

Anomalous frequency shifts in a one-dimensional trapped Bose gasM. Valiente¹ and V. Pastukhov²¹*Institute for Advanced Study, Tsinghua University, Beijing 100084, China*²*Department for Theoretical Physics, Ivan Franko National University of Lviv, 12 Drahomanov Street, Lviv-5, 79005, Ukraine*

(Received 21 February 2019; published 13 May 2019)

We consider a system of interacting bosons in one dimension at a two-body resonance. This system, which is weakly interacting, is known to give rise to effective three-particle interactions, whose dynamics is similar to that of a two-dimensional Bose gas with two-body interactions, and exhibits an identical scale anomaly. We consider the experimentally relevant scenario of a harmonically trapped system. We solve the three-body problem exactly and evaluate the shifts in the frequency of the lowest compressional mode with respect to the dipole mode, and find that the effect of the anomaly is to increase the mode's frequency. We also consider the weak-coupling regime of the trapped many-boson problem and find, within the local density approximation, that the frequency of the lowest compressional mode is also shifted upward in this limit. Moreover, the anomalous frequency shifts are enhanced by the higher particle number to values that should be observable experimentally.

DOI: [10.1103/PhysRevA.99.053607](https://doi.org/10.1103/PhysRevA.99.053607)**I. INTRODUCTION**

Systems of trapped ultracold atoms [1] provide a versatile platform for the observation of intriguing few- and many-body phenomena. Since the observation of the first Bose-Einstein condensates with alkali atoms over two decades ago [2,3], much progress has been made, and both bosonic and fermionic ultracold atomic gases can nowadays be manipulated and controlled with unprecedented accuracy. This includes the tunability of their effective interaction strengths via magnetic [4] and even orbital [5] Feshbach resonances, or external confinement [6,7] and their effective dimensional reduction to one and two dimensions [8–11]. Among the most interesting phenomena that have been observed using ultracold atoms are the observation [12–14] of Efimov states [15], whose low-energy, model-independent description requires the introduction of three-body forces [16,17], the recent observation of the effects of a quantum anomaly in a two-dimensional Fermi gas [18,19], the realization of an antiferromagnetic few-spin Heisenberg chain [20] in the strongly interacting limit [21], and the observation of quantum droplets stabilized by quantum fluctuations [22,23].

Among those systems addressable within the capabilities of current ultracold atomic experiments, reduced one-dimensional systems are especially appealing for a number of reasons. First, there exists a variety of models that are exactly solvable by means of the Bethe ansatz [24]. Among these, the Lieb-Liniger model [25], consisting of many bosons interacting via zero-range potentials, is of particular relevance to ultracold atoms, since it faithfully describes a variety of interaction and temperature regimes, as has been shown experimentally time and time again [10,11,26,27]. Second, quantum fluctuations are very relevant in one-dimensional systems and lead to the breakdown of Fermi liquid theory for fermions, which are in this case described by the Luttinger

liquid theory [28] at low energies. Moreover, the distinction between bosons and fermions is rather blurred in one dimension, with Luttinger liquid theory describing bosons as well, while in the hard-core [29] and even in certain soft-core [30] limits, fermions and bosons are related via duality transformations, i.e., Bose-Fermi mapping theorems.

When we consider bosons in one dimension, the Lieb-Liniger model is to be regarded as an effective field theory at low energies [31,32]. In the weakly repulsive case, it describes systems with (negative) scattering lengths that are much larger in magnitude than the interaction's effective range [33]. Effective range effects come into play as a next-to-leading-order effect in the ground state. This effect, however, identically vanishes if the scattering length diverges, rendering the many-body system noninteracting to all orders in the two-body effective interactions [34]. Typically, one assumes that the noninteracting picture is accurate even in the many-particle limit [6,35], which has been shown to be correct within experimental uncertainty in Ref. [26], which, however, focused entirely on the opposite, strongly interacting limit. Interestingly, even in the ground state, next-to-leading-order effects do contribute to the physics of the few- and many-boson problems, and these come in the form of three-body contact interactions [34,36–40]. When the effective three-body forces are attractive, multiparticle bound states may be formed [36,37,39–43], while for three-body repulsion [34,38] a quantum anomaly in the form of logarithmic corrections is found for three or more particles [34,41,42,44]. References [41,42] focused on the anomaly within this context. Remarkably, this is completely equivalent to the anomaly found with two-body interactions in two dimensions at low energies [18,19,45–48]. This can be understood by observing the kinematic equivalence—identical Schrödinger equations—between three particles in one dimension with three-body contact interactions and two particles in two dimensions with

two-body contact interactions. Such relations between systems across different dimensions also exist in other, related contexts, such as the equivalence between five bosons with four-body interactions in one dimension and three bosons with two-body interactions in three dimensions [49], implying a five-body Efimov effect in one dimension, or the kinematic equivalence between two fermions in a spin-triplet state (odd-wave channel) in one dimension and the s -wave two-body problem in three dimensions [32]. Similar arguments were used in Ref. [50], which described a generalized Efimov effect in one dimension.

With all the above theoretical works and the great recent interest in effective three-body interactions in one dimension, it only remains to be seen whether the effects of such forces can be observed experimentally. Since most experiments feature weak harmonic confinement along the effective dimension of the system, with the notable exception of box-like traps [51], measuring the excitation frequency shifts in the lowest compressional mode is arguably the most suitable way of probing the effects of the three-body interactions. Before that, however, it is necessary to see whether these shifts may be sizable even in the weak-coupling limit. This is the goal of the present work.

Here, we begin by reviewing the regularization and renormalization of the effective three-body interaction, both in the position and momentum representations, which we use to solve the trapped three-body problem analytically. With the exact solution at hand, we find the excitation frequency of the lowest compressional mode, which is shifted due to the quantum anomaly, and compare the exact results with the sum-rule approach of Ref. [35], simplified by the generalized virial theorem, which we also derive. We then study the weak-coupling limit of the trapped many-body problem, within the mean-field [52] and local density approximations [53], and extract the anomalous frequency shifts in this limit.

II. HAMILTONIAN OF THE SYSTEM

We consider one-dimensional non-relativistic identical bosons with mass m interacting via pairwise potentials. The system is trapped in a harmonic well with frequency ω . We assume that the two-body scattering length a_2 is negative and much larger in absolute value than all other length scales in the system. In this way, two particles at low energies become effectively noninteracting, since the Lieb-Liniger interaction strength $g = -2\hbar^2/ma_2$, associated with a two-body effective potential $g\delta(x_1 - x_2)$ [25], vanishes identically. Since two-body interactions are in reality not pointlike, residual interactions remain in the system at either higher energies or for more than two particles. For identical bosons, the lowest-order effects of the interaction when the boson-boson scattering length diverges are due to an effective three-body force [34]. Its bare (unrenormalized) form is given by

$$V_3^{\text{LO}}(x_1, x_2, x_3) = g_3 \delta(x_1 - x_2) \delta(x_2 - x_3). \quad (1)$$

The total Hamiltonian with the above interaction is therefore given by

$$H = H_0 + \sum_i V_{\text{trap}}(x_i) + \sum_{i < j < l} V_3^{\text{LO}}(x_i, x_j, x_l), \quad (2)$$

where

$$H_0 = -\frac{\hbar^2}{2m} \sum_{i=1}^N \partial_{x_i}^2, \quad (3)$$

$$V_{\text{trap}}(x_i) = \frac{1}{2} m \omega^2 x_i^2. \quad (4)$$

III. THREE-BODY PROBLEM

Here, we study the three-boson problem with effective three-body forces given by Eq. (1). We begin by reviewing the three-body problem in free space, whose solution via regularization-renormalization can be worked out analytically in both the momentum [34,39–42] and position [37,38] representations, with an emphasis on the repulsive side of the three-body interaction. We then solve the three-body problem in a harmonic trap analytically, taking advantage of the kinematic equivalence between three bosons in one dimension with three-body interactions and two bosons in two dimensions with two-body interactions [47], at low energies [41].

A. Renormalization of the bare interaction

The bare lowest-order interaction, V_3^{LO} , in Eq. (1) is too singular in one dimension and requires regularization-renormalization.

Since the trap introduces no new singularities, it is sufficient to renormalize the interacting theory in free space. We solve the three-body scattering problem first in momentum space by using the Lippmann-Schwinger equation for the transition matrix (T matrix) $T(z)$, which reads

$$T(z) = V_3^{\text{LO}} + V_3^{\text{LO}} G_0(z) T(z), \quad (5)$$

where $G_0(z) = (z - H_0)^{-1}$ is the noninteracting Green's function. Since the three-body interaction conserves total momentum $K = k_1 + k_2 + k_3$, and the system is Galilean relativistic, we rid ourselves of the center of mass and set its momentum $K = 0$. In this frame, the Lippmann-Schwinger equation (5) becomes two dimensional, and we find

$$\begin{aligned} \langle k'_1 k'_2 || T(z) || k_1 k_2 \rangle \\ = g_3 + \int \frac{dq_1 dq_2}{(2\pi)^2} \frac{g_3 \langle q_1 q_2 || T(z) || k_1 k_2 \rangle}{z - \frac{\hbar^2}{m} (q_1^2 + q_2^2 + q_1 q_2)}, \end{aligned} \quad (6)$$

where we have defined the reduced T matrix via

$$\langle k'_1 k'_2 k'_3 | T(z) | k_1 k_2 k_3 \rangle = 2\pi \delta(K - K') \langle k'_1 k'_2 || T(z) || k_1 k_2 \rangle, \quad (7)$$

and where we have set $K = k_1 + k_2 + k_3 = 0$. Obviously, we have chosen plane-wave normalization as $\langle k' | k \rangle = 2\pi \delta(k - k')$. Since the Fourier transform of the interaction (1) is a constant, we find that $\langle k'_1 k'_2 || T(z) || k_1 k_2 \rangle \equiv t_3(z)$ is a constant (i.e., independent of momentum states and only energy dependent), and Eq. (6) is readily solved as

$$t_3(z) = \frac{1}{g_3^{-1} - \mathcal{I}(z)}, \quad (8)$$

where we have defined

$$\mathcal{I}(z) = \int \frac{dq_1 dq_2}{(2\pi)^2} \frac{1}{z - \frac{\hbar^2}{m} (q_1^2 + q_2^2 + q_1 q_2)}. \quad (9)$$

Clearly, the integral $\mathcal{I}(z)$ in Eq. (9) is ultraviolet (UV) divergent. We must therefore regularize it in the UV in preparation for its renormalization. To do this, we first change variables in Eq. (9) to Jacobi coordinates. We define $q_x = (q_1 - q_2)/\sqrt{2}$ and $q_y = \sqrt{2/3}[q_3 - (q_1 + q_2)/2] = -\sqrt{3/2}(q_1 + q_2)$, where in the last equality we have used $q_3 = -q_1 - q_2$ ($K = 0$). We further transform the integral to polar coordinates via $q_x = q \cos \phi$ and $q_y = q \sin \phi$, and set a hard cutoff Λ in the hyperradial momentum integral, obtaining

$$\begin{aligned} \mathcal{I}(z) &= \frac{1}{\sqrt{3}} \int_0^\Lambda \frac{dq}{2\pi} \frac{q}{z - \frac{\hbar^2}{2m}q^2} \\ &= \frac{m}{2\pi\sqrt{3}\hbar^2} \left[\ln \left(\frac{k^2}{\Lambda^2} \right) - i\pi \right], \end{aligned} \quad (10)$$

where we have used $z = E + i0^+$ and where $k^2 = 2mE/\hbar^2$. Using Eqs. (8) and (10), we find

$$t_3(z) = \frac{1}{g_3^{-1} - \frac{m}{2\pi\sqrt{3}\hbar^2} \left[\ln \left(\frac{k^2}{\Lambda^2} \right) - i\pi \right]}. \quad (11)$$

Since we shall consider repulsive three-body interactions, we set a UV momentum scale Q_* such that the Landau pole of the T matrix occurs at energy $E = -\hbar^2 Q_*^2/2m$, which would correspond to the trimer's energy in the case of attractive interactions [36]. We note that the effective theory is valid at energies much lower than $\hbar^2 Q_*^2/2m$, and therefore Q_*^{-1} sets the length scale under which the theory ceases to be physically correct. The T matrix has a pole at the bound-state energy $-\hbar^2 Q_*^2/2m$, i.e., the denominator in Eq. (11) vanishes at that energy. This allows the bare coupling constant g_3 to be renormalized as

$$\frac{1}{g_3} = \frac{m}{\pi\sqrt{3}\hbar^2} \ln \left(\frac{Q_*}{\Lambda} \right), \quad (12)$$

which renormalizes the T matrix, rendering it finite and cutoff independent. It takes the form

$$t_3(z) = \frac{2\pi\sqrt{3}\frac{\hbar^2}{m}}{\ln \left(\frac{Q_*^2}{k^2} \right) + i\pi}. \quad (13)$$

As is clearly observed in Eq. (13) above, scale invariance at the classical level is broken by quantum renormalization effects, i.e., the T matrix exhibits a quantum anomaly [41].

We now investigate the problem in the position representation, which is necessary for the exact solution of the trapped three-body problem. Here, instead of regularizing the problem first, we may use the noninteracting Hamiltonian (3) supplemented with a short-range boundary condition. The simplest way to study this is to investigate the Landau pole or the trimer. First, we perform a change of variables to Jacobi coordinates. We eliminate the center-of-mass coordinate $X = (x_1 + x_2 + x_3)/3$, and set the total momentum to zero. Then we define, as in the momentum representation, $x = (x_1 - x_2)/\sqrt{2}$ and $y = \sqrt{2/3}[x_3 - (x_1 + x_2)/2]$, and go to polar coordinates via $x = r \cos \theta$ and $y = r \sin \theta$. For negative energies, the Schrödinger equation for the hyperradial

wave function $R(r)$ reads

$$R''(r) + \frac{1}{r}R'(r) - \frac{2m|E|}{\hbar^2}R(r) = 0. \quad (14)$$

The unnormalized singular solution at negative energies is simply given by

$$R(r) = K_0(2m|E|r/\hbar^2), \quad (15)$$

where K_0 is the zeroth-order modified Bessel function of the second kind [54]. We set now the location of the Landau pole to $E = -\hbar^2 Q_*^2/2m$, and using the short distance expansion of K_0 , we obtain the desired boundary condition

$$R(r) = -\ln \left(\frac{Q_* e^\gamma r}{2} \right) + O(\ln(Q_* r)(Q_* r)^2), \quad (16)$$

where γ is Euler's gamma constant. We also link the value of the UV scale to the three-body scattering length a_3 via [34,52]

$$(Q_* a_3)^2 = 8e^{-2\gamma}, \quad (17)$$

since we will use the latter, instead of Q_* , in the many-body problem.

B. Exact solution of the three-body problem in a harmonic trap

We now present the solution to the three-body problem in the presence of harmonic confinement. This is equivalent [55] to the problem of a single particle of mass m in two dimensions in a harmonic trap with frequency ω with a contact interaction, whose short-range boundary condition is given by Eq. (16), as we shall see shortly. In Jacobi coordinates (X, x, y) , the noninteracting Hamiltonian reads

$$H_0 = H_{\text{c.m.}} + H_r, \quad (18)$$

with

$$H_{\text{c.m.}} = -\frac{\hbar^2}{6m} \partial_x^2 + \frac{3}{2} m \omega^2 X^2, \quad (19)$$

$$H_r = -\frac{\hbar^2}{2m} (\partial_x^2 + \partial_y^2) + \frac{1}{2} m \omega^2 (x^2 + y^2). \quad (20)$$

For the sake of completeness, the bare three-body interaction is transformed as

$$V_3^{\text{LO}} = g_3 \delta(\sqrt{2}x) \delta(\sqrt{3/2}y) = \frac{g_3}{\sqrt{3}} \delta(x) \delta(y). \quad (21)$$

The center-of-mass motion is governed by Hamiltonian (19), which corresponds to a particle of mass $3m$ with frequency ω , and is readily solved. For the relative motion, together with the boundary condition (16), we go to polar coordinates and define

$$R(r) = \exp \left(-\frac{r^2}{2a_\parallel^2} \right) F(r), \quad (22)$$

where $a_\parallel = \sqrt{\hbar/m\omega}$ is the harmonic length. Defining $s = (r/a_\parallel)^2$ and $F(r) = u(s)$, the Schrödinger equation for angular momentum $m_z = 0$ reads

$$s u''(s) + (1-s) u'(s) + \frac{\mathcal{E} - 1}{2} u(s) = 0, \quad (23)$$

where $\mathcal{E} = E/\hbar\omega$ is the dimensionless form of the (relative) energy eigenvalues. The singular solution that leaves $R(r)$,

Eq. (22), normalizable is given by the following confluent hypergeometric function U [54]:

$$u(s) = U\left(\frac{1-\mathcal{E}}{2}, 1, s\right). \quad (24)$$

At short distances ($s \ll 1$), the confluent hypergeometric function behaves as

$$U(a, 1, s) \propto -\ln s - \psi(a) - 2\gamma + O(s \ln s), \quad (25)$$

where ψ is the digamma function. Comparing the boundary condition (16) with the short-range form of $u(s)$, Eqs. (24) and (25), we obtain the eigenvalue equation

$$\psi\left(\frac{1-\mathcal{E}}{2}\right) = \ln\left[\left(\frac{Q_* a_{\parallel}}{2}\right)^2\right]. \quad (26)$$

IV. TAN'S CONTACT AND THE VIRIAL THEOREM

We here derive the virial theorem for our system, which will be useful in the next section. This involves the so-called Tan's contact C_3 [41,42,52,56–58], relating the large-momentum tail of the momentum distribution, or the short-distance correlation functions, to a number of physical quantities. In our case, the virial theorem reads

$$E = 2\langle V_{\text{trap}} \rangle - \frac{\hbar^2}{4\pi m} \int dx C_3(x), \quad (27)$$

where E is the energy and $C_3(x)$ is the local three-body contact, given by

$$C_3(x) = \frac{1}{3\sqrt{3}} \left(\frac{mg_3}{\hbar^2}\right)^2 \langle [\phi^\dagger(x)]^3 [\phi(x)]^3 \rangle. \quad (28)$$

Above, ϕ^\dagger and ϕ are bosonic creation and annihilation operators in the position representation, while g_3 is the bare coupling constant given by Eq. (12).

To derive Eq. (27), we use a method analogous to Ref. [59], although this can be derived using Tan's original method particularized to two dimensions [60,61]. We begin by calling $\Psi(x_1, \dots, x_N)$ the normalized N -boson ground state of Hamiltonian (2) with energy E . We rescale the coordinates of the particles as $x_i \rightarrow \lambda x_i$ ($i = 1, \dots, N$) in Ψ , and evaluate how the expectation value of the energy, $E(\lambda)$, in the rescaled state Ψ_λ , changes due to this transformation. The rescaled state must be multiplied by a constant in order to remain normalized, as

$$\Psi_\lambda(x_1, \dots, x_N) = \lambda^{N/2} \Psi(\lambda x_1, \dots, \lambda x_N). \quad (29)$$

The rescaled wave function Ψ_λ does not satisfy the same short-distance boundary condition, Eq. (16), as the eigenstate $\Psi = \Psi_1$. In order to fix this, we must rescale the UV momentum scale Q_* as $Q_*^{-1} \rightarrow \lambda Q_*^{-1}$. The expectation value of the energy in this state reads

$$E(\lambda) = \lambda^2 \langle H_0 \rangle_1 + \lambda^{-2} \langle V_{\text{trap}} \rangle_1 + \lambda^2 \langle V_3^{\text{LO}} \rangle_\lambda, \quad (30)$$

where $\langle \cdot \rangle_\lambda$ stands for expectation value in state Ψ_λ . Using the fact that $E(\lambda)$ is minimized for $\lambda = 1$, corresponding to the true ground state of the system, we now differentiate $E(\lambda)$ in Eq. (30) with respect to λ and obtain

$$2\langle H_0 \rangle - 2\langle V_{\text{trap}} \rangle + 2\langle V_3^{\text{LO}} \rangle + \frac{\partial_\lambda g_3(\lambda \Lambda)|_{\lambda=1}}{3!} \int dx \langle [\phi^\dagger(x)]^3 [\phi(x)]^3 \rangle = 0, \quad (31)$$

where we have set once more $\langle \cdot \rangle = \langle \cdot \rangle_1$, and where we have used the relation between first- and second-quantized operators

$$\sum_{i < j < l} \langle \delta(x_i - x_j) \delta(x_i - x_l) \rangle = \frac{1}{3!} \int dx \langle [\phi^\dagger(x)]^3 [\phi(x)]^3 \rangle. \quad (32)$$

Substituting Eq. (12) into (31) and then setting $\lambda = 1$ in Eq. (30) [note that $E(1) = E$], we finally obtain the virial theorem, Eq. (27).

Since we shall also need the adiabatic relation, which was derived in Ref. [52], we simply quote it here, particularized to the trapped case

$$-\frac{\partial E}{\partial \ln Q_*} = \frac{\hbar^2}{2\pi m} \int dx C_3(x). \quad (33)$$

V. EXCITATIONS IN THE FEW-BODY LIMIT

With the exhaustive analysis of the three-body problem performed above, we are ready to calculate the excitation frequencies in the three-body sector. This will give us the order of magnitude of the shifts in frequencies due to the three-body interaction in the few-body limit, where the local density approximation (LDA) is not valid. This regime is experimentally relevant, because quasi-one-dimensional geometries consisting of arrays of tubes with few ($N \sim 8-11$) particles have been routinely prepared for at least a decade now [26].

In order to observe the effects of emergent three-particle interactions in one dimension discussed above, one of the best established methods is to measure excitation frequencies in a harmonic (or near-harmonic) trap [26]. For three particles, the lowest mode, in the noninteracting picture, that is affected by interactions corresponds to $E = 3\hbar\omega$, corresponding to an excitation of the relative motion. This is called the lowest compressional mode [35], with a noninteracting excitation frequency $\omega_C^{(0)} = 2\omega$. The dipole mode [35] is not affected by the interactions since it corresponds to an excitation of the center-of-mass motion, which is decoupled from the relative motion in a harmonic trap, and has energy $E = 2\hbar\omega$. Here, we shall explore the shifts in the ratio $R = \omega_C^2/\omega_D^2$ between the square frequency of the lowest compressional and the dipole modes due to the effective three-body interactions. The shift is defined as $\Delta = R - \left(\frac{\omega_C^{(0)}}{\omega_D}\right)^2 = R - 4$.

Before discussing the exact solution to Eq. (26), we study its weak-coupling limit. To this end, we use the Laurent expansion of the digamma function $\psi(a)$ near its poles $a = 0$ and $a = -1$ [54]. These read

$$\psi(a) = -\frac{1}{a} - \gamma + \frac{\pi^2}{6}a + O(a^2), \quad a \rightarrow 0, \quad (34)$$

$$\psi(a) = -\frac{1}{a+1} + 1 - \gamma + \left(1 + \frac{\pi^2}{6}\right)(a+1) + O((a+1)^2), \quad a \rightarrow -1. \quad (35)$$

Using $a = (1 - \mathcal{E})/2$ in the above relations, together with the eigenvalue equation (26), we find for the ground and second (first for bosons) excited states

$$\mathcal{E}_0 = 1 + g_\omega - \frac{\gamma}{2}g_\omega^2 + O(g_\omega^3), \quad (36)$$

$$\mathcal{E}_2 = 3 + g_\omega + \frac{1 - \gamma}{2}g_\omega^2 + O(g_\omega^3). \quad (37)$$

Above, we have defined the dimensionless coupling constant g_ω as

$$g_\omega = \frac{1}{\ln\left(\frac{Q_* a_\parallel}{2}\right)}. \quad (38)$$

We now see that, to lowest order in the interaction coupling constant (38), the excitation frequency of the lowest compressional mode is given by

$$\frac{\omega_C}{\omega} = \mathcal{E}_2 - \mathcal{E}_0 \approx 2 + \frac{1}{2}g_\omega^2. \quad (39)$$

As for the dipole mode, which is not affected by interactions, we simply have $\omega_D = \omega$. Therefore, the weak-coupling limit of the shift Δ in the ratio R , with respect to the noninteracting case, is given by

$$\Delta = 2g_\omega^2 + O(g_\omega^3). \quad (40)$$

We also use another route to obtain Eq. (40) by combining sum rules with the theory of Tan's contact (see Sec. IV). An upper bound to the ratio R is obtained via the expression (see Ref. [35])

$$\omega_C^2 = -2 \frac{\langle x^2 \rangle}{\frac{d\langle x^2 \rangle}{d\omega^2}}. \quad (41)$$

Using the Hellmann-Feynman theorem, we can relate the expectation value of x^2 in the ground state with the derivative of the ground-state energy E_0 (with respect to the center-of-mass zero-point energy) with respect to ω^2 , as

$$\frac{dE_0}{d\omega^2} = \frac{3}{2}m\langle x^2 \rangle. \quad (42)$$

The virial theorem, Eq. (27), combined with the adiabatic relation [52], Eq. (33), gives us another relation between the energy and $\langle x^2 \rangle$, as

$$\langle x^2 \rangle = \frac{1}{3m\omega^2} \left[E_0 - \frac{1}{2} \frac{dE_0}{d(\ln Q_*)} \right]. \quad (43)$$

Simple algebraic manipulations yield

$$R = 4 \frac{E_0(g_\omega) + \frac{1}{2}g_\omega^2 E_0'(g_\omega)}{E_0 - \frac{1}{2}g_\omega^3 E_0'(g_\omega) - \frac{1}{4}g_\omega^4 E_0''(g_\omega)}. \quad (44)$$

Using it to lowest order in g_ω , we find that

$$\Delta \approx 2 \frac{g_\omega^2}{E_0} \frac{dE_0}{dg_\omega} \Big|_{g_\omega=0}. \quad (45)$$

Inserting Eq. (36) into Eq. (45), we find

$$\Delta = 2g_\omega^2 + O(g_\omega^3), \quad (46)$$

in perfect agreement with the result obtained using the excitation energy directly, Eq. (40).

In Fig. 1, we show the value of the ratio $R = (\omega_C/\omega)^2$ as a function of the coupling constant g_ω , Eq. (38), calculated

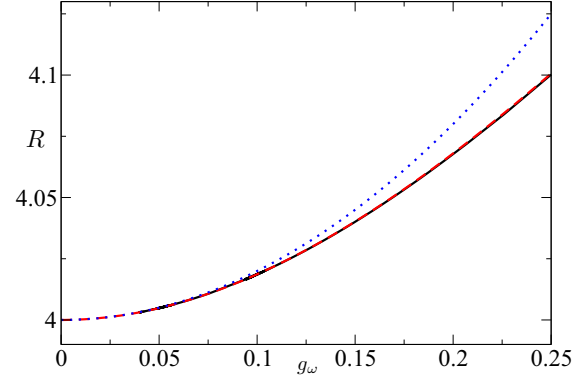


FIG. 1. Ratio $R = (\omega_C/\omega)^2$ between the frequencies of the lowest compressional mode and the dipole mode in the three-body problem. Black solid, red dashed, and blue dotted lines correspond, respectively, to R calculated via the excitation spectrum, Eq. (26), the sum-rule approach, Eq. (44), and the weak-coupling limit, Eq. (40).

using the exact excitation spectrum from the solutions of Eq. (26), the sum-rule approach, Eq. (44), and the weak-coupling limit, Eqs. (40) and (45). We see that in the very weak coupling regime, the shift $\Delta = R - 4$ is approximately quadratic in the coupling constant. Moreover, the agreement between the results using the excitation spectrum and the sum rules is excellent. We clearly see that the frequency ω_C is shifted upward with respect to its noninteracting value $\omega_C^{(0)} = 2\omega$. This is to be contrasted with the frequency shift due to repulsive Lieb-Liniger interactions which, for the few-body problem at weak coupling, is downward [55]. Therefore, the observation of a positive energy shift in this limit would be a clear signature of the role of effective three-body forces in 1D Bose systems.

VI. MANY-BODY PROBLEM

We now study the many-particle limit. Since the effective three-body repulsive interactions appear to be weak in all the physical settings studied so far [34,38], we can safely assume that mean-field theory will be at least qualitatively valid. In Ref. [52], one of us derived the mean-field, Bogoliubov, and beyond mean-field corrections to the ground-state energy of the Bose gas with repulsive three-body interactions in the thermodynamic limit. The ground-state energy per particle E/N takes the form

$$\frac{E}{N} = - \frac{\pi \hbar^2 n^2}{2\sqrt{3}m \ln(na_3)} \left[1 - \frac{4 \times 3^{1/4}}{\sqrt{-\pi \ln(na_3)}} + \frac{\frac{\ln[-\ln(na_3)]}{2} - C_E}{\ln(na_3)} \right] + O(n^2 \ln^{-2}(na_3)), \quad (47)$$

where $n = N/L$ is the density of the system, and C_E is a numerical constant whose value [52] is inconsequential for our purposes. Below, we employ the local density approximation (LDA) to study the large- N limit, in the mean-field approximation of Eq. (47), when the system is placed in a harmonic trap, and calculate the frequency of the lowest compressional mode to this order.

A. Local density approximation

Here we deal with the trapped many-body problem within the LDA. We present a fully analytical treatment in the weak-coupling limit, which introduces the quantum anomaly self-consistently. To lowest order in the interaction, the energy per particle E/N of the homogeneous many-body problem at zero temperature is given by [see Eq. (47)]

$$\frac{E}{N} = -\frac{\pi\hbar^2}{2\sqrt{3}m}n^2\left[\frac{1}{\ln(na_3)} + O(\ln^{-3/2}(na_3))\right]. \quad (48)$$

The LDA consists of setting the chemical potential $\mu = \mu_\ell + V_{\text{trap}}$ [53], where μ_ℓ is the local chemical potential, obtained by making the density explicitly dependent on position, i.e., by replacing $n \rightarrow n(x)$. We have, to lowest order

$$\mu = -\frac{\sqrt{3}\pi\hbar^2}{2m}\frac{[n(x)]^2}{\ln[n(x)a_3]} + \frac{1}{2}m\omega^2x^2. \quad (49)$$

In the weak-coupling limit, the logarithmic term above varies slowly. We can therefore rewrite the exact density $n(x)$ as a mean-field part plus a classical fluctuation

$$n(x) = n_0 + \frac{\partial_x\phi(x)}{\pi}, \quad (50)$$

with n_0 the average density of the system and ϕ the fluctuating field, well known from bosonization [62], and insert it into the dimensionless coupling constant $-\ln^{-1}[n(x)a_3]$ to obtain

$$-\frac{1}{\ln[n(x)a_3]} \approx -\frac{1}{\ln(n_0a_3)}\left[1 - \frac{\partial_x\phi}{\pi n_0 \ln(n_0a_3)}\right]. \quad (51)$$

The second term in the right-hand side of Eq. (51) is of higher order than Eq. (49) and can be dropped in this approximation. Therefore, to lowest order in the three-body interaction, Eq. (49) is simplified to

$$\mu = -\frac{\sqrt{3}\pi\hbar^2}{2m}\frac{[n(x)]^2}{\ln(n_0a_3)} + \frac{1}{2}m\omega^2x^2, \quad (52)$$

with the average density n_0 to be obtained self-consistently via

$$n_0 = \langle n(x) \rangle. \quad (53)$$

For further convenience, we define the interaction coupling constant $G(\omega)$ as

$$G(\omega) = -\frac{\sqrt{3}\pi}{2}\frac{1}{\ln(n_0a_3)}, \quad (54)$$

where its dependence on the trap's frequency ω has been made explicit because it is relevant for the calculation of the frequency of the lowest compressional mode. As usual, the chemical potential can be eliminated from Eq. (52) by setting the density to zero beyond a certain distance Γ (the Thomas-Fermi radius) and we have $\mu = m\omega^2\Gamma^2/2$. Finally, from Eq. (52) we have the density profile

$$n(x) = \frac{1}{\sqrt{2G(\omega)}a_\parallel^2}\sqrt{1 - \left(\frac{x}{\Gamma}\right)^2}. \quad (55)$$

Above, we see that the density profile coincides with that of a scale invariant system, such as the free Fermi gas or the Tonks-Girardeau gas. However, the important difference

is the dependence of the coupling constant $G = G(\omega)$ on the frequency of the trap via the self-consistency condition (53). This fact will introduce an anomalous deviation of the excitation frequency with respect to the scale invariant case (which has $R = 4$). We use now the particle number normalization condition

$$N = \int_{-\Gamma}^{\Gamma} dx n(x), \quad (56)$$

to eliminate the Thomas-Fermi radius Γ (or, equivalently, the chemical potential μ) in favor of N . We have

$$\Gamma^2 = \frac{2\sqrt{2G(\omega)}}{\pi}a_\parallel^2N. \quad (57)$$

With this, it is simple to extract the squared-mean radius of the system

$$\langle x^2 \rangle = \frac{1}{N} \int_{-\Gamma}^{\Gamma} dx x^2 n(x) = \frac{\sqrt{G(\omega)}N}{\sqrt{2\pi}}a_\parallel^2. \quad (58)$$

and the mean density n_0 from Eq. (53),

$$n_0 = \frac{2^{13/4}}{3\pi^{3/2}}\frac{N^{1/2}}{[G(\omega)]^{1/4}}a_\parallel^{-1}. \quad (59)$$

Inserting Eq. (54) into Eq. (59) above, and multiplying both resulting sides by a_3 , gives us the following equation for the dimensionless parameter n_0a_3

$$n_0a_3 = B\left(\frac{a_3}{a_\parallel}\right)[- \ln(n_0a_3)]^{1/4}, \quad (60)$$

with

$$B = \frac{2^{7/2}N^{1/2}}{3^{9/8}\pi^{7/4}}. \quad (61)$$

Equation (60) is solved by

$$(n_0a_3)^4 = \frac{B^4}{4}\left(\frac{a_3}{a_\parallel}\right)^4 W\left(\frac{4a_\parallel^4}{B^4a_3^4}\right), \quad (62)$$

where W is Lambert's function [63]. Inserting Eq. (62) into Eq. (54), we obtain the following for the coupling constant:

$$G(\omega) = \frac{2\pi\sqrt{3}}{W\left(\frac{4a_\parallel^4}{B^4a_3^4}\right)}. \quad (63)$$

The ratio $R = (\omega_C/\omega_D)^2$ is readily obtained using Eq. (41), and we have

$$R = 4\left[1 - \frac{G(\omega)}{2\pi\sqrt{3} + G(\omega)}\right]^{-1}, \quad (64)$$

which, to lowest order in $G(\omega)$, takes the form

$$R = 4 + \frac{2}{\pi\sqrt{3}}G(\omega) + O(G(\omega)^2). \quad (65)$$

The above relation shows that for repulsive three-body interactions, the role of the anomaly is to shift the frequency of the lowest compressional mode upward not only in the few-body limit, but also in the mean-field, many-body limit. Moreover, the shift $\Delta = R - 4$ is, in this case, of linear order in the coupling constant. This is to be contrasted with the few-body limit, where we showed that the shift, though positive as

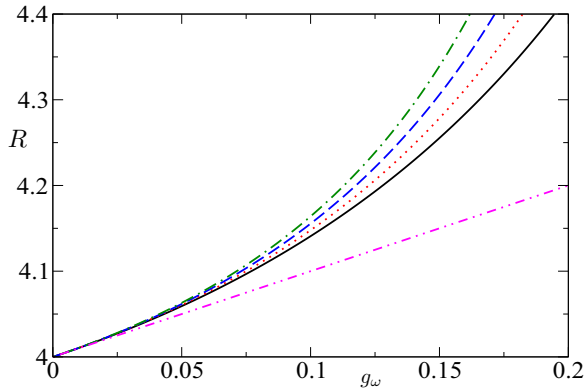


FIG. 2. Ratio $R = (\omega_c/\omega_D)^2$ between the frequencies of the lowest compressional mode and the dipole mode in the mean-field approximation, as a function of the few-body coupling constant g_ω . Black solid, red dotted, blue dashed, and green dashed-dotted lines correspond, respectively, to particle numbers $N = 250, 500, 1000$, and 2000 . The magenta dashed-double dotted line is the extreme weak-coupling result $R \approx 4 + g_\omega$.

well, is quadratic in the few-body coupling constant g_ω . In the very weak coupling limit, Eq. (65) becomes independent of the particle number and reads simply $R \approx 4 + g_\omega$. In Fig. 2, we plot the ratio R as a function of the few-body coupling constant, Eq. (38), for different particle numbers. As seen there, for very small values of g_ω , all curves collapse and they exhibit their N dependence as the coupling becomes stronger. From Fig. 2, it is also clear that larger particle numbers enhance the anomalous frequency shift. For instance, comparing Fig. 1 with Fig. 2, we observe that in order to achieve a shift $\Delta \approx 0.1$ in the three-body problem, we need $g_\omega \approx 0.25$, while with 250–2000 particles $g_\omega \approx 0.075$ is sufficient. With the experimental values reported for the trapping frequencies in Ref. [26], together with the theoretical value of a_3 in the

example of Ref. [34], we may expect a shift of around 3–4% in the value of R from the scale invariant limit $R = 4$. Therefore, if the experimental uncertainties for R are reduced in the weak-coupling limit and, especially, if the atom numbers can be increased from the few-body limit with $N \sim 10$ [26], the anomalous shifts should be clearly observable.

VII. CONCLUSIONS

We have considered one-dimensional bosons with very large two-body scattering lengths trapped in a harmonic well. We have studied the few-body limit by solving the three-body problem exactly, and obtained the shifts, which are always positive, in the frequency of the lowest compressional mode due to the emergent three-body forces among the bosons. We have shown that the sum-rule approach to obtaining the excitation frequencies is in excellent agreement with the exact results. We have also studied the weak-coupling regime of the many-body problem in a harmonic trap, and included the effect of the anomaly via a self-consistent version of the local density approximation. We have shown that the frequency shifts are also upward and largely enhanced by higher particle numbers.

The observation of the effects discussed here is within experimental reach, and has been for at least a decade now. Since for infinite scattering length, however, it was assumed that the system was simply noninteracting, before the development of and interest in the theory of emergent three-body forces in one dimension [34,36–43], the focus of experimental research has never been around this limit [26,27] where the three-body force is dominant. An experiment focusing around the limit of infinite scattering length in one dimension should be able to observe the effects of the three-body anomaly: any statistically significant upward frequency shift around the limit of infinite scattering length will be due to the anomaly. These effects could be observed either with single-component bosons [34] or with two-component bosons [38] as trimer interactions.

-
- [1] I. Bloch, J. Dalibard, and W. Zwerger, *Rev. Mod. Phys.* **80**, 885 (2008).
 - [2] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, *Science* **269**, 198 (1995).
 - [3] K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, *Phys. Rev. Lett.* **75**, 3969 (1995).
 - [4] C. Chin, R. Grimm, P. Julienne, and E. Tiesinga, *Rev. Mod. Phys.* **82**, 1225 (2010).
 - [5] R. Zhang, Y. Cheng, H. Zhai, and P. Zhang, *Phys. Rev. Lett.* **115**, 135301 (2015).
 - [6] M. Olshanii, *Phys. Rev. Lett.* **81**, 938 (1998).
 - [7] E. Haller, M. J. Mark, R. Hart, J. G. Danzl, L. Reichsöllner, V. Melezhik, P. Schmelcher, and H.-C. Nägerl, *Phys. Rev. Lett.* **104**, 153203 (2010).
 - [8] T. Kinoshita, T. Wenger, and D. S. Weiss, *Science* **305**, 1125 (2004).
 - [9] M. Greiner, I. Bloch, O. Mandel, T. W. Hänsch, and T. Esslinger, *Phys. Rev. Lett.* **87**, 160405 (2001).
 - [10] E. Haller *et al.*, *Nature* **466**, 597 (2010).
 - [11] P. Krüger, S. Hofferberth, I. E. Mazets, I. Lesanovsky, and J. Schmiedmayer, *Phys. Rev. Lett.* **105**, 265302 (2010).
 - [12] T. Kraemer *et al.*, *Nature* **440**, 315 (2006).
 - [13] S. Knoop *et al.*, *Nat. Phys.* **5**, 227 (2009).
 - [14] M. Zaccanti *et al.*, *Nat. Phys.* **5**, 586 (2009).
 - [15] V. Efimov, *Phys. Lett. B* **33**, 563 (1970).
 - [16] P. F. Bedaque, H.-W. Hammer, and U. van Kolck, *Phys. Rev. Lett.* **82**, 463 (1999).
 - [17] P. F. Bedaque, H.-W. Hammer, and U. van Kolck, *Nucl. Phys. A* **646**, 444 (1999).
 - [18] M. Holten, L. Bayha, A. C. Klein, P. A. Murthy, P. M. Preiss, and S. Jochim, *Phys. Rev. Lett.* **121**, 120401 (2018).
 - [19] T. Peppler, P. Dyke, M. Zamorano, I. Herrera, S. Hoinka, and C. J. Vale, *Phys. Rev. Lett.* **121**, 120402 (2018).

- [20] S. Murmann, F. Deuretzbacher, G. Zürn, J. Bjerlin, S. M. Reimann, L. Santos, T. Lompe, and S. Jochim, *Phys. Rev. Lett.* **115**, 215301 (2015).
- [21] A. G. Volosniev, D. V. Fedorov, A. S. Jensen, M. Valiente, and N. T. Zinner, *Nat. Comms.* **5**, 5300 (2014).
- [22] C. R. Cabrera, L. Tanzi, J. Sanz, B. Naylor, P. Thomas, and L. Tarruell, *Science* **359**, 301 (2018).
- [23] G. Semeghini *et al.*, *Phys. Rev. Lett.* **120**, 235301 (2018).
- [24] M. A. Cazalilla, R. Citro, T. Giamarchi, E. Orignac, and M. Rigol, *Rev. Mod. Phys.* **83**, 1405 (2011).
- [25] E. H. Lieb and W. Liniger, *Phys. Rev.* **130**, 1605 (1963).
- [26] E. Haller *et al.*, *Science* **325**, 1224 (2009).
- [27] B. Yang, Y.-Y. Chen, Y.-G. Zheng, H. Sun, H.-N. Dai, X.-W. Guan, Z.-S. Yuan, and J.-W. Pan, *Phys. Rev. Lett.* **119**, 165701 (2017).
- [28] F. D. M. Haldane, *J. Phys. C: Solid State Phys.* **14**, 2585 (1981).
- [29] M. Girardeau, *J. Math. Phys.* **1**, 516 (1960).
- [30] T. Cheon and T. Shigehara, *Phys. Rev. Lett.* **82**, 2536 (1999).
- [31] E. Epelbaum, H.-W. Hammer, and Ulf-G. Meißner, *Rev. Mod. Phys.* **81**, 1773 (2009).
- [32] M. Valiente and N. T. Zinner, *Few-body Syst.* **56**, 845 (2015).
- [33] V. E. Barlette, M. M. Leite, and S. Adhikari, *Eur. J. Phys.* **21**, 435 (2000).
- [34] M. Valiente, [arXiv:1902.01643v1](https://arxiv.org/abs/1902.01643v1).
- [35] C. Menotti and S. Stringari, *Phys. Rev. A* **66**, 043610 (2002).
- [36] L. Pricoupenko, *Phys. Rev. A* **97**, 061604(R) (2018).
- [37] L. Pricoupenko, *Phys. Rev. A* **99**, 012711 (2019).
- [38] G. Guijarro, A. Pricoupenko, G. E. Astrakharchik, J. Boronat, and D. S. Petrov, *Phys. Rev. A* **97**, 061605(R) (2018).
- [39] Y. Sekino and Y. Nishida, *Phys. Rev. A* **97**, 011602(R) (2018).
- [40] Y. Nishida, *Phys. Rev. A* **97**, 061603(R) (2018).
- [41] J. E. Drut, J. R. McKenney, W. S. Daza, C. L. Lin, and C. R. Ordóñez, *Phys. Rev. Lett.* **120**, 243002 (2018).
- [42] W. S. Daza, J. E. Drut, C. L. Lin, and C. R. Ordóñez, [arXiv:1808.07011v1](https://arxiv.org/abs/1808.07011v1).
- [43] J. R. McKenney and J. E. Drut, *Phys. Rev. A* **99**, 013615 (2019).
- [44] A. Pricoupenko and D. S. Petrov, *Phys. Rev. A* **97**, 063616 (2018).
- [45] E. Taylor and M. Randeria, *Phys. Rev. Lett.* **109**, 135301 (2012).
- [46] J. Hofmann, *Phys. Rev. Lett.* **108**, 185303 (2012).
- [47] S. R. Beane, *Phys. Rev. A* **82**, 063610 (2010).
- [48] H. Hu, B. C. Mulkerin, U. Toniolo, L. He, and X.-J. Liu, *Phys. Rev. Lett.* **122**, 070401 (2019).
- [49] Y. Nishida and D. T. Son, *Phys. Rev. A* **82**, 043606 (2010).
- [50] S. Moroz, J. P. D’Incao, and D. S. Petrov, *Phys. Rev. Lett.* **115**, 180406 (2015).
- [51] C. Eigen, J. A. P. Glidden, R. Lopes, N. Navon, Z. Hadzibabic, and R. P. Smith, *Phys. Rev. Lett.* **119**, 250404 (2017).
- [52] V. Pastukhov, *Phys. Lett. A* **383**, 894 (2019).
- [53] F. Dalfovo, Stefano Giorgini, L. P. Pitaevskii, and S. Stringari, *Rev. Mod. Phys.* **71**, 463 (1999).
- [54] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Dover Publications, New York, 1972).
- [55] T. Busch, B.-G. Englert, K. Rzazewski, and M. Wilkens, *Found. Phys.* **28**, 549 (1998).
- [56] S. Tan, *Ann. Phys.* **323**, 2952 (2008).
- [57] S. Tan, *Ann. Phys.* **323**, 2971 (2008).
- [58] S. Tan, *Ann. Phys.* **323**, 2987 (2008).
- [59] F. Werner, *Phys. Rev. A* **78**, 025601 (2008).
- [60] M. Valiente, N. T. Zinner, and K. Mølmer, *Phys. Rev. A* **84**, 063626 (2011).
- [61] M. Valiente, N. T. Zinner, and K. Mølmer, *Phys. Rev. A* **86**, 043616 (2012).
- [62] Z. Ristivojevic and K. A. Matveev, *Phys. Rev. B* **89**, 180507(R) (2014).
- [63] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knut, *Adv. Comput. Math.* **5**, 329 (1996).