Efficient representation of Gaussian states for multimode non-Gaussian quantum state engineering via subtraction of arbitrary number of photons

Christos N. Gagatsos[1] and Saikat Guha[1,2]
[1]College of Optical Sciences, University of Arizona, 1630 East University Boulevard, Tucson, Arizona 85719, USA
[2]Department of Electrical and Computer Engineering, University of Arizona, 1230 East Speedway Boulevard, Tucson, Arizona 85719, USA

(Received 12 February 2019; published 13 May 2019)

We consider a complete description of a multi-mode bosonic quantum state in the coherent-state basis (which in this paper is denoted as the "K" function), which—up to a phase—is the square root of the well-known Husimi Q representation. We express the K function of any N-mode Gaussian state as a function of its covariance matrix and displacement vector, and also that of a general continuous-variable cluster state in terms of the modal squeezing and graph topology of the cluster. This formalism lets us characterize the non-Gaussian state left over when one measures a subset of modes of a Gaussian state using photon number resolving detection, the fidelity of the obtained non-Gaussian state with any target state, and the associated heralding probability, all analytically. We show that this probability can be expressed as a Hafnian, reinterpreting the output state of a circuit claimed to demonstrate quantum supremacy termed Gaussian boson sampling. As an example application of our formalism, we propose a method to prepare a two-mode coherent-cat-basis Bell state with fidelity close to unity and success probability that is fundamentally higher than that of a well-known scheme that splits an approximate single-mode cat state—obtained by photon number subtraction on a squeezed vacuum mode—on a balanced beam splitter. This formalism could enable exploration of efficient generation of cat-basis entangled states, which are known to be useful for quantum error correction against photon loss.

DOI: 10.1103/PhysRevA.99.053816

I. INTRODUCTION

Gaussian states of bosonic modes—quantum states of light that can be prepared using quadrature squeezed light and passive linear optics—form an important set of quantum states the elegant mathematical description of which [1] and feasibility of experimental production [2] make Gaussian quantum information processing a major success [3]. However, it is well known that Gaussian states and Gaussian measurements (homodyne and heterodyne detection) do *not* constitute a universal set, i.e., resources that would allow universal quantum computation [4]. Moreover, various important protocols for quantum enhanced information processing cannot be performed when restricted to Gaussian states, Gaussian unitaries, and Gaussian measurements alone. Such no-go theorems have appeared for universal quantum computing [5], entanglement distillation [6–8], optimal cloning of coherent states [9], optimal discrimination of coherent states [10–12], receivers for optical communications and quantum repeaters [13], quantum error correction [14], and quantum-enhanced sensing [15].

Therefore, having access to non-Gaussian states becomes imperative in pretty much any application of quantum enhanced photonic information processing. Introducing non-Gaussianity into an optical system can be challenging. For example, large $\chi^{(3)}$ nonlinearities are very difficult to be implemented at optical frequencies, and obtaining a strong-enough non-Gaussian interaction through a $\chi^{(2)}$ medium with a depleted pump [16] is hard. An alternative way to inject non-Gaussianity is to utilize detection-induced, often probabilistic, methods such as photon number subtraction. Theoretical, numerical, and experimental studies [17–26] have shown that photon subtraction on a single-mode Gaussian (squeezed vacuum) state yields approximations of coherent cat states and has validated the non-Gaussian character of photon subtracted multimode states. Further, photon subtraction has been shown to enhance entanglement [27–29] and the fidelity of continuous-variable teleportation as was originally shown [27] and also later studied [30].

Evaluating the state obtained after subtracting m photons from a state $|\psi\rangle$, i.e., $\hat{a}^m|\psi\rangle$ using the photon number (Fock) basis $\{|n\rangle\}$ and even methods using the Husimi Q representation of the state $|\psi\rangle$, leads to onerous calculations. This is because one has to calculate expressions such as $\hat{a}^m|n\rangle$ and $\hat{a}^{\dagger m}|\alpha\rangle$ within difficult-to-handle summations and integrals, where $\hat{a}$ is the modal photon annihilation operator. Similar difficulties apply when using the Wigner representation. Furthermore, taking into account the deviations of the photon subtracted state from $\hat{a}^m|\psi\rangle$ pursuant to actual experimental methods of implementing such operation using a beam splitter and photon number resolving (PNR) detectors creates additional complexities. Despite photon number subtraction being a very promising tool for non-Gaussian state engineering, this analytical difficulty has stood in the way of theoretical progress in the field.

In this paper, we remedy the above situation by expressing the state on the coherent basis [31,32]. Specifically, we utilize the positive P_+ representation of a quantum state, which is essentially expressing a general density operator in the coherent-state overcomplete basis [33]. This representation always exists unlike the Glauber-Sudarshan $P_{\rm GS}$ function, which is not always defined, especially for squeezed states which are of interest in creation of cat states. The P_+ representation has been utilized for the numerical and analytical study of Fokker-Planck equations of dynamical systems [16,33–38], Ising systems [39], and single-mode quantum information analyses [40]. Formulas for P_+ have been given for Gaussian states [41] but only for the cases where the Glauber-Sudarshan $P_{\rm GS}$ function is well defined.

We first define the P_+ function of an N-mode pure Gaussian state, which we call the K function. It is a unique representation of any pure state, and can be interpreted as a square root of the Husimi Q function up to a phase. The latter mathematical observation allows us to derive clean, closed-form, and easy to use formulas for the P_+ representation (called K in this paper), for any Gaussian state. We begin with developing a closed-form expression of the K function of a general N-mode Gaussian state. This lets us analytically characterize non-Gaussian states created by photon number detection and/or photon number subtraction on a subset of modes of any N-mode Gaussian state in an analytic integral form. We show that this reproduces—in a rather simple set of steps—the theory behind *Gaussian boson sampling*, where it was argued that sampling from the photon number distribution of a random N-mode entangled Gaussian state is a classically hard computational task as was proven in [42] and also subsequently studied [43,44]. As an example application of our formalism, we consider the problem of engineering coherent cat-basis entangled cluster states. We propose a method to prepare a two-mode cat-basis Bell state by subtracting photons from both modes of a Gaussian two-mode entangled squeezed state. We show that the fidelity versus success probability tradeoff of our method is higher than that of the conventional method—that of splitting an approximate single-mode cat state, obtained by photon number subtraction on a squeezed vacuum mode, on a balanced beam splitter. The above analysis would be extremely cumbersome (and not scalable to a larger entangled state) if done in the traditional way in the photon number basis. We expect generalization of the above to enable exploration of efficient generation of cat-basis cluster states, which have recently emerged as a very powerful resource for quantum error correction against photon losses, with applications both to photonic quantum repeaters as well as to superconducting quantum computing [45–47].

II. THE K FUNCTION OF A PURE GAUSSIAN STATE

We work in units of $\hbar = 1$, where the N-mode vacuum state's covariance matrix (CM) is $V_0 = I/2$, with I being the N-mode identity operator. Coherent states of N modes $|\vec{\alpha}\rangle$ are not mutually orthogonal. Yet they form an overcomplete basis. In other words, they resolve the identity operator, viz.,

$$I = \frac{1}{(2\pi)^N}\int d^{2N}\vec{x}_\alpha |\vec{\alpha}\rangle\langle\vec{\alpha}|, \tag{1}$$

where $\vec{x}_\alpha^T = (\vec{q}_\alpha^T, \vec{p}_\alpha^T)$, and the volume element $d^{2N}\vec{x}_\alpha = dq_{\alpha_1}\dots dq_{\alpha_N} dp_{\alpha_1}\dots dp_{\alpha_N}$. We take $\alpha_i = (q_{\alpha_i} + ip_{\alpha_i})/\sqrt{2}$. Using Eq. (1), we can express any N-mode pure state $|\Psi_0\rangle$ as

$$\begin{aligned}|\Psi_0\rangle &= \frac{1}{(2\pi)^N}\int d^{2N}\vec{x}_\alpha \langle\vec{\alpha}|\Psi_0\rangle|\vec{\alpha}\rangle\\ &= \int d^{2N}\vec{x}_\alpha\, K(\vec{x}_\alpha)|\vec{\alpha}\rangle,\end{aligned} \tag{2}$$

where we call $K(\vec{x}_\alpha) = 1/(2\pi)^N\langle\vec{\alpha}|\Psi_0\rangle$ the K function of the state $|\Psi_0\rangle$. When compared to the Q function $Q(\vec{x}_\alpha) = 1/(2\pi)^N|\langle\vec{\alpha}|\Psi_0\rangle|^2$, the K function resembles something that could be called *the square root* of the Q function. However, one has to be careful as $\langle\vec{\alpha}|\Psi_0\rangle$ is a complex number and its square root will contain a phase that if omitted will produce wrong results since it depends on $\vec{x}_\alpha$.

Let us assume that $|\Psi_0\rangle$ is a zero-mean Gaussian state, such that $Q(\vec{x}_\alpha) = 1/(2\pi)^N\langle\vec{\alpha}|\Psi_0\rangle\langle\Psi_0|\vec{\alpha}\rangle$ is a Gaussian function. In order to calculate the K function, one must break up the Gaussian Q function's exponent into two conjugate parts, yielding a Gaussian K function. This step becomes easier if instead of working with Cartesian coordinates $(\vec{q}_\alpha, \vec{p}_\alpha)$ we move to complex coordinates $(\vec{\alpha}, \vec{\alpha}^*)$ with a $\pi/4$ phase-space rotation. After we finish the calculation we rotate back to Cartesian coordinates.

Let us now consider a general N-mode Gaussian pure state $|\Psi\rangle = D(\vec{\beta})|\Psi_0\rangle$, where $D(\vec{\beta})$ is the displacement operator. With $|\Psi_0\rangle$ expressed in its K-function form (2), it is straightforward to evaluate the K function of $|\Psi\rangle$ since $D(\vec{\beta})|\vec{\alpha}\rangle = \exp(\vec{\beta}\vec{\alpha}^* - \vec{\beta}^*\vec{\alpha})|\vec{\alpha} + \vec{\beta}\rangle$.

Using the above method, we show that any N-mode pure Gaussian state with CM V and displacement vector $\vec{x}_\beta^T = (\vec{q}_\beta^T, \vec{p}_\beta^T)$ [48] can be written as follows (see Secs. 2 and 3 of the Appendix for the complete derivation):

$$|\Psi\rangle = \int d^{2N}\vec{x}_\alpha\, K(\vec{x}_\alpha)G(\vec{x}_\alpha, \vec{x}_\beta)|\vec{\alpha}\rangle, \tag{3}$$

where

$$K(\vec{x}_\alpha) = \frac{\exp\left[-\frac{1}{2}\vec{x}_\alpha^T\mathcal{B}\vec{x}_\alpha\right]}{(2\pi)^N(\det\Gamma)^{1/4}}, \tag{4}$$

$$G(\vec{x}_\alpha, \vec{x}_\beta) = \exp\left[\tfrac{1}{4}\left(\vec{x}_\alpha^T\ \vec{x}_\beta^T\right)\mathcal{D}(\vec{x}_\alpha\ \vec{x}_\beta)\right], \tag{5}$$

with $\Gamma = V + I/2$, and

$$\mathcal{B} = \frac{1}{2}\begin{pmatrix} A + \frac{i}{2}(C + C^T) & C - \frac{i}{2}(A - B)\\ C^T - \frac{i}{2}(A - B) & B - \frac{i}{2}(C + C^T)\end{pmatrix}, \tag{6}$$

$$\mathcal{D} = \begin{pmatrix} 0 & 2\mathcal{B} + \mathcal{X}\\ 2\mathcal{B} - \mathcal{X} & -2\mathcal{B}\end{pmatrix}, \tag{7}$$

$$\mathcal{X} = \begin{pmatrix} I & iI\\ -iI & I\end{pmatrix}, \tag{8}$$

where $A = A^T, B = B^T,$ and C are defined as the blocks of the CM Γ defined as follows [49]:

$$\Gamma^{-1} = \begin{pmatrix} A & C\\ C^T & B\end{pmatrix}. \tag{9}$$

III. PHOTON SUBTRACTION FROM A GENERAL MULTIMODE GAUSSIAN STATE

Subtraction of m photons from a single-mode quantum state $|\psi\rangle$ can be implemented by transmitting $|\psi\rangle$ through a beam splitter of transmissivity τ (chosen to be close to 1) while detecting the low-transmissivity output of the beam splitter with a PNR detector. If the detector registers m photons, the transmitted state projects to $\mathcal{P}_{-m}[|\psi\rangle]$, which is an approximation of the m-photon subtracted state $\hat{a}^m|\psi\rangle$. Since $\hat{a}^m$ is not a unitary, photon subtraction only succeeds probabilistically.

Let us consider subtracting a vector $\vec{m} = (m_1, \dots, m_N)$ photons from an N-mode pure Gaussian state $|\Psi\rangle$ using an array of beam splitters of transmissivities τ_i, and PNR detectors. The postsubtraction state will be denoted $\mathcal{P}_{-\vec{m}}[|\Psi\rangle]$, implying m_i photons were subtracted from the ith mode, $i = 1, 2, \dots, N$. Using the K function of $|\Psi\rangle$ (3), we see that $\mathcal{P}_{-\vec{m}}$ acts only on the coherent states (see Sec. 1 of the Appendix), i.e., $\mathcal{P}_{-\vec{m}}[|\vec{\alpha}\rangle]$, which assumes a simple form, $\mathcal{P}_{-\vec{m}}[|\vec{\alpha}\rangle] = \prod_{i=1}^N c_i|\alpha_i\sqrt{\tau_i}\rangle$, $c_i = [(-\sqrt{1-\tau_i})^{m_i}]/[\sqrt{m_i!}]\alpha^{m_i}e^{-(1-\tau_i)|\alpha_i|^2/2}$.

The photon subtracted state $|\Psi_{-\vec{m}}\rangle$ is given as

$$|\Psi_{-\vec{m}}\rangle = \frac{1}{\sqrt{P}}\prod_{i=1}^{N}\frac{(-\sqrt{1-\tau_i})^{m_i}}{\sqrt{m_i!}}\int d^{2N}\vec{x}_\alpha\, K(\vec{x}_\alpha) \times G(\vec{x}_\alpha, \vec{x}_\beta)e^{-\frac{(1-\tau_i)}{4}|\vec{x}_\alpha|^2} \times \left(\frac{q_{\alpha_i}+ip_{\alpha_i}}{\sqrt{2}}\right)^{m_i}|\sqrt{\tau_i}\vec{\alpha}\rangle, \tag{10}$$

where $P = \langle\Psi_{-\vec{m}}|\Psi_{-\vec{m}}\rangle$ is the probability of success of the N-mode vector photon subtraction. P is a $4N$-dimensional integral with the elementary volume $d^{2N}\vec{x}_\alpha d^{2N}\vec{x}_\gamma$ ($\vec{x}_\gamma$ are the coordinates of $\langle\Psi_{-\vec{m}}|$), with a Gaussian kernel, and polynomial terms $(q_{\alpha_i}+ip_{\alpha_i})^{m_i}(q_{\gamma_i}-ip_{\gamma_i})^{m_i}$. This kind of integrals can be analytically calculated (see Sec. 4 of the Appendix).

If one wishes to use photon subtraction to produce a desired non-Gaussian multimode entangled state $|C\rangle$ (for example a cat-basis Bell state that we consider later), one can evaluate analytically the fidelity $F = |\langle C|\Psi_{-\vec{m}}\rangle|^2$ between the desired state $|C\rangle$ and the actual state obtained, $|\Psi_{-\vec{m}}\rangle$, if $|C\rangle$ is expressed in its K-function form. For cat states, for example, which are superpositions of coherent states $|\vec{\gamma}\rangle$, the fidelity calculation will require us to calculate the amplitude $|\langle\vec{\gamma}|\Psi_{-\vec{m}}\rangle|$, which again is a $4N$-dimensional integral, with Gaussian kernel and polynomial terms $(q_{\alpha_i}+ip_{\alpha_i})^{m_i}$, which can be analytically calculated (see Sec. 5 of the Appendix).

For the rest of this paper we will restrict our attention to zero-mean states, to keep the exposition simple. Including nonzero means is a trivial extension. Further, we will assume that all the beam splitters employed for photon subtraction on an N-mode Gaussian state have the same transmissivity, τ.

IV. GAUSSIAN BOSON SAMPLING AND NON-GAUSSIAN STATE ENGINEERING

Consider a pure N-mode Gaussian state $|\Psi\rangle$, the first $M < N$ modes of which are detected using PNR detectors, obtaining the outcome $\vec{n} = (n_1, \dots, n_M)$. It is simple to show that the resulting state $|\Phi\rangle$ on the unmeasured modes is given by (see Sec. 6 of the Appendix)

$$|\Phi\rangle = \frac{1}{\sqrt{P_M}}\prod_{i=1}^{M}\frac{1}{\sqrt{2^{n_i}n_i!}}\int d^{2N}\vec{x}_\alpha K(\vec{x}_\alpha)e^{-\frac{1}{4}x_{\alpha_i}^2} \times (q_{\alpha_i}+ip_{\alpha_i})^{n_i}|\alpha_{M+1}, \dots, \alpha_N\rangle, \tag{11}$$

where we used $|\alpha\rangle = \exp(-|\alpha|^2/2)\sum_n \alpha^n/(\sqrt{n!})|n\rangle$. The probability P_M of detecting the photon number pattern $\vec{n}$ and hence heralding the state $|\Phi\rangle$ can be calculated by setting $\langle\Phi|\Phi\rangle = 1$.

Gaussian boson sampling is the special case of $M = N$, where all N modes are detected [43,44]. The success probability of detecting a photon-number pattern $\vec{n}$, $P_{\vec{n}} = |\langle\vec{n}|\Psi\rangle|^2 = |\langle n_1 \dots n_N|\Psi\rangle|^2$ can be evaluated using our formalism and shown to be (see Sec. 7 of the Appendix)

$$P_{\vec{n}} = \frac{1}{\det\mathcal{H}\sqrt{\det\Gamma}\prod_{i=1}^N n_i!2^{n_i}}\left|\mathcal{I}_{\vec{n}}\right|^2, \tag{12}$$

where

$$\mathcal{I}_{\vec{n}} = \int d^{2N}\vec{x}_\alpha R(\vec{x}_\alpha)\prod_{i=1}^{N}\left(q_{\alpha_i}+ip_{\alpha_i}\right)^{n_i}, \tag{13}$$

$$R(\vec{x}_\alpha) = \frac{\sqrt{\det\mathcal{H}}}{(2\pi)^N}e^{-\frac{1}{2}\vec{x}_\alpha^T\mathcal{H}\vec{x}_\alpha}, \tag{14}$$

and $\mathcal{H} = \mathcal{B} + I/2$. Since $\mathcal{H} = \mathcal{H}^T$ and its real part is positive definite (see Sec. 8 of the Appendix), Eq. (14) is a proper Gaussian distribution. Therefore, Eq. (13) is the mean value $\langle f_1^{n_1}\dots f_N^{n_N}\rangle$, where $f_i = q_{\alpha_i}+ip_{\alpha_i}$, under the distribution of Eq. (14). Using Wick's theorem [50,51] we can write it as

$$\mathcal{I}_{\vec{n}} = \begin{cases} 0 & \Sigma = \text{odd}, \\ \mathrm{Hf}(F) & \Sigma = \text{even}, \end{cases} \tag{15}$$

where $\Sigma = \sum_{i=1}^N n_i$ and $\mathrm{Hf}(F)$ is the Hafnian of the matrix F with elements $F_{ij} = \langle f_i f_j\rangle$, $1 \leqslant i, j \leqslant \Sigma$.

V. PHOTON SUBTRACTION FROM MULTIMODE SQUEEZED CLUSTER STATES

Continuous variable quantum computing is a field that explores the use of multimode entangled squeezed states for all-photonic quantum computing. Such Gaussian cluster states of thousands of modes have been prepared experimentally [2,52,53]. It is known, however, that Gaussian cluster states by themselves are not a resource sufficient for universal quantum processing. Photon number detection being the most practical "de-Gaussification" tool, and given that it is known that approximate cat states can be prepared using photon number subtraction from a single-mode squeezed vacuum, we will explore the creation of cat-basis cluster (graph) states by photon number subtraction on Gaussian cluster states.

Let us consider the Gaussian graph state $|G\rangle$ which is the result of the unitary evolution of an N-mode vacuum state under the unitary $\hat{U}_r = \exp(-ir\hat{H})$ the generating Hamiltonian of which is

$$\hat{H} = -\frac{i}{2}\sum_{i,j}^{N}G_{ij}(\hat{a}_i^\dagger\hat{a}_j^\dagger - \hat{a}_i\hat{a}_j), \tag{16}$$

where $\hat{a}_i$ and $\hat{a}_i^\dagger$ are the annihilation and creation operators of the ith mode, respectively. The state $|G\rangle$ is a squeezed entangled state among its N modes. The information about which modes are entangled is described by the graph (a symmetric matrix) G. We assume that the squeezing parameter $r > 0$ is the same for all modes [54]. In the limit $r \to \infty$, $|G\rangle$ is a continuous variable cluster state if G is a full rank matrix [55]. For the same r, we will consider a matrix G which is its own inverse, i.e., $G^2 = I$. Under this assumption on G, we show that (see Sec. 9 of the Appendix)

$$\mathcal{B} = \frac{1}{2}I + \frac{1}{2}\tanh r \begin{pmatrix} -G & iG \\ iG & G \end{pmatrix}. \tag{17}$$

To demonstrate the power of our method, as a first example we consider a two-mode squeezed vacuum state (TMSV), from which we subtract five photons per mode (ten in total). We calculate the photon subtracted state $|\Psi_{-5,-5}\rangle$, the probability of success P_5, and the fidelity $F_5 = |\langle C|\Psi_{-5,-5}\rangle|^2$, where

$$|C\rangle = \frac{1}{N_+}(|\gamma,\gamma\rangle + |-\gamma,-\gamma\rangle), \tag{18}$$

with normalization $|N_+|^2 = 2[1 + e^{-2(q_\gamma^2+p_\gamma^2)}]$. We compare the state $|\Psi_{-5,-5}\rangle$ with the specific state of Eq. (18), because both states are parity $(-1)^{\hat{n}_1+\hat{n}_2}$ eigenstates with eigenvalue $+1$. If the $K(\vec{x}_\alpha)$ function is known, then the state $|\Psi_{-5,-5}\rangle$ is known from Eq. (10) for zero displacement. The only thing required to find the $K(\vec{x}_\alpha)$ is the matrix $\mathcal{B}$ [56], which is given by Eq. (17) for

$$G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{19}$$

which describes the graph corresponding to the TMSV, as can also be seen by Eq. (16). The probability P_5 and the fidelity F_5 are given by

$$P_5 = \frac{(1-\tau^2)^{10}\tanh^{10} r}{\cosh^2 r}p(\mu), \tag{20}$$

$$F_5 = \frac{2e^{-(q_\gamma^2+p_\gamma^2-\frac{z}{2})}(1-\tau)^5\tanh^2 r}{\left[1+e^{-2(q_\gamma^2+p_\gamma^2)}\right]\sqrt{P_5}(\det\Gamma)^{\frac{1}{4}}}w(z), \tag{21}$$

where $p(\mu) = [(1+\mu^2)(1+24\mu^2+76\mu^2+24\mu^6+\mu^8)]/[(1-\mu^2)^{11}]$, $w(z) = 1+(5z)/(2)+(5z^2)/(4)+(5z^3)/(24)+(5z^4)/(384)+(z^5)/(3840)$, $\mu = \tau\tanh r$, and $z = (q_\gamma - ip_\gamma)\mu$. For example for $q_\gamma = 0.5$, $p_\gamma = 0$, $\tau = 0.01$, and $r = 1$ we get $P_5 = 0.025$ and $F_5 = 0.979$. Note that in the above example the analytical complexity would not have changed if we decided to subtract more (e.g., ten photons) from each mode, whereas a traditional Fock basis calculation would become completely intractable.

As a second example we consider two ways to produce the cat-basis Bell state $|C\rangle$.

(i) We subtract two photons from a single-mode squeezed state, and the resulting state is known to be an approximation of the cat state $|\delta\rangle + |-\delta\rangle$, which if then split in a 50-50 beam splitter is known to produce the state $|C\rangle$ with $\delta = \sqrt{2}\gamma$ [57].

(ii) We subtract one photon from each of the two modes of a TMSV.

In both scenarios two photons are subtracted in total. Also, the beam splitter used in scenario (i) is a Hadamard gate, and if this is used to mix a position-squeezed state with a momentum-squeezed state we get a TMSV (see Fig. 1). We set $p_\gamma = 0$ and we calculate the probabilities of success $P_{(i)}$, $P_{(ii)}$ and the fidelities $F_{(i)}$, $F_{(ii)}$ for scenarios (i) and (ii) to the desired state $|C\rangle$, as

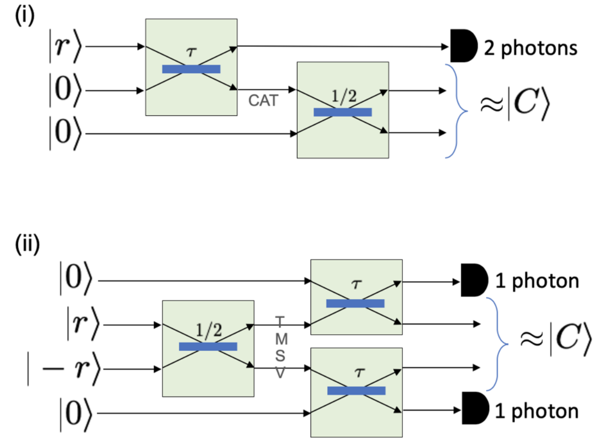

FIG. 1. Scenario (i): Two-photon subtraction from a single-mode squeezed state creates a state close to a single-mode coherent cat state. This when split on a balanced beam splitter creates a state that approximates the two-mode coherent cat-basis entangled state $|C\rangle$. Scenario (ii): An approximation to $|C\rangle$ is created by subtracting one photon from each mode of a two-mode squeezed vacuum state.

$$P_{(i)} = \frac{(\tanh r - \mu)^2(1+2\mu^2)}{2\cosh r(1-\mu^2)^{\frac{5}{2}}}, \tag{22}$$

$$P_{(ii)} = \frac{(\tanh r - \mu)^2(1+\mu^2)}{\cosh^2 r(1-\mu^2)^3}, \tag{23}$$

$$F_{(i)} = \frac{2e^{q_\gamma^2(1+\mu)}\left(q_\gamma^2\mu+1\right)^2(1-\mu^2)^{\frac{5}{2}}}{\left(e^{q_\gamma^2}+1\right)(1+2\mu^2)}, \tag{24}$$

$$F_{(ii)} = \frac{e^{q_\gamma^2(1+\mu)}\left(q_\gamma^2\mu+2\right)^2(1-\mu^2)^3}{2\left(e^{2q_\gamma^2}+1\right)(1+\mu^2)}. \tag{25}$$

Comparative results for these two scenarios are shown in Figs. 2 and 3. To produce a cat-basis Bell state $|C\rangle$ with a small amplitude, scenario (ii) is better than scenario (i) in both fidelity and probability of success. As the amplitude of $|C\rangle$ increases, the situation begins to change: scenario (i) favors high fidelity, at the expense of smaller probability of success compared to scenario (ii), for example, for $q_\gamma = 0.1$, $p_\gamma = 0$, $r = 0.9$, $\tau = 0.4$, $P_{(i)} = 0.179$, $P_{(ii)} = 0.249$, $F_{(i)} = 0.999$, and $F_{(ii)} = 0.999$ and for $q_\gamma = 1$, $p_\gamma = 0$, $r = 0.9$, $\tau = 0.4$, $P_{(i)} = 0.093$, $P_{(ii)} = 0.126$, $F_{(i)} = 0.990$, and $F_{(ii)} = 0.806$. It is of similar ease to find expressions for P_5, F_5, $P_{(i)}$, $P_{(ii)}$, $F_{(i)}$, and $F_{(ii)}$ for $p_\gamma \neq 0$ (generality is not lost by assuming real amplitude).

VI. MIXED GAUSSIAN STATES

A mixed Gaussian state $\hat{\rho}$ can be written as $\hat{\rho} = \hat{U}\hat{\rho}_{\text{th}}\hat{U}^\dagger$, where $\hat{\rho}_{\text{th}}$ is a thermal state and $\hat{U}$ is a Gaussian unitary. Using the Glauber-Sudarshan P_{GS} function of the thermal state $P_{\text{GS,th}} \equiv P_{\text{th}}(\vec{x}_\alpha)$, we have $\hat{\rho} = \int d^{2N}\vec{x}_\alpha P_{\text{th}}(\vec{x}_\alpha)|\Psi\rangle\langle\Psi|$ where

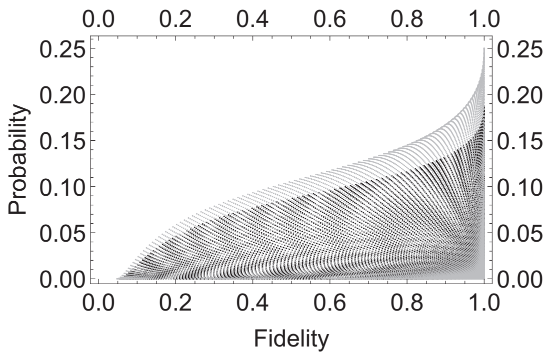

FIG. 2. Black dots correspond to scenario (i) while gray dots correspond to scenario (ii). Each dot corresponds to $q_\gamma = 0.1,\ p_\gamma = 0$; $r,\ \tau$ are taking values in [0.01,1] with step 0.01. Scenario (ii) is superior to scenario (i) as it can achieve higher fidelity with higher probability of success.

$|\Psi\rangle = U|\vec{\alpha}\rangle$. The state $|\Psi\rangle$ can be expressed using Eq. (3) and therefore $\hat{\rho}$ is expressed in the coherent-state basis as two integrals coming from $|\Psi\rangle$ and $\langle\Psi|$, which are convoluted into a third integral over $\vec{x}_\alpha$ with $P_{\rm th}(\vec{x}_\alpha)$.

Concerning mixed Gaussian states, things become even easier if an initial pure Gaussian state $|\Psi_0\rangle$ goes through a pure loss channel. We remind the reader that under a pure loss channel every mode of the state $|\Psi_0\rangle$ is coupled with $|0\rangle$ (the environment) via a beam splitter of transmittance τ_i, where $i = 1, \dots, N$ counts the modes, i.e., the loss does not have to be uniform across the N modes. Then the environment's output is traced out. The single-mode pure loss channel is described by the Kraus operators [58]:

$$\hat{A}_l = \sqrt{\frac{(1-\tau)^l}{l!}}\tau^{\hat{n}/2}\hat{a}^l, \tag{26}$$

and the final state is

$$\hat{\rho} = \sum_{l_1,\dots,l_n=0}^{\infty} \hat{A}_{l_1}\dots\hat{A}_{l_N}|\Psi_0\rangle\langle\Psi_0|\hat{A}^\dagger_{l_N}\dots\hat{A}^\dagger_{l_1}. \tag{27}$$

Here we observe that if $|\Psi_0\rangle$ is expressed on the coherent basis the operators $\tau^{\hat{n}/2}\hat{a}^l$ in Eq. (26), will act on coherent states,

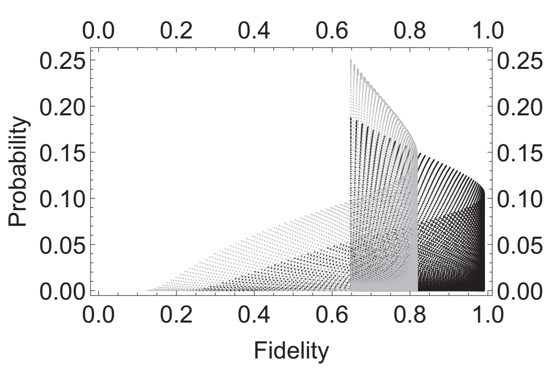

FIG. 3. Black dots correspond to scenario (i) while gray dots correspond to scenario (ii). Each dot corresponds to $q_\gamma = 1,\ p_\gamma = 0$; $r,\ \tau$ are taking values in [0.01,1] with step 0.01. Scenario (i) can achieve higher fidelity. However, for high fidelity the probability of success is smaller compared to smaller coherent cat states.

resulting in manageable expressions. For further simplicity we assume the same transmittance rate τ per mode (even though this assumption can be easily dropped). The final state will be

$$\begin{aligned}\hat{\rho} = \int & d^{2N}\vec{\alpha}d^{2N}\vec{\beta}K(\vec{\alpha})K^*(\vec{\beta})\\ &\times \exp\bigg[-\frac{1-\tau}{2}(|\vec{\alpha}|^2+|\vec{\beta}|^2)\\ &+(1-\tau)\vec{\beta}^{*T}\vec{\alpha}\bigg]|\sqrt{\tau}\vec{\alpha}\rangle\langle\sqrt{\tau}\vec{\beta}|,\end{aligned} \tag{28}$$

an expression which can be useful, for example, in an analysis of a Gaussian boson sampling with a pure loss scheme.

VII. CONCLUSIONS

We have derived a general representation of Gaussian states in the coherent-state basis, and showed that it opens the door to analytical and thorough investigations of non-Gaussian states prepared via photon subtraction and partial PNR detection of Gaussian states. We showed a simplified analysis of Gaussian boson sampling as a special case of our formalism. As a specific example application of our formalism, we considered cat-basis cluster creation by multimode photon subtraction on entangled Gaussian states. We showed that by subtracting photons simultaneously from both modes of a two-mode squeezed vacuum state a coherent cat-basis Bell state can be produced with higher fidelity and probability of success, compared to the well-known method of first creating a cat state via photon number subtraction of a single-mode squeezed vacuum followed by linear-optical manipulation. The question of whether more general coherent cat-basis graph states—known to be an excellent resource for quantum error correction against photon loss—can be systematically engineered from Gaussian cluster states and photon subtraction is left open for future work. We anticipate that our formalism will prove a powerful tool for non-Gaussian cluster state engineering [59–61], which is a subject of intense interest in designing scalable solutions for all-photonic quantum computing and other forms of quantum-enhanced photonic information processing such as all-photonic quantum repeaters, where photonic cluster states replace quantum memories [62,63], and optical-domain quantum machine learning via receivers powered with cluster states [64].

Note added. Recently, it came to our attention [65] that similar phase-space methods have been developed [66,67] practically concurrently.

ACKNOWLEDGMENTS

C.N.G. was supported by the Army Research Office STIR program, Contract No. W911NF-18-1-0377. S.G. acknowledges Xanadu Quantum Technologies for supporting multiple useful discussions on this topic. The authors acknowledge Daiqin Su, Krishna Kumar Sabapathy, Hari Krovi, Raf Alexander, and Kaushik Seshadreesan for valuable discussions.

APPENDIX

1. Photon subtraction from a coherent state using a beam splitter

Subtraction of m photons from a mode of a state $|\Psi\rangle$ can be implemented with a beam splitter of transmittance τ. The beam splitter couples the mode in which the photon subtraction will take place with vacuum. Then, if the photon number resolution measurement (PNRM) registers m photons, the resulting state is $|\Psi_{-m}\rangle$ as shown in Fig. 4. Since a measurement is involved, this procedure is probabilistic and heralded. Because of the probabilistic nature of photon subtraction the final state needs to be normalized. The absolute square of the normalization is the probability of finding m photons in the PNRM. This probability is also called the *probability of success*.

Since we expand $|\Psi\rangle$ on a coherent basis, when subtracting photons from some mode of $|\Psi\rangle$, the beam splitter will couple a coherent state with vacuum. If $\hat{a}_1,\ \hat{a}_2$ and $\hat{b}_1,\ \hat{b}_2$ are the input and output annihilation operators, respectively, we have

$$\begin{pmatrix}\hat{b}_1\\ \hat{b}_2\end{pmatrix}=\begin{pmatrix}\sqrt{\tau} & \sqrt{1-\tau}\\ -\sqrt{1-\tau} & \sqrt{\tau}\end{pmatrix}\begin{pmatrix}\hat{a}_1\\ \hat{a}_2\end{pmatrix}. \tag{A1}$$

Therefore, if the global two-mode input state is $|\alpha,0\rangle$ the final state is $|\sqrt{\tau}\alpha,-\sqrt{1-\tau}\alpha\rangle$. The conditional state on the upper output port, upon finding m photons in the PNRM, is

$$\langle m|-\sqrt{1-\tau}\alpha\rangle|\alpha\sqrt{\tau}\rangle=\frac{(-\sqrt{1-\tau})^m}{\sqrt{m!}}\alpha^m e^{-(1-\tau)\frac{|\alpha|^2}{2}}|\alpha\sqrt{\tau}\rangle, \tag{A2}$$

therefore we can write

$$\hat{P}_{-m}[|\alpha\rangle]=c_s|\alpha_{-m}\rangle, \tag{A3}$$

where

$$c_s=\frac{(-\sqrt{1-\tau})^m}{\sqrt{m!}}\alpha^m e^{-(1-\tau)\frac{|\alpha|^2}{2}}, \tag{A4}$$

$$|\alpha_{-m}\rangle=|\alpha\sqrt{\tau}\rangle \tag{A5}$$

and the probability of success is given by $P=|c_s|^2$. Subtracting photons from a coherent state yields the same amplitude-damped coherent state $|\alpha\sqrt{\tau}\rangle$ regardless of the PNRM result. Therefore, for applications there is not much meaning in subtracting photons from coherent states. However, it is highly convenient for mathematical manipulation of photon subtraction written on a coherent basis.

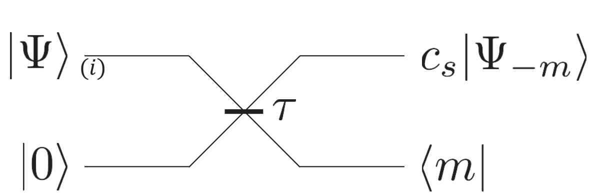

FIG. 4. The ith mode of a state $|\Psi\rangle$ is mixed with vacuum in a beam splitter with transmittance τ. If a photon number resolution measurement registers m photons in the lower output port, then m photons have been subtracted from the ith mode of the input state.

2. Coherent basis representation of pure Gaussian states without displacement

We define $\vec{x}_\alpha=\binom{\vec{q}_\alpha}{\vec{p}_\alpha}$ and we work with $\hbar=1$. Using the unit resolution on coherent states,

$$\begin{aligned}\frac{1}{\pi^N}\int d^{2N}\vec{\alpha}\ |\vec{\alpha}\rangle\langle\vec{\alpha}|&=\frac{1}{(2\pi)^N}\int d^N\vec{q}_\alpha d^N\vec{p}_\alpha\ |\vec{\alpha}\rangle\langle\vec{\alpha}|\\ &=\frac{1}{(2\pi)^N}\int d^{2N}\vec{x}_\alpha\ |\vec{\alpha}\rangle\langle\vec{\alpha}|\\ &=I,\end{aligned} \tag{A6}$$

for any state $|\Psi\rangle$ we can write

$$|\Psi\rangle=\frac{1}{(2\pi)^N}\int d^{2N}\vec{x}_\alpha\ \langle\vec{\alpha}|\Psi\rangle|\vec{\alpha}\rangle=\int d^{2N}\vec{x}_\alpha\ K(\vec{x}_\alpha)|\vec{\alpha}\rangle, \tag{A7}$$

where we define

$$K(\vec{x}_\alpha)=\frac{1}{(2\pi)^N}\langle\vec{\alpha}|\Psi\rangle. \tag{A8}$$

which up to some constant is the *the square root* of the $Q(\vec{x}_\alpha)$ representation:

$$Q(\vec{x}_\alpha)=\frac{1}{(2\pi)^N}\langle\vec{\alpha}|\Psi\rangle\langle\Psi|\vec{\alpha}\rangle, \tag{A9}$$

therefore we can write

$$\frac{1}{(2\pi)^N}Q(\vec{x}_\alpha)=|K(\vec{x}_\alpha)|^2\Rightarrow K(\vec{x}_\alpha)=\frac{1}{(2\pi)^{N/2}}Q_{1/2}(\vec{x}_\alpha), \tag{A10}$$

such that

$$Q_{1/2}(\vec{x}_\alpha)Q^*_{1/2}(\vec{x}_\alpha)=Q(\vec{x}_\alpha). \tag{A11}$$

Equations (A10) and (A11) imply that to find $Q_{1/2}(\vec{x}_\alpha)$ we have to separate the $Q(\vec{x}_\alpha)$ representation into a product of two conjugate parts. In that way, if the state $|\Psi\rangle=|\Psi_0\rangle$ is a Gaussian state with zero displacement, we can express $K(\vec{x}_\alpha)$ as a function of the states' CM. The $Q(\vec{x}_\alpha)$ representation of a Gaussian state with CM V is

$$Q(\vec{x}_\alpha)=\frac{1}{(2\pi)^N\sqrt{\det\Gamma}}\exp\left[-\frac{1}{2}\vec{x}_\alpha^T\Gamma^{-1}\vec{x}_\alpha\right], \tag{A12}$$

where

$$\Gamma=V+\tfrac{1}{2}I, \tag{A13}$$

where I is the identity matrix of appropriate dimensions. Any CM is a real symmetric matrix, and as per Eq. (A13) Γ is a real symmetric matrix. The inverse of a real symmetric matrix is again real and symmetric, therefore in block form the matrix Γ^{-1} is

$$\Gamma^{-1}=\begin{pmatrix}A & C\\ C^T & B\end{pmatrix}, \tag{A14}$$

where $A=A^T$ and $B=B^T$ and $A,\ B,$ and C real. It is more convenient if we change coordinates in the following manner:

$$\vec{z}=R\vec{x}_\alpha, \tag{A15}$$

where

$$\vec{z}_\alpha=\begin{pmatrix}\vec{\alpha}\\ \vec{\alpha}^*\end{pmatrix}, \tag{A16}$$

$$\vec{\alpha}=\frac{1}{\sqrt{2}}(\vec{q}_\alpha+i\vec{p}_\alpha), \tag{A17}$$

$$\vec{\alpha}^* = \frac{1}{\sqrt{2}}(\vec{q}_\alpha - i\vec{p}_\alpha), \tag{A18}$$

$$\vec{x}_\alpha = \begin{pmatrix} \vec{q}_\alpha \\ \vec{p}_\alpha \end{pmatrix}, \tag{A19}$$

$$R = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ -iI & iI \end{pmatrix}. \tag{A20}$$

Note that R is unitary, i.e., $RR^\dagger = I$.

To break Eq. (A12) into two conjugate parts, we must express the term $\vec{x}_\alpha^T \Gamma^{-1} \vec{x}_\alpha$ which appears in its exp(.) as a summation of two conjugate terms. To this end we express Γ in the $\vec{z}_\alpha$ basis:

$$\vec{x}^T \Gamma^{-1} \vec{x} = \vec{z}^\dagger R^\dagger \Gamma^{-1} R \vec{z} = \vec{z}^\dagger \tilde{\Gamma}^{-1} \vec{z}, \tag{A21}$$

where

$$\tilde{\Gamma}^{-1} = R^\dagger \Gamma^{-1} R \tag{A22}$$

is the transformed Γ^{-1} in the $\vec{z}_\alpha$ basis. From Eqs. (A22) and (A20) we have

$$\begin{aligned}\tilde{\Gamma}^{-1} &= \frac{1}{2}\begin{pmatrix} A + B - i(C - C^T) & A - B + i(C + C^T) \\ A - B - i(C + C^T) & A + B + i(C - C^T) \end{pmatrix} \\ &= \begin{pmatrix} \tilde{A} & \tilde{C} \\ \tilde{C}^* & \tilde{A}^* \end{pmatrix}. \end{aligned} \tag{A23}$$

Therefore we can write

$$\vec{z}_\alpha^\dagger \tilde{\Gamma}^{-1} \vec{z}_\alpha = \begin{pmatrix} \vec{\alpha}^{*T} & \vec{\alpha}^T \end{pmatrix} \begin{pmatrix} \tilde{A} & \tilde{C} \\ \tilde{C}^* & \tilde{A}^* \end{pmatrix} \begin{pmatrix} \vec{\alpha} \\ \vec{\alpha}^* \end{pmatrix} \tag{A24}$$

$$\begin{aligned} &= \vec{\alpha}^{*T} \tilde{A} \vec{\alpha} + \vec{\alpha}^{*T} \tilde{C} \vec{a}^* + \vec{\alpha}^T \tilde{C}^* \vec{a} + \vec{\alpha}^T \tilde{A}^* \vec{\alpha}^* \\ &= \vec{z}^\dagger \tilde{\mathcal{B}} \vec{z} + \vec{z}^\dagger \tilde{\mathcal{B}}^\dagger \vec{z}. \end{aligned} \tag{A25}$$

Equation (A25) shows that we can readily derive the two conjugate terms where

$$\tilde{\mathcal{B}} = \frac{1}{2}\begin{pmatrix} \tilde{A} & \tilde{C} \\ 0 & \tilde{A}^* \end{pmatrix}. \tag{A26}$$

Going back to Cartesian coordinates $\vec{x}_\alpha$ we get the matrix $\mathcal{B}$:

$$\Gamma^{-1} = R\tilde{\Gamma}^{-1}R^\dagger = R\tilde{\mathcal{B}}R^\dagger + R\tilde{\mathcal{B}}^\dagger R^\dagger = \mathcal{B} + \mathcal{B}^\dagger, \tag{A27}$$

where

$$\mathcal{B} = R\tilde{\mathcal{B}}R^\dagger = \frac{1}{2}\begin{pmatrix} A + \frac{i}{2}(C + C^T) & C - \frac{i}{2}(A - B) \\ C^T - \frac{i}{2}(A - B) & B - \frac{i}{2}(C + C^T) \end{pmatrix} \tag{A28}$$

where we have used Eqs. (A20) and (A26). Therefore, given the CM V of a pure Gaussian state $|\Psi_0\rangle$, we can find Γ^{-1} and from that we can immediately write $\mathcal{B}$ and the expansion on a coherent basis is

$$K(\vec{x}_\alpha) = \frac{1}{(2\pi)^N} \frac{1}{(\det \Gamma)^{1/4}} \exp\left[-\frac{1}{2}\vec{x}_\alpha^T \mathcal{B} \vec{x}_\alpha\right]. \tag{A29}$$

3. Coherent basis representation of pure Gaussian states with displacement

A displaced pure Gaussian state $|\Psi\rangle$ can be derived by applying a displacement $D(\vec{x}_\beta)$ and multiple-mode squeezing $S(\vec{r})$ (phases can be absorbed into the squeezing operator) [3] onto a multiple-mode vacuum state $|\vec{0}\rangle$:

$$|\Psi\rangle = D(\vec{\beta})S(\vec{r})|\vec{0}\rangle \Rightarrow |\Psi\rangle = D(\vec{\beta})|\Psi_0\rangle, \tag{A30}$$

where $|\Psi_0\rangle$ is the state for which we worked out its coherent basis expansion in Sec. 2. Therefore we have

$$\begin{aligned} |\Psi\rangle &= \frac{1}{(2\pi)^N}\int d^{2N}\vec{x}_\alpha\ \langle\vec{\alpha}|D(\vec{\beta})|\Psi_0\rangle|\vec{\alpha}\rangle \\ &= \frac{1}{(2\pi)^N}\int d^{2N}\vec{x}_\alpha\ \langle\vec{0}|D(-\vec{\alpha})D(\vec{\beta})|\Psi_0\rangle|\vec{\alpha}\rangle \qquad \text{(A31)} \\ &= \frac{1}{(2\pi)^N}\int d^{2N}\vec{x}_\alpha\ \langle\vec{\alpha} - \vec{\beta}|\Psi_0\rangle e^{\frac{1}{2}\vec{\beta}\vec{\alpha}^* - \frac{1}{2}\vec{\beta}^*\vec{\alpha}}|\vec{\alpha}\rangle, \qquad \text{(A32)} \end{aligned}$$

where in the last step we have used $D(-\vec{\alpha})D(\vec{\beta}) = e^{\frac{1}{2}\vec{\beta}\vec{\alpha}^* - \frac{1}{2}\vec{\beta}^*\vec{\alpha}}D(\vec{\beta} - \vec{\alpha})$, which acts on $\langle\vec{0}|$ and therefore the sign of the displacement should be inversed. In Eq. (A32), $\langle\vec{\alpha} - \vec{\beta}|\Psi_0\rangle$ is known from Eq. (A29). Additionally, by defining

$$\mathcal{X} = \begin{pmatrix} I & iI \\ -iI & I \end{pmatrix}, \tag{A33}$$

Eq. (A32) is written

$$\begin{aligned} |\Psi\rangle &= \frac{1}{(2\pi)^N}\frac{1}{(\det\Gamma)^{1/4}}\int d^{2N}\vec{x}_\alpha \exp\left[-\frac{1}{2}(\vec{x}_\alpha - \vec{x}_\beta)^T\mathcal{B}(\vec{x}_\alpha - \vec{x}_\beta)\right]\exp\left(\frac{1}{4}\vec{x}_\alpha^T\mathcal{X}\vec{x}_\beta - \frac{1}{4}\vec{x}_\beta^T\mathcal{X}\vec{x}_\alpha\right) \qquad \text{(A34)} \\ &= \frac{1}{(2\pi)^N}\frac{1}{(\det\Gamma)^{1/4}}\int d^{2N}\vec{x}_\alpha \exp\left(-\frac{1}{2}\vec{x}_\alpha^T\mathcal{B}\vec{x}_\alpha\right)\exp\left[\frac{1}{4}\begin{pmatrix}\vec{x}_\alpha^T & \vec{x}_\beta^T\end{pmatrix}\begin{pmatrix} 0 & 2\mathcal{B} + \mathcal{X} \\ 2\mathcal{B} - \mathcal{X} & -2\mathcal{B}\end{pmatrix}\begin{pmatrix}\vec{x}_\alpha \\ \vec{x}_\beta\end{pmatrix}\right]. \qquad \text{(A35)} \end{aligned}$$

From Eqs. (A29) and (A35) we have

$$|\Psi\rangle = \int d^{2N}\vec{x}_\alpha\ K(\vec{x}_\alpha)G(\vec{x}_\alpha, \vec{x}_\beta)|\vec{\alpha}\rangle, \tag{A36}$$

where

$$G(\vec{x}_\alpha, \vec{x}_\beta) = \exp\left[\tfrac{1}{4}(\vec{x}_\alpha^T\ \vec{x}_\beta^T)\mathcal{D}(\vec{x}_\alpha\ \vec{x}_\beta)\right], \tag{A37}$$

with

$$\mathcal{D} = \begin{pmatrix} 0 & 2\mathcal{B} + \mathcal{X} \\ 2\mathcal{B} - \mathcal{X} & -2\mathcal{B}\end{pmatrix}. \tag{A38}$$

4. Probability of success

The photon subtracted state is

$$\begin{aligned} |\Psi_{-\vec{m}}\rangle = \frac{1}{\sqrt{P}}\prod_{i=1}^{N}\frac{(-\sqrt{1-\tau_i})^{m_i}}{\sqrt{m_i!}}\int d^{2N}\vec{x}_\alpha\ K(\vec{x}_\alpha)G(\vec{x}_\alpha, \vec{x}_\beta) \\ \times e^{-\frac{(1-\tau_i)}{4}|\vec{x}_\alpha|^2}\left(\frac{q_{\alpha_i} + ip_{\alpha_i}}{\sqrt{2}}\right)^{m_i}|\sqrt{\tau_i}\vec{\alpha}\rangle, \end{aligned} \tag{A39}$$

therefore the probability of success is given by the condition $\langle\Psi_{-\vec{m}}|\Psi_{-\vec{m}}\rangle = 1$. Therefore we have

$$P = \prod_{i=1}^{N} \frac{(1-\tau_i)^{m_i}}{m_i!} \int d^{2N}\vec{x}_\alpha d^{2N}\vec{x}_\gamma\, K(\vec{x}_\alpha)G(\vec{x}_\alpha,\vec{x}_\beta)K^*(\vec{x}_\gamma) \times G^*(\vec{x}_\gamma,\vec{x}_\beta)e^{-\frac{(1-\tau_i)}{4}(|\vec{x}_\alpha|^2+|\vec{x}_\gamma|^2)}\left(\frac{q_{\alpha_i}+ip_{\alpha_i}}{\sqrt{2}}\right)^{m_i} \times \left(\frac{q_{\gamma_i}-ip_{\gamma_i}}{\sqrt{2}}\right)^{m_i}\langle\sqrt{\tau_i}\vec{\gamma}|\sqrt{\tau_i}\vec{\alpha}\rangle. \tag{A40}$$

By writing

$$\langle\sqrt{\tau_i}\vec{\gamma}|\sqrt{\tau_i}\vec{\alpha}\rangle = \exp\left(-\tfrac{1}{4}\tau_i\vec{x}_\alpha^T\vec{x}_\alpha - \tfrac{1}{4}\tau_i\vec{x}_\gamma^T\vec{x}_\gamma + \tfrac{1}{2}\tau_i\vec{x}_\gamma^T\mathcal{X}\vec{x}_\alpha\right), \tag{A41}$$

Eq. (A40) gives

$$P = \prod_{i=1}^{N} \frac{(1-\tau_i)^{m_i}}{2^{m_i}m_i!} \int d^{2N}\vec{x}_\alpha d^{2N}\vec{x}_\gamma\, K(\vec{x}_\alpha)G(\vec{x}_\alpha,\vec{x}_\beta)K^*(\vec{x}_\gamma) \times G^*(\vec{x}_\gamma,\vec{x}_\beta)e^{-\frac{(1-\tau_i)}{4}(|\vec{x}_\alpha|^2+|\vec{x}_\gamma|^2)-\frac{1}{4}\tau_i\vec{x}_\alpha^T\vec{x}_\alpha-\frac{1}{4}\tau_i\vec{x}_\gamma^T\vec{x}_\gamma+\frac{1}{2}\tau_i\vec{x}_\gamma^T\mathcal{X}\vec{x}_\alpha} \times \left(q_{\alpha_i}+ip_{\alpha_i}\right)^{m_i}\left(q_{\gamma_i}-ip_{\gamma_i}\right)^{m_i}. \tag{A42}$$

Equation (A42) is a Gaussian integral [represented by the $e^{-\frac{(1-\tau_i)}{4}(|\vec{x}_\alpha|^2+|\vec{x}_\gamma|^2)-\frac{1}{4}\tau_i\vec{x}_\alpha^T\vec{x}_\alpha-\frac{1}{4}\tau_i\vec{x}_\gamma^T\vec{x}_\gamma+\frac{1}{2}\tau_i\vec{x}_\gamma^T\mathcal{X}\vec{x}_\alpha}$, $K(\vec{x}_\alpha)$, and $K^*(\vec{x}_\gamma)$, kernels] with linear terms [represented by $G(\vec{x}_\alpha,\vec{x}_\beta)$ and $G^*(\vec{x}_\gamma,\vec{x}_\beta)$] and polynomial terms $(q_{\alpha_i}+ip_{\alpha_i})^{m_i}(q_{\gamma_i}-ip_{\gamma_i})^{m_i}$. The way to calculate this analytically and efficiently is to use the identity

$$\left(q_{\alpha_i}+ip_{\alpha_i}\right)^{m_i}\left(q_{\gamma_i}-ip_{\gamma_i}\right)^{m_i} = \frac{d^{m_i}}{d\lambda_i^{m_i}}\frac{d^{m_i}}{d\mu_i^{m_i}}e^{\lambda_i(q_{\alpha_i}+ip_{\alpha_i})+\mu_i(q_{\gamma_i}-ip_{\gamma_i})}\Big|_{\lambda_i=\mu_i=0}. \tag{A43}$$

Using Eq. (A43), we cast Eq. (A42) into a Gaussian integral, i.e., there is only an exponential and no polynomial terms, with extra lineal terms $\lambda_i(q_{\alpha_i}+ip_{\alpha_i})+\mu_i(q_{\gamma_i}-ip_{\gamma_i})$ in the exponential. Then one should take the m_ith-order derivatives on the result of the Gaussian integral with respect to λ_i and μ_i at $\lambda_i=\mu_i=0$.

5. Fidelity

For any state of the form

$$|\phi\rangle = \sum_i c_i|\vec{\gamma}^{(i)}\rangle, \tag{A44}$$

$\sum_i|c_i|^2 = 1$ and $|\vec{\gamma}^{(i)}\rangle = |\gamma_1^{(i)}\gamma_1^{(i)}\dots\gamma_N^{(i)}\rangle$. Note that a special example of $|\phi\rangle$ is the coherent cat state used in the main paper. The fidelity $F = |\langle\phi|\Psi_{-\vec{m}}\rangle|^2$ requires the calculation of $\langle\vec{\gamma}|\Psi_{-\vec{m}}\rangle$. From Eq. (A39) we have

$$\langle\vec{\gamma}|\Psi_{-\vec{m}}\rangle = \frac{1}{\sqrt{P}}\prod_{i=1}^{N}\frac{(-\sqrt{1-\tau_i})^{m_i}}{\sqrt{m_i!}} \times \int d^{2N}\vec{x}_\alpha\, K(\vec{x}_\alpha)G(\vec{x}_\alpha,\vec{x}_\beta)e^{-\frac{(1-\tau_i)}{4}|\vec{x}_\alpha|^2} \times \left(\frac{q_{\alpha_i}+ip_{\alpha_i}}{\sqrt{2}}\right)^{m_i}\langle\vec{\gamma}|\sqrt{\tau_i}\vec{\alpha}\rangle, \tag{A45}$$

where the probability of success P should be calculated first as per Sec. 4 of the Appendix. We have

$$\langle\vec{\gamma}|\sqrt{\tau_i}\vec{\alpha}\rangle = \exp\left(-\tfrac{1}{4}\tau_i\vec{x}_\alpha^T\vec{x}_\alpha - \tfrac{1}{4}\vec{x}_\gamma^T\vec{x}_\gamma + \tfrac{1}{2}\sqrt{\tau_i}\vec{x}_\gamma^T\mathcal{X}\vec{x}_\alpha\right), \tag{A46}$$

therefore Eq. (A45) is written as

$$\langle\vec{\gamma}|\Psi_{-\vec{m}}\rangle = \frac{1}{\sqrt{P}}\prod_{i=1}^{N}\frac{(-\sqrt{1-\tau_i})^{m_i}}{\sqrt{2^{m_i}m_i!}}\int d^{2N}\vec{x}_\alpha\, K(\vec{x}_\alpha)G(\vec{x}_\alpha,\vec{x}_\beta) \times e^{-\frac{(1-\tau_i)}{4}|\vec{x}_\alpha|^2-\frac{1}{4}\tau_i\vec{x}_\alpha^T\vec{x}_\alpha-\frac{1}{4}\vec{x}_\gamma^T\vec{x}_\gamma+\frac{1}{2}\sqrt{\tau_i}\vec{x}_\gamma^T\mathcal{X}\vec{x}_\alpha} \times \left(q_{\alpha_i}+ip_{\alpha_i}\right)^{m_i}. \tag{A47}$$

Equation (A47), similarly to P in Sec. 4 of the Appendix, is a Gaussian integral with linear terms, and polynomial terms $(q_{\alpha_i}+ip_{\alpha_i})^{m_i}$ which can be injected into the exponential of Eq. (A47) by using the identity

$$\left(q_{\alpha_i}+ip_{\alpha_i}\right)^{m_i} = \frac{d^{m_i}}{d\lambda_i^{m_i}}e^{\lambda_i(q_{\alpha_i}+ip_{\alpha_i})}\Big|_{\lambda_i=0}. \tag{A48}$$

That way Eq. (A47) will become a Gaussian integral, upon which we take m_ith-order derivatives with respect to λ_i at $\lambda_i = 0$.

6. The conditional state and its normalization

We set zero displacements, therefore we work with the N-mode Gaussian state $|\Psi_0\rangle$. Upon finding a pattern $\{n_1,\dots,n_M\}$, $M<N$ at the PNRMs at each one of the M modes, the conditional state $|\Phi\rangle$ is

$$|\Phi\rangle = \frac{1}{\sqrt{P_M}}\langle n_1,\dots,n_M|\Psi_0\rangle = \frac{1}{\sqrt{P_M}}\prod_{i=1}^{M}\frac{1}{\sqrt{2^{n_i}n_i!}}\int d^{2N}\vec{x}_\alpha K(\vec{x}_\alpha)e^{-\frac{1}{4}x_{\alpha_i}^2} \times \left(q_{\alpha_i}+ip_{\alpha_i}\right)^{n_i}|\alpha_{M+1},\dots,\alpha_N\rangle. \tag{A49}$$

The probability P_M is given by the normalization $\langle\Phi|\Phi\rangle = 1$:

$$P_M = \prod_{i=1}^{M}\frac{1}{2^{n_i}n_i!}\int d^{2N}\vec{x}_\alpha d^{2N}\vec{x}_\gamma K(\vec{x}_\alpha)K^*(\vec{x}_\gamma) \times e^{-\frac{1}{4}|x_\alpha|^2-\frac{1}{4}|x_\gamma|^2+\sum_{k,l=M+1}^{N}x_{\gamma_k}\mathcal{X}_{kl}x_{\alpha_l}} \times \left(q_{\alpha_i}+ip_{\alpha_i}\right)^{n_i}\left(q_{\gamma_i}-ip_{\gamma_i}\right)^{n_i} \tag{A50}$$

where we have used $\langle\gamma|\alpha\rangle = \exp(-|\gamma|^2/2-|\alpha|^2/2+\gamma^*\alpha)$ and $|\vec{x}_{\alpha,\gamma}|^2 = \sum_{k=1}^{N}x_{\alpha_i,\gamma_i}^2$, and $\mathcal{X}_{kl}$ are the matrix elements of $\mathcal{X}$ of Eq. (A33) for dimensions $(N-M)\times(N-M)$. The same method using ancillary variables λ_i as in Sec. 4 of the Appendix can be applied to calculate P_M of Eq. (A50).

7. The probability distribution $P_{\vec{n}}$

We set zero displacements. The probability of finding a pattern $\{n_1,\dots,n_N\}$ at each one of all the N modes is

$$P_{\vec{n}} = |\langle\vec{n}|\Psi_0\rangle|^2 = |\langle n_1\dots n_N|\Psi_0\rangle|^2. \tag{A51}$$

From Eq. (A29) and using $\langle n|\alpha\rangle = \exp(-|\alpha|^2/2)\alpha^{n*}/(\sqrt{n!})$ we get

$$\begin{aligned}\langle n_1 \dots n_N|\Psi_0\rangle &= \frac{1}{(2\pi)^N(\det\Gamma)^{1/4}}\prod_{i=1}^{M}\frac{1}{\sqrt{2^{n_i}n_i!}}\\ &\quad\times\int d^{2N}\vec{x}_\alpha e^{-\frac{1}{2}\vec{x}_\alpha^T\mathcal{B}\vec{x}_\alpha-\frac{1}{4}\vec{x}_\alpha^T\vec{x}_\alpha}(q_{\alpha_i}+ip_{\alpha_i})^{n_i}\\ &= \frac{1}{(2\pi)^N(\det\Gamma)^{1/4}}\prod_{i=1}^{M}\frac{1}{\sqrt{2^{n_i}n_i!}}\\ &\quad\times\int d^{2N}\vec{x}_\alpha e^{-\frac{1}{2}\vec{x}_\alpha^T\mathcal{H}\vec{x}_\alpha}(q_{\alpha_i}+ip_{\alpha_i})^{n_i},\end{aligned}\tag{A52}$$

where $\mathcal{H} = \mathcal{B} + I/2$. As it is shown in Sec. 8 of the Appendix, $\mathcal{H}$ is symmetric with positive definite real part. Therefore, the function

$$R(\vec{x}_\alpha) = \frac{\sqrt{\det\mathcal{H}}}{(2\pi)^N}e^{-\frac{1}{2}\vec{x}_\alpha^T\mathcal{H}\vec{x}_\alpha}\tag{A53}$$

represents a Gaussian distribution. In that way, Eq. (A52) is written as

$$\begin{aligned}&\langle n_1 \dots n_N|\Psi_0\rangle\\ &= \frac{1}{\sqrt{\det\mathcal{H}}(\det\Gamma)^{1/4}}\prod_{i=1}^{M}\frac{1}{\sqrt{2^{n_i}n_i!}}\int d^{2N}\vec{x}_\alpha R(\vec{x}_\alpha)(q_{\alpha_i}+ip_{\alpha_i})^{n_i}\\ &= \frac{1}{\sqrt{\det\mathcal{H}}(\det\Gamma)^{1/4}}\frac{1}{\sqrt{2^{n_1}n_1!\dots 2^{n_N}n_N!}}\langle f_1^{n_1}\dots f_N^{n_N}\rangle,\end{aligned}\tag{A54}$$

where $f_i = q_{\alpha_i} + ip_{\alpha_i}$. Mean values of the form $\langle f_1^{n_1}\dots f_N^{n_N}\rangle$ represent Hafnians via Wick's theorem as argued in the main paper. Equation (A54) yields a complex number result, the absolute square of which is the probability $P_{\vec{n}}$ of Eq. (A51).

8. The $\mathcal{H}$ matrix is symmetric and its real part is positive definite

From Eq. (A28) and given that $A^T = A$ and $B^T = B$ we can readily see that $\mathcal{B}^T = \mathcal{B}$. Therefore $\mathcal{H} = \mathcal{B} + I/2$ is symmetric as well. The real part of $\mathcal{H}$ is

$$\begin{aligned}\mathrm{Re}(\mathcal{H}) &= \frac{1}{2}(\mathcal{H}+\mathcal{H}^\dagger) = \frac{1}{2}(\mathcal{H}+\mathcal{H}^*) = \frac{1}{2}\left[\begin{pmatrix}A & C\\ C^T & B\end{pmatrix}+I\right]\\ &= \frac{1}{2}(\Gamma^{-1}+I).\end{aligned}\tag{A55}$$

Since any CM V is positive definite, denoted as $V > 0$, then $\Gamma = V + I/2 > 0 \Rightarrow \Gamma^{-1} > 0$ since the inverse of a positive definite matrix is also positive definite.

9. $\mathcal{B}$ matrix for multimode squeezed states

The Hamiltonian

$$\hat{H} = -\frac{i}{2}\sum_{i,j}^{N}G_{ij}(\hat{a}_i^\dagger\hat{a}_j^\dagger - \hat{a}_i\hat{a}_j)\tag{A56}$$

generates the unitary $\hat{U}_r = \exp(-ir\hat{H})$ which corresponds to the symplectic matrix:

$$S_r = \begin{pmatrix}e^{rG} & 0\\ 0 & e^{-rG}\end{pmatrix}.\tag{A57}$$

Therefore the CM is

$$V = \frac{1}{2}S_rS_r^T = \frac{1}{2}\begin{pmatrix}e^{2rG} & 0\\ 0 & e^{-2rG}\end{pmatrix}\tag{A58}$$

and from Eq. (A28) we get

$$\mathcal{B} = \frac{1}{2}I + \frac{1}{2}\begin{pmatrix}-\tanh Gr & i\tanh Gr\\ i\tanh Gr & \tanh Gr\end{pmatrix}.\tag{A59}$$

In Eq. (A59) the matrix $G = G^T$ is in the argument of tanh(.) which denotes

$$\tanh Gr = \frac{e^{2rG}-I}{e^{2rG}+I}.\tag{A60}$$

For a self-inverse matrix $G = G^{-1}$, i.e., $G^2 = I$, we expand e^{2rG} in Taylor series. In this way we get

$$e^{2rG} - I = I\cosh 2r + G\sinh 2r - I,\tag{A61}$$

$$e^{2rG} + I = I\cosh 2r + G\sinh 2r + I.\tag{A62}$$

From Eqs. (A60), (A61), and (A62), we have

$$\tanh Gr = \tanh^2 r\left(I + G\frac{1}{\tanh r}\right)(I + G\tanh r)^{-1}.\tag{A63}$$

We have that

$$I = (I + G\tanh r)(I - G\tanh r)\cosh^2 r\tag{A64}$$

$$\Rightarrow (I + G\tanh r)^{-1} = (I - G\tanh r)\cosh^2 r.\tag{A65}$$

Equations (A63) and (A65) give

$$\tanh Gr = G\tanh r.\tag{A66}$$

From Eqs. (A59) and (A66) we find

$$\mathcal{B} = \frac{1}{2}I + \frac{1}{2}\tanh r\begin{pmatrix}-G & iG\\ iG & G\end{pmatrix}.\tag{A67}$$

[1] A. Ferraro, S. Olivares, and M. G. A. Paris, *Gaussian States in Quantum Information*, Napoli Series on Physics and Astrophysics (Bibliopolis, 2005).

[2] J.-i. Yoshikawa, S. Yokoyama, T. Kaji, C. Sornphiphatphong, Y. Shiozawa, K. Makino, and A. Furusawa, APL Photonics **1**, 060801 (2016).

[3] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Rev. Mod. Phys. **84**, 621 (2012).

[4] S. Lloyd and S. L. Braunstein, Phys. Rev. Lett. **82**, 1784 (1999).

[5] S. D. Bartlett, B. C. Sanders, S. L. Braunstein, and K. Nemoto, Phys. Rev. Lett. **88**, 097904 (2002).

[6] J. Eisert, S. Scheel, and M. B. Plenio, Phys. Rev. Lett. **89**, 137903 (2002).

[7] J. Fiurášek, Phys. Rev. Lett. **89**, 137904 (2002).

[8] G. Giedke and J. I. Cirac, Phys. Rev. A **66**, 032316 (2002).

[9] N. J. Cerf, O. Krüger, P. Navez, R. F. Werner, and M. M. Wolf, Phys. Rev. Lett. **95**, 070501 (2005).

[10] M. Takeoka and M. Sasaki, Phys. Rev. A **78**, 022320 (2008).
[11] K. Tsujino, D. Fukuda, G. Fujii, S. Inoue, M. Fujiwara, M. Takeoka, and M. Sasaki, Phys. Rev. Lett. **106**, 250503 (2011).
[12] C. Wittmann, U. L. Andersen, M. Takeoka, D. Sych, and G. Leuchs, Phys. Rev. Lett. **104**, 100505 (2010).
[13] R. Namiki, O. Gittsovich, S. Guha, and N. Lütkenhaus, Phys. Rev. A **90**, 062316 (2014).
[14] J. Niset, J. Fiurášek, and N. J. Cerf, Phys. Rev. Lett. **102**, 120501 (2009).
[15] C. N. Gagatsos, D. Branford, and A. Datta, Phys. Rev. A **94**, 042342 (2016).
[16] M. Wolinsky and H. J. Carmichael, Phys. Rev. Lett. **60**, 1836 (1988).
[17] M. Dakna, T. Anhut, T. Opatrný, L. Knöll, and D.-G. Welsch, Phys. Rev. A **55**, 3184 (1997).
[18] V. Averchenko, C. Jacquard, V. Thiel, C. Fabre, and N. Treps, New J. Phys. **18**, 083042 (2016).
[19] H. Takahashi, K. Wakui, S. Suzuki, M. Takeoka, K. Hayasaka, A. Furusawa, and M. Sasaki, Phys. Rev. Lett. **101**, 233605 (2008).
[20] R. Tualle-Brouri, A. Ourjoumtsev, A. Dantan, P. Grangier, M. Wubs, and A. S. Sørensen, Phys. Rev. A **80**, 013806 (2009).
[21] P. Marek, H. Jeong, and M. S. Kim, Phys. Rev. A **78**, 063811 (2008).
[22] Y.-S. Ra, C. Jacquard, A. Dufour, C. Fabre, and N. Treps, Phys. Rev. X **7**, 031012 (2017).
[23] A. Dufour, C. Jacquard, Y.-S. Ra, C. Fabre, and N. Treps, Photon subtraction from a multimode squeezed vacuum state, *Quantum Information and Measurement (QIM) 2017* (Optical Society of America, 2017), p. QT4B.2.
[24] S. M. Barnett, G. Ferenczi, C. R. Gilson, and F. C. Speirits, Phys. Rev. A **98**, 013809 (2018).
[25] S. Glancy and H. M. de Vasconcelos, J. Opt. Soc. Am. B **25**, 712 (2008).
[26] Y.-S. Ra, A. Dufour, M. Walschaers, C. Jacquard, T. Michel, C. Fabre, and N. Treps, arXiv:1901.10939 (2019).
[27] T. Opatrný, G. Kurizki, and D.-G. Welsch, Phys. Rev. A **61**, 032302 (2000).
[28] A. Kitagawa, M. Takeoka, M. Sasaki, and A. Chefles, Phys. Rev. A **73**, 042310 (2006).
[29] C. Navarrete-Benlloch, R. García-Patrón, J. H. Shapiro, and N. J. Cerf, Phys. Rev. A **86**, 012328 (2012).
[30] K. P. Seshadreesan, J. P. Dowling, and G. S. Agarwal, Phys. Scr. **90**, 074029 (2015).
[31] V. Bargmann, Commun. Pure Appl. Math. **14**, 187 (1961).
[32] J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968), Chap. 7.
[33] P. D. Drummond and C. W. Gardiner, J. Phys. A **13**, 2353 (1980).
[34] S.-Y. Zhu and N. Lu, Phys. Lett. A **137**, 191 (1989).
[35] R. Schack and A. Schenzle, Phys. Rev. A **44**, 682 (1991).
[36] A. Gilchrist, C. W. Gardiner, and P. D. Drummond, Phys. Rev. A **55**, 3014 (1997).
[37] H. -y. Fan and M. Xiao, Phys. Lett. A **244**, 256 (1998).
[38] M. Olsen and A. Bradley, Opt. Commun. **282**, 3924 (2009).
[39] D. W. Barry and P. D. Drummond, Phys. Rev. A **78**, 052108 (2008).
[40] V. V. Dodonov, O. V. Man'ko, and V. I. Man'ko, Phys. Rev. A **49**, 2993 (1994).
[41] *Quantum Detection and Estimation Theory*, edited by C. W. Helstrom, Mathematics in Science and Engineering Vol. 123 (Elsevier, New York, USA, 1976), Chap. 5, pp. 119–159.
[42] A. P. Lund, A. Laing, S. Rahimi-Keshari, T. Rudolph, J. L. O'Brien, and T. C. Ralph, Phys. Rev. Lett. **113**, 100502 (2014).
[43] C. S. Hamilton, R. Kruse, L. Sansoni, S. Barkhofen, C. Silberhorn, and I. Jex, Phys. Rev. Lett. **119**, 170501 (2017).
[44] N. Quesada, J. Chem. Phys. **150**, 164113 (2019).
[45] M. H. Michael, M. Silveri, R. T. Brierley, V. V. Albert, J. Salmilehto, L. Jiang, and S. M. Girvin, Phys. Rev. X **6**, 031006 (2016).
[46] L. Li, C.-L. Zou, V. V. Albert, S. Muralidharan, S. M. Girvin, and L. Jiang, Phys. Rev. Lett. **119**, 030502 (2017).
[47] V. V. Albert, S. O. Mundhada, A. Grimm, S. Touzard, M. H. Devoret, and L. Jiang, arXiv:1801.05897 (2018).
[48] We work in the *qqpp* representation;, i.e., the upper left (lower right) block of the CM concerns position (momentum), while the off-diagonal blocks hold information of correlations thereof.
[49] Since the CM V is symmetric, Γ and Γ^{-1} will be symmetric.
[50] A. Zvonkin, Math. Comput. Model. **26**, 281 (1997).
[51] J.-G. Luque and J.-Y. Thibon, Adv. Appl. Math. **29**, 620 (2002).
[52] M. Chen, N. C. Menicucci, and O. Pfister, Phys. Rev. Lett. **112**, 120505 (2014).
[53] J. Zhang, J. J. Wang, R. G. Yang, K. Liu, and J. R. Gao, Opt. Express **25**, 27172 (2017).
[54] This assumption can be dropped just by having a symmetric matrix G the elements of which are the different values of the squeezing parameter.
[55] N. C. Menicucci, S. T. Flammia, and P. van Loock, Phys. Rev. A **83**, 042335 (2011).
[56] The matrix Γ^{-1} can be easily found to be $\Gamma^{-1} = \mathcal{B} + \mathcal{B}^{\dagger}$, from which we calculate $\det \Gamma = \cosh^4 r$.
[57] T. C. Ralph, A. Gilchrist, G. J. Milburn, W. J. Munro, and S. Glancy, Phys. Rev. A **68**, 042319 (2003).
[58] J. S. Ivan, K. K. Sabapathy, and R. Simon, Phys. Rev. A **84**, 042311 (2011).
[59] F. Arzani, A. Ferraro, and V. Parigi, Phys. Rev. A **99**, 022342 (2019).
[60] M. Walschaers, S. Sarkar, V. Parigi, and N. Treps, Phys. Rev. Lett. **121**, 220501 (2018).
[61] K. K. Sabapathy, H. Qi, J. Izaac, and C. Weedbrook, arXiv:1809.04680 (2018).
[62] K. Azuma, K. Tamaki, and H.-K. Lo, Nat. Commun. **6**, 6787 (2015).
[63] M. Pant, H. Krovi, D. Englund, and S. Guha, Phys. Rev. A **95**, 012304 (2017).
[64] Q. Zhuang and Z. Zhang, arXiv:1901.09566 (2019).
[65] Xanadu Quantum Technologies (private communication).
[66] D. Su, C. R. Myers, and K. K. Sabapathy, arXiv:1902.02323 (2019).
[67] D. Su, C. R. Myers, and K. K. Sabapathy, arXiv:1902.02331 (2019).