Phase transition in a noisy Kitaev toric code model

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Many aspects of the well-known mapping between the partition function of a classical spin model and the quantum entangled state have been studied in recent years. However, the consequences of the existence of a classical (critical) phase transition on the corresponding quantum state have been mostly ignored. In this paper, we consider this problem for an important example of the Kitaev toric code model which has been shown to correspond to the two-dimensional (2D) Ising model though a duality transformation. We show that the temperature on the classical side is mapped to bit-flip noise on the quantum side. It is then shown that a transition from a coherent superposition of a given quantum state to a noncoherent mixture corresponds exactly to paramagnetic-ferromagnetic phase transition in the Ising model. To identify such a transition further, we define an order parameter to characterize the decoherence of such a mixture and show that it behaves similar to the order parameter (magnetization) of the 2D Ising model, a behavior that is interpreted as a *robust coherence* in the toric code model. Furthermore, we consider other properties of the noisy toric code model exactly at the critical point. We show that there is a relative stability to noise for the toric code state at the critical noise which is revealed by a relative reduction in susceptibility to noise.

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I. INTRODUCTION

Among the well-known connections from statistical mechanics to quantum information theory [1–15], a fascinating correspondence between partition functions of classical spin models and quantum entangled states has attracted much attention [16–18]. In 2007, it was shown that the partition function of a classical spin model can be written as an inner product of a product state and an entangled state [16]. Such mappings led to a cross-fertilization between quantum information theory and statistical mechanics [19,20]. Specifically, it has been shown that measurement-based quantum computation on quantum entangled states [21,22] is related to computational complexity of classical spin models [23,24]. In this way, a concept of the completeness was defined where the partition function of a classical spin model generates the partition function of all classical models [25–30] (see also [31,32] for recent developments in this direction). Most such studies were based on a specific mapping between classical-quantum models. However, we have recently introduced a canonical relation as a duality mapping where any given Calderbank-Shor-Steane (CSS) quantum state can be mapped, via hypergraph representations, to an arbitrary classical spin model [33].

On the other hand, the problem of phase transition in classical spin models has attracted much attention in the past and is therefore a well-studied phenomenon [34]. Simply, in the high-temperature phase such models exhibit no net magnetization due to the symmetric behavior of dynamical variables. Upon decreasing the temperature, this symmetry is spontaneously broken at a critical temperature $T_{\rm cr}$, and a nonzero magnetization appears in the system. The behaviors of such systems at the critical point are characterized by nonanalytic properties of the leading thermodynamics functions such as magnetic susceptibility. Such nonanalytic behavior is characterized by a set of critical exponents which fully identify the symmetry-breaking property (or universality class) of the particular phase transition [35].

Now, since there is a correspondence between such classical spin models and entangled quantum states, one would have to wonder what the consequences of such phase transitions are on the quantum states. It is our intention to take a step in this direction by considering the well-known ferromagnetic phase transition in a 2D Ising model and its consequences on the Kitaev toric code (TC) [36], which we have previously shown to be related via a duality mapping [33]. The TC state is of particular interest since it has a topological order [37,38] with a robust nature [39–42] as well as an important application in quantum error correction [43,44]. On the other hand the 2D Ising model is a well-known model in standard statistical mechanics which allows an exact solution. Therefore, one can hope that exploration of such a mapping between the partition function of the Ising model and the TC code can open an avenue for many possible studies related to topological properties of the TC state.

Subsequently, we consider the TC in the presence of an independent bit-flip noise where the Pauli operators X are applied to each qubit with probability p. We consider the effect of the bit-flip noise in a coherent superposition of two specific quantum states in the TC. Then we define an order parameter that can characterize decoherence of the above quantum state. Interestingly, we show that such an order parameter is mapped to the magnetization of the Ising model. Therefore, we conclude that there is a phase transition from a

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coherent superposition to a noncoherent mixture of the above quantum state at a critical probability of $p_{\rm cr}$ corresponding to critical temperature of the 2D Ising model. We interpret such a behavior as a robust coherence in the TC model. On the other hand, it is well known that criticality can be marked by interesting behavior at the transition point. Therefore, we define a quantity as susceptibility to noise in the noisy TC model and we show that, at the critical noise of $p_{\rm cr}$, the susceptibility shows a relative reduction which is indicative of critical stability which has been pointed out before [33].

Finally, we note that the TC under a bit-flip noise has already been studied in the context of error threshold [6]. In fact, it is proved that the degeneracy of the ground state of the TC can be protected against bit-flip noise by an active error correcting protocol if the rate of noise is below a threshold. However, our problem here is different as we consider the effect of the bit-flip noise in a coherent superposition of two specific quantum states in the TC instead of the ground state considered in [6]. More importantly, we do not consider any error correcting protocol as we are only interested in studying the natural robustness of the coherence of the above quantum state against the bit-flip noise. Accordingly, while the problem of error threshold is mapped to an Ising model with random couplings [6], our problem is mapped to a ferromagnetic Ising model with different physics.

This paper is structured as follows. In Sec. II, we give an introduction to the TC model including its ground state and excitations. In Sec. III, we first review the duality mapping from the partition function of the 2D Ising model to the TC state and specifically show how such a problem is related to a TC state under a bit-flip noise. Then, we introduce a similar mapping for the magnetization of the Ising model. In Sec. IV, we provide our main result where we introduce a decoherence process for a coherent superposition of two quantum states in the TC and we find a singular phase transition to the noncoherent phase which is mapped to the ferromagnetic phase transition in the 2D Ising model. In Sec. V, we introduce a susceptibility to noise which reveals a relative (critical) stability of the toric code state against bit-flip noise at the transition point.

II. REVIEW OF THE KITAEV TC MODEL

TC is the first well-known topological quantum code which was introduced by Kitaev in 2003 [36]. Since we will consider behavior of this model under noise, here we give a brief review on the TC model which is specifically defined on a 2D square lattice with periodic boundary condition (i.e., on a torus). To this end, consider a $L \times L$ square lattice where qubits live on edges of the lattice. Corresponding to each vertex and face of the lattice, two stabilizer operators are defined in the following form:

$$B_f = \prod_{i\in\partial f} X_i \ , \ A_v = \prod_{i\in v} Z_i \tag{1}$$

where $i \in \partial f$ refers to all qubits living on edges of the face f and $i \in v$ refers to qubits living on edges incoming to the vertex v [see Fig. 1(a)]. The above operators are in fact generators of the stabilizer group of the TC where each product of them is also a stabilizer. For example, if we represent each

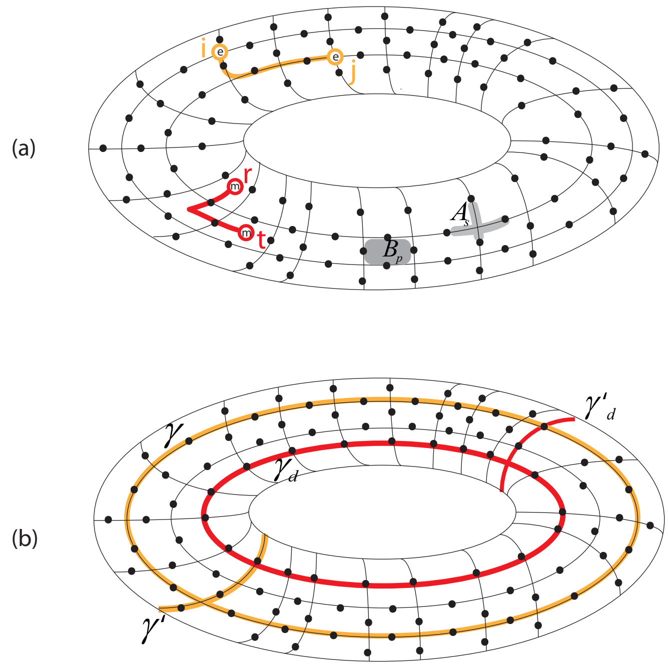

FIG. 1. (a) A 2D square lattice on a torus where qubits live on edges of the lattice. Each vertex (face) operator A_v (B_f) is defined corresponding to each vertex (face) of the lattice. (b) There are two different directions on the lattice where two nontrivial loops can be defined. The two nontrivial loops on the edges of the lattice are denoted by γ and γ' while on the dual lattice they are denoted by γ_d and γ'_d.

face operator of B_f as a loop around the boundary of the corresponding face, each product of them will also have a loop representation. In this way, corresponding to each kind of loop in the lattice, there will be an X-type stabilizer.

On a torus topology, there are two relations between these operators in the form of $\prod_f B_f = I$ and $\prod_v A_v = I$ where I refers to the identity operator. In this way, the number of independent stabilizers is equal to $2L - 2$. By the fact that $[A_v, B_f] = 0$, it is simple to show that the following state is an eigenstate of all face and vertex operators with eigenvalue $+1$:

$$\begin{aligned} |K\rangle &= \frac{1}{\sqrt{2^{(|f|-1)}}} \prod_f (I + B_f)|0\rangle^{\otimes 2L} \\ &= \frac{1}{\sqrt{2^{(|v|-1)}}} \prod_v (I + A_v)|+\rangle^{\otimes 2L} \end{aligned} \tag{2}$$

where $\prod_f$ refers to the product of all independent face operators and $\prod_v$ refers to the product of all independent vertex operators. $|f|$ and $|v|$ refer to the number of faces and vertices, respectively. $|0\rangle$ and $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ are positive eigenstates of Pauli operators Z and X, respectively. The stabilizer space of the toric code is fourfold degenerate and thus there are three other stabilizer states which are generated by nonlocal operators. In fact, one can consider two nontrivial loops around the torus in two different directions [see Fig. 1(b)]. Then two operators corresponding to nontrivial loops γ and γ' are defined in the following form:

$$\Gamma_x = \prod_{i\in\gamma} X_i, \quad \Gamma'_x = \prod_{i\in\gamma'} X_i \tag{3}$$

where $i \in \gamma$ and $i \in \gamma'$ refers to all qubits living in these loops. In this way, the following four quantum states will be the bases of stabilizer space:

$$|\psi_{\mu,\nu}\rangle = (\Gamma_x)^{\mu}(\Gamma'_x)^{\nu}|K\rangle, \tag{4}$$

where $\mu, \nu = 0, 1$ refer to exponents of nontrivial loop operators. In addition to the above nonlocal operators, there are also two other nonlocal operators constructed by Z operators. Such operators correspond to two loops γ_d and γ'_d around the torus on a dual lattice in the form of $\Gamma_z = \prod_{i\in\gamma_d} Z_i$ and $\Gamma'_z = \prod_{i\in\gamma'_d} Z_i$ [see Fig. 1(b)]. One can check that these operators can characterize four different bases of stabilizer space where expectation values of these operators are different for the bases (4).

Another important property of the TC state is related to excitations of the model. To this end, consider two vertices of the lattice denoted by i, j where a string, denoted by S_{ij}, can connect these two vertices [see Fig. 1(a)]. Then we apply the Pauli operators X on all qubits belonging to the string S_{ij} where we denote the corresponding string operator by S^x_{ij}. It is clear that such an operator commutes with all vertex operators A_v instead of A_i and A_j, which are the two end points of the string S_{ij}. Such an excited state can also be interpreted as two charge anions at the two end points of S_{ij}. Charge anions are generated as pairs and one can move one of them in the lattice by applying a chain of Pauli operators. Furthermore, we can also define string operators of the Pauli operators Z. To this end, consider two faces r and t where a string S_{rt} can connect them [see Fig. 1(a)]. One can define a string operator S^z_{rt} which is a product of Z operators on the S_{rt}. Similarly, such an operator does not commute with two face operators B_r and B_t at the two end points of the S_{rt}, and it is interpreted as two flux anions at the end points of the S_{rt}.

We should emphasize that the TC model can also be defined on other lattices with different topologies. The most important difference between different topologies is related to degeneracy of the stabilizer space. Specifically in this paper, we consider a two-dimensional square lattice with an open boundary condition (see Fig. 2). Vertex and face operators are defined similarly to Eq. (1). However, note that vertex operators corresponding to vertices of the boundary of the lattice are three-body local. It is simple to check that unlike the TC on the torus there is only one constraint on vertex operators, $\prod_v A_v = I$, and no constraint on face operators. In this way, the degeneracy of stabilizer space will be equal to 2. It is also interesting to consider excitation of this model. Unlike the TC on a torus, here one can find flux anions in odd numbers. In fact if we apply a Z operator on a qubit on the boundary of the lattice it will only generate one flux anion in the neighboring face. The other flux anion always lives on the boundary of the lattice. In other words, the corresponding string operator has two end points with one on the boundary and another inside the lattice. We denote such a string operator by S_{0i} where zero refers to a qubit on the boundary and i refers to a qubit inside the lattice (see Fig. 2).

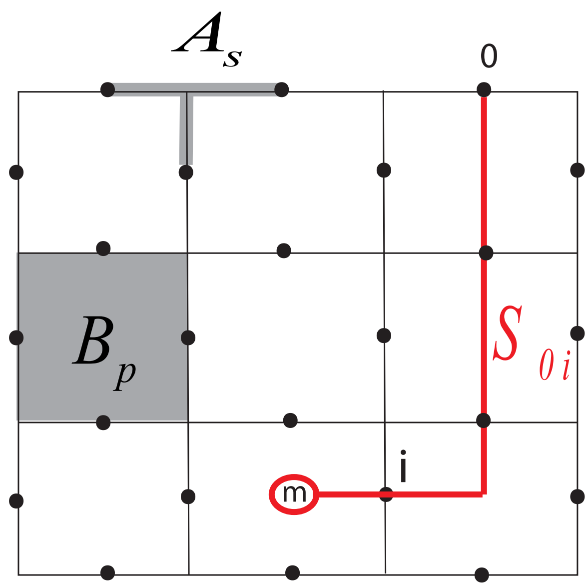

FIG. 2. TC model on a 2D square lattice with open boundary. The vertex operators corresponding to vertices on the boundary are three-local. If a string starts in the boundary the corresponding operator generates only one flux anion at the end point of the string in the lattice.

III. MAPPING THE ISING MODEL TO A NOISY TC MODEL

It is well known that the partition function of a classical spin model can be mapped to an inner product of a product state and an entangled state [16]. We have recently provided such a mapping using a duality transformation for CSS states which are mapped to classical spin models [33]. In this section, we first review such mapping between the TC state and 2D Ising model. We also show how a change of variable allows the temperature in the Ising model to be transformed to bit-flip noise in the TC state. Next, we extend such mapping to the magnetization of the 2D Ising model where it is also mapped to a specific quantity in the noisy toric TC model.

A. Mapping for the partition function

We start with the partition function of a 2D Ising model which is defined on a 2D square lattice with an open boundary condition where we suppose all spins in the boundary are fixed to a value of +1. The partition function will be in the following form:

$$\mathcal{Z} = \sum_{\{\sigma_i\}} e^{\beta J \sum_{\langle i,j\rangle} \sigma_i\sigma_j} \tag{5}$$

where $\sigma_i = \{\pm 1\}$ refers to spin variables which live on vertices of the lattice which we call vertex spins, J refers to coupling constants, and $\beta = \frac{1}{k_B T}$. Now, we define new spin variables ξ_l which live on edges of the square lattice which we call edge spins. In Fig. 3, we show these new spins by green circles. We also define the value of each edge spin ξ_l in the form of $\xi_l = \sigma_i\sigma_j$ where σ_i and σ_j are two vertex spins which live on two end points of the edge l. In the next step, we rewrite the partition function of Eq. (5) in terms of the edge

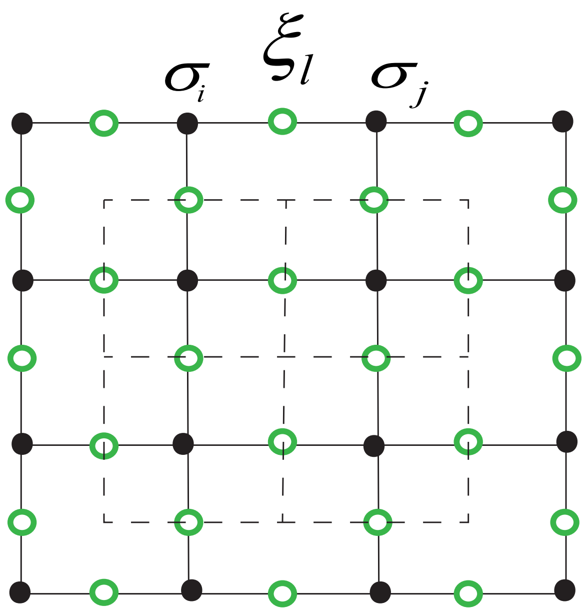

FIG. 3. Spin variables of the 2D Ising model are denoted by black circles. Corresponding to each edge of the lattice a new spin variable is defined and is denoted by the green circle. The dashed lattice is the dual of the initial lattice where faces and vertices of the initial lattice correspond to vertices and faces of the dual lattice, respectively.

spins ξ_l in the following form:

$$\mathcal{Z} = \sum_{\{\xi_l\}} e^{\beta J \sum_l \xi_l} \prod_f \delta\left(\prod_{l\in\partial f} \xi_l\right) \tag{6}$$

where $l \in \partial f$ refers to edges around the face of f and we have added delta functions corresponding to each face of the lattice in order to satisfy constraints between edge spins. In other words, since $\xi_l = \sigma_i\sigma_j$, it is clear that the product of spin variables corresponding to each face of the lattice will be equal to 1. We should emphasize that one can find another representation for constraints based on the dual lattice which is the same as the square lattice. As it is shown in Fig. 3, each face of the square lattice corresponds to a vertex of the dual lattice. In this way, constraints in the form of $\prod_f \delta(\prod_{l\in\partial f}\xi_l)$ in Eq. (6) can be replaced by $\prod_{v_d} \delta(\prod_{l\in v_d}\xi_l)$ where v_d refers to a vertex of the dual lattice. Here, we use such dual representation for finding the quantum formalism of the 2D Ising model.

Now, we rewrite each delta function in the form of $\delta(\prod_{l\in v_d}\xi_l) = \frac{1+\prod_{l\in v_d}\xi_l}{2}$. Next, the partition function will be written in a quantum language in the following form:

$$\mathcal{Z} = \langle\alpha|G\rangle \tag{7}$$

where $|\alpha\rangle = (e^{\beta J}|0\rangle + e^{-\beta J}|1\rangle)^{\otimes N_d}$ and $|G\rangle = \prod_{v_d} \frac{I+\prod_{l\in v_d} Z_l}{2}$ $(|0\rangle + |1\rangle)^{\otimes N_d}$ where N_d is the number of edges of the dual lattice. By comparison with Eq. (2) and by the fact that $\prod_{l\in v_d} Z_l$ is in fact a vertex operator A_v on the dual lattice, it is clear that $|G\rangle$ is the same as the toric code state $|K\rangle$ on the dual lattice up to a correction in the normalization factor. Finally, the partition function will be in the form of

$$\mathcal{Z} = \sqrt{2^{|f_d|}}\langle\alpha|K\rangle, \tag{8}$$

where $|f_d|$ is the number of faces of the dual lattice. In this way, the partition function of the 2D Ising model on a square lattice is related to a TC state on the dual lattice with qubits which live on the edges.

Now, let us define a new quantity p which is related to Boltzmann weight in the form of $p = \frac{e^{-2\beta J}}{1+e^{-2\beta J}}$. Since βJ is a quantity between zero and infinity, it is concluded that $0 < p < \frac{1}{2}$. In terms of this new quantity, the partition function can be rewritten in the following form:

$$\mathcal{Z} = \frac{1}{[p(1-p)]^{\frac{N_d}{2}}} W(p), \tag{9}$$

where

$$W(p) = 2^{N_d-1}\, {}^{N_d\otimes}\langle 0| \prod_i ((1-p)I + pX_i)|K\rangle. \tag{10}$$

We now show that the $W(p)$ can be interpreted as an important quantity in a noisy TC state. To this end, we consider a probabilistic bit-flip noise on the TC state where an X operator is applied on each qubit with a probability of p. We consider the density matrix of the model after applying a quantum channel corresponding to the bit-flip noise. Such a noise leads to different patterns of errors constructed by X operators on qubits and we denote such an error by $\hat{\mathcal{E}}(X)$. The probability of such an error is equal to $W_{\mathcal{E}}(p) = p^{|\mathcal{E}|}(1-p)^{N_d-|\mathcal{E}|}$ where $|\mathcal{E}|$ refers to the number of qubits which have been affected by the noise and N_d is the total number of qubits. The effect of bit-flip noise on an arbitrary N-qubit quantum state, denoted by density matrix ρ, can be presented by a quantum channel in the following form:

$$\Phi(\rho) = \sum_{\mathcal{E}} W_{\mathcal{E}}(p)\hat{\mathcal{E}}(X)\rho\hat{\mathcal{E}}(X). \tag{11}$$

Now, we come back to the relation of $W(p)$ in Eq. (10) and suppose p is the probability of bit-flip noise. If we expand $\prod_i[(1-p)I + pX_i]$ in this equation, we will have a superposition of all possible errors with the corresponding probability in the following form:

$$\prod_i[(1-p)I + pX_i] = \sum_{\mathcal{E}} W_{\mathcal{E}}(p)\hat{\mathcal{E}}(X). \tag{12}$$

On the other hand, the toric code state on the dual lattice is in the form of $|K\rangle = \frac{1}{\sqrt{2^{|f_d|}}}\prod_f (I + B_f)|0\rangle^{\otimes N_d}$ where the operator $\prod_f(I + B_f)$ is also a superposition of all possible X-type loop operators if we interpret the identity operator as a loop operator with a zero length (an interpretation that will be supposed in the remainder of the paper). Therefore, it will be easy to see that $W(p)$ is equal to the probability of generating loops in the noisy TC state. By this fact, Eq. (9) is a relation between the partition function of the Ising model on a 2D square lattice and the probability of generating loops in the noisy TC state.

B. Mapping for the magnetization

Equation (9) for the partition function of the 2D Ising model shows that there might be a deeper correspondence

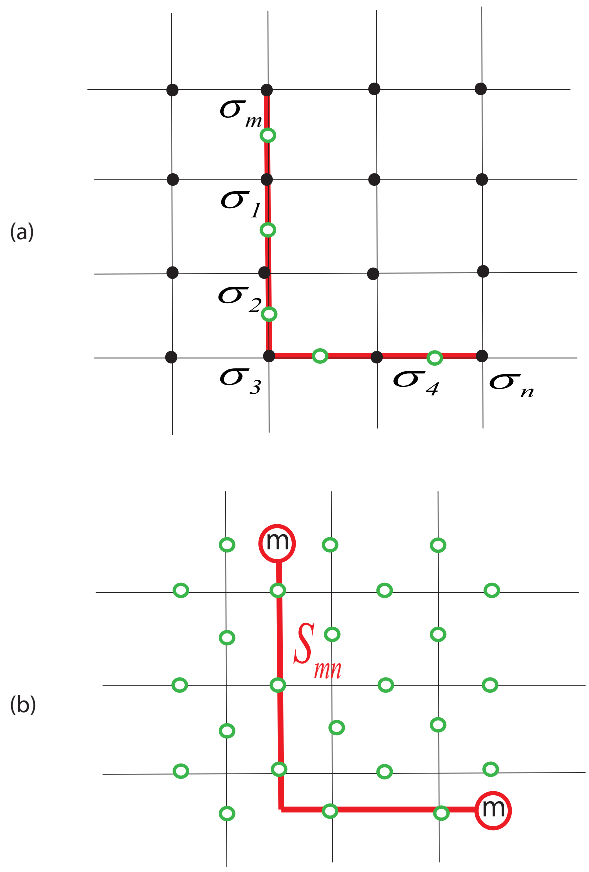

FIG. 4. (a) There is a chain of spins σ_1, σ_2, σ_3, and σ_4 between two spins σ_m and σ_n. (b) After change of variables and converting to quantum language, there will be a string operator corresponding to the initial string S_{mn}.

between the 2D Ising model and a noisy TC state. However, in order to better understand such a correspondence it will be useful to consider such a mapping for other important quantities in the 2D Ising model. One of the most important candidates is the magnetization of the 2D Ising model which plays an important role in characterization of phase transition in the 2D Ising model.

We start by considering the mean value of the product of two arbitrary spins in the 2D Ising model, i.e., the correlation function. The correlation function is formally given by

$$\langle \sigma_m \sigma_n \rangle = \frac{1}{\mathcal{Z}} \sum_{\{\sigma_i\}} \sigma_m \sigma_n e^{\beta J \sum_{\langle i,j \rangle} \sigma_i \sigma_j}. \tag{13}$$

Now, let us consider two particular spins as is shown in Fig. 4. Then we consider an arbitrary string of spins denoted by S_{mn} between two spins σ_m and σ_n which we denote by $\sigma_1, \dots, \sigma_4$. Since $\sigma_i^2 = 1$, we can write $\sigma_m \sigma_n = (\sigma_m \sigma_1)(\sigma_1 \sigma_2)(\sigma_2 \sigma_3)(\sigma_3 \sigma_4)(\sigma_4 \sigma_n)$. Next, we write Eq. (13) in the following form:

$$\langle \sigma_m \sigma_n \rangle = \frac{\sum_{\{\sigma_i\}} (\sigma_m \sigma_1)(\sigma_1 \sigma_2)(\sigma_2 \sigma_3)(\sigma_3 \sigma_4)(\sigma_4 \sigma_n) e^{\beta J \sum_{\langle i,j \rangle} \sigma_i \sigma_j}}{\mathcal{Z}}. \tag{14}$$

By this new form, we can use edge spins that we defined in the previous section to rewrite the above relation in terms of edge spins ξ_l. We will have

$$\langle \sigma_m \sigma_n \rangle = \frac{1}{\mathcal{Z}} \sum_{\{\xi_l\}} \left[\prod_{l \in S_{mn}} \xi_l \right] e^{\beta J \sum_l \xi_l} \prod_f \delta \left(\prod_{l \in \partial f} \xi_l \right) \tag{15}$$

where $l \in S_{mn}$ refers to all edges belonging to the string of S_{mn}. Similar to the procedure that we performed for the partition function of the 2D Ising model, we can rewrite the above relation in a quantum language where we replace edge spins with the Pauli matrices Z and by using definition of TC states we will have

$$\langle \sigma_m \sigma_n \rangle = \frac{\langle \alpha | \prod_{l \in S_{mn}} Z_l | K \rangle}{\langle \alpha | K \rangle} = \frac{\langle \alpha | S^z_{mn} | K \rangle}{\langle \alpha | K \rangle} \tag{16}$$

where S^z_{mn} is a Z-type string operator between vertices of m and n and $S^z_{mn}|K\rangle$ is an excited state including two flux anions in vertices of m and n.

The above process can also be applied for the mean value of an arbitrary spin of the 2D Ising model denoted by $\langle \sigma_i \rangle$ which is in fact the order parameter of this model. Since we considered a 2D Ising model with an open boundary condition where all spins in the boundary of the lattice are fixed to the value of $+1$, the expectation value of an arbitrary spin σ_i is in fact equal to the correlation function between that spin and another spin on the boundary of the lattice denoted by $\sigma_0 = +1$. Therefore, we can use the above formalism for the correlation function to find a quantum formalism for $\langle \sigma_i \rangle$. As we show in Fig. 5, we consider a string of spins $\sigma_1, \dots, \sigma_k$ between a particular spin σ_m and a spin $\sigma_0 = +1$ on the boundary of the lattice. In this way, we will have $\sigma_m = (\sigma_m \sigma_1)(\sigma_1 \sigma_2) \dots (\sigma_{k-1} \sigma_k)(\sigma_k \sigma_0)$ and the order parameter will be in the following form:

$$\langle \sigma_m \rangle = \frac{\sum_{\{\sigma_i\}} (\sigma_m \sigma_1)(\sigma_1 \sigma_2) \dots (\sigma_{k-1} \sigma_k)(\sigma_k \sigma_0) e^{\beta J \sum_{\langle i,j \rangle} \sigma_i \sigma_j}}{\mathcal{Z}}. \tag{17}$$

Then we use a change of variable to edge spins ξ_l and finally the above equation is rewritten in a quantum language in the following form:

$$\langle \sigma_m \rangle = \frac{\langle \alpha | \prod_{l \in S_{m0}} Z_l | K \rangle}{\langle \alpha | K \rangle} = \frac{\langle \alpha | S^z_{m0} | K \rangle}{\langle \alpha | K \rangle} \tag{18}$$

where S_{m0} refers to a string with one of its end points on the boundary of the lattice and the other on the face of m. Furthermore, since S^z_{m0} is a Z-type string operator, the quantum state of $S^z_{m0}|K\rangle$ is an excited state of the TC model with only one flux anion in one of the end points of the string.

In order to relate the above result to a noisy TC model, we use the transformation from Boltzmann weights in the $|\alpha\rangle$ to probability of noise p according to $p = \frac{e^{-2\beta J}}{1+e^{-2\beta J}}$. In this way, according to Eq. (9) in the previous section, it is clear that the denominator in Eq. (18) is the same as the partition function up to a factor $\sqrt{2^{|f_d|}}$ and will in fact be equal to $\frac{W(p)}{\sqrt{2^{|f_d|}}[p(1-p)]^{\frac{N_d}{2}}}$. Furthermore, we need to find an interpretation for the numerator of Eq. (18). To this end, we note that after transformation to probability of noise p the numerator

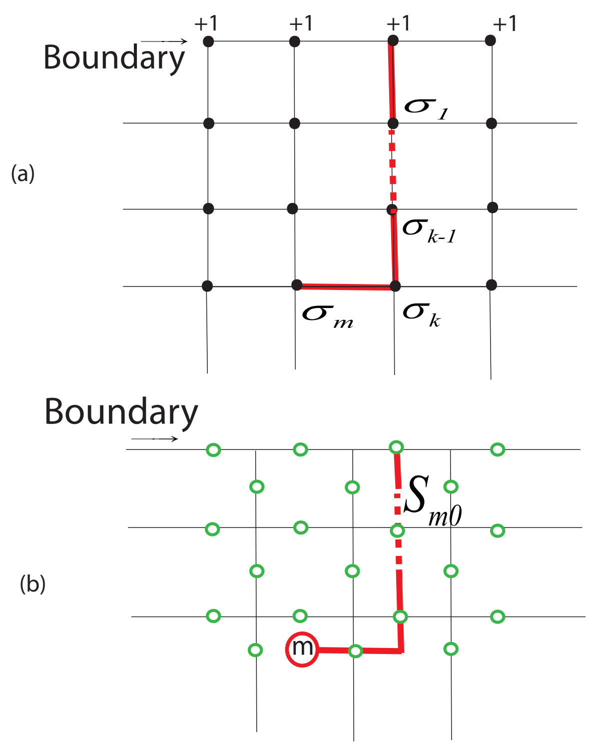

FIG. 5. (a) There is a chain of spins between the boundary of the lattice and spin σ_m. (b) After change of variables and converting to the quantum language, there will be a string operator corresponding to the string of S_{0m} which corresponds to a flux anion in the TC model with an open boundary.

is written in the following form:

$$\langle\alpha|S^z_{m0}|K\rangle = \frac{{}^{N\otimes}\langle 0|(pX+(1-p)I)^{\otimes N} S^z_{m0}|K\rangle}{\sqrt{2^{|f_d|}}[p(1-p)]^{\frac{N}{2}}} \tag{19}$$

where we have replaced the N_d by N for simplifying the notation. As noted previously, $[pX+(1-p)I]^{\otimes N}$ is equal to superposition of all possible bit-flip errors. On the other hand, in the $S^z_{m0}|K\rangle = S^z_{m0}\prod_f (I+B_f)|0\rangle^{\otimes N}$, $\prod_f (I+B_f)$ is equal to superposition of all X-type loop operators with the same weight $+1$. However, the string operator S^z_{m0} does not commute with X-type loop operators that cross the string of S_{m0} an odd number of times. Therefore, one can conclude that in the state of $S^z_{m0}|K\rangle$ we will have a superposition of all X-type loop operators with two different weights $+1$ for loops that cross the string S_{m0} an even number of times and -1 for loops that cross the S_{m0} an odd number of times. By this fact, the numerator will be related to the total probability, denoted by $W_+(p)$, that noise leads to loops with even crossings with the S_{m0} minus total probability, denoted by $W_-(p)$, that noise leads to loops with odd crossings with S_{m0}. Finally, Eq. (18) will be in the following form:

$$\langle\sigma_m\rangle = \frac{W_+(p)-W_-(p)}{W_+(p)+W_-(p)} \tag{20}$$

where we have replaced $W(p)$ in the denominator by $W_+(p)+W_-(p)$ and we have removed a factor of $\frac{1}{\sqrt{2^{|f_d|}}[p(1-p)]^{\frac{N}{2}}}$ from denominator and nominator.

According to Eq. (20) we have found a quantum analog for the magnetization of the 2D Ising model. By using this equation and Eq. (9) for the partition function, we are ready to consider an interesting problem for a noisy TC model. In other words, since in the 2D Ising model it is well known that there is a phase transition at a critical temperature $T_{\rm cr}$, one can ask if there is a phase transition in the noisy TC model at a corresponding probability of $p_{\rm cr}$. In the next section we use the above mappings to show that there is a transition from a coherent to a noncoherent phase in a noisy TC model.

IV. PHASE TRANSITION IN A NOISY TC MODEL

In this section, we give the main result of this paper where we introduce a phase transition in a noisy TC model. To this end, we note that the magnetization in Eq. (20), which is in fact the order parameter of the 2D Ising model, shows a second-order phase transition. Therefore, we expect the same behavior for the quantity in the right-hand side of Eq. (20) which should be an order parameter for characterization of a type of phase transition in the noisy TC model. In order to understand such a phase transition, first we need to find a physical interpretation for the right-hand side of Eq. (20). To this end, we come back to the noisy model and we consider the effect of the bit-flip noise on a particular initial state. Suppose that the initial state is an eigenstate of string operator S^z_{m0}. It is a simple matter to check that such a state will be in the following form:

$$|\psi_+\rangle = \frac{1}{\sqrt{2}}(|K\rangle + S^z_{mo}|K\rangle). \tag{21}$$

The above state is in fact a coherent superposition of a vacuum state $|K\rangle$, where there is no anion, and a two anion state $S^z_{m0}|K\rangle$ where one lives on the boundary zero and the other lives on the face of m. Our main purpose is to investigate the decoherence process of this coherent superposition as a result of noise. We expect that by increasing the probability of the bit flip noise the initial state converts to a complete mixture of $|K\rangle\langle K|$ and $S^z_{m0}|K\rangle\langle K|S^z_{m0}$. However, the actual trend, as a function of noise probability, that such a transition to decoherence occurs is an important consideration. For example, is the transition a gradual one or is there a sharp (second-order) phase transition? How can one characterize such a transition. Next, we set out to show that such a transition is in fact sharp and can be characterized by an order parameter which measures the amount of coherence in the system.

In order to consider the above decoherence process, we divide all errors $\mathcal{E}(X)$ in Eq. (11) to three parts. By the fact that each error of $\mathcal{E}(X)$ can be represented as a pattern of string operators X on the lattice, we consider three kinds of strings which are schematically shown in Fig. 6. The first are open strings where there are charge anions in the end points of those strings. The second are closed strings (loops) that cross the string of S_{m0} an odd number of times and the third are closed strings that cross the string of S_{m0} an even number of times. It is simple to check that the effect of errors corresponding to the open strings, denoted by E_1, on the state of $|\psi_+\rangle$ leads to generation of charge anions on end points of open strings and takes the initial state to other excited states. The effect of the second kind of errors, denoted by E_2, is interesting where

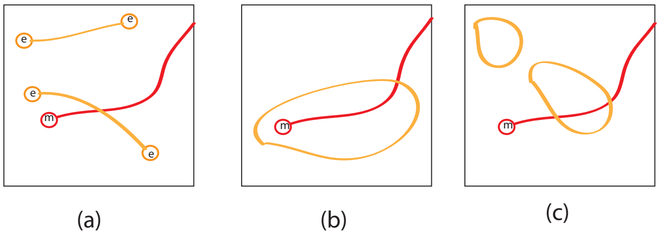

FIG. 6. The red string refers to the string operator S^z_{m0} with a flux anion in the end point. (a) The first set of errors E_1 which correspond to open strings of X operators with two charge anions in the end points. (b) The second set of errors E_2 which correspond to loops of X operators which cross the S_{m0} an odd number of times. (c) The third set of errors E_3 which correspond to loops of X operators which cross the S_{m0} an even number of times.

it takes $|\psi_+\rangle$ to $\frac{1}{\sqrt{2}}(|K\rangle - S^z_{mo}|K\rangle)$, denoted by $|\psi_-\rangle$ which is orthogonal to $|\psi_+\rangle$, because loop operators of the second type anticommute with S^z_{m0}. Finally the effect of the third kind of errors, denoted by E_3, is trivial as it takes $|\psi_+\rangle$ to $|\psi_+\rangle$. In this way, Eq. (11), when we insert $\rho = |\psi_+\rangle\langle\psi_+|$, can be written in the following form:

$$\Phi(|\psi_+\rangle\langle\psi_+|) = \sum_{\mathcal{E}\in E_1} W_{\mathcal{E}}(p)|\psi_{\mathcal{E}}\rangle\langle\psi_{\mathcal{E}}| + W_-(p)|\psi_-\rangle\langle\psi_-| + W_+(p)|\psi_+\rangle\langle\psi_+| \tag{22}$$

where

$$W_-(p) = \sum_{\mathcal{E}\in E_2} W_{\mathcal{E}}(p),$$

$$W_+(p) = \sum_{\mathcal{E}\in E_3} W_{\mathcal{E}}(p),$$

and

$$|\psi_{\mathcal{E}}\rangle = \mathcal{E}(X)|\psi\rangle,$$

and E_1, E_2, and E_3 refer to the above three kinds of errors, respectively. W_+ and W_- are the total probability of generating loops which cross the S_{m0} an even and odd number of times, respectively. In order to find a better interpretation for the above state, note that in the final quantum state there is a mixture of $|\psi_+\rangle$ with probability of $W_+(p)$ and $|\psi_-\rangle$ with probability of $W_-(p)$. In other words, while the state of $|\psi_+\rangle = \frac{1}{\sqrt{2}}(|K\rangle + S^z_{mo}|K\rangle)$ is a coherent superposition of states $|K\rangle$ and $S^z_{m0}|K\rangle$, the effect of bit-flip noise has led to generating a noncoherent mixture of them which can be represented in the corresponding subspace in the following form:

$$\frac{1}{2}\begin{pmatrix} (W_+ + W_-) & (W_+ - W_-) \\ (W_+ - W_-) & (W_+ + W_-) \end{pmatrix}. \tag{23}$$

Now, we need to define a parameter to characterize the amount of decoherence in the above mixture in terms of p. According to Eq. (23), the following parameter is a well-defined measure for the above decoherence:

$$O(p) = \frac{W_+(p) - W_-(p)}{W_+(p) + W_-(p)}. \tag{24}$$

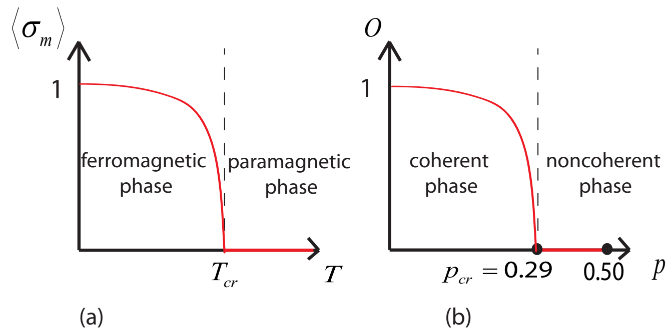

FIG. 7. (a) A schematic representation of the well-known diagram of the order parameter in the 2D Ising model where the magnetization goes to zero at $T_{\rm cr}$. (b) By comparing Eqs. (20) and (24), the expectation value of the string operator S^z_{m0} shows the same behavior with a phase transition at $p_{\rm cr} = 0.29$.

This quantity can be also interpreted as the expectation value of S^z_{m0} in a subspace which is spanned by $|\psi_+\rangle$ and $|\psi_-\rangle$. For example, when $W_+ = W$ and $W_- = 0$ the order parameter O is equal to 1 which indicates a coherent superposition, and when $W_+ = W_-$ it is equal to zero, indicating a complete mixture where Eq. (23) becomes proportional to the identity operator. Now, referring to Eq. (20), we see that O is identical to the order parameter of the 2D Ising model. On the other hand, it is well known that the order parameter in the 2D Ising model characterizes the nature of ferromagnetic phase transition where at a critical temperature $T_{\rm cr}$ the system shows a second-order phase transition from an ordered phase $\langle\sigma_m\rangle \neq 0$ to a disordered phase $\langle\sigma_m\rangle = 0$. Therefore, by using Eq. (24), it is concluded that there is a phase transition in the noisy TC state where a relatively coherent state, $O \neq 0$, goes to a complete mixture, $O = 0$, at a well-defined (and relatively large) noise value of $p_{\rm cr}$ which is easily calculated from the critical temperature of the 2D Ising model to be $p_{\rm cr} = 0.29$.

The picture that emerges is very interesting. One would expect that increasing bit-flip noise on a coherent superposition would lead to decoherence. However, we have shown that the system remains relatively robust to such an effect for small values of noise probability $p \ll p_{\rm cr}$ and that the transition can be characterized by an order parameter which shows a sharp (second-order) phase transition to decoherence at a relatively large value of noise value. Figure 7 schematically shows such a behavior as increasing p from zero to half leads to decoherence at the value of $p_{\rm cr} = 0.29$. This interesting and unexpected property indicates a *robust coherence* which might be related to topological order of the TC state, a point that we will come back to in Sec. VI.

V. SUSCEPTIBILITY AND CRITICAL STABILITY AT PHASE TRANSITION

Although the existence of a phase transition in a physical system is very important by itself, the critical point which separates two different phases of the system is also a key matter which should specifically be considered. Therefore, we intend to look for other consequences of criticality on the classical side for the quantum model. In this section we consider this problem and specifically show that the ground state of the TC model under the noise displays an interesting behavior precisely at the critical point p_{cr}.

We consider an important issue that has been previously emphasized in the general context of CSS states, i.e., relative *critical stability*, which occurs at p_{cr} [33]. Here, we investigate such a concept in terms of a susceptibility to noise which is defined for an initial state affected by a bit-flip noise. To this end, we use a familiar quantity in the quantum information theory called fidelity. In other words, if we consider the state of $|K\rangle$ as an initial state, the fidelity of this state and the final state after applying noise will be in the following form:

$$F(p) = \langle K|\Phi(\rho)|K\rangle \tag{25}$$

where $\Phi(\rho)$ is the final state after applying the bit-flip channel to the initial state $\rho = |K\rangle\langle K|$. Now it is clear that if $F(p)$ is small (large) it means that susceptibility to noise is high (low) because the final state is very different from (similar to) the initial state. In other words, there is an inverse relation between susceptibility to noise and fidelity. Therefore, we define the following quantity to measure the susceptibility:

$$\chi(p) = -\log[F(p)]. \tag{26}$$

It will be interesting to give a physical picture to this quantity. To this end, we interpret $F(p)$ in an anionic picture for the TC state. In fact, when a bit-flip noise is applied to a qubit of the lattice, it generates two charged anions. Therefore, the effect of probabilistic bit-flip noise on all qubits can lead to two events, generation of pair anions and walking anions in the lattice. It is clear that as long as there is an anion in the lattice the system is in an excited mode and fidelity is zero, i.e., susceptibility is infinite. The only possible way that the system comes back to the ground state is that anions fuse to each other and annihilate. In other words, anions must generate complete loops in the lattice. This consideration seems to indicate that $F(p)$ should be equal to the probability of generating loops which is the same as $W(p)$.

In order to explicitly prove that $F(p) = W(p)$, we use the definition of Eq. (25) for $F(p)$. On one hand, we know that $|K\rangle = \frac{1}{\sqrt{2^{|f|}}}\prod_f (I + B_f)|0\rangle^{\otimes N}$ and $\prod_f (I + B_f)$ is equal to summation of all possible X-type loop operators in the lattice. On the other hand, $\Phi(\rho)$ in Eq. (25) is equal to $\sum_{\mathcal{E}} W_{\mathcal{E}}(p)\hat{\mathcal{E}}(X)\rho\hat{\mathcal{E}}(X)$ which is equal to a summation of all possible X chains with the corresponding probability. In this way, when such a summation inserts between two states $|K\rangle$ in Eq. (25), all X chains in the summation lead to error and convert the inner product to zero, except for the X chains corresponding to loops which convert the inner product to 1. Therefore, there will be a summation of probability of generating loops and it is equal to $W(p)$.

Now, we note that $W(p)$ was related to the partition function of the 2D Ising model according to Eq. (9) and since $F(p) = W(p)$ we have

$$\mathcal{Z} = \frac{1}{[p(1-p)]^{\frac{N}{2}}} F(p). \tag{27}$$

Therefore, one can find the fidelity in the form of $[p(1-p)]^{\frac{N}{2}}\mathcal{Z}$. Since $p \leqslant \frac{1}{2}$, it is concluded that $[p(1-p)]^{\frac{N}{2}} \to 0$ for large N. In this way, it seems that $F(p)$ should be zero for any nonzero value of p and therefore susceptibility is infinite for any generic noise probability. However, at the critical point, the partition function displays a nonanalytic behavior, where fluctuations become relevant. A fluctuation correction to the partition function can be written as [35]

$$\mathcal{Z} \approx \exp(-\beta A)\sqrt{2\pi k T^2 C_v}, \tag{28}$$

where A is the Helmholtz free energy and C_v is the heat capacity which behaves as $C_v \sim | T - T_{\text{cr}} |^{-\alpha}$, near the critical point. Fidelity is clearly equal to unity for zero noise (or temperature), but it is also a strongly decreasing function of p as can be seen from Eq. (27). However, the divergence of heat capacity at the critical point will cause a relative increase of the value of fidelity (or relative decrease of susceptibility) as the critical point is approached, thus leading to a *relative stability* [33].

There are two points about critical stability that should be emphasized. The first is that the concept of susceptibility that we defined here is not the usual susceptibility in classical phase transitions. For example, in the Ising model, heat capacity can be interpreted as susceptibility of the system to an infinitesimal change of temperature. However, in our case, the susceptibility measures stability of the initial state to the whole of the noise, not an infinitesimal change of the probability of the noise. In other words, our susceptibility is not defined as a derivative, but as the response of the ground state to a noise probability of a finite value p.

As the second point, we emphasize that the critical probability of p_{cr} should not be regarded as a threshold for stability of the toric code state. In contrast, it is a relative stability that occurs only at a particular noise probability, p_{cr}. It is different from the role of p_{cr} in the previous section where it was a threshold for maintaining coherence of a particular initial state.

A physical picture may help to clarify what is happening in both situations. As p increases the possibility of forming larger loops also increases. It is at p_{cr} where the possibility of having loops of the order of system size, N, appear. This is related to the fact that correlation length diverges at the critical point. These system size loops are prime candidates for increasing the value of W_-, as they are prime candidates for crossing the string operators once, when compared to smaller strings which typically do not cross or cross twice (see Fig. 6). This explains how the order parameter suddenly drops to zero at the critical point as a significant W_- cancels out the already reduced W_+. Also note that for $p > p_{\text{cr}}$ both values of W_+ and W_- are significantly small and equal, leading to zero order parameter. On the other hand, the emergence of such system size loops which can occur only at p_{cr} give a relative increase to the value of $W(p) = W_+ + W_-$, thus explaining the relative stability near (at) the critical point.

VI. DISCUSSION

Although some time has passed since the introduction of quantum formalism for the partition function of classical spin models, it seems that such mappings are richer than what had been previously supposed. In particular, here we studied a neglected aspect of such mappings related to the phase transition on the classical side. In particular, we observed the existence of a sharp (second-order) phase transition from a coherent phase to a mixed phase in the noisy TC model as well as a relative critical stability at the transition point. These seem like important physical properties which might find relevance in the applications of quantum states in general. As a closing remark, we would like to emphasize that the large value of the critical noise $P_{\mathrm{cr}} = 0.29$ can be interpreted as a robust coherence of the initial state against bit-flip noise. On the other hand, since string operators and loop operators in the TC model correspond to processes of generating and fusing anions, it seems that the robust coherence is in fact related to anionic properties. This point in addition to a relative stability at the critical noise support a conjecture that such behaviors might be related to topological order of the TC state. Since the critical stability has also been observed in other topological CSS states [33], it will be interesting to consider the existence of the robust coherence in such models, a problem we intend to address in future studies. Also, recently, more exotic 2D topological phases [45] have been proposed and studied, and since dualities between these phases and statistical mechanical models have also been introduced [46] it will be interesting to consider our approach for such models along the lines of our approach via possible consequences of criticality.

We finally note that we have specifically looked at the mapping for a toric code on a square lattice with open boundary conditions. However, it is easy to see that any other lattice structure and/or boundary condition could have been used while the mapping to the corresponding Ising model with the same lattice structure and boundary condition would have still been valid [33]. Therefore, since the Ising transition is independent of such details, the same conclusions, e.g., robust coherence, would still hold.

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[1] J. Geraci and D. A. Lidar, On the exact evaluation of certain instances of the Potts partition function by quantum computers, Commun. Math. Phys. **279**, 735 (2008).

[2] D. A. Lidar and O. Biham, Simulating Ising spin glasses on a quantum computer, Phys. Rev. E **56**, 3661 (1997).

[3] J. Geraci and D. A. Lidar, Classical Ising model test for quantum circuits, New J. Phys. **12**, 075026 (2010).

[4] R. D. Somma, C. D. Batista, and G. Ortiz, Quantum Approach to Classical Statistical Mechanics, Phys. Rev. Lett. **99**, 030603 (2007).

[5] F. Verstraete, M. Wolf, D. Perez-Garcia, and J. Cirac, Criticality, the Area Law, and the Computational Power of Projected Entangled Pair States, Phys. Rev. Lett. **96**, 220601 (2006).

[6] E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, Topological quantum memory, J. Math. Phys. **43**, 4452 (2002).

[7] H. G. Katzgraber, H. Bombin, and M. A. Martin-Delgado, Error Threshold for Color Codes and Random Three-Body Ising Models, Phys. Rev. Lett. **103**, 090501 (2009).

[8] S. Popescu, A. J. Short, and A. Winter, Entanglement and the foundations of statistical mechanics, Nat. Phys. **2**, 754 (2006).

[9] A. Montakhab and A. Asadian, Multipartite entanglement and quantum phase transitions in the one-, two-, and three-dimensional transverse-field Ising model, Phys. Rev. A **82**, 062313 (2010).

[10] C. Gogolin and J. Eisert, Equilibration, thermalisation, and the emergence of statistical mechanics in closed quantum systems, Rep. Prog. Phys. **79**, 056001 (2016).

[11] J. Goold, M. Huber, A. Riera, L. del Rio, and P. Skrzypczyk, The role of quantum information in thermodynamics: A topical review, J. Phys. A **49**, 143001 (2016).

[12] F. J. Wegner, Duality in generalized Ising models and phase transitions without local order parameters, J. Math. Phys. **12**, 2259 (1971).

[13] S. Vijay, J. Haah, and L. Fu, Fracton topological order, generalized lattice gauge theory, and duality, Phys. Rev. B **94**, 235157 (2016).

[14] D. J. Williamson, Fractal symmetries: Ungauging the cubic code, Phys. Rev. B **94**, 155128 (2016).

[15] D. Vodola, D. Amaro, M. A. Martin-Delgado, and M. Müller, Twins Percolation for Qubit Losses in Topological Color Codes, Phys. Rev. Lett. **121**, 060501 (2018).

[16] M. Van den Nest, W. Dür, and H. J. Briegel, Classical Spin Models and the Quantum-Stabilizer Formalism, Phys. Rev. Lett. **98**, 117207 (2007).

[17] M. Van den Nest, W. Dür, R. Raussendorf, and H. J. Briegel, Quantum algorithms for spin models and simulable gate sets for quantum computation, Phys. Rev. A **80**, 052334 (2009).

[18] W. Dür and M. Van den Nest, Quantum Simulation of Classical Thermal States, Phys. Rev. Lett. **107**, 170402 (2011).

[19] G. De las Cuevas, A quantum information approach to statistical mechanics, J. Phys. B **46**, 243001 (2013).

[20] G. De las Cuevas, W. Dür, M. Van den Nest, and M. A. Martin-Delgado, Quantum algorithms for classical lattice models, New J. Phys. **13**, 093021 (2011).

[21] R. Raussendorf and H. J. Briegel, A One-Way Quantum Computer, Phys. Rev. Lett. **86**, 5188 (2001).

[22] H. J. Briegel, D. E. Browne, W. Dür, R. Raussendorf, and M. Van den Nest, Measurement-based quantum computation, Nat. Phys. **5**, 19 (2009).

[23] S. Bravyi and R. Raussendorf, Measurement-based quantum computation with the toric code states, Phys. Rev. A **76**, 022304 (2007).

[24] H. Bombin and M. A. Martin-Delgado, Statistical mechanical models and topological color codes, Phys. Rev. A **77**, 042322 (2008).

[25] M. Van den Nest, W. Dür, and H. J. Briegel, Completeness of the Classical 2D Ising Model and Universal Quantum Computation, Phys. Rev. Lett. **100**, 110501 (2008).

[26] V. Karimipour and M. H. Zarei, Algorithmic proof for the completeness of the two-dimensional Ising model, Phys. Rev. A **86**, 052303 (2012).

[27] G. De las Cuevas, W. Dür, H. J. Briegel, and M. A. Martin-Delgado, Unifying All Classical Spin Models in a Lattice Gauge Theory, Phys. Rev. Lett. **102**, 230502 (2009).

[28] Y. Xu, G. De las Cuevas, W. Dür, H. J. Briegel, and M. A. Martin-Delgado, The U(1) lattice gauge theory universally connects all classical models with continuous variables, including background gravity, J. Stat. Mech. (2011) P02013.

[29] V. Karimipour and M. H. Zarei, Completeness of classical ϕ^4 theory on two-dimensional lattices, Phys. Rev. A **85**, 032316 (2012).

[30] M. H. Zarei and Y. Khalili, Systematic study of the completeness of two-dimensional classical ϕ^4 theory, Int. J. Quantum Inform. **15**, 1750051 (2017).

[31] G. De las Cuevas and T. S. Cubitt, Simple universal models capture all classical spin physics, Science **351**, 1180 (2016).

[32] T. S. Cubitt, A. Montanaro, and S. Piddock, Universal quantum Hamiltonians, Proc. Natl. Acad. Sci. USA **115**, 9497 (2018).

[33] M. H. Zarei and A. Montakhab, Dual correspondence between classical spin models and quantum CSS states, Phys. Rev. A **98**, 012337 (2018).

[34] H. E. Stanley, *Phase Transitions and Critical Phenomena* (Clarendon, Oxford, 1971).

[35] R. K. Pathria, *Statistical Mechanics*, International Series in Natural Philosophy Vol. 45 (Pergamon, Oxford, 1986).

[36] A. Y. Kitaev, Fault-tolerant quantum computation by anyons, Ann. Phys. (NY) **303**, 2 (2003).

[37] X.-G. Wen and Q. Niu, Ground-state degeneracy of the fractional quantum Hall states in the presence of a random potential and on high-genus Riemann surfaces, Phys. Rev. B **41**, 9377 (1990).

[38] X.-G. Wen, Topological orders and edge excitations in fractional quantum Hall states, Adv. Phys. **44**, 405 (1995).

[39] S. Trebst, P. Werner, M. Troyer, K. Shtengel, and C. Nayak, Breakdown of a Topological Phase: Quantum Phase Transition in a Loop Gas Model with Tension, Phys. Rev. Lett. **98**, 070602 (2007).

[40] S. Dusuel, M. Kamfor, R. Orus, K. P. Schmidt, and J. Vidal, Robustness of a Perturbed Topological Phase, Phys. Rev. Lett. **106**, 107203 (2011).

[41] M. H. Zarei, Robustness of topological quantum codes: Ising perturbation, Phys. Rev. A **91**, 022319 (2015).

[42] M. H. Zarei, Strong-weak coupling duality between two perturbed quantum many-body systems: CSS codes and Ising-like systems, Phys. Rev. B **96**, 165146 (2017).

[43] J. R. Wootton and J. K. Pachos, Bringing Order Through Disorder: Localization of Errors in Topological Quantum Memories, Phys. Rev. Lett. **107**, 030503 (2011).

[44] B. J. Brown, D. Loss, J. K. Pachos, C. N. Self, and J. R. Wootton, Quantum memories at finite temperature, Rev. Mod. Phys. **88**, 045005 (2016).

[45] Y.-Z. You, Z. Bi, A. Rasmussen, K. Slagle, and C. Xu, Wave Function and Strange Correlator of Short-Range Entangled States, Phys. Rev. Lett. **112**, 247202 (2014).

[46] R. Vanhove, M. Bal, D. J. Williamson, N. Bultinck, J. Haegeman, and F. Verstraete, Mapping Topological to Conformal Field Theories Through Strange Correlators, Phys. Rev. Lett. **121**, 177203 (2018).