Approximate stabilizer rank and improved weak simulation of Clifford-dominated circuits for qudits

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Bravyi and Gosset [S. Bravyi and D. Gosset, Phys. Rev. Lett. **116**, 250501 (2016)] recently gave classical simulation algorithms for quantum circuits dominated by Clifford operations. These algorithms scale exponentially with the number of T gates in the circuit, but polynomially in the number of qubits and Clifford operations. Here we extend their algorithm to qudits of odd prime dimension. We generalize their approximate stabilizer rank method for weak simulation to qudits and obtain the scaling of the approximate stabilizer rank with the number of single-qudit magic states. We also relate the canonical form of qudit stabilizer states to Gauss sum evaluations and give an $O(n^3)$ algorithm for calculating the inner product of two n-qudit stabilizer states.

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I. INTRODUCTION

With the prospect of noisy intermediate-scale quantum computers with 50–100 qubits appearing in the next decade [1,2], determining the minimal classical cost of simulation of quantum computers has recently received much attention [3–7]. The Gottesman-Knill theorem shows that Clifford circuits are efficiently classically simulatable [8]. Adding any non-Clifford gate creates a universal gate set.[1] One such choice for a non-Clifford gate is the T gate: $T|j\rangle = e^{ij\pi/4}|j\rangle$, $j \in \{0, 1\}$ [11]. Bravyi and Gosset gave a classical algorithm for simulation of quantum circuits that scales exponentially with the number of T gates in the circuit but polynomially with the number of qubits and Clifford gates [4]. This algorithm was further developed in [12].

What is supplied by the addition of T gates to a Clifford circuit? The fault-tolerant implementation of Clifford + T circuits substitutes magic states for each T gate [13,14]. Colloquially, T gates add "magic" to a Clifford circuit. Magic is supplied by contextuality, a longstanding source of puzzles and paradoxes in the foundations of quantum mechanics [15].

The relationship of magic to contextuality also provides a connection to quasiprobability representations of quantum mechanics [16,17]. Specifically, the positivity of a quasiprobability representation is equivalent to the absence of contextuality, and such positive states, operations, and measurements admit efficient classical simulation in some cases [18,19]. Classical statistical theories with an imposed uncertainty principle can reproduce these positive quasiprobabilistic theories for Gaussian states and qudits with $d > 2$ [20,21].

Pashayan *et al.* gave an algorithm allowing a positive quasiprobability description to include some negativity [22]. Comparing the algorithms of Bravyi and Gosset [4] and Pashayan [22] should shed more light on the relationship between magic, contextuality, and negativity. However, quasiprobability representations for qubits are distinct from their d-dimensional cousins [23–25]. The desire to understand the relationship between magic, contextuality, and negativity therefore motivates extension of the algorithm of Bravyi and Gosset to qudits with dimension greater than 2. In the present paper we extend the algorithm of Bravyi and Gosset to qudits of odd prime dimension.

The structure of the paper is as follows. In Secs. II and III we briefly introduce the necessary background. In Sec. IV we give the nonorthogonal decomposition of the magic state. In Sec. V we give results on the approximate stabilizer rank and weak simulation algorithm for qudits. In Sec. VI we briefly compare our algorithm to that of [22].

II. QUDIT PAULI GROUP AND CLIFFORD GATES

The Pauli and Clifford groups were first generalized beyond qubits by Gottesman [26]. Assuming henceforth that d is an odd prime, we define the Heisenberg-Weyl operators

$$D_{\vec{x}} = \tau^{xz} X^x Z^z, \tag{1}$$

where $X|j\rangle = |j \oplus 1\rangle$ ($\oplus$ denotes addition modulo d), $Z|j\rangle = \omega^j|j\rangle$, $\vec{x} = (x, z)$ [x and z are integers modulo d and $\omega = \exp(2\pi i/d)$], and $\tau = e^{(d+1)\pi i/d} = \omega^{2^{-1}}$. The Heisenberg-Weyl operators form a group whose product rule follows from the Heisenberg-Weyl commutation relation $\omega XZ = ZX$,

$$D_{\vec{x}_1} D_{\vec{x}_2} = \tau^{\langle \vec{x}_1 \cdot \vec{x}_2 \rangle} D_{\vec{x}_1 + \vec{x}_2}, \tag{2}$$

where $\langle \vec{x}_1 \cdot \vec{x}_2 \rangle$ is the symplectic inner product $\langle \vec{x}_1 \cdot \vec{x}_2 \rangle = z_1 x_2 - x_1 z_2$.

The generators of the Clifford group on qudits are P, H, and controlled-NOT (CNOT), where $P|j\rangle = \omega^{j(j-1)/2}|j\rangle$, $H|j\rangle = d^{-1/2}\sum_k \omega^{jk}|k\rangle$, and CNOT $|j, k\rangle = |j, k \oplus j\rangle$. We can also write any single-qudit Clifford unitary as $C_{F,\vec{\chi}} = D_{\vec{\chi}} U_F$, where $\vec{\chi} = (x, z)$ and F is a 2×2 matrix with entries modulo d. We will make particular use of matrices $C_{\gamma,\vec{\chi}} = D_{\vec{\chi}} U_\gamma$ for $U_\gamma|k\rangle = \tau^{\gamma k^2}|k\rangle$. The order of $C_{\gamma,\vec{\chi}}$ is d. The Clifford group is reviewed in more detail in Appendix A.

[1] An elegant recent presentation of this result in group-theoretic terms is given in [9] and is briefly summarized in [10].

Qudit stabilizer states can be prepared from a logical basis state by a qudit Clifford circuit. The Gottesman-Knill theorem generalizes to qudits and qudit stabilizer computations allow efficient classical simulation [26]. Qudit stabilizer states possess canonical forms in the logical basis just as in the qubit case [27–29].

The remaining generalization we require is an efficient classical algorithm for obtaining the inner product of two stabilizer states. This is required by the algorithm of Bravyi and Gosset and the qubit case was given in [4]. We give an $O(n^3)$ algorithm for the inner product of two n-qudit stabilizer states based on Gauss sums in Appendix F.

The qudit T gate was defined in [10,30] as a diagonal gate U_T that maps Pauli operators to Clifford operators. Its action is specified by the image of $X = D_{(1,0)}$ under U_T. Magic states are then eigenvectors of this image. Let the eigenstate of X with eigenvalue ω^k be $|+_k\rangle$; then the magic states are $U_T\,|+_k\rangle$. This approach is that taken by Howard and Vala in [30].

The image of X under U_T can be written (up to a phase) as $C = XP^{\gamma}Z^{\xi}$ for γ and ξ integers modulo d. The effect of nonzero ξ is simply to reorder the eigenvectors and hence we can choose $\xi = 0$. Similarly, the eigenvectors for $\gamma > 1$ and $\gamma = 1$ are related by application of $P^{\gamma-1}$, a Clifford operator. We can therefore specialize to the case $\gamma = 1$ and $\xi = 0$, and the gate with action

$$C_d = M_d X M_d^{\dagger} = \begin{cases} e^{2\pi i/9}XP, & d = 3 \\ \omega^{-\bar{3}}XP, & d > 3, \end{cases} \tag{3}$$

where $\bar{3}$ indicates the multiplicative inverse of 3 modulo d. This is the gate defined by Campbell *et al.* in [10]. The qudit magic states are reviewed in more detail in Appendix B.

The definition of magic states allows one to replace a Clifford + T circuit with a Clifford circuit with injected magic states [13,14]. This construction was extended to qudits in [30] and we review it in Appendix D. In Sec. III we will review the Bravyi-Gosset algorithm for qubits, which we will generalize to qudits.

III. BRAVYI-GOSSET ALGORITHM

Bravyi and Gosset gave algorithms for both weak and strong simulation in [4,12]. A strong simulation outputs the probability of measuring output x from a given Clifford + T circuit. A weak simulation algorithm generates samples from the probability distribution over outputs of a given Clifford + T circuit. Here we review the weak simulation algorithm. A brief summary of relevant features of the strong simulation algorithm is given in Appendix C.

The key advantage of weak simulation is that one can sample from a $\tilde{P}_{\rm out}(x)$ that is close enough to the actual $P_{\rm out}(x)$. Bravyi and Gosset devised a method to approximate the t-qubit magic state $|A^{\otimes t}\rangle$, where $|A\rangle = 2^{-1/2}(|0\rangle + e^{i\pi/4}\,|1\rangle)$, with a superposition of fewer than 2^t stabilizer states.

The approximate stabilizer rank χ' is defined as the minimal stabilizer rank (defined in [31] and reviewed in Appendix C) of a state $|\psi\rangle$ that satisfies $|\langle\psi|A^{\otimes t}\rangle| \geqslant 1-\delta$. A close approximation to the tensor product of magic states means a close approximation to the action of a Clifford + T circuit realized by magic state injection [4]. Therefore, $\tilde{P}_{\rm out}(x)$ will be close enough to $P_{\rm out}(x)$ if δ is small enough.

The sampling procedure given by Bravyi and Gosset relies on standard computations of stabilizers. The extension of such computations to $d > 2$ have long been well understood [26]. We will therefore refer the reader to [4] for details of these procedures which, *mutatis mutandis*, can be applied in the qudit case and focus on the approximate stabilizer rank.

We begin by reviewing the approximate stabilizer rank construction from [4]. From the magic state $|A\rangle$ defined above one can construct the equivalent magic state

$$|H\rangle = e^{-\pi i/8}PH|A\rangle = \cos(\pi/8)\,|0\rangle + \sin(\pi/8)\,|1\rangle\,. \tag{4}$$

The state $|H\rangle$ can be decomposed into a sum of nonorthogonal stabilizer states as

$$|H\rangle = \frac{1}{2\cos\pi/8}(|\tilde{0}\rangle + |\tilde{1}\rangle), \tag{5}$$

where $|\tilde{0}\rangle = |0\rangle$ and $|\tilde{1}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Then $|H^{\otimes t}\rangle$ can be rewritten as

$$|H^{\otimes t}\rangle = \frac{1}{[2\cos(\pi/8)]^t}\sum_{x\in\mathcal{F}_2^t}|\tilde{x}\rangle\,. \tag{6}$$

The weak simulation algorithm reduces the number of stabilizer states required by approximating $|H^{\otimes t}\rangle$. This approximation $|H^{\otimes t*}\rangle$ is constructed by taking a subspace $\mathcal{L}$ of $\mathcal{F}_2^t$:

$$|H^{\otimes t*}\rangle = \frac{1}{[2\cos(\pi/8)]^t}\sum_{x\in\mathcal{L}}|\tilde{x}\rangle\,. \tag{7}$$

The stabilizer rank of this approximation state is the number of elements in $\mathcal{L}$, which is 2^k. The random subspace $\mathcal{L}$ is chosen so that $|H^{\otimes t*}\rangle$ satisfies

$$\langle H^{\otimes t*}\rangle\, H^{\otimes t} \leqslant 1-\delta. \tag{8}$$

It is useful to discuss the subspaces of $\mathcal{F}_2^t$ in the language of d-ary linear codes. Here $\mathcal{L}$ is a k-dimensional binary linear code which can be specified by k generators of length t. These generators can be written in a standard form as a $k\times t$ matrix $\{1_k|G\}$, where 1_k is the $k\times k$ identity matrix and G is a $k\times(t-k)$ matrix. Sampling random subspaces of $\mathcal{F}_2^t$ is therefore equivalent to sampling matrices G.

The algorithm of Bravyi and Gosset achieves an improved scaling of $\cos(\pi/8)^{-2t} \simeq 2^{0.23t}$ for weak simulation over $2^{0.47t}$ for strong simulation. In Secs. IV and V we will see more details of how to bound the scaling while we extend this approximate rank and weak simulation scheme to qudits.

IV. NONORTHOGONAL DECOMPOSITIONS OF QUDIT MAGIC STATES

The qudit magic state we want to decompose is an eigenvalue-1 eigenstate of the Clifford operator C_d as defined by Eq. (3). We choose a stabilizer state $|\tilde{0}\rangle$ with nonzero inner product with the magic state and act on it with powers of C_d to obtain d stabilizer states $\{|\tilde{j}\rangle = C_d^j\,|\tilde{0}\rangle,\ j = 0,\dots,d-1\}$. We know that these stabilizer states are distinct because if any pair were equal then the original state $|\tilde{0}\rangle$ would be an eigenstate of the Clifford operator and hence a magic state. The sum of these d states forms a decomposition of the magic state (up to

TABLE I. Matrices M_d, optimal value of p, and approximate stabilizer rank scaling comparison for $d = 2, 3, 5, 7$. Here $\kappa = -2\log_d \alpha$, so $d^{\kappa t} = \alpha^{-2t}$. Here the ω for $d = 5$ and $d = 7$ rows are $e^{2\pi i/5}$ and $e^{2\pi i/7}$, respectively.

d	M_d	p	$\lvert\alpha(d)\rvert$	$\lvert\alpha(d)\rvert$	$d^{\kappa t}$
2	$\mathrm{diag}(1, e^{i\pi/4})$	0	$\cos\pi/8$	0.92388	$2^{0.23t}$
3	$\mathrm{diag}(e^{2\pi i/9}, 1, e^{-2\pi i/9})$	0	$\frac{1+2\cos(2\pi/9)}{3}$	0.84403	$3^{0.31t}$
5	$\mathrm{diag}(\omega^{-2}, \omega, \omega^{-1}, \omega^{-2}, \omega^{-1})$	4	$\frac{3+2\cos(2\pi/5)}{5}$	0.723607	$5^{0.40t}$
7	$\mathrm{diag}(\omega^3, \omega^{-2}, 1, \omega^3, \omega^1, \omega^2, 1)$	3	$\frac{1+6\cos(2\pi/7)}{7}$	0.677277	$7^{0.40t}$

a possible global phase). Because C_d has order d, this state is by construction an eigenvalue-1 eigenstate of C_d.

The d stabilizer states in the decomposition form an orbit around the magic state. This construction was discussed previously in [32]. There are $d(d+1)$ single-qudit stabilizer states [33], partitioned into $d+1$ orbits, each orbit giving a decomposition of the magic state. Every state in each orbit has the same overlap with the magic state

$$\langle\tilde{j}\rangle M_d = \langle\tilde{j}|\, C_d^\dagger C_d \,|M_d\rangle = \langle\widetilde{j+1}\rangle M_d, \tag{9}$$

where the qudit magic state is $|M_d\rangle = M_d\,|+\rangle$. This property is a generalization of $\langle\tilde{0}\rangle H = \langle\tilde{1}\rangle H = \cos\pi/8$ for the qubit case. The overlaps of the elements of the nonorthogonal basis are given by $|\langle\tilde{0}|\tilde{j}\rangle| = \frac{1}{\sqrt{d}}$ for all j, i.e.,

$$|\langle\tilde{j}|\tilde{k}\rangle|^2 = \frac{1+(d-1)\delta_{j,k}}{d}. \tag{10}$$

This expression is that for states in a symmetric, informationally complete, positive-operator-valued measure (SIC-POVM), and the construction here is similar to the generation of such states from a fiducial state [34,35]. Here we only obtain d states however. See Appendix G for the evaluation of the phase of $\langle\tilde{j}|\tilde{k}\rangle$.

The states $|+_p\rangle = Z^p\,|+\rangle$ are representatives of the d orbits, each of which generated by C_d. This is because $C_d^a\,|+_p\rangle \neq |+_q\rangle$ for any a, p, and q, which follows simply from the action of C_d in the logical basis. Further, C_d applies phases quadratic in j to $|j\rangle$ followed by a shift. This cannot be equal to a state generated from $|+\rangle$ by any power of Z, which can only apply phases linear in j to $|j\rangle$.

From the orbit representatives we can determine the inner product of the states in the orbit with the magic state. This is given by

$$\alpha = \langle +|\, Z^{-p}\,|M_d\rangle = \langle +|\, Z^{-p} M_d\,|+\rangle = \frac{1}{d}\mathrm{Tr}(Z^{-p}M_d). \tag{11}$$

This is a cubic Gauss sum which can be written

$$\alpha = \frac{\omega^{\frac{1}{d}\binom{d}{4}-p}}{d}\sum_{l=0}^{d-1}\omega^{\bar{6}l[l^2+\psi(p,d)]} \quad d > 3. \tag{12}$$

For the $d = 3$ case, the magnitude and phase of this cubic Gauss sum, and $\phi(p, d)$, are computed in Appendix E. The sum is real, although not necessarily positive. Although we do not obtain a closed form for this sum, we can compute the integer value of p which maximizes its absolute value for a given d. These values are tabulated for small d in Table I. The complete form of the nonorthogonal decomposition is

$$|M_d\rangle = \pm\frac{\omega^{\frac{1}{d}\binom{d}{4}-p}}{d|\alpha|}\sum_j C_d^j\,|\tilde{0}\rangle, \tag{13}$$

which is the generalization of Eq. (5) to arbitrary d.

V. WEAK SIMULATION AND APPROXIMATE STABILIZER RANK

In order to get an approximation for $|M^{\otimes t}\rangle$, we can follow the method of Bravyi and Gosset for the qubit case, taking a k-dimensional subspace of $\mathcal{F}_d^t$:

$$|M^{\otimes t*}\rangle = |\mathcal{L}\rangle = \frac{1}{\sqrt{d^k Z(\mathcal{L})}}\sum_{x\in\mathcal{L}}|\tilde{\vec{x}}\rangle. \tag{14}$$

Here we label the state by $\mathcal{L}\subset\mathcal{F}_d^t$, a k-dimensional code subspace of $\mathcal{F}_d^t$, and $Z(\mathcal{L})$ is a normalization factor. Comparison with Eq. (13) shows that $Z(F_d) = d|\alpha|^2$. We require

$$|\langle\mathcal{L}|M^{\otimes t}\rangle|^2 = \frac{d^k|\alpha|^{2t}}{Z(\mathcal{L})} \geqslant 1-\delta \tag{15}$$

for a given δ, where the first equality follows from Eq. (9) and where

$$Z(\mathcal{L}) = \sum_{\vec{x}\in\mathcal{L}}\langle\tilde{0}^{\otimes t}|\, C_{\vec{x}}\,|\tilde{0}^{\otimes t}\rangle \tag{16}$$

for $C_{\vec{x}} = C^{x_1}\otimes C^{x_2}\otimes\cdots\otimes C^{x_t}$, $x_i\in\mathcal{F}_d$.

Selection of the subspace $\mathcal{L}$ depends on two factors. First, we choose the dimension of $\mathcal{L}$ by setting k:

$$k = \lceil 1 - 2t\log_d|\alpha| - \log_d\delta\rceil. \tag{17}$$

Note that the maximum precision that can be required from the method for given t is obtained by setting $k = t$, so $\delta_{\max} = 2^{-t(1+2\log_d|\alpha|)+1}$.

Next we find an $\mathcal{L}$ for which $Z(\mathcal{L})$ is not too large. The probability of obtaining a small enough $Z(\mathcal{L})$ can be analyzed as in [4] by evaluating the expectation value of $Z(\mathcal{L})$ over all possible $\mathcal{L}\in\mathcal{F}_d^t$:

$$\begin{aligned} E(Z(\mathcal{L})) &= 1 + \sum_{\vec{x}\in\mathcal{F}_d^t/\{0\}}\langle\vec{\tilde{0}}^t|\, C_{\vec{x}}\,|\vec{\tilde{0}}^t\rangle E(I_{\mathcal{L}}(\vec{x})) \\ &= 1 + \frac{(d^k-1)}{(d^t-1)}[Z(F_d)^t - 1] \\ &= 1 + \frac{d^k-1}{d^t-1}(d^t|\alpha|^{2t}-1) \\ &\leqslant 1 + d^k|\alpha|^{2t}. \end{aligned} \tag{18}$$

Here $I_{\mathcal{L}}(\vec{x})$ is a indicator function, i.e., it is equal to 1 when $x \in \mathcal{L}$ and 0 otherwise. The second equals sign stands because the expectation value of $I_{\mathcal{L}}(x)$ for a fixed x is $\frac{d^k-1}{d^t-1}$ and

$$\sum_{x\in\mathcal{F}_d^t/\{0\}} \langle\vec{\tilde{0}}^t|\, C_{\vec{x}}\, |\vec{\tilde{0}}^t\rangle = \sum_{x\in\mathcal{F}_d^t} \langle\vec{\tilde{0}}|\, C_{\vec{x}}\, |\vec{\tilde{0}}\rangle - 1$$

$$= \left(\langle\tilde{0}| \sum_{x=0}^{d-1} C^x\, |\tilde{0}\rangle\right)^t - 1. \qquad (19)$$

From Eq. (17) we have $d^k|\alpha|^{2t} = O(1)$, so $E(Z(\mathcal{L})) = O(1)$. Therefore, from Markov's inequality we obtain

$$\text{Prob}[Z(\mathcal{L}) \leqslant (1 + d^k|\alpha|^{2t})(1+\delta)]$$
$$> 1 - \frac{E(Z(\mathcal{L}))}{(1 + d^k|\alpha|^{2t})(1+\delta)}$$
$$\geqslant 1 - \frac{1}{1+\delta} > \delta. \qquad (20)$$

Randomly choosing δ^{-1} subspaces gives an $\mathcal{L}$ such that

$$Z(\mathcal{L}) \leqslant (1 + d^k|\alpha|^{2t})(1+\delta), \qquad (21)$$

hence satisfying Eq. (15), with high probability.

The upper bound for the approximate stabilizer rank of a t-qudit magic state given by the above method is

$$\chi'(t) = d^k = O(\delta^{-1}|\alpha|^{-2t}). \qquad (22)$$

In the qubit case an explicit sum formula was given for $Z(\mathcal{L})$ with 2^k terms and hence the cost of evaluating $Z(\mathcal{L})$ is $O(2^k)$. What is the cost of evaluating $Z(\mathcal{L})$ for arbitrary d? In Appendix G we give an explicit formula for $Z(\mathcal{L})$ as a sum of products and hence the cost of evaluating $Z(\mathcal{L})$ for arbitrary d is $O(d^{k+1})$.

VI. DISCUSSION

The motivation to study the qudit generalizations of stabilizer rank algorithms such as those in [4,12] is to enable comparison with other simulation algorithms. In [22], the authors apply Monte Carlo sampling on trajectories of the quasiprobability representation to estimate the probability of a measurement outcome. They find that the hardness of this strong simulation depends on the total negativity (negativity of the inputs, gates, and measurements) of the circuit. Specifically, the cost of the algorithm scales with the square of the total negativity.

For Clifford + T circuits that are gadgetized so that the circuit is realized by Clifford gates with magic state injection, the negativity of the circuit only comes from the ancilla inputs of magic states. If we apply the method of [22] to the gadgetized circuit with an input of t-qutrit magic states, the cost scales as $3^{0.84t}$. This result is obtained by calculating the negativity of a single-qutrit magic state.

In the present paper we obtain a scaling of $3^{0.32t}$ for weak simulation of qutrit Clifford + T circuits. This shows that weak simulation using the approximate rank method has superior scaling to strong simulation using the method of [22]. A stabilizer rank based strong simulation algorithm for qudits would require new results on an exact stabilizer rank of qudit magic states, a topic for future work. Recent progress in extending the qubit case has been reported in [12], and improvements to Pashayan's algorithm using a discrete system generalization of the stationary phase approximation were given in [36].

It should be noted that one should not think of weak simulation as easy and strong simulation as hard. The difficulty of weak and strong simulations is a property of the distribution being sampled or computed. In some cases, such as quantum supremacy, we expect the difficulties of weak and strong simulations to coincide [3].

If we consider negativity and stabilizer rank as two measures of quantumness, we can see that they differ. Bravyi *et al.* [31] conjectured that the magic state has the smallest stabilizer rank out of the nonstabilizer states. However, the quasiprobability of the magic state has the largest negativity. In fact, Howard and Campbell also noticed this disagreement between stabilizer rank and robustness of magic [37]. It is worth noting the differences between stabilizer rank and approximate stabilizer rank. Namely, the approximate stabilizer rank seems to agree with other measures of quantumness such as negativity or robustness of magic in that it reaches a maximum at the magic state and a minimum on stabilizer states. The exact stabilizer rank does not share these properties. This makes the investigation of the difference between exact and approximate stabilizer ranks interesting.

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APPENDIX A: QUDIT CLIFFORD GROUP

We recall that d is an odd prime. In a d-dimensional system the Pauli operators X and Z are defined as

$$X = \sum_{j\in F_d} |j \oplus 1\rangle\langle j|\,, \quad Z = \sum_{j\in F_d} \omega^j\, |j\rangle\langle j|\,, \qquad (A1)$$

where $\omega = \exp(2\pi i/d)$. These operators obey the Heisenberg-Weyl commutation relation

$$\omega XZ = ZX. \qquad (A2)$$

In d dimensions the Weyl-Heisenberg displacement operators are defined by

$$D_{\vec{x}} = \tau^{xz} X^x Z^z, \qquad (A3)$$

where $\vec{x} = (x, z)$ and $\tau = e^{(d+1)\pi i/d} = \omega^{2^{-1}}$. The qubit Pauli operators are recovered from this expression for $d = 2$, with $D_{(1,0)} = X$, $D_{(0,1)} = Z$, and $D_{(1,1)} = -Y$. The Heisenberg-Weyl operators form a group with multiplication rule

$$D_{\vec{x}_1} D_{\vec{x}_2} = \tau^{\langle\vec{x}_1\cdot\vec{x}_2\rangle} D_{\vec{x}_1+\vec{x}_2}, \qquad (A4)$$

where $\langle\vec{x}_1 \cdot \vec{x}_2\rangle$ is the symplectic inner product

$$\langle\vec{x}_1 \cdot \vec{x}_2\rangle = z_1x_2 - x_1z_2. \qquad (A5)$$

For $d > 2$ the Weyl-Heisenberg operators are unitary but not generally Hermitian.

In the qubit case, the Clifford gates map Pauli operators to Pauli operators. In the qudit case Clifford gates map Weyl-Heisenberg operators to one another. The generators of the Clifford group are defined so that the Hadamard gate maps $X \to Z$ and the phase gate maps $X \to XZ$. The generators of the single-qubit Clifford group are

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, \quad P = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}. \tag{A6}$$

The d-dimensional Clifford operators are generated by

$$P = \sum_{j\in F_d} \omega^{j(j-1)/2} |j\rangle\langle j|, \quad H = \sum_{j,k} \omega^{jk} |j\rangle\langle k| /\sqrt{d}, \tag{A7}$$

and

$$\text{CNOT} = \sum_j |j\rangle\langle j| \otimes X^j. \tag{A8}$$

The single-qudit Clifford group is isomorphic to the semidirect product group of $SL(2, Z_d)$ and $(Z_d)^2$ [35,38].[2] We can represent the Clifford group using a 2×2 matrix F and a 2 vector $\vec{\chi}$, both with entries in Z_d:

$$\mathcal{C} = \{C_{(F|\vec{\chi})} \mid F \in SL(2, Z_d), \vec{\chi} \in {Z_d}^2\}. \tag{A9}$$

Specifically, a Clifford unitary is given as

$$C_{(F|\vec{\chi})} = D_{\vec{\chi}} U_F, \tag{A10}$$

where if

$$F = \begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}, \quad \vec{\chi} = \begin{bmatrix}x\\ z\end{bmatrix}, \tag{A11}$$

then

$$U_F = \frac{1}{\sqrt{d}} \sum_{j,k=0}^{d-1} \tau^{\beta^{-1}(\alpha k^2 - 2jk + \delta j^2)} |j\rangle\langle k| \tag{A12}$$

if $\beta \neq 0$ and

$$U_F = \sum_{k=0}^{d-1} \tau^{\alpha\gamma k^2} |\alpha k\rangle\langle k| \tag{A13}$$

if $\beta = 0$ [35].

The multiplication rule is

$$C_{(F_1|\vec{\chi}_1)} C_{(F_2|\vec{\chi}_2)} = \tau^{\langle\vec{\chi}_1 \cdot F\vec{\chi}_2\rangle} C_{(F_1F_2|\vec{\chi}_1 + F_1\vec{\chi}_2)}. \tag{A14}$$

The action of the Clifford operators on the Heisenberg-Weyl operators in this representation can be given as

$$C_{(F|\vec{\chi})} D_{\vec{x}} C^\dagger_{(F|\vec{\chi})} = \omega^{\vec{\chi}\cdot\vec{x}} D_{F\vec{x}}. \tag{A15}$$

In particular, we are interested in Clifford operations defined by matrices of the form

$$F_\gamma = \begin{bmatrix}1 & 0\\ \gamma & 1\end{bmatrix} \tag{A16}$$

[2] $SL(2, Z_d)$ is the group of 2×2 matrices with entries from Z_d and determinant 1.

and we introduce the notation

$$C_{\gamma,\vec{\chi}} = C_{\begin{bmatrix}1 & 0\\ \gamma & 1\end{bmatrix}, \begin{bmatrix}x\\ z\end{bmatrix}} \tag{A17}$$

for $\vec{\chi} = (x, z)^T$. From Table I in Ref. [35] the order of any element $C_{\gamma,\vec{\chi}}$ is d. Clearly X, P, and Z are order d. For $d = 2$, H is order 2 and for $d > 2$, H is order 4.

The generators H and P are given by

$$F_H = \begin{pmatrix}0 & d-1\\ 1 & 0\end{pmatrix}, \quad \vec{\chi}_H = (0, 0)^T, \tag{A18}$$

which follows from $HXH^\dagger = Z$ and $HZH^\dagger = X^{-1}$, and

$$F_P = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \quad \vec{\chi}_P = (0, (d-1)/2)^T. \tag{A19}$$

These expressions for H and P allow us to construct the F and $\vec{\chi}$ for any single-qudit Clifford operation expressed as a word on the generators H and P.

APPENDIX B: QUDIT MAGIC STATES AND T GATES

To go beyond Clifford group computation it is useful to introduce the Clifford hierarchy, which classifies unitary operators by their action on the Pauli group. The Clifford hierarchy was defined by Gottesman and Chuang in [39]:

$$\mathcal{C}(k+1) = \{U \mid UPU \in \mathcal{C}(k), P \in \mathcal{P}\} \quad (k \geqslant 0). \tag{B1}$$

The first level of the Clifford hierarchy is the Pauli group $\mathcal{C}(1) = \mathcal{P}$. The Clifford group is the second level of the hierarchy, unitary operators that map the Pauli group to itself. Note that elements of the Pauli group are themselves elements of the first level of the Clifford hierarchy. The third level of the Clifford hierarchy are operators that map Pauli operators to Clifford operators. The qubit T gate is such an operator because $TXT^\dagger = PHP^2H$, a non-Pauli element of the second level of the Clifford hierarchy.

Bravyi and Kitaev first proposed qubit magic states in [13]. They define magic states as the image of $|H\rangle$ and $|T\rangle$ under single-qubit Clifford gates, where $|H\rangle$ is defined by Eq. (4) and $|T\rangle$ by

$$|T\rangle = \cos\beta\,|0\rangle + \sin\beta\, e^{i\pi/4}\,|1\rangle \tag{B2}$$

for $\cos(2\beta) = \frac{1}{\sqrt{3}}$. Here $|H\rangle$ is the eigenstate of the Hadamard gate H and $|T\rangle$ is the eigenstate of the product of Hadamard and phase gate PH.

Any magic state is equivalent as a resource to any other state obtainable from it by a Clifford operation. We can define magic states more generally as the eigenstates of Clifford operations and obtain them as follows. Taking any H-type magic state $|H\rangle$, we have

$$UHU^\dagger U|H\rangle = UH|H\rangle = \lambda U|H\rangle, \tag{B3}$$

where λ is the eigenvalue of H and U is a Clifford gate. This means that $U|H\rangle$ is the eigenstate of a new Clifford operator $UHU^\dagger$. The same is true for T-type magic states.

Campbell *et al.* [10] used this relationship between magic states and eigenvectors of Clifford operators to extend the definition of magic states to qudits [10]. Concurrently, equivalent extensions were obtained by Howard and Vala [30].

1. Qudit *T* gates

Campbell *et al.* [10] define sets of gates $\mathcal{M}_d^m$ containing all gates M with the following properties: M is diagonal, $M^{d^m}=1$, $\det M=1$ so that $M\in \mathrm{SU}(d)$, and M is in the third but not the second level of the Clifford hierarchy. Within this set of gates is the canonical $\mathcal{M}_d$ gate

$$M_d=\sum_j \exp(i2\lambda_j\pi/d^m)|j\rangle\langle j|, \qquad \text{(B4)}$$

which is defined so that it maps the X operator to a Clifford operator proportional to XP,

$$C_d=M_dXM_d^\dagger=\begin{cases} e^{2\pi i/9}XP, & d=3\\ \omega^{-\bar{3}}XP, & d>3.\end{cases} \qquad \text{(B5)}$$

Here $\bar{3}$ is the multiplicative inverse of 3 modulo d. This Clifford operator has order d.

This condition, as well as the condition $\det M=1$, gives the following form for the λ_j (see Appendix A of [10]):

$$\lambda_j=d^{m-2}\left[d\binom{j}{3}-j\binom{d}{3}+\binom{d+1}{4}\right]. \qquad \text{(B6)}$$

The parameter m determines the order d^m of the operator M. For $d=3$ the form above is valid when $m\geqslant 2$. For $d>3$ it is valid when $m\geqslant 1$.

By definition M maps X, a generalized Pauli operator, to a non-Pauli Clifford operator and so is in the third, but not the second, level of the Clifford hierarchy. We can therefore think of M as a generalized T gate.

From the definition of the matrix M in (B4) we have, for $d=3$ and $m=2$,

$$M_3=\mathrm{diag}\big(e^{i2\pi/9},1,e^{-i2\pi/9}\big) \qquad \text{(B7)}$$

and

$$M_5=\mathrm{diag}\big(e^{-4\pi i/5},e^{2\pi i/5},e^{-2\pi/i5},e^{-4\pi i/5},e^{-2\pi i/5}\big) \qquad \text{(B8)}$$

for $d=5$ and $m=1$, where $\omega=e^{2\pi i/5}$. The qudit version of the T gate M is further generalized in [30], which we will discuss below.

The T gate is also sometimes called the $\pi/8$ gate because

$$T=e^{-i\pi/8}\begin{pmatrix} e^{i\pi/8} & 0\\ 0 & e^{-i\pi/8}\end{pmatrix}. \qquad \text{(B9)}$$

Howard and Vala developed the qudit versions of this gate concurrently with the development of qudit magic states by Campbell *et al.* [10,30]. The results are equivalent and we give the details of the relationship between them here.

Howard and Vala parametrized the set of diagonal gates on a single qudit as follows:

$$U_v=U(v_0,v_1,\dots,v_{p-1})=\sum_{j=0}^{d-1}\omega^{v_k}|k\rangle\langle k| \quad (v_k\in Z_d). \qquad \text{(B10)}$$

All diagonal gates fix $D_{(0,1)}$ and so their action is completely determined by $U_vD_{(1,0)}U_v^\dagger=U_vXU_v^\dagger$. This parallels the development of Campbell *et al.*, who considered the action of their canonical gate M on the operator X and insisted that the result of that action was proportional to XP.

Howard and Vala proceeded more generally, computing the action of these diagonal matrices

$$U_vD_{(x|z)}U_v^\dagger=D_{(x|z)}\sum_k\omega^{(v_{k+1}-v_k)}|k\rangle\langle k|. \qquad \text{(B11)}$$

Given U_v is diagonal, only $U_vD_{(1|0)}{U_v}^\dagger$ is nontrivial.

Howard and Vala then considered the case that U_v is in the third level of the Clifford hierarchy so that the image of X can be written [cf. Eq. (18) in [30]]

$$U_vX{U_v}^\dagger=\omega^{\epsilon'}C_{\gamma',(1,z')^T}, \qquad \text{(B12)}$$

where $\epsilon',\gamma',z'\in Z_d$. The right-hand side here is the most general form allowed because Eq. (B11) implies that the image of X must be X times a diagonal Clifford operator and the most general form of a diagonal Clifford operator has $\vec{\chi}=(0,1)$ and $\beta=0$ and $\alpha=1$. Combining Eqs. (B11) and (B12), one obtains [cf. Eq. (19) in [30]]

$$X\sum_k\omega^{(v_{k+x}-v_k)}|k\rangle\langle k|=\omega^{\epsilon'}C_{\gamma,(1,z')^T}. \qquad \text{(B13)}$$

Howard and Vala then solved for U_v with these three parameters:

$$v_k=\bar{12}k\{\gamma'+k[6z'+(2k-3)\gamma']\}+k\epsilon'. \qquad \text{(B14)}$$

This analysis is equivalent to that performed in Ref. [10], Appendix A.

The $d=3$ case as usual presents some special difficulties. In the Campbell *et al.* analysis one must choose $m=2$ for λ as there are no Clifford operators with $m=1$ and $d=3$ [10].

The set of operators U_v for $d=3$ is given by

$$U_v=\sum_{k=0}^{2}\xi^{v_k}|k\rangle\langle k|, \qquad \text{(B15)}$$

where $\xi=e^{2\pi i/9}$. The v_k are given by

$$v=(v_0,v_1,v_2)=(0,6z'+2\gamma'+3\epsilon',6z'+\gamma'+6\epsilon'), \qquad \text{(B16)}$$

where all operations can be taken modulo 9. The determinant of U_v for $d=3$ can be computed from the definition

$$\det U_v=\exp\left(\frac{2\pi i}{9}\sum_k=0^2v_k\right)=\exp\left(\frac{2\pi i}{3}(z'+\gamma')\right),$$

showing that U_v is not in SU(3) for $d=3$.

We can relate the diagonal operators U_v defined by Howard and Vala and the operators M defined by Campbell *et al.* as follows. Writing

$$M=\sum_{k=0}^{d-1}\exp\left(\frac{2\pi i}{d^m}\lambda_k\right)|k\rangle\langle k|=\sum_{k=0}^{d-1}\omega^{\lambda_k/d^{m-1}}|k\rangle\langle k| \qquad \text{(B17)}$$

and

$$U_v=\sum_{k=0}^{d-1}\omega^{v_k}|k\rangle\langle k|, \qquad \text{(B18)}$$

we wish to compare

$$\frac{\lambda_k}{d^{m-1}}=\frac{1}{d}\left[d\binom{k}{3}-k\binom{d}{3}+\binom{d+1}{4}\right] \qquad \text{(B19)}$$

and

$$v_k = \overline{12}k\{\gamma' + k[6z' + (2k-3)\gamma']\} + k\epsilon'. \quad \text{(B20)}$$

These are both cubic in k, so we can find the particular U_v that corresponds to M by equating the coefficients. We begin by setting $k = 0$ to find the constant term. We immediately obtain

$$v_0 = 0, \quad \frac{\lambda_0}{d} = \frac{1}{d}\binom{d+1}{4}. \quad \text{(B21)}$$

We conclude that U_v and M will only be equivalent up to a global phase determined by this convention.

Equating the cubic terms yields $\gamma' = 1$. Equating the quadratic terms gives

$$z' - \frac{\gamma'}{2} = d - 1, \quad \text{(B22)}$$

so $z' = (d-1)/2$. Finally, equating the linear terms gives

$$\epsilon' = \overline{12}(6d - 2d^2 - 1). \quad \text{(B23)}$$

We may therefore relate $U_v(z', \gamma', \epsilon')$ and M for arbitrary $d > 3$ as follows:

$$M_d = \omega^{\frac{1}{d}\binom{d+1}{4}} U_v((d-1)/2, 1, \overline{12}(6d - 2d^2 - 1)). \quad \text{(B24)}$$

The first two cases of this equivalence are for $d = 5$ and $d = 7$ and, up to a global phase, are as given in Eqs. (70) and (71) of [30].

The case of $d = 3$ is distinct ($\overline{12}$ does not exist modulo 3), but from the definition of U_v for $d = 3$ given in Eqs. (B15) and (B16) we have

$$M_3 = e^{2\pi i/9} U_v(1, 1, 0). \quad \text{(B25)}$$

This is, up to a global phase, as given in Eq. (69) of [30].

2. Qudit magic states

The gates M also allow us to find eigenstates of C_M as follows. Define the state $|M_k\rangle = M|+_k\rangle$, where $|+_k\rangle$ is the eigenstate of X with eigenvalue ω^k. We can calculate

$$\begin{aligned} C_M |M_k\rangle &\propto MXM^\dagger |M_k\rangle \\ &= MXM^\dagger M |+_k\rangle \\ &= \omega^k M |+_k\rangle \\ &= \omega^k |M_k\rangle. \end{aligned} \quad \text{(B26)}$$

Given Eq. (B12), Howard and Vala recovered the definition of the magic states of Campbell *et al.* and showed that these magic states $U_v|+\rangle$ are eigenstates of $C_{\gamma',(1,z')^T}$ with eigenvalue $\omega^{-\epsilon'}$:

$$\begin{aligned} C_{\gamma',(1,z')^T} U_v |+\rangle &= \omega^{-\epsilon'} U_v D_{(1|0)} U_v{}^\dagger U_v |+\rangle \\ &= \omega^{-\epsilon'} U_v D_{(1|0)} |+\rangle \\ &= \omega^{-\epsilon'} U_v |+\rangle. \end{aligned} \quad \text{(B27)}$$

APPENDIX C: STRONG SIMULATION FOR QUBITS

We review here the strong simulation algorithm given by Bravyi and Gosset in [4]. Let t be the number of T gates in the n-qubit quantum circuit we wish to classically simulate. The first step is to replace every T gate in the circuit by Clifford gates and an ancilla input of a magic state $|A\rangle$, defined in [13] as

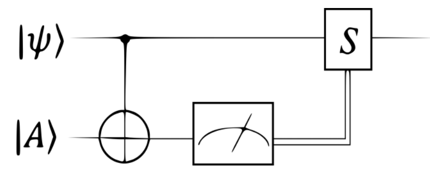

FIG. 1. Gadget to implement a T gate using an ancilla magic state $|A\rangle$ as defined in [14]. Using this gadget, universal quantum computation can be achieved using a Clifford circuit with injected magic states.

$$|A\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi/4}|1\rangle). \quad \text{(C1)}$$

This is accomplished using the gadget shown in Fig. 1 [14]. The number of ancilla qubits is t. We consider an initial state $|0^{\otimes n}\rangle$ for the Clifford + T circuit and $|0^{\otimes n}\rangle \otimes |A^{\otimes t}\rangle$ for the gadgetized circuit.

At the end of the computation we will measure w of the n qubits in the logical basis. This measurement with outcome x (where x is a bit string of length w), postselected to the case where all ancilla measurements have result 0, is represented by a projector $\Pi(x) = |x\rangle\langle x| \otimes \mathbf{1} \otimes |0^t\rangle\langle 0^t|$. The strong simulation algorithm classically computes the probability of this measurement outcome after acting with a Clifford circuit V, which is our original (non-Clifford) circuit with all T gates replaced by the gadget of Fig. 1. Therefore, we can express the probability of obtaining output x as

$$P(x) = 2^t \langle 0^n A^t | V^\dagger \Pi V | 0^n A^t\rangle. \quad \text{(C2)}$$

The factor of 2^t here compensates for the fact that we postselected the measurement outcomes of the t ancilla qubits.

We define a t-qubit projection operator $\Pi_G = \langle 0^n | V^\dagger \Pi V | 0^n\rangle$. This projector maps states onto a stabilizer subspace. Then Eq. (C2) becomes

$$P(x) = 2^t \langle 0^n A^t | V^\dagger \Pi V | 0^n A^t\rangle = 2^{-u}\langle A^t | \Pi_G | A^t\rangle, \quad \text{(C3)}$$

where u is an integer that depends on the number of qubits we are measuring out of n and the dimension of the stabilizer subspace Π_G is mapping onto.

If we can expand $|A^t\rangle$ into a sum of stabilizer states, then we can express $P(x)$ as a sum of inner products of t-qubit stabilizer states, which can be computed in $O(t^3)$ time [4,8,31,40]. The fewer stabilizer states in the expansion of $|A^t\rangle$, the more efficient the algorithm.

Stabilizer rank is defined as the minimal number of stabilizer states needed to write a pure state as a linear combination of stabilizer states. The value of $\chi(t)$ is trivially upper bounded by 2^t because logical basis states are stabilizer states, and $\chi(t)$ is also believed to be lower bounded by an exponential in t. For practical purposes we can achieve progress through a series of constructive upper bounds.

In [31], Bravyi *et al.* found a stabilizer rank upper bound by obtaining $\chi_A(6) \leqslant 7$ for $|A^6\rangle$ and dividing the t-qubit state into a product of six-qubit states. Therefore, $\chi_A(t)$ has a upper bound $7^{t/6} \simeq 2^{0.47t}$. If we denote the stabilizer rank for the tensor product of t single-qubit magic states $|A^t\rangle$ by $\chi_A(t)$, the cost of classically computing $P(x)$ by taking inner products as described above is $O(t^3\chi_A(t)^2)$.

The quadratic dependence on stabilizer rank can be improved by a Monte Carlo method, developed by Bravyi and Gosset, to approximate the norm of a tensor product of magic states projected on a stabilizer subspace:

$$|\langle A^t|\,\Pi_G\,|A^t\rangle| = \|\Pi_G\,|A^t\rangle\|^2 = \|\psi\|^2, \tag{C4}$$

therefore enabling one to calculate $P(x)$ with cost $O(t^3\chi_A(t))$, linear in stabilizer rank. This concludes our summary of the strong simulation algorithm of Bravyi and Gosset.

APPENDIX D: QUDIT T GATE GADGET

We also require a gadget that substitutes a qudit T gate by an injected qudit magic state and Clifford gates. The qudit gadget was introduced by Howard and Vala and is shown in Fig. 2.

Howard and Vala also generalized the qubit T-gate gadget to qudits for their magic state construction [30]. We reproduce their gadget here in the interest of making the paper self-contained.

In order to project a qudit state onto the eigenstate of operator P with eigenvalue ω^k, the projection operator can be written as

$$\Pi_{(P|k)} = \frac{1}{d}(I + \omega^{-k}P + \omega^{-2k}P^2 + \cdots + \omega^{-(d-1)k}P^{d-1}). \tag{D1}$$

By analogy with the qubit case, we need a gadget that allows us to implement the qudit U_v gate by injecting magic states. It is straightforward to check that

$$\text{CSUM}^{-1}\Pi_{(0,0|1,d-1)[0]}(|\psi\rangle|\psi_{U_v}\rangle) = U_v|\psi\rangle|0\rangle \tag{D2}$$

performs this task for a given arbitrary state $|\psi\rangle$, where $|\psi_{U_v}\rangle = U_v\,|+\rangle$ is the magic state and Π is a rank-p projector defined by

$$\Pi_{(0,0|1,d-1)[0]} = \frac{1}{d}[I + Z\otimes Z^{-1} + \cdots + (Z\otimes Z^{-1})^{d-1}]. \tag{D3}$$

This projection is equivalent to measuring the $Z\otimes Z^{-1}$ observable to get eigenvalue 1. If we get eigenvalue ω^k, we perform an X^{-k} on the first qudit state to recover it back to the 1 eigenspace. In fact, this gadget works for implementing any diagonal gate U by injecting the state $U\,|+\rangle$.

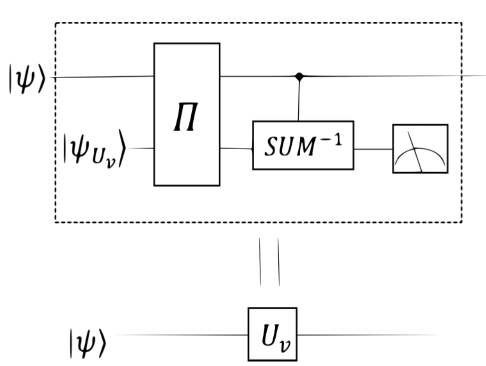

FIG. 2. Gadget for qudit U_v gate.

APPENDIX E: MAGNITUDE AND PHASE OF THE MAGIC STATE INNER PRODUCT WITH ORBIT REPRESENTATIVES OF NONORTHOGONAL DECOMPOSITIONS

Here we compute Eq. (12). We begin with $d = 3$. In this case we need only tabulate the inner product for three values of p,

$$\begin{aligned}\langle +|\,Z^{-p}\,|M_3\rangle &= \langle +|\,Z^{-p}M_3\,|+\rangle\\ &= \frac{1}{d}\mathrm{Tr}(Z^{-p}M_3)\\ &= \frac{1}{d}\big(e^{2\pi i/9} + e^{-2\pi i p/3} + e^{2\pi i/3(2p-1/3)}\big),\end{aligned} \tag{E1}$$

giving

$$\begin{aligned}\langle +|\,Z^0\,|M_3\rangle &= \frac{1}{3}\left[1 + 2\cos\left(\frac{2\pi}{9}\right)\right],\\ \langle +|\,Z^{-1}\,|M_3\rangle &= \frac{1}{3}e^{i\pi/3}\left[2\cos\left(\frac{\pi}{9}\right) - 1\right],\\ \langle +|\,Z^{-2}\,|M_3\rangle &= \frac{1}{3}e^{2i\pi/3}\left[1 + 2\cos\left(\frac{4\pi}{9}\right)\right].\end{aligned} \tag{E2}$$

The largest magnitude overlap is obtained for $p = 0$.

Now we consider generally prime $d > 3$. Given the expression for M_d we can write

$$\langle +|\,Z^{-p}\,|M_d\rangle = \frac{\omega^{\frac{1}{d}\binom{d+1}{4}}}{d}\sum_{j=0}^{d-1}\omega^{\phi(j)}, \tag{E3}$$

where $\phi(j)$ is a cubic in j given by

$$\phi(j) = \binom{j}{3} - \frac{j}{d}\binom{d}{3} - pj. \tag{E4}$$

The evaluation of cubic Gauss sums is not as straightforward as for quadratic Gauss sums. However, we can obtain a closed form for the phase of the sum, up to a sign, by depressing the cubic to remove the quadratic term. In this case this is particularly simple,

$$\begin{aligned}\phi'(j) &= \phi(j+1)\\ &= \binom{j+1}{3} - \frac{j+1}{d}\binom{d}{3} - pj - p\\ &= \bar{6}j[j^2 - 1 - (d-1)(d-2) - 6p] - \frac{1}{d}\binom{d}{3} - p\\ &= \bar{6}j[j^2 - \psi(d,p)] - \frac{1}{d}\binom{d}{3} - p,\end{aligned} \tag{E5}$$

where

$$\psi(d,p) = d^2 - 3d + 3 + 6p. \tag{E6}$$

Then

$$\langle +|\,Z^{-p}\,|M_d\rangle = \frac{\omega^{\frac{1}{d}\binom{d+1}{4} - \frac{1}{d}\binom{d}{3} - p}}{d}\sum_{j=0}^{d-1}\omega^{\bar{6}j[j^2-\psi(d,p)]}. \tag{E7}$$

The magnitude of this expression can be determined from the sum, which is real:

$$
\begin{aligned}
S &= \frac{1}{d}\sum_{j=0}^{d-1}\omega^{\bar{6}j[j^2-\psi(d,p)]}\\
&= \frac{1}{d}+\frac{1}{d}\sum_{j=1}^{(d-1)/2}\omega^{\bar{6}j[j^2-\psi(d,p)]}+\frac{1}{d}\sum_{j=(d+1)/2}^{d-1}\omega^{\bar{6}j[j^2-\psi(d,p)]}\\
&= \frac{1}{d}+\frac{1}{d}\sum_{j=1}^{(d-1)/2}(\omega^{\bar{6}j[j^2-\psi(d,p)]}+\omega^{\bar{6}(d-j)[(d-j)^2-\psi(d,p)]})\\
&= \frac{1}{d}+\frac{1}{d}\sum_{j=1}^{(d-1)/2}(\omega^{\bar{6}j[j^2-\psi(d,p)]}+\omega^{-\bar{6}j[j^2-\psi(d,p)]})\\
&= \frac{1}{d}+\frac{2}{d}\sum_{j=1}^{(d-1)/2}\cos\frac{2\pi}{d}\bar{6}j[j^2-\psi(d,p)]. \qquad \text{(E8)}
\end{aligned}
$$

While this shows that the sum is real, it does not guarantee that it is positive and hence the phase of the inner product, up to a sign, is given by

$$\omega^{\frac{1}{d}\binom{d+1}{4}-\frac{1}{d}\binom{d}{3}-p}=\omega^{\frac{1}{d}\binom{d}{4}-p}. \qquad \text{(E9)}$$

APPENDIX F: CANONICAL FORMS FOR QUDIT STABILIZER STATES AND THE INNER PRODUCT ALGORITHM

A qubit stabilizer state can be written in the form [27,29]

$$|\psi\rangle = 2^{-m/2}\sum_{x\in\mathcal{A}}(-1)^{q(x)}i^{l(x)}|x\rangle, \qquad \text{(F1)}$$

where $l(x)$ is a linear form and $q(x)$ takes the quadratic form $q(x)=\sum_{i\neq j}q_{ij}x_ix_j+c_ix_i$, with q_{ij} and c_i constants in Z_2. In addition, $\mathcal{A}$ is an affine space defined as $\mathcal{A}=\{Gu+h \mid u\in {Z_2}^m, h\in {Z_2}^n\}$, with G an $n\times m$ matrix with entries in Z_2.

To prove that this canonical form holds true for all qubit stabilizer states, one only need to make sure that every state in this form is the eigenstate of a stabilizer operator, as shown in [27]. It also suffices to verify that any of the $\{H, P, \text{CNOT}\}$ gates preserves the form, only changing the coefficients of $q(x)$, $l(x)$, and affine space A. This proof is given in [29].

The normal form was generalized to arbitrary dimensions in [28]. The stabilizer canonical form for qudits is

$$|\psi\rangle \propto \sum_{u\in Z_d^k}\omega^{q_d(u)+q_n(u)}|Gu+h\rangle, \qquad \text{(F2)}$$

where $q_n(u)=\sum_{i\neq j}q_{ij}u_iu_j$ and $q_d(u)=\sum_{i=1}^k q_i\frac{u_i(u_i-1)}{2}+l_iu_i$, with $q_{ij},q_i,l_i\in Z_d$. The state has support in a k-dimensional affine space

$$
\begin{aligned}
\vec{x} &= Gu+h=\text{span}(g^1,\dots,g^k)\oplus h\\
&= u_1g^1\oplus u_2g^2\oplus\cdots\oplus u_kg^k\oplus h, \qquad \text{(F3)}
\end{aligned}
$$

where G is an $n\times k$ matrix and has each of its columns being $g_1,\dots,g_k$ with entries in Z_d, while h is an $n\times 1$ vector that has entries in Z_d. The division of the phase into two quadratic terms reflects the action of the phase and Hadamard gates, respectively. States of this form were shown to be the +1 eigenstate of some Pauli (Weyl-Heisenberg) operator in [28].

This quadratic form on the exponent can also be represented in matrix form

$$q_d(u)+q_n(u)=2^{-1}u^TQu+Lu, \qquad \text{(F4)}$$

where 2^{-1} is taken modulo d. Here Q is a $k\times k$ matrix with its diagonal terms being q_i and off-diagonal terms being q_{ij} and L is a $1\times n$ matrix where each term corresponds to l_i-q_i.

We will give another proof that this form is preserved under Clifford operations using the properties of quadratic Gauss sums. We give this proof in order to develop the techniques we will use in the inner product algorithm for qudit stabilizer states.

We consider the single-qudit case first. We will prove that the form

$$\frac{1}{\sqrt{d}}\sum_{j\in Z_d^m}\omega^{fj(j-1)/2+gj}|j+y\rangle \qquad \text{(F5)}$$

is preserved under the action of the single-qudit Clifford generators where f and g belong to Z_d and y is a shift vector that also belongs to Z_d. We are studying the single-qudit case here, so m is either 0 or 1. When $m=0$, this is simply a computational basis state.

Acting with diagonal Clifford gates on (F5) such as P or Z will only change the coefficients f and g in this expression. Similarly, acting with powers of the X gate will only shift y, again preserving the quadratic form of the exponents.

It only remains to check the Hadamard gate:

$$
\begin{aligned}
&H\frac{1}{\sqrt{d}}\sum_{j\in Z_d^m}\omega^{fj(j-1)/2+gj}|j+y\rangle\\
&\quad=\frac{1}{d}\sum_k\omega^{yk}\left(\sum_{j\in Z_d^m}\omega^{fj(j-1)/2+(k+g)j}\right)|k\rangle. \qquad \text{(F6)}
\end{aligned}
$$

If $m=0$, the quantity in large parentheses is simply a phase factor without the sum. Then this form reverts to (F5) with $f=0 \bmod d$. If $m=1$, we recognize the quantity in the large parentheses as a Gauss sum. There are again two cases. If $f=0 \bmod d$, then we have

$$\sum_{j\in Z_d}\omega^{(k+g)j}=d\delta_{k+g,0}. \qquad \text{(F7)}$$

Then (F6) reverts to (F5) as in the $m=0$ case, i.e., a computational basis state.

If $f\neq 0 \bmod d$, to compute this Gauss sum, we first complete the square

$$
\begin{aligned}
\sum_j\omega^{fj(j-1)/2+(k+g)j} &= \sum_j\omega^{(f/2)[j^2-j+2(k+g)j\bar{f}]}\\
&= \omega^{-\bar{2}f[\bar{f}(k+g)-\bar{2}]^2}\sum_j\omega^{\bar{2}f[j-\bar{2}+(k+g)\bar{f}]^2}\\
&= \omega^{-\bar{2}f[\bar{f}(k+g)-\bar{2}]^2}\sum_n e^{2\pi i\bar{2}fn^2/d}, \qquad \text{(F8)}
\end{aligned}
$$

where $\bar{2}$ and $\bar{f}$, meaning that $2\bar{2} \equiv 1 \bmod d$ and $f\bar{f} \equiv 1 \bmod d$. The value of this Gauss sum is well known,

$$\sum_n e^{2\pi i \bar{2} f n^2/d} = \begin{cases} \left(\frac{\bar{2}f}{d}\right)\sqrt{d}, & d \equiv 1(\text{mod}4) \\ i\left(\frac{\bar{2}f}{d}\right)\sqrt{d}, & d \equiv 3(\text{mod}4), \end{cases} \tag{F9}$$

where $(\frac{\bar{2}f}{d})$ is the Legendre symbol. Hence

$$\sum_j \omega^{fj(j-1)/2+(k+g)j} \propto \omega^{-\bar{f}k(k-1)/2-\bar{2}[(2g+1)\bar{f}-1]k}. \tag{F10}$$

The new coefficients $-\bar{f}$ and $-\bar{2}[(2g+1)\bar{f}-1]$ here are still in Z_d. This means that the general form $\sum_k \omega^{fk(k-1)/2+gk}|k\rangle$ of single-qudit stabilizer states is preserved under the action of any Clifford operations.

For multiqudit states, we have the same affine space property as the qubit case except that the additions are modulo d. Before we give the proof, we need to show that the quadratic forms given in terms of the basis vectors of the affine space $\vec{u}$ and the qudit vector itself $\vec{x}$ are equivalent. Changing the arguments only changes the coefficients of the quadratic form. Given Eq. (F4), we further assume the quadratic and linear matrices in terms of x being $\tilde{Q}$ and $\tilde{L}$:

$$\begin{aligned} \omega^{x^T\tilde{Q}x+\tilde{L}x} &= \omega^{(u^TG^T+h^T)\tilde{Q}(Gu+h)+\tilde{L}(Gu+h)} \\ &\propto \omega^{u^TG^T\tilde{Q}Gu+(2h^T\tilde{Q}G+\tilde{L}G)u}. \end{aligned} \tag{F11}$$

From this equation we can see the relationship between Q, L and $\tilde{Q}, \tilde{L}$: $Q = G^T\tilde{Q}G$ and $L = 2h^T\tilde{Q}G + \tilde{L}G$.

Now we use Van den Nest's method [29] to prove that the canonical form (F2) is preserved under the action of CSUM, P, and H. The $\text{CSUM}_{i\to j}$ gate shifts the affine space by mapping $|a\rangle|b\rangle$ to $|a\rangle|a\oplus b\rangle$, without changing the phases. As in the qubit case, we only need to add the ith row of the matrix G to the jth row,

$$\text{CSUM}\sum_{u\in Z_d^m} \omega^{q_d(u)+q_n(u)}|Gu+h\rangle = \sum_{u\in Z_d^m} \omega^{q_d(u)+q_n(u)}|G'u+h\rangle, \tag{F12}$$

where G' differs from G by $\tilde{g}_j \to \tilde{g}_i \oplus \tilde{g}_j$.

Acting with P on qudit i results in the state

$$P_i|\psi\rangle \propto \sum_{x\in A} \omega^{q_n(x)+q_d(x)}\omega^{x_i(x_i-1)/2}|x\rangle, \tag{F13}$$

which again leaves the canonical form unchanged. The Hadamard gate requires some work. Without loss of generality, we assume that H acts on the first qudit

$$H_1|\psi\rangle \propto \sum_{v=0}^{d-1}\sum_u \omega^{q_n(u)+q_d(u)+v(\vec{g}_1u+h_1)}|v, \bar{G}u+\bar{h}\rangle, \tag{F14}$$

where $(\vec{g}_1)^T$ is the first row of G and $\bar{G}$ is the rest of it. If $\bar{G}$ is still full rank after taking out $(\vec{g}_1)^T$, we obtain the new G' as

$$\begin{pmatrix} 1 & \vec{0}^T \\ \vec{0} & \bar{G} \end{pmatrix}. \tag{F15}$$

Therefore, we now have $m+1$ basis vectors and v becomes the new u_1. The term $v(\bar{g}_1u+t_1)$ in the phase can be absorbed in the quadratic form $q_n(u)$. So this is of the canonical form (F2).

If $\bar{G}$ is rank $m-1$ after taking out $(\tilde{g}_1)^T$, then the columns of $\bar{G}$ are not linearly independent. In this case one of the u_i is redundant and we want it to be summed out in order to get back to the canonical form. Without loss of generality, let us assume that $u_1 = \sum_{i=2}^m r_i\bar{g}_i$, therefore $\bar{G}u+\bar{h} = \sum_{i=2}^m(u_i + r_i)\bar{g}_i + \bar{h}$. If we define $u_i' \equiv u_i + r_i$ for $i = 2$ to m $(\bar{u})$ and $u_1' \equiv v$, $q_n(\bar{u})$ and $q_d(\bar{u})$ can be written in terms of $\bar{u}'$ with different coefficients from q_n and q_d, say, $q_n'(\bar{u}')$ and $q_d'(\bar{u}')$, together with some constant factor which can be neglected. Then Eq. (F14) becomes

$$\begin{aligned} H_1|\psi\rangle &\propto \sum_{v=0}^{d-1}\sum_u \omega^{q_n(u)+q_d(u)+v(\tilde{g}_1^Tu+h_1)}|v, \bar{G}u+\bar{h}\rangle \\ &\propto \sum_{v=0}^{d-1}\sum_{u_2',\dots,u_m'}\sum_{u_1} \omega^{q_n(u)+q_d(u)+v(\tilde{g}_1^Tu+h_1)}\left|v, \sum_{i=2}^m u_i'\bar{g}_i + \bar{h}\right\rangle \\ &= \sum_{u_1',u_2',\dots,u_m'} \omega^{q_n'(\bar{u}')+q_d'(\bar{u}')+u_1'[\sum_{i=2}^m g_{1i}(u_i'-r_i)+\bar{h}_1]} \\ &\quad\times\left(\sum_{u_1}\omega^{q_n(u_1)+q_d(u_1)+vg_{11}u_1}\right)\left|v, \sum_{i=2}^m u_i'\bar{g}_i+\bar{h}\right\rangle. \end{aligned} \tag{F16}$$

Here the large parentheses contain the Gauss sum we computed earlier. Then we can drop the prime for the u and absorb the result of the Gauss sum and $u_1'[\sum_{i=2}^m g_{1i}(u_i'-r_i)+\bar{h}_1]$ into the q_n' and q_d' functions. Finally, we arrive at the same form but with different coefficients. Hence, the canonical form is preserved under the action of all Clifford gates.

We now use this canonical form and the Gauss sum techniques to provide an $O(n^3)$ algorithm for the computation of the inner products of two-qudit stabilizer states.

Inner product of two-qudit stabilizer states

The inner product between two-qubit stabilizer states can be computed efficiently in $O(n^3)$ [4,8,31,40]. However, a corresponding algorithm for qudits has not yet been given, although most aspects of the theory of stabilizer states have been generalized [26,28]. We will now describe an $O(n^3)$ algorithm that computes the inner product of two-qudit stabilizer states based on the Gauss sum techniques we discussed in Appendix E.

As discussed above, the quadratic forms in terms of the basis vector of the affine space $\vec{u}$ and the qudit vector itself $\vec{x}$ are equivalent. Therefore, Eq. (F2) is equivalent to

$$|\psi\rangle \propto \sum_{x\in A}\omega^{\tilde{q}_n(x)+\tilde{q}_d(x)}|x\rangle, \tag{F17}$$

where A is the affine space defined by $Gu+h$ in Eq. (F3).

Assume we have two-qudit stabilizer states $|\psi_1\rangle$ and $|\psi_2\rangle$, which take the above form (F17) with subindices 1 and 2,

$$\begin{aligned}\langle\psi_2||\psi_1\rangle &= d^{-(k_1+k_2)/2}\sum_{x_1\in A_1}\sum_{x_2\in A_2}\omega^{\tilde{q}_1(x_1)-\tilde{q}_2(x_2)}\langle x_2\rangle\, x_1\\ &= d^{-(k_1+k_2)/2}\sum_{x\in A_1\cap A_2}\omega^{\tilde{q}_1(x)-\tilde{q}_2(x)}\\ &= d^{-(k_1+k_2)/2}\sum_{x\in A_1\cap A_2}\omega^{\tilde{q}(x)}\\ &= d^{-(k_1+k_2)/2}\sum_{u\in F_d^k}\omega^{q(u)},\end{aligned}\tag{F18}$$

where $\tilde{q}_1 = \tilde{q}_{1d} + \tilde{q}_{1n}$, $\tilde{q}_2 = \widetilde{q_{2d}} + \tilde{q}_{2n}$, $\tilde{q} = \tilde{q}_1 - \tilde{q}_2$, k is the dimension of $A_1 \cap A_2$, and q is the quadratic form in the new basis of $A_1 \cap A_2$. The new basis of the affine space $A_1 \cap A_2$, as well as the new quadratic form associated with it, can be calculated with the same method used by Bravyi and Gosset in Appendixes B and C for qubits [4], with cost $O(n^3)$.

What remains in Eq. (F18) is a Gauss sum, which we again rewrite in the form

$$\sum_{u\in F_d^k}\omega^{u^TQu+Lu},\tag{F19}$$

where the exponent is given by Eq. (F4). We can diagonalize Q and factor this sum into a product of k Gauss sums over F_d. We obtain a transformation matrix P that gives

$$P^TQP = \Lambda,\tag{F20}$$

where Λ is the diagonal matrix with entries $\lambda_1, \ldots, \lambda_k$.

Then if we further define $u = Pu'$, we obtain

$$\begin{aligned}\sum_{u\in F_d^k}\omega^{u^TQu+Lu} &= \sum_{u'\in F_d^k}\omega^{u'^TP^TQPu'+LPu'}\\ &= \sum_{u'\in F_d^k}\omega^{u'^T\Lambda u'+LPu'}\\ &= \prod_{i=1}^{k}\sum_{u_i\in F_d}\omega^{\lambda_i u_i'^2+l_i'u_i'},\end{aligned}\tag{F21}$$

where $l_i' = \sum_j p_{ji}l_j$. This is a product of k Gauss sums, as given in Eqs. (F8)–(F10).

Each Gauss sum only takes $O(1)$ time, so the product of k of them takes time $O(k)$. The scaling of this algorithm is determined by the complexity of Gaussian elimination, $O(k^3)$ because Q has rank k. Therefore, together with the first step to obtain $A_1 \cap A_2$, the algorithm takes $O(n^3)$ time overall in the worst case.

APPENDIX G: EVALUATION OF $Z(\mathcal{L})$

The quantity $Z(\mathcal{L})$ is given by Eq. (16),

$$Z(\mathcal{L}) = \sum_{x\in\mathcal{L}}\langle\tilde{0}^t|\, C_{\vec{x}}\,|\tilde{0}^t\rangle = \sum_{x\in\mathcal{L}}\prod_{l=1}^{t}\langle\tilde{0}|\, C^{x_l}\,|\tilde{0}\rangle = \sum_{x\in\mathcal{L}}\prod_{l=1}^{t}\langle\tilde{0}||\tilde{x}_l\rangle\,.\tag{G1}$$

We can see that this quantity is a function of the values $\langle\tilde{0}|\tilde{1}\rangle, \ldots, \langle\tilde{0}|d \tilde{-} 1\rangle$. We label the phase of $\langle\tilde{0}|\tilde{j}\rangle$ by β_j for all j, where $\beta_0 = 1$. Using Eq. (10), $Z(\mathcal{L})$ can be rewritten in the form

$$Z(\mathcal{L}) = \sum_{x\in\mathcal{L}}\prod_{l=1}^{t}\langle\tilde{0}||\tilde{x}_l\rangle = \sum_{x\in\mathcal{L}}\prod_{l=1}^{t}\beta_{x_l}\sqrt{\frac{1+(d-1)\delta_{0,x_l}}{d}} = \sum_{x\in\mathcal{L}}\frac{\prod_{l=1}^{t}\beta_{x_l}}{d^{(t-|x|)/2}} = \sum_{x\in\mathcal{L}}\frac{\prod_{j=1}^{d-1}\beta_j^{|x|_j}}{d^{(t-|x|)/2}},\tag{G2}$$

where $|x|$ is the Hamming weight of codeword x in code $\mathcal{L}$, i.e., the number of nonzero elements in the codeword; $|x|_j$ means the number of digits in string x that equals j. If we regard $\mathcal{L}$ as a linear code, then the qubit case $Z(\mathcal{L})$ is exactly the weight enumerator of the code. In the qudit case, $Z(\mathcal{L})$ depends on the Hamming weight as well as the β_j. Now let us calculate an explicit expression for the β_j.

For the $d = 3$ case, we specifically obtain $\beta_1 = e^{\pi i/18}$ and $\beta_2 = e^{-\pi i/18}$. For the $d > 3$ case, we assume our initial stabilizer state $|\tilde{0}\rangle = Z^p\,|+\rangle$. In addition,

$$\beta_j = \sqrt{d}\,\langle\tilde{0}\rangle\,\tilde{j} = \sqrt{d}\,\langle\tilde{0}|\,C^j\,|\tilde{0}\rangle\,,\tag{G3}$$

where the C for the Campbell *et al.* choice of $|M_d\rangle$ is simply $\omega^{-\bar{3}}XP$ according to Eq. (3). We can calculate $(XP)^j$ as

$$(XP)^j = \sum_k \omega^{\sum_{l=0}^{j-1}\binom{k+l}{2}}\,|k+j\rangle\langle k| = \omega^{\bar{6}(j^3-3j^2+2j)}\sum_k \omega^{\bar{2}(jk^2+(j^2-2j)k)}\,|k+j\rangle\langle k|\,.\tag{G4}$$

Therefore, we can rewrite C^j as

$$C^j = \omega^{-\bar{3}j}(XP)^j = \omega^{\bar{6}(j^3-3j^2)}\sum_k \omega^{\bar{2}(jk^2+(j^2-2j)k)}\,|k+j\rangle\langle k|\,.\tag{G5}$$

Then we can calculate β_j as

$$\begin{aligned}\beta_j &= \sqrt{d}\,\langle +|\,Z^{-a}C^jZ^a\,|+\rangle \\ &= \frac{\omega^{\bar{6}(j^3-3j^2)}}{\sqrt{d}}\,\langle k''|\sum_{k''}\omega^{-pk''}\sum_k \omega^{\bar{2}(jk^2+(j^2-2j)k)}\,|k+j\rangle\,\langle k|\sum_{k'}\omega^{pk'}\,|k'\rangle \\ &= \frac{\omega^{\bar{6}(j^3-3j^2-6pj)}}{\sqrt{d}}\sum_k \omega^{\bar{2}[jk^2+(j^2-2j)k]}. \end{aligned} \tag{G6}$$

This is a quadratic Gauss sum times a phase. Using Eqs. (F8) and (F9) for $f = j$ and $k + g = \bar{2}(j^2 - j)$, we obtain

$$\sum_k \omega^{\bar{2}[jk^2+(j^2-2j)k]} = \omega^{-\bar{2}^3 j(j-2)^2}\left(\frac{2j}{d}\right) = \omega^{-\bar{2}^3 j(j-2)^2} i\left(\frac{2j}{d}\right). \tag{G7}$$

The final expression of β_j in terms of p is

$$\begin{aligned}\beta_j &= \omega^{\bar{6}j^3-\bar{2}j^2-pj}\omega^{-\bar{2}^3 j(j-2)^2}\left(\frac{2j}{d}\right) \\ &= \omega^{\bar{6}j^3-\bar{2}j^2-pj}\omega^{-\bar{2}^3 j(j-2)^2} i\left(\frac{2j}{d}\right) \\ &= \begin{cases}\omega^{(\bar{6}-\bar{2}^3)j^3-(p+\bar{2})j}\left(\frac{2j}{d}\right), & d \equiv 1(\mathrm{mod}\,4) \\ \omega^{(\bar{6}-\bar{2}^3)j^3-(p+\bar{2})j} i\left(\frac{2j}{d}\right), & d \equiv 3(\mathrm{mod}\,4),\end{cases}\end{aligned} \tag{G8}$$

where again $(\frac{\bar{2}j}{d})$ is the Legendre symbol.

[1] S. Boixo, S. V. Isakov, V. N. Smelyanskiy, R. Babbush, N. Ding, Z. Jiang, M. J. Bremner, J. M. Martinis, and H. Neven, Characterizing quantum supremacy in near-term devices, Nat. Phys. **14**, 595 (2018).

[2] C. Neill, P. Roushan, K. Kechedzhi, S. Boixo, S. V. Isakov, V. Smelyanskiy, A. Megrant, B. Chiaro, A. Dunsworth, K. Arya *et al.*, A blueprint for demonstrating quantum supremacy with superconducting qubits, Science **360**, 195 (2018).

[3] S. Boixo, S. V. Isakov, V. N. Smelyanskiy, and H. Neven, Simulation of low-depth quantum circuits as complex undirected graphical models, arXiv:1712.05384.

[4] S. Bravyi and D. Gosset, Improved Classical Simulation of Quantum Circuits Dominated by Clifford Gates, Phys. Rev. Lett. **116**, 250501 (2016).

[5] I. L. Markov, A. Fatima, S. V. Isakov, and S. Boixo, Quantum supremacy is both closer and farther than it appears, arXiv:1807.10749.

[6] M. Smelyanskiy, N. P. D. Sawaya, and A. Aspuru-Guzik, qHiPSTER: The quantum high performance software testing environment, arXiv:1601.07195.

[7] T. Häner and D. S. Steiger, 0.5 petabyte simulation of a 45-qubit quantum circuit, *Proceedings of the International Conference for High Performance Computing, Networking, Storage and Analysis, SC '17* (ACM, New York, NY, 2017), pp. 33:1–33:10.

[8] S. Aaronson and D. Gottesman, Improved simulation of stabilizer circuits, Phys. Rev. A **70**, 052328 (2004).

[9] G. Nebe, E. M. Rains, and N. J. A. Sloane, The invariants of the Clifford groups, Designs Codes Cryptogr. **24**, 99 (2001).

[10] E. T. Campbell, H. Anwar, and D. E. Browne, Magic-State Distillation in All Prime Dimensions Using Quantum Reed-Muller Codes, Phys. Rev. X **2**, 041021 (2012).

[11] P. O. Boykin, T. Mor, M. Pulver, V. Roychowdhury, and F. Vatan, *Proceedings of the 40th Annual Symposium on Foundations of Computer Science* (IEEE, Piscataway, 1999), pp. 486–494.

[12] S. Bravyi, D. Browne, P. Calpin, E. Campbell, D. Gosset, and M. Howard, Simulation of quantum circuits by low-rank stabilizer decompositions, arXiv:1808.00128.

[13] S. Bravyi and A. Kitaev, Universal quantum computation with ideal Clifford gates and noisy ancillas, Phys. Rev. A **71**, 022316 (2005).

[14] X. Zhou, D. W. Leung, and I. L. Chuang, Methodology for quantum logic gate construction, Phys. Rev. A **62**, 052316 (2000).

[15] M. Howard, J. Wallman, V. Veitch, and J. Emerson, Contextuality supplies the 'magic' for quantum computation, Nature (London) **510**, 351 (2014).

[16] C. Ferrie and J. Emerson, Frame representations of quantum mechanics and the necessity of negativity in quasi-probability representations, J. Phys. A: Math. Theor. **41**, 352001 (2008).

[17] R. W. Spekkens, Negativity and Contextuality are Equivalent Notions of Nonclassicality, Phys. Rev. Lett. **101**, 020401 (2008).

[18] A. Mari and J. Eisert, Positive Wigner Functions Render Classical Simulation of Quantum Computation Efficient, Phys. Rev. Lett. **109**, 230503 (2012).

[19] V. Veitch, N. Wiebe, C. Ferrie, and J. Emerson, Efficient simulation scheme for a class of quantum optics experiments with non-negative physical Wigner representation, New J. Phys. **15**, 013037 (2013).

[20] S. D. Bartlett, T. Rudolph, and R. W. Spekkens, Reconstruction of gaussian quantum mechanics from Liouville mechanics with an epistemic restriction, Phys. Rev. A **86**, 012103 (2012).

[21] R. W. Spekkens, in *Quantum Theory: Informational Foundations and Foils*, edited by G. Chiribella and R. W. Spekkens, Fundamental Theories of Physics Vol. 181 (Springer, Berlin, 2016), pp. 83–135.

[22] H. Pashayan, J. J. Wallman, and S. D. Bartlett, Estimating Outcome Probabilities of Quantum Circuits Using Quasiprobabilities, Phys. Rev. Lett. **115**, 070501 (2015).

[23] L. Kocia and P. Love, Discrete Wigner formalism for qubits and noncontextuality of Clifford gates on qubit stabilizer states, Phys. Rev. A **96**, 062134 (2017).

[24] L. Kocia and P. Love, Measurement contextuality and Planck's constant, New J. Phys. **20**, 073020 (2018).

[25] L. Kocia and P. Love, The non-disjoint ontic states of the Grassmann ontological model, transformation contextuality, and the single qubit stabilizer subtheory, J. Phys. A: Math. Theor. **52** 095303 (2019).

[26] D. Gottesman, in *Quantum Computing and Quantum Communications*, edited by C. P. Williams (Springer, Berlin, 1999), pp. 302–313.

[27] J. Dehaene and B. De Moor, Clifford group, stabilizer states, and linear and quadratic operations over GF(2), Phys. Rev. A **68**, 042318 (2003).

[28] E. Hostens, J. Dehaene, and B. De Moor, Stabilizer states and Clifford operations for systems of arbitrary dimensions and modular arithmetic, Phys. Rev. A **71**, 042315 (2005).

[29] M. Van den Nest, Classical simulation of quantum computation, the Gottesman-Knill theorem, and slightly beyond, Quant. Inf. Comp. **10**, 0258 (2010).

[30] M. Howard and J. Vala, Qudit versions of the qubit $\pi/8$ gate, Phys. Rev. A **86**, 022316 (2012).

[31] S. Bravyi, G. Smith, and J. A. Smolin, Trading Classical and Quantum Computational Resources, Phys. Rev. X **6**, 021043 (2016).

[32] M. Howard, Maximum nonlocality and minimum uncertainty using magic states, Phys. Rev. A **91**, 042103 (2015).

[33] W. K. Wootters, A Wigner-function formulation of finite-state quantum mechanics, Ann. Phys. (NY) **176**, 1 (1987).

[34] C. A. Fuchs, M. C. Hoang, and B. C. Stacey, The SIC question: History and state of play, Axioms **6**, 21 (2017).

[35] H. Zhu, SIC POVMs and Clifford groups in prime dimensions, J. Phys. A: Math. Theor. **43**, 305305 (2010).

[36] L. Kocia and P. Love, Stationary phase method in discrete wigner functions and classical simulation of quantum circuits, arXiv:1810.03622.

[37] M. Howard and E. Campbell, Application of a Resource Theory for Magic States to Fault-Tolerant Quantum Computing, Phys. Rev. Lett. **118**, 090501 (2017).

[38] D. M. Appleby, Properties of the extended Clifford group with applications to SIC-POVMs and MUBs, arXiv:0909.5233.

[39] D. Gottesman and I. L. Chuang, Demonstrating the viability of universal quantum computation using teleportation and single-qubit operations, Nature (London) **402**, 390 (1999).

[40] H. J. Garcia, I. L. Markov, and A. W. Cross, Efficient inner-product algorithm for stabilizer states, arXiv:1210.6646.