This decrease corresponds to the quantum-jump probability [18,19].

Note that the output-mode amplitude α_k is determined by the atom-cavity amplitude α_g satisfying Eqs. (6)–(8). Introducing the position operator, state vector, and amplitude for the output mode as

$$\tilde{b}(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} b(k)e^{ikz}dk, \tag{10}$$

$$|z\rangle = \tilde{b}^\dagger(z)|0\rangle, \tag{11}$$

$$\int_{-\infty}^{\infty}\tilde{\alpha}_z|z\rangle dz = \int_{-\infty}^{\infty}\alpha_k|k\rangle dk, \tag{12}$$

the position amplitude is determined by α_g as follows:

$$\tilde{\alpha}_z = \begin{cases}\sqrt{2\kappa_{\rm ex}}\alpha_g(t-z) & \cdots \quad 0 < z < t,\\ 0 & \cdots \quad \text{otherwise.}\end{cases} \tag{13}$$

The pulse shape of the generated photon is proportional to the atom-cavity amplitude α_g. Thus, we can control the pulse shape by controlling the atom-cavity state using, e.g., the vSTIRAP technique [6].

IV. UPPER BOUND FOR THE SINGLE-PHOTON GENERATION EFFICIENCY

First, from Eqs. (6)–(8), we obtain

$$\frac{dN}{dt} = -2\gamma|\alpha_e|^2 - 2\kappa|\alpha_g|^2 \Rightarrow 2\gamma I_e + 2\kappa I_g \approx 1, \tag{14}$$

where we have introduced $N = |\alpha_u|^2 + |\alpha_g|^2 + |\alpha_e|^2$, $I_g = \int_0^T |\alpha_g(t)|^2dt$, and $I_e = \int_0^T |\alpha_e(t)|^2dt$, and also assumed $N(0) = 1$ and $N(T) \approx 0$ for a sufficiently long time T. Thus the success probability P_S is given by

$$P_S = \int_{-\infty}^{\infty}|\tilde{\alpha}_z(T)|^2dz = 2\kappa_{\rm ex}I_g = \frac{\kappa_{\rm ex}}{\kappa}(1 - 2\gamma I_e). \tag{15}$$

The last expression has a simple physical meaning: the first factor is the escape efficiency $\eta_{\rm esc}$ and the second term in the second factor comes from the excited-state decay. I_g and I_e are evaluated as follows. From Eq. (8),

$$I_e = \int_0^T \frac{|\dot{\alpha_g}(t) + \kappa\alpha_g(t)|^2}{g^2}dt \approx \frac{I_g'}{g^2} + \frac{\kappa^2}{g^2}I_g, \tag{16}$$

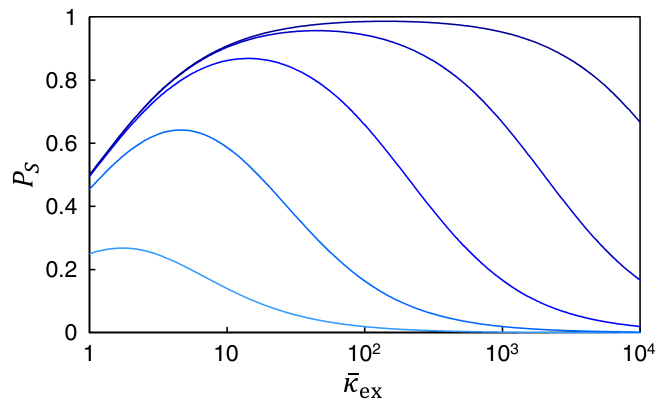

FIG. 3. Success probability P_S in Eq. (19). This holds when I_g' is negligible. The dimensionless parameter $\bar{\kappa}_{\rm ex}$ is defined by $\bar{\kappa}_{\rm ex} = \kappa_{\rm ex}/\kappa_{\rm in}$. Five curves correspond to the cases where $C_{\rm in} = 1$, 10, 10^2, 10^3, and 10^4 from the bottom.

where we have used $|\alpha_g(0)|^2 = 0$ and $|\alpha_g(T)|^2 \approx 0$, and also set $I_g' = \int_0^T |\dot{\alpha}_g(t)|^2dt$. Equations (14) and (16) lead to

$$I_g = \frac{C}{\kappa(1+2C)}\left(1 - \frac{I_g'}{\kappa C}\right), \tag{17}$$

$$I_e = \frac{1}{2\gamma}\left[1 - \frac{2C}{1+2C}\left(1 - \frac{I_g'}{\kappa C}\right)\right]. \tag{18}$$

Thus, P_S is upper bounded as follows:

$$P_S = \frac{\kappa_{\rm ex}}{\kappa}\frac{2C}{1+2C}\left(1 - \frac{I_g'}{\kappa C}\right) \leqslant \left(1 - \frac{\kappa_{\rm in}}{\kappa}\right)\left(1 - \frac{1}{1+2C}\right),$$

where we have used $I_g' \geqslant 0$ by its definition. The equality approximately holds when $I_g' \ll \kappa C$. This can actually be achieved in some cases [5,20]. Notably, it is known that photon storage with cavity-QED systems without internal loss also has a similar upper bound, $2C/(2C+1)$, on the success probability [21,22]. This, together with the results for quantum gate operations [13,14], implies the universality of the upper bound.

The first and second factors of the upper bound are $\eta_{\rm esc}$ and $\eta_{\rm in}$, respectively. There is a tradeoff relation between $\eta_{\rm esc}$ and $\eta_{\rm in}$ with respect to $\kappa_{\rm ex}$. To see this, it is notable that P_S in the case where I_g' is negligible is expressed only with two dimensionless parameters, $\bar{\kappa}_{\rm ex} = \kappa_{\rm ex}/\kappa_{\rm in}$ and $C_{\rm in}$, as follows:

$$P_S = \left(1 + \frac{1}{\bar{\kappa}_{\rm ex}}\right)^{-1}\left(1 + \frac{1+\bar{\kappa}_{\rm ex}}{2C_{\rm in}}\right)^{-1}. \tag{19}$$

Examples for various values of $C_{\rm in}$ are shown in Fig. 3. The above tradeoff relation results in the maxima of P_S. By maximizing P_S in Eq. (19) with respect to $\bar{\kappa}_{\rm ex}$ analytically, we obtain Eqs. (1) and (2). From Fig. 3, it is also found that the maximization with respect to $\kappa_{\rm ex}$ is robust against the deviation of $\kappa_{\rm ex}$ from $\kappa_{\rm ex}^{\rm opt}$.

The approximate lower bound in Eq. (1) can be derived more directly using the arithmetic-geometric mean inequality as follows:

$$P_F \geqslant \frac{\kappa_{\rm in}}{\kappa} + \frac{1}{2C+1} - \frac{\kappa_{\rm in}}{\kappa}\frac{1}{2C+1} \approx \frac{\kappa_{\rm in}}{\kappa} + \frac{\kappa\gamma}{g^2} \geqslant \sqrt{\frac{2}{C_{\rm in}}},$$

where $C \gg 1$ have been assumed. Note that κ is canceled out by multiplying the two terms. A similar technique has been applied to the derivation of an upper bound on the success probability of a quantum gate operation based on cavity QED [13].

V. TYPICAL OPTICAL CAVITY-QED SYSTEMS

For optical cavity-QED systems where a single atom or ion is coupled to a single cavity mode [5–10], the cavity-QED parameters are expressed as follows [3]:

$$g = \sqrt{\frac{\mu_{g,e}^2 \omega_{g,e}}{2\epsilon_0 \hbar A_{\rm eff} L}}, \tag{20}$$

$$\kappa_{\rm in} = \frac{c}{2L}\alpha_{\rm loss}, \tag{21}$$

$$r_g \gamma = \frac{\mu_{g,e}^2 \omega_{g,e}^3}{6\pi\epsilon_0 \hbar c^3}, \tag{22}$$

where ϵ_0 is the permittivity of vacuum, $\mu_{g,e}$ and $\omega_{g,e}$ are the dipole moment and frequency of the $|g\rangle - |e\rangle$ transition, respectively, L is the cavity length, $A_{\rm eff}$ is the effective cross-section area of the cavity mode at the atomic position, and $\alpha_{\rm loss}$ is the one-round-trip cavity internal loss. Substituting them into the definition of $C_{\rm in}$, we obtain

$$2C_{\rm in} = r_g \frac{1}{\alpha_{\rm loss}} \frac{1}{\tilde{A}_{\rm eff}}, \tag{23}$$

where $\tilde{A}_{\rm eff} = A_{\rm eff}/\sigma$ is the effective cavity-mode area normalized by the atomic absorption cross section $\sigma = 3\lambda^2/(2\pi)$ ($\lambda = 2\pi c/\omega_{g,e}$). Note that L and $\mu_{g,e}$ are canceled out. Thus, the single-photon generation efficiency is limited by the three dimensionless quantities: the one-round-trip internal loss $\alpha_{\rm loss}$, the normalized cavity-mode area $\tilde{A}_{\rm eff}$, and a branching ratio r_g.

So far, we have not counted photons generated by the "reexcitation" process, where the atom is reexcited after its decay to $|u\rangle$ via spontaneous emission. If we count such photons, as in the ion-trap experiment in Ref. [7], the success probability will become higher. In the following, however, we show that even in this case, the success probability is upper bounded in a similar manner.

Taking the quantum jump $\mathcal{J}_u$ into account, we obtain the formal solution of the master equation (3) [18]:

$$\rho_c(t) = \mathcal{V}_c(t,0)\rho_0 \tag{24}$$

$$= \mathcal{V}_{\mathcal{H}}(t,0)\rho_0 + \int_0^t \mathcal{V}_c(t,t')\mathcal{J}_u\mathcal{V}_{\mathcal{H}}(t',0)\rho_0 dt', \tag{25}$$

where ρ_c denotes the density operator conditioned on no quantum jumps of $\mathcal{J}_g$, $\mathcal{J}_o$, and $\mathcal{J}_{\rm in}$, ρ_0 is the initial density operator, and $\mathcal{V}_{\mathcal{H}}$ and $\mathcal{V}_c$ are the quantum dynamical semigroups defined as follows:

$$\frac{d}{dt}\mathcal{V}_{\mathcal{H}}(t,t') = \mathcal{L}_{\mathcal{H}}(t)\mathcal{V}_{\mathcal{H}}(t,t'), \quad \frac{d}{dt}\mathcal{V}_c(t,t') = \mathcal{L}_c(t)\mathcal{V}_c(t,t'),$$

where $\mathcal{L}_c = \mathcal{L}_{\mathcal{H}} + \mathcal{J}_u$ is the Liouville operator for the conditioned time evolution. The decrease of the trace of ρ_c corresponds to the failure probability due to $\mathcal{J}_g$, $\mathcal{J}_o$, and $\mathcal{J}_{\rm in}$ [18,19].

Note that $\rho_{\mathcal{H}}(t) = \mathcal{V}_{\mathcal{H}}(t,0)\rho_0$ can be expressed as $\rho_{\mathcal{H}} = |\psi\rangle\langle\psi|$ with $|\psi\rangle$ given by Eqs. (5)–(9). Thus,

$$\rho_c(t) = |\psi(t)\rangle\langle\psi(t)| + 2\gamma r_u \int_0^t |\alpha_e(t')|^2 \mathcal{V}_c(t,t')\rho_0 dt'. \tag{26}$$

The success probability P_S is formulated as

$$P_S = \int_{-\infty}^{\infty} \langle g|_c\langle 0|\langle z|\rho_c(T)|g\rangle|0\rangle_c|z\rangle dz. \tag{27}$$

Using Eqs. (24)–(27) and Eq. (15), P_S is expressed as

$$P_S = \int_{-\infty}^{\infty} dz \langle g|_c\langle 0|\langle z|\mathcal{V}_c(t,0)\rho_0|g\rangle|0\rangle_c|z\rangle \tag{28}$$

$$= 2\kappa_{\rm ex}\int_0^T dt|\alpha_g(t)|^2 + 2\gamma r_u \int_0^T dt|\alpha_e(t)|^2 \times \int_{-\infty}^{\infty} dz\langle g|_c\langle 0|\langle z|\mathcal{V}_c(T,t)\rho_0|g\rangle|0\rangle_c|z\rangle. \tag{29}$$

The second term, which is denoted by $P_{\rm rep}$, is the contribution of the reexcitation.

Here we assume the following inequality:

$$\int_{-\infty}^{\infty} dz\langle g|_c\langle 0|\langle z|\mathcal{V}_c(T,t)\rho_0|g\rangle|0\rangle_c|z\rangle \leqslant \int_{-\infty}^{\infty} dz\langle g|_c\langle 0|\langle z|\mathcal{V}_c(T,0)\rho_0|g\rangle|0\rangle_c|z\rangle = P_S. \tag{30}$$

This assumption is natural because $\mathcal{V}_c(T,t)$ should be designed to maximize P_S at $t=0$. Then, Eqs. (29) and (30) result in

$$P_S \leqslant \frac{2\kappa_{\rm ex} I_g}{1 - 2\gamma r_u I_e}. \tag{31}$$

Substituting Eqs. (17) and (18) into Eq. (31), we obtain the upper bound on P_S in the case of the reexcitation:

$$P_S \leqslant \frac{\kappa_{\rm ex}}{\kappa}\frac{2C}{1+2C}\frac{1-\frac{I_g'}{\kappa C}}{1 - r_u + r_u\frac{2C}{1+2C}\left(1-\frac{I_g'}{\kappa C}\right)} \tag{32}$$

$$\leqslant \left(1-\frac{\kappa_{\rm in}}{\kappa}\right)\left(1-\frac{1}{1+2C}\right)\sum_{n=0}^{\infty}\left(\frac{r_u}{1+2C}\right)^n, \tag{33}$$

where we have used $0 \leqslant 1 - I_g'/(\kappa C) \leqslant 1$ [23].

In a similar manner to deriving Eq. (1), we obtain

$$P_F \geqslant \frac{2}{1+\sqrt{1+2C_{\rm in}/(1-r_u)}}, \tag{34}$$

where $\kappa_{\rm ex}$ is set to $\kappa_{\rm in}\sqrt{1+2C_{\rm in}/(1-r_u)}$. Thus in the case of the reexcitation, $C_{\rm in}$ in Eqs. (1) and (2) is replaced with $C_{\rm in}/(1-r_u)$. From Eq. (34), it seems that the lower bound on P_F becomes zero by $r_u \to 1$. However, this is not the case because r_u and $C_{\rm in}$ are not independent. Instead of Eq. (23), we should examine the following quantity:

$$\frac{2C_{\rm in}}{1-r_u} = \frac{1-r_u-r_o}{1-r_u}\frac{1}{\alpha_{\rm loss}}\frac{1}{\tilde{A}_{\rm eff}} \leqslant \frac{1}{\alpha_{\rm loss}}\frac{1}{\tilde{A}_{\rm eff}}, \tag{35}$$

where we have used $r_g = 1 - r_u - r_o$. The equality in Eq. (35) holds when $r_o = 0$. Thus, it turns out that even if we count photons generated by the reexcitation process, the single-photon generation efficiency is limited by the one-round-trip internal loss $\alpha_{\rm loss}$ and the normalized cavity-mode area $\tilde{A}_{\rm eff}$.

Using Eqs. (30) and (18), the contribution of the reexcitation $P_{\rm rep}$ is upper bounded as

$$P_{\rm rep} \leqslant 2\gamma r_u I_e P_S \leqslant \frac{1}{1+2C} + \frac{2C}{1+2C}\frac{I_g'}{\kappa C}. \qquad (36)$$

Thus, the contribution of the reexcitation is negligible when $C \gg 1$ and $I_g' \ll \kappa C$.

VI. CONCLUSION AND OUTLOOK

By analytically solving the master equation for a general cavity-QED model, we have derived an upper bound on the efficiency of single-photon generation based on cavity QED in a unified way. We have treated cavity internal loss explicitly, which results in a tradeoff relation between the internal generation efficiency and the escape efficiency with respect to the cavity external loss rate $\kappa_{\rm ex}$. By optimizing $\kappa_{\rm ex}$, we have derived a lower bound on the failure probability. The lower bound is inversely proportional to the square root of the internal cooperativity $C_{\rm in}$. This means that $C_{\rm in}$ is a suitable figure of merit for cavity-QED systems used for single-photon generation. The optimal value of $\kappa_{\rm ex}$ has also been given explicitly.

For typical optical cavity-QED systems, the lower bound is given by the one-round-trip internal loss, the cavity-mode area normalized by the atomic absorption cross section, and a branching ratio. The reexcitation process, where the atom is reexcited after its decay to the initial ground state via spontaneous emission, has also been examined. As a result, it has turned out that the single-photon generation efficiency is limited in a similar manner, even including photons generated by the reexcitation. Its bound is expressed with the one-round-trip internal loss and the normalized cavity-mode area.

The lower bound is achieved in the limit that the variation of the system is sufficiently slow. When the short generation time is desirable, optimization of the control parameters will be necessary. This problem is left for future work.

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