States of charged quantum fields and their statistical properties in the presence of critical potential steps

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The evolution of charged quantum fields under the action of constant nonuniform electric fields is studied. To this end we construct density operators of the quantum fields with different initial conditions. Then we study some reductions of the density operators, for example, reductions to electron or positron subsystems, reduction induced by measurements, and spatial reduction to the left or to the right subsystems of final particles. We calculate von Neumann entropy for the corresponding reduced density operators, estimating in such a way an information loss. We illustrate the obtained results by calculations in a specific background of a strong constant electric field between two infinite capacitor plates separated by a finite distance L.

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I. INTRODUCTION

Problems of quantum field theory with external backgrounds violating the vacuum stability have been studied systematically for a long time. Recently, they have drawn special attention due to new real possible applications in astrophysics and physics of nanostructures. In these areas one often encounters a situation where the effects of vacuum instability (in particular, due to the presence of potential steps, that is, inhomogeneous electric fields) and finite temperature are combined. Astrophysical objects such as black holes and neutron stars can generate huge electromagnetic fields in their vicinity. The Coulomb barrier at the quark star surface of a hot strange star may be a powerful source of e^+e^- pairs, which are created in extremely strong constant electric fields (dozens of times higher that the critical field E_c) of the barrier, and they flow away from the star (see [1] for the review). Such emission may be a good observational signature of bare strange stars. The existence of critical electric fields on the quark star surfaces was also predicted in Ref. [2] in the transition at very high densities, from the normal nuclear matter phase at the core to the color-flavor-locked phase of quark matter at the inner core of hybrid stars. The possibility of existence of critical electromagnetic fields at the core surface of a neutron star was indicated in Ref. [3]. Critical electric fields are expected to appear in the late phases of gravitational collapse and from cosmological horizons, with a consequent process of pair creation by vacuum polarization (see, e.g., reviews in [4–6]). There is a close connection between particle creation by strong electrostatic potentials, in particular, by steps and barriers, and the Unruh effect, which is the phenomenon of particle emission from black holes and cosmological horizons. Particle creation from the vacuum by external fields (the generation of electron-hole pairs by the electric field or Zener tunneling) has become an observable effect in physics of graphene and similar nanostructures (e.g., in topological insulators and Weyl semimetals); this area is currently under intense development (see the reviews in [7–9] as well as the recent article [10] and references therein).

Note that the cases with homogeneous time-dependent electric fields are considered in most of these articles. The effect of pair production from the vacuum by time-dependent electric fields was considered in a number of works, starting with the pioneer work of Schwinger [11], followed by that of Nikishov [12,13], Brezin and Itzykson [14], and many others. Later a nonperturbative formulation of QED with so-called t-electric potential steps (time-dependent potentials of special form) was developed in Ref. [15] and applied to various physical problems (see, e.g., Refs. [16–19]). In particular, quantum entanglement in the Schwinger effect of Dirac or the Klein-Gordon field due to the t-electric steps, between a subsystem and the rest of the system, as measured by the von Neumann entropy of the reduced density matrix, was calculated [20] (see as well Refs. [21,22]). For more information on the subject see recent reviews in [4,9,23,24], where the progress on particle creation due to time-dependent field configurations is described and a number of important applications of such fields are considered.

However, the case where external backgrounds are represented by strong time-independent nonuniform electric fields concentrated in restricted space areas is much closer to a real experimental situation. We refer to such backgrounds as x-electric potential steps. There are theoretical articles where fields of this type are considered; see, for example, Refs. [13,25–33]. In the recent work [34] a consistent

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nonperturbative (with respect to an external electric field in zeroth order in the radiation interaction) formulation of QED with x-electric potential steps strong enough to violate the vacuum stability was constructed. In Refs. [34,35] some quantum effects related to a violation of the vacuum instability by x-electric potential steps were calculated. The particle creation effect is crucial for understanding the conductivity of graphene, especially in the so-called nonlinear regime. In this regime it is natural to consider a constant voltage applied between two electrodes. Possible experimental configurations for testing the pair creation by a linear step of finite length were proposed in Ref. [36]. For the case of a constant voltage between two electrodes the evidence of the existence of electron-hole pair creation was obtained in graphene by its indirect influence on the graphene conductivity [37]. The first experimental observation of graphene optical emission induced by the intense terahertz pulse was recently reported [38]. The experimental data are in a good agreement with the theory of Landau-Zener interband transitions. The impact of Zener tunneling on the charge-transport properties of graphene in the high-field regime was studied theoretically in Ref. [39]. It is shown that the inclusion of both Zener tunneling and electron-electron relaxation improves the agreement with the measurements performed in graphene in the high-field regime at low doping. The p-n junctions and sharp n-n junctions can also play the role of potential steps. For these steps the Klein tunneling was observed by several experimental groups (but only for the kinetic energies of electron that exclude the possibility of pair production) (see, e.g., the review in [7]). It should also be noted that in the context of strong interactions and quantum chromodynamics, a similar phenomenon may play a role in the discussion of particle production in heavy-ion collisions or in the decay of hadronic strings in the process of hadronization (see Ref. [9] for a review).

In this article we study the evolution of different initial states of charged quantum fields in x-electric critical potential steps, using the above-mentioned formulation of QED [34]. To this end, we construct density operators for different initial states of the system of quantum fields. We consider pure initial states and thermal (mixed) initial states. Corresponding final states are studied using three types of reductions. Since x-electric potential steps cause a natural division of created particles in subsystems of electrons and positrons substantially separated spatially, we first consider reductions to electron or positron subsystems. In the background under consideration, it is interesting to calculate reductions to the left and right parts of the whole system and compare the obtained states with states resulting from the previously mentioned reductions. Finally, we study reductions due to possible measurements of a number of final particles. The latter kind of reductions can also occur due to some decoherence processes, such as collisions with some external sources (e.g., with impurities in the graphene). To study the loss of the information in all the reductions, we calculate von Neumann entropy for reduced density operators. In two first reduction cases this entropy can also be identified with a measure of quantum entanglement between the corresponding quantum subsystems.

The article is organized as follows. In Sec. II we recall basic points of QED with x-electric potential steps. In Sec. III we present density operators for different initial states of charged quantum fields. All the above-mentioned reductions are presented in Sec. IV. The corresponding von Neumann entropy is calculated in Sec. V. Special generating functionals allow us to construct density matrices for different initial conditions by choosing appropriate sources presented in Sec. III, and their normal forms are placed in Appendix A. In Appendix B we briefly consider the case when the initial state of the system is given by a pure state with a definite number of particles. Some useful operatorial relations are given in Appendix C.

II. QED WITH x-ELECTRIC POTENTIAL STEPS

The general theory of quantization of charged fields in the presence of critical potential steps that we use was formulated by Gitman and Gavrilov in Ref. [34]. They constructed a special self-consistent QED with x-electric potential steps utilizing the so-called generalized Furry picture. In the framework of this QED it is possible to take into account the external electric field exactly in zeroth order in the radiation interaction when the analytical solutions of the Dirac equation in the corresponding field are known. Here we repeat some crucial moments of this theory. In this article we generally adapt the notation used in Ref. [34]; we utilize the system of units where $c = \hbar = 1$.

We work in $(d = D + 1)$-dimensional Minkowski spacetime parametrized by coordinates X,

$$X = (X^\mu, \mu = 0, 1, \ldots, D) = (t, x, \mathbf{r}_\perp),$$
$$X^0 = t, \quad x = X^1, \quad \mathbf{r}_\perp = (X^2, \ldots, X^D), \tag{1}$$

which correspond to an x-electric potential step of the form

$$A^\mu(X) = (A^0(x), A^j = 0, j = 1, 2, \ldots, D), \tag{2}$$

so that the magnetic field B is zero and the electric field E reads

$$\mathbf{E}(X) = \mathbf{E}(x) = (E_x(x), 0, \ldots, 0),$$
$$E_x(x) = -A_0'(x) = E(x). \tag{3}$$

The electric field (3) is directed along the x axis, is inhomogeneous in the x direction, and does not depend on time t. The main property of any x-electric potential step is

$$A_0(x) \overset{x\to\pm\infty}{\longrightarrow} A_0(\pm\infty), \quad E(x) \overset{|x|\to\infty}{\longrightarrow} 0, \tag{4}$$

where $A_0(\pm\infty)$ are constant quantities, which means that the electric field under consideration is switched off at spatial infinity. In addition, it is supposed that the first derivative of the scalar potential $A^0(x)$ does not change its sign for any $x \in R$ and that there exist points x_L and x_R $(x_\mathrm{R} > x_\mathrm{L})$ such that for $x \in S_\mathrm{L} = (-\infty, x_\mathrm{L}]$ and for $x \in S_\mathrm{R} = [x_\mathrm{R}, \infty)$ the electric field is already switched off, so

$$A_0(x)|_{x\in S_\mathrm{L}} = A_0(-\infty), \quad E(x)|_{x\in S_\mathrm{L}} = 0,$$
$$A_0(x)|_{x\in S_\mathrm{R}} = A_0(+\infty), \quad E(x)|_{x\in S_\mathrm{R}} = 0, \tag{5}$$

whereas the electric field is not zero in the region $S_\mathrm{int} = (x_\mathrm{L}, x_\mathrm{R})$ (note that both x_L and x_R can tend to infinity). An example of an x-potential step can be found in Fig. 1. There

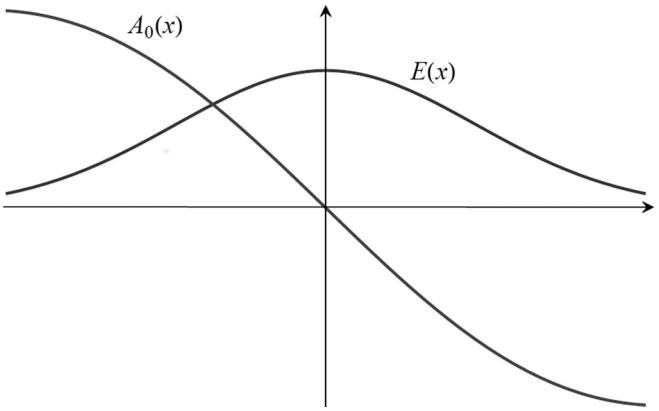

FIG. 1. Example of an x-electric potential step.

are two types of electric steps, noncritical and critical, which are distinguished by their magnitudes

$$\mathbb{U} = \begin{cases} \mathbb{U} < 2m, & \text{noncritical steps} \\ \mathbb{U} > 2m, & \text{critical steps,} \end{cases} \tag{6}$$

where $\mathbb{U} = U_{\rm R} - U_{\rm L}$, $U_{\rm L\,(R)}$ is the left (right) asymptotic potential energy, $U_{\rm L} = U(x \to -\infty)$, $U_{\rm R} = U(x \to +\infty)$, $U(x) = -eA_0(x)$ is the potential energy of the electron in the x-electric potential step, and m is the electron mass. We are mostly interested in critical steps, which can produce pairs from the vacuum.

One of the most important points of QED with critical potential steps is that the whole space of quantum numbers $n \in \Omega$ [which are the full energy of particle p_0, momenta $\mathbf{p}$, and spin σ, $n = (p_0, \mathbf{p}, \sigma)$] can be divided into five different ranges Ω_i, $i = 1, \ldots, 5$, $n_i \in \Omega_i$, where the solutions of the corresponding Dirac equation have similar forms. The full operator of the Dirac field can be presented as a sum of operators defined for each particular range Ω_i,

$$\hat{\Psi}(X) = \sum_{i=1}^{5} \hat{\Psi}_i(X). \tag{7}$$

The explicit forms of operators $\hat{\Psi}_i(X)$ are given in Appendix A and a graphical representation of the quantum ranges Ω_i can be found in Fig. 2. Detailed consideration of each range Ω_i was carried out in Ref. [34]; here we repeat only the most important points. In these ranges there exist two types of solutions of the Dirac equation, ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$, $\zeta = \pm$. Those solutions satisfy the following asymptotic conditions:

$$\begin{aligned}
&{}_\zeta\psi_n(X) = \exp(-ip_0t + i\mathbf{p}_\perp\mathbf{r}_\perp)\,{}_\zeta\varphi_n(x), \quad {}_\zeta\varphi_n(x) = {}_\zeta\varphi_n^{\rm L}(x), \quad x \in S_{\rm L}, \quad \left[\hat{p}_x^2 - (p_0 - U_{\rm L})^2 + \pi_\perp^2\right]{}_\zeta\varphi_n^{\rm L}(x) = 0, \quad \pi_\perp^2 = m^2 + \mathbf{p}_\perp^2, \\
&{}^\zeta\psi_n(X) = \exp(-ip_0t + i\mathbf{p}_\perp\mathbf{r}_\perp)\,{}^\zeta\varphi_n(x), \quad {}^\zeta\varphi_n(x) = {}^\zeta\varphi_n^{\rm R}(x), \quad x \in S_{\rm R}, \quad \left[\hat{p}_x^2 - (p_0 - U_{\rm R})^2 + \pi_\perp^2\right]{}^\zeta\varphi_n^{\rm R}(x) = 0, \quad \hat{p}_x = -i\partial_x.
\end{aligned} \tag{8}$$

Nontrivial solutions ${}^\zeta\psi_n(X)$ and ${}_\zeta\psi_n(X)$ exist only for quantum numbers n that obey the relations

$$(p_0 - U_{\rm R})^2 > \pi_\perp^2 \quad \text{for } {}^\zeta\psi_n(X), \tag{9}$$

$$(p_0 - U_{\rm L})^2 > \pi_\perp^2 \quad \text{for } {}_\zeta\psi_n(X) \tag{10}$$

and correspond to states with definite momenta $p^{\rm R}$ and $p^{\rm L}$,

$$\begin{aligned}
&\hat{p}_x\,{}_\zeta\psi_n(X) = p^{\rm L}\,{}_\zeta\psi_n(X), \quad x \to -\infty, \quad p^{\rm L} = \zeta\sqrt{(p_0 - U_{\rm L})^2 - \pi_\perp^2}, \\
&\hat{p}_x\,{}^\zeta\psi_n(X) = p^{\rm R}\,{}^\zeta\psi_n(X), \quad x \to +\infty, \quad p^{\rm R} = \zeta\sqrt{(p_0 - U_{\rm R})^2 - \pi_\perp^2}.
\end{aligned} \tag{11}$$

The ranges Ω_1 and Ω_5 exist for any step, critical or noncritical, and are defined by the inequalities

$$\begin{aligned}
&p_0 \geqslant U_R + \pi_\perp \quad \text{if } n \in \Omega_1, \\
&p_0 \leqslant U_L - \pi_\perp \quad \text{if } n \in \Omega_5
\end{aligned} \tag{12}$$

for a given $\pi_\perp$. Solutions ${}^\zeta\psi_n(X)$ can be interpreted as either a wave function of an electron for $n \in \Omega_1$ or a wave function of a positron for $n \in \Omega_5$ with the momenta $p^{\rm R}$ along the x axis, whereas solutions ${}_\zeta\psi_n(X)$ can be interpreted as either a wave function of an electron for $n \in \Omega_1$ or a wave function of a positron for $n \in \Omega_5$ with momenta $p^{\rm L}$ along the x axis.

The ranges Ω_2 and Ω_4 also exist for any step and include the quantum numbers $n \in \Omega_2$ that obey the inequalities

$$\begin{aligned}
&U_{\rm R} - \pi_\perp < p_0 < U_{\rm R} + \pi_\perp, \quad p_0 - U_{\rm L} > \pi_\perp \quad \text{if } 2\pi_\perp < \mathbb{U}, \\
&U_{\rm L} + \pi_\perp < p_0 < U_{\rm R} + \pi_\perp \quad \text{if } 2\pi_\perp < \mathbb{U}
\end{aligned} \tag{13}$$

and the quantum numbers $n \in \Omega_4$ that obey the inequalities

$$\begin{aligned}
&U_{\rm L} - \pi < p_0 < U_{\rm L} + \pi, \quad p_0 - U_{\rm R} < -\pi_\perp \quad \text{if } 2\pi_\perp < \mathbb{U}, \\
&U_{\rm L} - \pi < p_0 < U_{\rm R} - \pi \quad \text{if } 2\pi_\perp < \mathbb{U}.
\end{aligned} \tag{14}$$

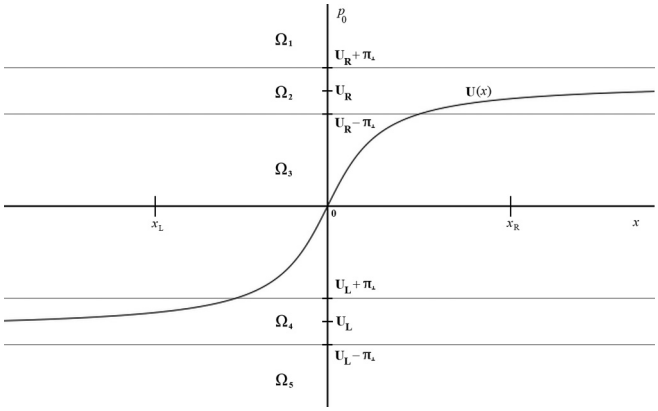

FIG. 2. Graphical representation of the quantum ranges Ω_i showing the potential energy of the electron $U(x) = -eA_0(x)$ in the electric field.

As a consequence of these inequalities there exist solutions ${}_\zeta\psi_n(X)$, $n \in \Omega_2$, with definite left asymptotics and ${}^\zeta\psi_n(X)$, $n \in \Omega_4$, with definite right asymptotics. Solutions ${}_\zeta\psi_n(X)$, $n \in \Omega_2$, and ${}^\zeta\psi_n(X)$, $n \in \Omega_4$, can be interpreted as wave functions of the electron and positron, respectively. Nontrivial solutions ${}^\zeta\psi_n(X)$, $n \in \Omega_2$, and ${}_\zeta\psi_n(X)$, $n \in \Omega_4$, do not exist, as the inequality (13) contradicts Eq. (9) and the inequality (14) contradicts Eq. (10).

The range Ω_3, the Klein zone, exists only for critical steps. The quantum numbers $p_\perp$ are restricted by the inequality $2\pi_\perp < U$ and for any of such $\pi_\perp$ quantum numbers p_0 obey the double inequality

$$U_L + \pi \leqslant p_0 \leqslant U_R - \pi. \tag{15}$$

In the range Ω_3 there exist the following sets of solutions:

$$\{{}_\zeta\psi_n(X)\}, \{{}^\zeta\psi_n(X)\}, \quad n \in \Omega_3, \ \zeta = \pm. \tag{16}$$

However, the one-particle interpretation of these solutions based on the energy spectrum in a similar way as has been done in the ranges Ω_1 and Ω_5 becomes inconsistent. Indeed, it is enough to see the following contradiction: From the point of view of the left asymptotic area S_L, only electron states are possible in the range Ω_3, whereas from the point of view of the right asymptotic area S_R, only positron states are possible in this range. For the detailed consideration of this fact in the framework of QED, see Sec. VII of Ref. [34].

In what follows, operators a_n and $a_n^\dagger$, and b_n and $b_n^\dagger$ denote operators of creation and annihilation of particles (electrons) and antiparticles (positrons), respectively, for each range Ω_i. Operators

$$\begin{aligned} {}^-a_n(\text{in}) &= {}^-a_n, \quad {}_+a_n(\text{in}) = {}_+a_n, \\ {}_-b_n^\dagger(\text{in}) &= {}_-b_n^\dagger, \quad {}^+b_n(\text{in}) = {}^+b_n \end{aligned} \tag{17}$$

and their conjugates correspond to the initial electrons and positrons, while operators

$$\begin{aligned} {}^+a_n(\text{out}) &= {}^+a_n, \quad {}_-a_n(\text{ out}) = {}_-a_n, \\ {}_+b_n(\text{out}) &= {}_+b_n, \quad {}^-b_n(\text{ out}) = {}^-b_n \end{aligned} \tag{18}$$

correspond to the final electrons and positrons.

The formalism developed in Ref. [34] is applicable to any one-dimensional x-electric potential step as long as the condition (4) is satisfied. When the solutions ${}^\zeta\psi_n(X)$ and ${}_\zeta\psi_n(X)$ of the corresponding Dirac equation with such a potential can be found analytically in each spatial region S_L, S_R, and S_{int}, it is possible to use border conditions (gluing conditions) to calculate all main characteristics of vacuum instability (the number of particles created from the vacuum, the probability of the vacuum to remain a vacuum, etc.). The general procedure of such calculations can be found in Refs. [34,40,41], where several examples of exactly solvable cases are considered: the so-called L-constant field [40], the Sauter-like field [34], and the peak electric field given by an exponential step [41].

III. DENSITY OPERATORS WITH DIFFERENT INITIAL CONDITIONS

To obtain the density operators for the system under consideration, we introduce the special generating functionals $R(J)$, which are given in Appendix A. Choosing the appropriate sources J, we are able to obtain the explicit form for the density operators for different initial conditions.

A. Initial vacuum state

To obtain the density operator with an initial vacuum state, we set all $J = 0$ in $R(J)$, i.e., we set $J^{(i)}_{\pm,n} = J^{(i)}_n = 0$ in every partial generating functional $R^{(i)}(J)$. In this case, the general density operator with a vacuum initial state takes the form

$$R(J=0) = \hat\rho_v = \otimes \prod_{i=1}^{5} \hat\rho_v^{(i)}, \quad \hat\rho_v^{(i)} = \prod_{n\in\Omega_i} \hat\rho_{v,n}^{(i)}, \tag{19}$$

where the one-mode partial density operators $\hat\rho_{v,n}^{(i)}$ are (in terms of an in set of creation and annihilation operators[1])

$$\begin{aligned} \hat\rho_{v,n}^{(1)} &= :\exp[-{}_+a_n^\dagger {}_+a_n - {}^-a_n^\dagger {}^-a_n]:, \\ \hat\rho_{v,n}^{(5)} &= :\exp[-{}^+b_n^\dagger {}^+b_n - {}_-b_n^\dagger {}_-b_n]:, \\ \hat\rho_{v,n}^{(3)} &= :\exp[-{}^-a_n^\dagger {}^-a_n - {}_- b_n^\dagger {}_-b_n]:, \\ \hat\rho_{v,n}^{(2)} &= :\exp[-a_n^\dagger a_n]:, \quad \hat\rho_{v,n}^{(4)} = :\exp[-b_n^\dagger b_n]:. \end{aligned} \tag{20}$$

Taking into account the well-known Berezin formula [42]

$$|0\rangle\langle 0| = :\exp[-a^\dagger a]:, \tag{21}$$

one can see that the operators $\hat\rho_{v,n}^{(i)}$ are in fact partial vacuum projectors for the initial particles:

$$\begin{aligned} \hat\rho_{v,n}^{(i)} &= |0,\text{in}\rangle_n^{(i)}\,{}_n^{(i)}\langle 0,\text{in}|, \quad i = 1,3,5, \\ \hat\rho_{v,n}^{(2,4)} &= |0\rangle_n^{(2,4)}\,{}_n^{(2,4)}\langle 0|. \end{aligned} \tag{22}$$

One can show that the differential numbers of initial electrons and positrons [see Eqs. (17) and (18) for reference] in the state

[1]Here and in what follows colons $:\cdots:$ always denote the normal form with respect to the creation and annihilation operators inside them.

described by the operator $\hat{\rho}_v$ vanish for all n,

$$\begin{aligned}\mathrm{tr}\hat{\rho}_v\, {}^{-}a_n^{\dagger}\, {}^{-}a_n &= \mathrm{tr}\hat{\rho}_v\, {}_{+}a_n^{\dagger}\, {}_{+}a_n = \mathrm{tr}\hat{\rho}_v\, {}_{-}b_n^{\dagger}\, {}_{-}b_n \\ &= \mathrm{tr}\hat{\rho}_v\, {}^{+}b_n^{\dagger}\, {}^{+}b_n = 0, \quad n \in \Omega_{1,3,5}, \\ \mathrm{tr}\hat{\rho}_v\, a_n^{\dagger}a_n &= \mathrm{tr}\hat{\rho}_v b_n^{\dagger}b_n = 0, \quad n \in \Omega_{2,4}. \end{aligned} \tag{23}$$

The mean differential numbers of final electrons and positrons are different from zero in the range Ω_3. These numbers are equal to the number of pairs created from vacuum,

$$N_n^a = N_n^b = N_n^{\mathrm{cr}} = \mathrm{tr}\hat{\rho}_v\, {}^{+}a_n^{\dagger}\, {}^{+}a_n = |g({}_{-}|{}^{+})|^{-2}, \quad n \in \Omega_3, \tag{24}$$

where $g({}_{-}|{}^{+})$ are mutual decomposition coefficients of the solutions ${}_{-}\psi_n(X)$ and ${}^{+}\psi_n(X)$ [see Eqs. (A17) and (A18)].

B. Initial thermal state

Before writing the expressions for the density operator, we must recall that we consider the situation when the electric field is not zero only in the finite region $S_{\mathrm{int}} = (x_{\mathrm{L}}, x_{\mathrm{R}})$ situated between the planes $x = x_{\mathrm{L}}$ and $x = x_{\mathrm{R}}$. Outside of S_{int} for $x \in S_{\mathrm{L}} = (-\infty, x_{\mathrm{L}}]$ and for $x \in S_{\mathrm{R}} = [x_{\mathrm{R}}, \infty)$ particles are free (i.e., their movement is unbounded at least in one direction). It should be noted that usually quantum field theory deals with physical quantities that are presented by volume integrals on the hyperplane $t = \mathrm{const}$. The main contribution to these integrals is from regions S_{L} and S_{R}, where particles are free. This fact allows one to obtain the explicit form of kinetic energies for all particles (see details in Ref. [34]) and is used in what follows.

To obtain the density operator with the initial thermal state, we need to set the sources J as

$$\begin{aligned} J^{(i)}_{\pm,n} &= e^{-E^{\pm}_{n\in\Omega_i}}, \quad E_n^{\pm} = \beta(\varepsilon_n^{\pm} - \mu^{\pm}),\ \beta = \Theta^{-1},\ n \in \Omega_{1,3,5}, \\ J^{(i)}_{n} &= e^{-E_{n\in\Omega_i}}, \quad E_n = \beta(\varepsilon_n - \mu),\ n \in \Omega_{2,4}, \end{aligned} \tag{25}$$

where $\varepsilon_n^{\pm}$ and ε_n are the kinetic energies of particles and antiparticles with quantum numbers n; $\mu^{\pm}$ and μ are the corresponding chemical potentials and Θ is the absolute temperature.[2] For the sake of simplicity, in what follows we will suppose that all chemical potentials for electrons and positrons are equal. The density operator $\hat{\rho}_\beta$ can be written as

$$\hat{\rho}_\beta = \otimes \prod_{i=1}^{5} \hat{\rho}_\beta^{(i)}, \quad \hat{\rho}_\beta^{(i)} = \prod_{n\in\Omega_i} \hat{\rho}^{(i)}_{\beta,n}, \tag{26}$$

where the one-mode density operators $\hat{\rho}^{(i)}_{\beta,n}$ have the form

$$\begin{aligned} \hat{\rho}^{(1)}_{\beta,n} &= \left[Z_n^{(1)}\right]^{-1} \exp[-{}_{+}a_n^{\dagger} E_n^{+}\, {}_{+}a_n - {}^{-}a_n^{\dagger} E_n^{-}\, {}^{-}a_n], \\ \hat{\rho}^{(5)}_{\beta,n} &= \left[Z_n^{(5)}\right]^{-1} \exp[-{}^{+}b_n^{\dagger} E_n^{+}\, {}^{+}b_n - {}_{-}b_n^{\dagger} E_n^{-}\, {}_{-}b_n], \\ \hat{\rho}^{(3)}_{\beta,n} &= \left[Z_n^{(3)}\right]^{-1} \exp[-{}^{-}a_n^{\dagger} E_n^{+}\, {}^{-}a_n - {}_{-}b_n^{\dagger} E_n^{-}\, {}_{-}b_n], \\ \hat{\rho}^{(2)}_{\beta,n} &= \left[Z_n^{(2)}\right]^{-1} \exp[-a_n^{\dagger} E_n a_n], \\ \hat{\rho}^{(4)}_{\beta,n} &= \left[Z_n^{(4)}\right]^{-1} \exp[-b_n^{\dagger} E_n b_n]. \end{aligned} \tag{27}$$

The statistical sums $Z_n^{(i)}$ have the form

$$Z_n^{(1,3,5)} = (1 + e^{-E_n^{+}})(1 + e^{-E_n^{-}}), \quad Z_n^{(2,4)} = (1 + e^{-E_n}). \tag{28}$$

Note that the operators (27) can also be presented as

$$\hat{\rho}^{(i)}_{\beta,n} = \left[Z_n^{(i)}\right]^{-1} \exp\left\{-\beta\left[\hat{H}_n^{(i)} - \mu\hat{N}_n^{(i)}\right]\right\}, \tag{29}$$

where for $i = 1, 3, 5$ we have

$$\begin{aligned} \hat{H}_n^{(i)} &= \begin{cases} {}_{+}a_n^{\dagger}\varepsilon_n^{+}\, {}_{+}a_n + {}^{-}a_n^{\dagger}\varepsilon_n^{-}\, {}^{-}a_n, & n \in \Omega_1 \\ {}^{-}a_n^{\dagger}\varepsilon_n^{+}\, {}^{-}a_n + {}_{-}b_n^{\dagger}\varepsilon_n^{-}\, {}_{-}b_n, & n \in \Omega_3 \\ {}^{+}b_n^{\dagger}\varepsilon_n^{+}\, {}^{+}b_n + {}_{-}b_n^{\dagger}\varepsilon_n^{-}\, {}_{-}b_n, & n \in \Omega_5, \end{cases} \\ \mu\hat{N}_n^{(i)} &= \begin{cases} \mu^{+}\, {}_{+}a_n^{\dagger}\, {}_{+}a_n + \mu^{-}\, {}^{-}a_n^{\dagger}\, {}^{-}a_n, & n \in \Omega_1 \\ \mu^{+}\, {}^{-}a_n^{\dagger}\, {}^{-}a_n + \mu^{-}\, {}_{-}b_n^{\dagger}\, {}_{-}b_n, & n \in \Omega_3 \\ \mu^{+}\, {}^{+}b_n^{\dagger}\, {}^{+}b_n + \mu^{-}\, {}_{-}b_n^{\dagger}\, {}_{-}b_n, & n \in \Omega_5, \end{cases} \end{aligned} \tag{30}$$

while for $i = 2, 4$ these operators take the form

$$\begin{aligned} \hat{H}_n^{(i)} &= \begin{cases} a_n^{\dagger}\varepsilon_n a_n, & n \in \Omega_2 \\ b_n^{\dagger}\varepsilon_n b_n, & n \in \Omega_4, \end{cases} \\ \mu\hat{N}_n^{(i)} &= \begin{cases} \mu a_n^{\dagger} a_n, & n \in \Omega_2 \\ \mu b_n^{\dagger} b_n, & n \in \Omega_4. \end{cases} \end{aligned} \tag{31}$$

The density operators (29) in each range Ω_i are the density operators of the grand canonical ensemble at temperature Θ and with chemical potentials $\mu^{\pm}$ and μ. The differential mean distributions $N_n^{(i)}$, calculated with the help of density matrices $\hat{\rho}^{(i)}_{\beta,n}$, are well-known Fermi-Dirac and Bose-Einstein distributions. For example, the differential number of initial electrons in the range Ω_3 can be found as

$$N^{(3)}_{n,\beta,-}(\mathrm{in}) = \mathrm{tr}\hat{\rho}^{(3)}_{\beta,n}\hat{N}^{(3)}_{n,\beta,-}(\mathrm{in}) = \mathrm{tr}\hat{\rho}^{(3)}_{\beta,n}\, {}^{-}a_n^{\dagger}\, {}^{-}a_n = (e^{E_n^{-}} + 1)^{-1}, \quad n \in \Omega_3. \tag{32}$$

Other differential distributions can be calculated in the same way using the corresponding creation and annihilation operators and partial density operators.

IV. REDUCED DENSITY OPERATORS

A. Reduced density operators for electron and positron subsystems

In the general case, the states of the system under consideration at the final time instant contain both particles and antiparticles due to the pair creation by external fields and the structure of the initial state. However, we are often interested in physical quantities $F_{\pm}$ that describe only electrons $(+)$ or positrons $(-)$ at the final state of the system. The corresponding operators $\hat{F}_{\pm}$ are functions of either electron creation and annihilation operators a and $a^{\dagger}$ or positron ones b and $b^{\dagger}$. The mean values of these operators can be obtained from the so-called reduced density operators $\hat{\rho}_{\pm}$, defined as reduced traces of the general density matrix $\hat{\rho}$ over one of the subsystems (the positron or electron one, respectively):

$$\hat{\rho}_{\pm} = \mathrm{tr}_{\mp}\hat{\rho}, \quad \hat{\rho} = \otimes \prod_{i=1}^{5} \hat{\rho}^{(i)}, \quad \hat{\rho}^{(i)} = \prod_{n\in\Omega_i} \hat{\rho}_n^{(i)}. \tag{33}$$

[2]Here and later in the definition of von Neumann entropy we omit the Boltzmann constant k_B for the sake of convenience.

In the latter expression, the reduced traces $\mathrm{tr}_\mp$ of the operator $\hat\rho$ are defined as

$$\mathrm{tr}_+\hat\rho = \sum_{M=0}^{\infty}\sum_{\{m\}}\frac{1}{M!}\langle\Psi^a_{\{m\}_M}|\hat\rho|\Psi^a_{\{m\}_M}\rangle,$$
$$\mathrm{tr}_-\hat\rho = \sum_{M=0}^{\infty}\sum_{\{m\}}\frac{1}{M!}\langle\Psi^b_{\{m\}_M}|\hat\rho|\Psi^b_{\{m\}_M}\rangle, \tag{34}$$

where $|\Psi^{a\,(b)}_{\{m\}_M}\rangle$ are state vectors for electron (positron) states,

$$|\Psi^a_{\{m\}_M}\rangle = a^\dagger_{m_1}\cdots a^\dagger_{m_M}|0,\mathrm{out}\rangle_a,$$
$$|\Psi^b_{\{m\}_M}\rangle = b^\dagger_{m_1}\cdots b^\dagger_{m_M}|0,\mathrm{out}\rangle_b. \tag{35}$$

Here $|0,\mathrm{out}\rangle_a$ and $|0,\mathrm{out}\rangle_b$ are partial electron and positron vacua. Note that in the ranges Ω_1 and Ω_2 (where only electron states exist) $|0,\mathrm{out}\rangle^{(1,2)}_a = |0,\mathrm{out}\rangle^{(1,2)}$; similarly to this, in the ranges Ω_4 and Ω_5 we have $|0,\mathrm{out}\rangle^{(4,5)}_b = |0,\mathrm{out}\rangle^{(4,5)}$. In the Klein zone Ω_3, where both electron and positron states exist, the total vacuum is a a product of electron and positron partial vacua $|0,\mathrm{out}\rangle^{(3)}_a\otimes|0,\mathrm{out}\rangle^{(3)}_b = |0,\mathrm{out}\rangle^{(3)}$. Every partial electron and positron vacuum can be presented in turn as a product in quantum modes

$$|0,\mathrm{out}\rangle^{(i)}_a = \prod_{n\in\Omega_i}|0,\mathrm{out}\rangle^{(i)}_{a,n},\quad |0,\mathrm{out}\rangle^{(i)}_b = \prod_{n\in\Omega_i}|0,\mathrm{out}\rangle^{(i)}_{b,n}. \tag{36}$$

For this reason it is obvious enough that the reduced trace tr_+ completely traces out partial density operators $\hat\rho^{(1,2)}$ and leaves partial operators $\hat\rho^{(4,5)}$ unaffected. In the same manner the reduced trace tr_- traces out operators $\hat\rho^{(4,5)}$ and leaves $\hat\rho^{(1,2)}$ unchanged. Therefore, the reduced density operators $\hat\rho_\pm$ can be presented as

$$\hat\rho_+ = \hat\rho^{(1)}\otimes\hat\rho^{(2)}\otimes\hat\rho^{(3)}_+,\quad \hat\rho^{(3)}_+ = \mathrm{tr}_-\,\hat\rho^{(3)},$$
$$\hat\rho^{(3)}_+ = \sum_{M=0}^{\infty}\sum_{\{m\}}(M!)^{-1}\,{}^{(3)}_{b}\langle 0,\mathrm{out}|\,{}_+b_{m_M}\cdots{}_+b_{m_1}\hat\rho^{(3)}\,{}_+b^\dagger_{m_1}\cdots{}_+b^\dagger_{m_M}|0,\mathrm{out}\rangle^{(3)}_b,$$
$$\hat\rho_- = \hat\rho^{(4)}\otimes\hat\rho^{(5)}\otimes\hat\rho^{(3)}_-,\quad \hat\rho^{(3)}_- = \mathrm{tr}_+\hat\rho^{(3)},$$
$$\hat\rho^{(3)}_- = \sum_{M=0}^{\infty}\sum_{\{m\}}(M!)^{-1}\,{}^{(3)}_{a}\langle 0,\mathrm{out}|\,{}^+a_{m_M}\cdots{}^+a_{m_1}\hat\rho^{(3)}\,{}^+a^\dagger_{m_1}\ldots{}^+a^\dagger_{m_M}|0,\mathrm{out}\rangle^{(3)}_a. \tag{37}$$

The reduced density operators $\hat\rho^{(3)}_\pm$ can be obtained from the reduced generating functionals $\underline{R}^{(3)}_\pm$,

$$\underline{R}^{(3)}_\pm = \mathrm{tr}_\mp\underline{R}^{(3)}, \tag{38}$$

where the partial traces are defined in the same way as in Eq. (37).

Using the path-integral representation for traces (C6) in the representation (A50) for $\underline{R}^{(3)}$, we obtain that

$$\underline{R}^{(3)}_+ = \prod_{n\in\Omega_3}\left[Z^{(3)}_{+,n}\right]^{-1}:\exp\{-{}^+a^\dagger_n[1-K_+(J)]\,{}^+a_n\}:,$$
$$\underline{R}^{(3)}_- = \prod_{n\in\Omega_3}\left[Z^{(3)}_{-,n}\right]^{-1}:\exp\{-{}_+b^\dagger_n[1-K_-(J)]\,{}_+b_n\}:, \tag{39}$$
$$K_\pm(J) = D_\pm + C^\dagger(1+D_\mp)^{-1}C,\quad \left[Z^{(3)}_{\pm,n}\right]^{-1} = |w_n(-|-)|^{-2}(1+AB)(1+D_\mp),$$

where A, B, C, and $D_\pm$ are some functions of sources J and relative elementary amplitudes w_n given by Eq. (A50); $w_n(-|-)$ is the relative amplitude of the electron transition. The explicit form of the elementary amplitudes w_n is given by Eq. (A47). Choosing appropriate sources J in the same manner as it was done in Sec. III, we can obtain the corresponding partial density operators $\hat\rho^{(3)}_\pm$ for different initial conditions.

B. Measurement-induced reduction

We can also consider a reduction of density operators, which occurs due to measurement of a physical quantity, namely, the number of final particles, by some classical tool. This kind of reduction can also occur due to some decoherence processes, such as collisions with some external sources (e.g., with impurities in graphene). For us, there is no difference which of the mechanisms is implemented, so in what follows we talk about an intermediate measurement by a classical tool as a source of the decoherence.

We study the measurement-induced deformation of the density matrix for two initial conditions, namely, when the initial state of the system is a pure state and when the system initially is in a thermal equilibrium. Suppose that we are measuring the number of final particles (electrons or positrons) N in the state $\hat\rho$ of the system under consideration. According to von Neumann [43],

the density operator $\hat{\rho}$ after this measurement is reduced to the operator $\hat{\rho}_N$ of the form

$$\hat{\rho}_N = \sum_{\{s\}} W_s \hat{P}_s, \quad \hat{P}_s = |s, \text{out}\rangle\langle s, \text{out}|, \quad W_s = \langle s, \text{out}|\hat{\rho}|s, \text{out}\rangle, \tag{40}$$

where $|s, \text{out}\rangle$ are eigenstates of the operator $\hat{N}$ with the eigenvalues s that represent the total number of electrons and positrons in the state $|s, \text{out}\rangle$,

$$\begin{aligned}\hat{N}(\text{out})|s, \text{out}\rangle = s|s, \text{out}\rangle, |s, \text{out}\rangle = &\prod_{n\in\Omega_1} [{}^{+}a_n^{\dagger}]^{l_{n,1}} [{}_{-}a_n^{\dagger}]^{k_{n,1}} \prod_{n\in\Omega_2} [a_n^{\dagger}]^{l_{n,2}} \\ &\times \prod_{n\in\Omega_4} [b_n^{\dagger}]^{l_{n,4}} \prod_{n\in\Omega_5} [{}_{+}b_n^{\dagger}]^{l_{n,5}} [{}^{-}b_n^{\dagger}]^{k_{n,5}} \prod_{n\in\Omega_3} [{}^{+}a_n^{\dagger}]^{l_{n,3}} [{}_{+}b_n^{\dagger}]^{k_{n,3}} |0, \text{out}\rangle, \\ s = &\sum_{n\in\Omega_1} (l_{n,1} + k_{n,1}) + \sum_{n\in\Omega_2} (l_{n,2}) + \sum_{n\in\Omega_4} (l_{n,4}) + \sum_{n\in\Omega_5} (l_{n,5} + k_{n,5}) + \sum_{n\in\Omega_3} (l_{n,3} + k_{n,3}).\end{aligned} \tag{41}$$

Note that for Dirac particles $l_{n,i}, k_{n,i} = (0, 1)$. It is convenient to introduce partial density operators for each range Ω_i:

$$\begin{aligned}&\hat{\rho}_N^{(i)} = \prod_{n\in\Omega_i} \hat{\rho}_{N,n}^{(i)}, \quad \hat{\rho}_{N,n}^{(i)} = \sum_{\{s_i\}} W_{s,n}^{(i)} \hat{P}_{s,n}^{(i)}, \\ &\hat{P}_{s,n}^{(i)} = |s_i, \text{out}\rangle_n^{(i)} {}_n^{(i)}\langle s_i, \text{out}|, \quad W_{s,n}^{(i)} = {}^{(i)}\langle s_i, \text{out}|\hat{\rho}^{(i)}|s_i, \text{out}\rangle^{(i)}.\end{aligned} \tag{42}$$

This way the general density operator of the system can be presented as

$$\hat{\rho}_N = \otimes \prod_{i=1}^{5} \hat{\rho}_N^{(i)}. \tag{43}$$

The state vectors $|s_i, \text{out}\rangle_n^{(i)}$ introduced in Eq. (42) are different for each range Ω_i:

$$\begin{aligned}&|s_1, \text{out}\rangle_n^{(1)} = [{}^{+}a_n^{\dagger}]^{l_{n,1}} [{}_{-}a_n^{\dagger}]^{k_{n,1}} |0, \text{out}\rangle_n^{(1)}, \quad s_1 = l_{n,1} + k_{n,1}, \\ &|s_3, \text{out}\rangle_n^{(3)} = [{}^{+}a_n^{\dagger}]^{l_{n,3}} [{}_{+}b_n^{\dagger}]^{k_{n,3}} |0, \text{out}\rangle_n^{(3)}, \quad s_3 = l_{n,3} + k_{n,3}, \\ &|s_5, \text{out}\rangle_n^{(5)} = [{}_{+}b_n^{\dagger}]^{l_{n,5}} [{}^{-}b_n^{\dagger}]^{k_{n,5}} |0, \text{out}\rangle_n^{(5)}, \quad s_5 = l_{n,5} + k_{n,5}, \\ &|s_2, \text{out}\rangle_n^{(2)} = [a_n^{\dagger}]^{l_{n,2}} |0\rangle_n^{(2)}, \quad |s_4, \text{out}\rangle_n^{(4)} = [b_n^{\dagger}]^{l_{n,4}} |0\rangle_n^{(4)}, \quad s_{2\,(4)} = l_{n,2\,(4)}.\end{aligned} \tag{44}$$

The sum of all eigenvalues is equal to the total number of particles in the state $|s, \text{out}\rangle$, i.e., $\sum_{i=1}^{5} \sum_{n\in\Omega_i} s_i = s$. In what follows we also use the following notation for partial vacuum projectors for the Klein zone:

$$P_{v,n}^{(3)}(\text{in}) = |0, \text{in}\rangle_n^{(3)} {}_n^{(3)}\langle 0, \text{in}|, \quad P_{v,n}^{(i)}(\text{out}) = |0, \text{out}\rangle_n^{(i)} {}_n^{(i)}\langle 0, \text{out}|. \tag{45}$$

1. Initial vacuum state

Vacuum states in ranges $\Omega_{1,2,4,5}$ remain in vacuum and the measurement of the number of particles does not deform the partial density operators with vacuum initial conditions in these ranges. It is easy to show that in the Klein zone Ω_3 the initial vacuum state evolves as

$$|0, \text{in}\rangle_n^{(3)} = c_{\text{v},n}[1 - {}^{+}a_n^{\dagger} w_n(+ - |0)_{+}b_n^{\dagger}]|0, \text{out}\rangle_n^{(3)}, \quad c_{\text{v},n} = {}_n^{(3)}\langle 0, \text{out}|0, \text{in}\rangle_n^{(3)}, \tag{46}$$

where $w_n(+ - |0)$ is a relative amplitude of pair production. The corresponding partial density operator $\hat{\rho}_{v,n}^{(3)}$ with an initial vacuum state can be written as

$$\hat{\rho}_{v,n}^{(3)} = P_{v,n}^{(3)}(\text{in}) = |c_{\text{v},n}|^2 [1 - {}^{+}a_n^{\dagger} w_n(+ - |0)\, {}_{+}b_n^{\dagger}] P_{v,n}^{(3)}(\text{out}) [1 - {}_{+}b_n w_n(+ - |0)^* {}^{+}a_n]. \tag{47}$$

Performing a reduction procedure (40), we obtain

$$\hat{\rho}_{N,n}^{(3)} = |c_{\text{v},n}|^2 P_{v,n}^{(3)}(\text{out}) + |c_{\text{v},n}|^2 |w_n(+ - |0)|^2 {}^{+}a_n^{\dagger} {}_{+}b_n^{\dagger} P_{v,n}^{(3)}(\text{out})\, {}_{+}b_n\, {}^{+}a_n. \tag{48}$$

The first term of this expression corresponds to the situation where we find a vacuum state after the measurement and the second one corresponds to the situation where we find the state with an electron-positron pair. The coefficients $|c_{\text{v},n}|^2$ and $|c_{\text{v},n}|^2 |w_n(+ - |0)|^2$ are classical probabilities for each of the outcomes.

2. Initial thermal state

We can consider the measurement-induced reduction for the thermal initial state of the system. The partial density operators $\hat{\rho}_n^{(i)}$ are obtained from the generating functionals $R^{(i)}$ by setting the sources J as in Eq. (25). The following are the nonvanishing

weights $W^{(i)}$ from Eq. (42): the range Ω_1,

$$
\begin{aligned}
W^{(1)}_{1,n} &= {}^{(1)}_{n}\langle 0,\mathrm{out}|\hat{\rho}^{(1)}_{\beta,n}|0,\mathrm{out}\rangle^{(1)}_{n} = \left[Z^{(1)}_{n}\right]^{-1},\\
W^{(1)}_{2,n} &= {}^{(1)}_{n}\langle 0,\mathrm{out}|\,{}_{+}a_n\hat{\rho}^{(1)}_{\beta,n}\,{}_{+}a^{\dagger}_{n}|0,\mathrm{out}\rangle^{(1)}_{n} = \left[Z^{(1)}_{n}\right]^{-1}\tilde{C}_{++},\quad \tilde{C}_{++} = 1 + C_{++},\\
W^{(1)}_{3,n} &= {}^{(1)}_{n}\langle 0,\mathrm{out}|\,{}_{-}a_n\hat{\rho}^{(1)}_{\beta,n}\,{}_{-}a^{\dagger}_{n}|0,\mathrm{out}\rangle^{(1)}_{n} = \left[Z^{(1)}_{n}\right]^{-1}\tilde{C}_{--},\quad \tilde{C}_{--} = 1 + C_{--},\\
W^{(1)}_{4,n} &= {}^{(1)}_{n}\langle 0,\mathrm{out}|\,{}_{+}a_n\,{}_{-}a_n\hat{\rho}^{(1)}_{\beta,n}\,{}_{-}a^{\dagger}_{n}\,{}_{+}a^{\dagger}_{n}|0,\mathrm{out}\rangle^{(1)}_{n} = \left[Z^{(1)}_{n}\right]^{-1}[\tilde{C}_{++}\tilde{C}_{--} - C_{+-}C_{-+}];
\end{aligned}
\tag{49}
$$

the range Ω_5,

$$
\begin{aligned}
W^{(5)}_{1} &= {}^{(5)}_{n}\langle 0,\mathrm{out}|\hat{\rho}^{(5)}_{\beta,n}|0,\mathrm{out}\rangle^{(5)}_{n} = \left[Z^{(5)}_{n}\right]^{-1},\\
W^{(5)}_{2} &= {}^{(5)}_{n}\langle 0,\mathrm{out}|\,{}_{+}b_n\hat{\rho}^{(5)}_{\beta,n}\,{}_{+}b^{\dagger}_{n}|0,\mathrm{out}\rangle^{(5)}_{n} = \left[Z^{(5)}_{n}\right]^{-1}\tilde{D}_{++},\quad \tilde{D}_{++} = 1 + D_{++},\\
W^{(5)}_{3} &= {}^{(5)}_{n}\langle 0,\mathrm{out}|\,{}^{-}b_n\hat{\rho}^{(5)}_{\beta,n}\,{}^{-}b^{\dagger}_{n}|0,\mathrm{out}\rangle^{(5)}_{n} = \left[Z^{(5)}_{n}\right]^{-1}\tilde{D}_{--},\quad \tilde{D}_{--} = 1 + D_{--},\\
W^{(5)}_{4} &= {}^{(5)}_{n}\langle 0,\mathrm{out}|\,{}_{+}b_n\,{}^{-}b_n\hat{\rho}^{(5)}_{\beta,n}\,{}^{-}b^{\dagger}_{n}\,{}_{+}b^{\dagger}_{n}|0,\mathrm{out}\rangle^{(5)}_{n} = \left[Z^{(5)}_{n}\right]^{-1}[\tilde{D}_{++}\tilde{D}_{--} - D_{+-}D_{-+}];
\end{aligned}
\tag{50}
$$

and the range Ω_3,

$$
\begin{aligned}
W^{(3)}_{1} &= {}^{(3)}_{n}\langle 0,\mathrm{out}|\hat{\rho}^{(3)}_{\beta,n}|0,\mathrm{out}\rangle^{(3)}_{n} = \tilde{Z}^{(3)}_{n},\quad \tilde{Z}^{(3)}_{n} = \left[Z^{(3)}_{n}\right]^{-1}|c_{\mathrm{v},n}|^2(1+AB),\\
W^{(3)}_{2} &= {}^{(3)}_{n}\langle 0,\mathrm{out}|\,{}^{+}a_n\hat{\rho}^{(3)}_{\beta,n}\,{}^{+}a^{\dagger}_{n}|0,\mathrm{out}\rangle^{(3)}_{n} = \tilde{Z}^{(3)}_{n}D_{+},\\
W^{(3)}_{3} &= {}^{(3)}_{n}\langle 0,\mathrm{out}|\,{}_{+}b_n\hat{\rho}^{(3)}_{\beta,n}\,{}_{+}b^{\dagger}_{n}|0,\mathrm{out}\rangle^{(3)}_{n} = \tilde{Z}^{(3)}_{n}D_{-},\\
W^{(3)}_{4} &= {}^{(3)}_{n}\langle 0,\mathrm{out}|\,{}^{+}a_n\,{}_{+}b_n\hat{\rho}^{(3)}_{\beta,n}\,{}_{+}b^{\dagger}_{n}\,{}^{+}a^{\dagger}_{n}|0,\mathrm{out}\rangle^{(3)}_{n} = \tilde{Z}^{(3)}_{n}(D_{+}D_{-} + C^{\dagger}C).
\end{aligned}
\tag{51}
$$

C. Spatial reduction (left and right subsystems)

The x-electric potential steps provide the spatial separation of the whole system in two subsystems, the left subsystem and right subsystem, i.e., final particles to the left of potential step and final particles to the right of potential step. It is easy to imagine a situation when we are interested in measuring physical values only in left and right asymptotic areas. For example, we can suppose that measuring tools are situated only to the left of the potential step. In this case the general density operator must be averaged (reduced) over all unavailable states of the final right particles.

From the general theory [34] we know the following. In the range Ω_3 all the electrons (initial and final) are located on the left side of the potential step and all the positrons are on the right side. Therefore, one can see that a reduction over the left and right subsystems in the range Ω_3 coincides with a reduction over electron and positron subsystems, respectively, i.e.,

$$
\hat{\rho}^{(3)}_{\mathrm{left}} = \hat{\rho}^{(3)}_{+},\quad \hat{\rho}^{(3)}_{\mathrm{right}} = \hat{\rho}^{(3)}_{-}.
\tag{52}
$$

In the range Ω_2 there are only electrons on the left side of potential step, so $\hat{\rho}^{(2)}_{\mathrm{left}} = \hat{\rho}^{(2)}$. Similarly, there are only right positrons in the range Ω_4; therefore $\hat{\rho}^{(4)}_{\mathrm{right}} = \hat{\rho}^{(4)}$.

The range Ω_1 contains left and right electrons; the range Ω_5 contains left and right positrons. In these ranges we need to consider the reduction of partial generating functionals over the left or right final particles. Let us start with the range Ω_1. It is convenient to use the expression for the generating functional $\underline{R}^{(1)}$ given by Eq. (A39) to calculate the partial (right and left) trace over states with right electrons, i.e., states constructed with creation and annihilation operators ${}_{+}a^{\dagger}$ and ${}_{+}a$, or ${}_{-}a^{\dagger}$ and ${}_{-}a$, thus creating spatially reduced partial generating functionals as

$$
\begin{aligned}
\underline{R}^{(1)}_{\mathrm{left}} &= \mathrm{tr}^{(1)}_{\mathrm{right}}\underline{R}^{(1)},\quad \underline{R}^{(1)}_{\mathrm{right}} = \mathrm{tr}^{(1)}_{\mathrm{left}}\underline{R}^{(1)},\\
\mathrm{tr}_{\mathrm{right}}\hat{A} &= \sum_{M=0}^{\infty}\sum_{\{m\}}(M!)^{-1}\left\langle\Psi^{\mathrm{right}}_{\{m\}_M}\middle|A\middle|\Psi^{\mathrm{right}}_{\{m\}_M}\right\rangle,\\
\mathrm{tr}_{\mathrm{left}}\hat{A} &= \sum_{M=0}^{\infty}\sum_{\{m\}}(M!)^{-1}\left\langle\Psi^{\mathrm{left}}_{\{m\}_M}\middle|A\middle|\Psi^{\mathrm{left}}_{\{m\}_M}\right\rangle,
\end{aligned}
\tag{53}
$$

where $|\Psi^{\rm right}_{\{m\}_M}\rangle$ and $|\Psi^{\rm left}_{\{m\}_M}\rangle$ are state vectors for right and left electrons, respectively,

$$\begin{aligned}
\left|\Psi^{\rm right}_{\{m\}_M}\right\rangle &= {}^{+}a^{\dagger}_{m_1}\cdots{}^{+}a^{\dagger}_{m_M}|0,\,{\rm out}\rangle^{(1)}_{\rm right},\\
\left|\Psi^{\rm left}_{\{m\}_M}\right\rangle &= {}_{-}a^{\dagger}_{m_1}\cdots{}_{-}a^{\dagger}_{m_M}|0,\,{\rm out}\rangle^{(1)}_{\rm left},\\
|0,\,{\rm out}\rangle^{(1)}_{\rm right}&\otimes|0,\,{\rm out}\rangle^{(1)}_{\rm left}=|0,\,{\rm out}\rangle^{(1)}.
\end{aligned}\tag{54}$$

Note that partial left and right electron vacua can be factorized in quantum modes n and therefore generating functionals (53) can be factorized as well. Taking this into account and calculating the trace, we obtain

$$\begin{aligned}
\underline{R}^{(1)}_{{\rm left},\,n} &= (1+\tilde{C}_{++}){:}\exp\{{}_{-}a^{\dagger}_{n}L_{-}\,{}_{-}a_{n}\}{:},\quad L_{-}=C_{--}-C_{+-}(1+\tilde{C}_{++})^{-1}C_{-+},\\
\underline{R}^{(1)}_{{\rm right},\,n} &= (1+\tilde{C}_{--}){:}\exp\{{}^{+}a^{\dagger}_{n}L_{+}\,{}^{+}a_{n}\}{:},\quad L_{+}=C_{++}-C_{+-}(1+\tilde{C}_{--})^{-1}C_{-+}.
\end{aligned}\tag{55}$$

Similar results can be obtained for the range Ω_5:

$$\begin{aligned}
\underline{R}^{(5)}_{{\rm right},\,n} &= (1+\tilde{D}_{--}){:}\exp\{{}_{+}b^{\dagger}_{n}K_{+}\,{}_{+}b^{\dagger}_{n}\}{:},\quad K_{+}=D_{++}-D_{+-}(1+\tilde{D}_{--})^{-1}D_{-+},\\
\underline{R}^{(5)}_{{\rm left},\,n} &= (1+\tilde{D}_{++}){:}\exp\{{}^{-}b^{\dagger}_{n}K_{-}\,{}^{-}b_{n}\}{:},\quad K_{-}=D_{--}-D_{+-}(1+\tilde{D}_{++})^{-1}D_{-+}.
\end{aligned}\tag{56}$$

Now setting the sources J as discussed in Sec. III, one can obtain the density operators $\hat{\rho}^{(1,5)}_{\rm left\ (right)}$ with different initial conditions.

V. ENTROPY OF REDUCED DENSITY OPERATORS

It is known that the electron-positron pair is always produced in an entangled state. The process of pair production does not change the entropy of the whole system since its evolution is unitary. It is easy, however, to imagine a situation in which only a certain quantum subsystem is available for measurements; in this case, we must reduce the total density matrix over the states of the inaccessible subsystem. Reduction over one of the quantum subsystems makes part of the information unavailable [44]. Thus, the reduced density operators we have introduced in the preceding section always describe mixed states even when the initial state of the system is pure. This means that the entropy of particular subsystems can change as the subsystems become entangled due to pair creation. This entropy change can be used as a measure of information loss due to reduction or as a measure of entanglement between those subsystems. In what follows we show that the above-mentioned entropy change due to pair production can be quantified via the corresponding numbers of final particles, which depend on the choice of initial state and on the type of reduction, or, in other words, on the choice of quantum subsystem. By controlling the strength and duration of the electric field applied to the system, we have the theoretical ability to change the average number of pairs generated by the field and thus their entanglement. Another important task is to understand how a measurement of a physical value (for example, a measurement of the final mean number of particles) can change the entropy of the system. To address all these questions we calculate the entropy, corresponding to different reductions of the general density matrix for different initial conditions. As a measure of information loss due to reduction, we use the von Neumann entropy, defined as

$$S(\hat{\rho})=-{\rm tr}\hat{\rho}\ln\hat{\rho}.\tag{57}$$

A. Entropy corresponding to reduction over the subsystems of electrons and positrons

First, we calculate von Neumann entropy for reduced density matrices (37),

$$S(\hat{\rho}_{\pm})=-{\rm tr}\hat{\rho}_{\pm}\ln\hat{\rho}_{\pm},\tag{58}$$

where tr denotes the full trace of the operator, ${\rm tr}\hat{A}={\rm tr}_{-}{\rm tr}_{+}\hat{A}$. Using the definitions (37), let us transform the operator $\ln\hat{\rho}_{\pm}$ as follows:

$$\begin{aligned}
\ln\hat{\rho}_{+} &= \ln\hat{\rho}^{(1)}+\ln\hat{\rho}^{(2)}+\ln\hat{\rho}^{(3)}_{+},\\
\ln\hat{\rho}_{-} &= \ln\hat{\rho}^{(5)}+\ln\hat{\rho}^{(4)}+\ln\hat{\rho}^{(3)}_{-}.
\end{aligned}\tag{59}$$

Due to the fact that partial density matrices $\hat{\rho}^{(i)}$ and $\hat{\rho}^{(3)}_{\pm}$ are normalized (${\rm tr}\hat{\rho}^{(i)}={\rm tr}\hat{\rho}^{(3)}_{\pm}=1$), it is easy to show that Eq. (58) transforms into the sum of entropies

$$\begin{aligned}
S(\hat{\rho}_{+}) &= S(\hat{\rho}^{(1)})+S(\hat{\rho}^{(2)})+S(\hat{\rho}^{(3)}_{+}),\\
S(\hat{\rho}_{-}) &= S(\hat{\rho}^{(5)})+S(\hat{\rho}^{(4)})+S(\hat{\rho}^{(3)}_{-}).
\end{aligned}\tag{60}$$

We recall that in each range Ω_i the partial density operators $\hat{\rho}^{(i)}$ and $\hat{\rho}^{(3)}_{\pm}$ can be factorized in quantum modes n and each one-mode operator is also normalized. This allows us to further simplify the expressions (60) and write

$$\begin{aligned}
S(\hat{\rho}^{(i)}) &= \sum_{n\in\Omega_i}S(\hat{\rho}^{(i)}_{n})=-\sum_{n\in\Omega_i}{\rm tr}\hat{\rho}^{(i)}_{n}\ln\hat{\rho}^{(i)}_{n},\quad i=1,2,4,5,\\
S(\hat{\rho}^{(3)}_{\pm}) &= \sum_{n\in\Omega_3}S(\hat{\rho}^{(3)}_{\pm,n})=-\sum_{n\in\Omega_3}{\rm tr}\hat{\rho}^{(3)}_{\pm,n}\ln\hat{\rho}^{(3)}_{\pm,n}.
\end{aligned}\tag{61}$$

Now one can calculate the entropy for the density operators with different initial conditions.

1. Initial vacuum state

In this case partial density operators $\hat{\rho}^{(i)}_{n}$, $i=1,2,4,5$, are given by Eq. (20). It is easy to see that the corresponding entropies vanish, i.e., $S(\hat{\rho}^{(1,2,4,5)}_{n})=0$. The entropies of density operators $\hat{\rho}^{(3)}_{\pm,n}$ are equal and can be calculated [35] to have the form

$$S\big(\hat{\rho}^{(3)}_{\pm,n}\big)=-\big[\big(1-N^{\rm cr}_{n}\big)\ln\big(1-N^{\rm cr}_{n}\big)+N^{\rm cr}_{n}\ln N^{\rm cr}_{n}\big],\tag{62}$$

where $N_n^{\rm cr}$ is the mean differential number of pairs created from vacuum by the electric field (24). The total entropy can be found as a sum over all quantum numbers in Ω_3,

$$S(\hat\rho_\pm^{(3)}) = -\sum_{n\in\Omega_3}\left[\left(1-N_n^{\rm cr}\right)\ln\left(1-N_n^{\rm cr}\right)+N_n^{\rm cr}\ln N_n^{\rm cr}\right]. \tag{63}$$

2. Initial thermal state

The entropies corresponding to partial density operators $\hat\rho_{\beta,n}^{(i)}$ are

$$S\big(\hat\rho_{\beta,n}^{(1,5)}\big) = -\sum_{\zeta=\pm}\big\{\big[1-N_{n,\beta,\zeta}^{(1,5)}(\text{in})\big]\ln\big[1-N_{n,\beta,\zeta}^{(1,5)}(\text{in})\big]+N_{n,\beta,\zeta}^{(1,5)}(\text{in})\ln N_{n,\beta,\zeta}^{(1,5)}(\text{in})\big\}, \tag{64a}$$

$$S\big(\hat\rho_{\beta,n}^{(2,4)}\big) = -\big\{\big[1-N_{n,\beta}^{(2,4)}(\text{in})\big]\ln\big[1-N_{n,\beta}^{(2,4)}(\text{in})\big]+N_{n,\beta}^{(2,4)}(\text{in})\ln N_{n,\beta}^{(2,4)}(\text{in})\big\}, \tag{64b}$$

$$S\big(\hat\rho_{\beta,\pm,n}^{(3)}\big) = -\big\{\big[1-N_{n,\beta,\pm}^{(3)}(\text{out})\big]\ln\big[1-N_{n,\beta,\pm}^{(3)}(\text{out})\big]+N_{n,\beta,\pm}^{(3)}(\text{out})\ln N_{n,\beta,\pm}^{(3)}(\text{out})\big\}, \tag{64c}$$

where the mean differential numbers of particles from Eqs. (64a) and (64b) are given by

$$\begin{aligned} N_{n,\beta,\zeta}^{(1)}(\text{in}) &= \left(e^{E_n^\zeta}+1\right)^{-1}, \quad \zeta=\pm,\ n\in\Omega_{1,5},\\ N_{n,\beta}(\text{in}) &= (e^{E_n}+1)^{-1}, \quad n\in\Omega_{2,4}, \end{aligned} \tag{65}$$

and $N_{n,\beta,\pm}^{(3)}(\text{out})$ are the differential mean numbers of final electrons (+) and positrons (−) in the range Ω_3,

$$\begin{aligned} N_{n,\beta,+}^{(3)}(\text{out}) &= \mathrm{tr}\hat\rho_{+,n,\beta}^{(3)}\,{}_+a_n^\dagger\,{}_+a_n = N_n^{\rm cr}\big[1-N_{n,\beta,-}^{(3)}(\text{in})\big]+\big(1-N_n^{\rm cr}\big)N_{n,\beta,+}^{(3)}(\text{in}),\\ N_{n,\beta,-}^{(3)}(\text{out}) &= \mathrm{tr}\hat\rho_{-,n,\beta}^{(3)}\,{}_+b_n^\dagger\,{}_+b_n = N_n^{\rm cr}\big[1-N_{n,\beta,+}^{(3)}(\text{in})\big]+\big(1-N_n^{\rm cr}\big)N_{n,\beta,-}^{(3)}(\text{in}). \end{aligned} \tag{66}$$

The differential mean numbers $N_{n,\beta,\pm}^{(3)}(\text{in})$ in Eq. (66) can be calculated similarly to Eq. (32) using the corresponding creation and annihilation operators.

B. Entropy corresponding to measurement-induced reduction

The measurement reduced density operators $\hat\rho_N^{(i)}$ with different initial conditions are given by Eq. (42). Similarly to the preceding section, it is easy to show that von Neumann entropy can be presented as the sum over quantum modes n of partial entropies

$$S\big(\hat\rho_N^{(i)}\big) = \sum_{n\in\Omega_i} S\big(\hat\rho_{N,n}^{(i)}\big) = -\sum_{n\in\Omega_i}\mathrm{tr}\hat\rho_{N,n}^{(i)}\ln\hat\rho_{N,n}^{(i)}. \tag{67}$$

Therefore, to obtain the total entropy it is sufficient to calculate only the entropies $S(\hat\rho_{N,n}^{(i)})$ corresponding to partial density operators $\hat\rho_{N,n}^{(i)}$ and then perform the summation over all quantum numbers $n\in\Omega_i$.

1. Initial vacuum state

Let us calculate von Neumann entropy corresponding to density operator $\hat\rho_{N,n}^{(3)}$. We can show that the entropy for this case takes the form

$$\begin{aligned} S\big(\hat\rho_{N,n}^{(3)}\big) &= -[|c_{{\rm v},n}|^2\ln|c_{{\rm v},n}|^2+|c_{{\rm v},n}|^2|w_n(+-|0)|^2\ln|c_{{\rm v},n}|^2|w_n(+-|0)|^2],\\ |c_{{\rm v},n}|^2 &= 1-N_n^{\rm cr}, \quad |w_n(+-|0)|^2 = N_n^{\rm cr}\left(1-N_n^{\rm cr}\right)^{-1}, \end{aligned} \tag{68}$$

which leads us to the result

$$S\big(\hat\rho_{N,n}^{(3)}\big) = -\left[\left(1-N_n^{\rm cr}\right)\ln\left(1-N_n^{\rm cr}\right)+N_n^{\rm cr}\ln N_n^{\rm cr}\right]. \tag{69}$$

2. Initial thermal state

For this case the density operators $\hat\rho_{N,n}^{(1)}$, $\hat\rho_{N,n}^{(3)}$, and $\hat\rho_{N,n}^{(5)}$ have the form (42) with the weights W given by Eq. (49) in Ω_1, by Eq. (50) in Ω_5, and by Eq. (51) in Ω_3. It can be shown that entropies $S(\hat\rho_{N,n}^{(i)})$ take the form

$$S\big(\hat\rho_{N,n}^{(i)}\big) = -\sum_{l=1}^{4} W_l^{(i)}\ln W_l^{(i)}. \tag{70}$$

Sources J, given in Eq. (25) for the case of the thermal initial state are connected to differential mean numbers of initial particles by the relation

$$J^{(i)}_{\pm,n} = e^{-E^{\pm}_{n\in\Omega_i}} = N^{(i)}_{n,\beta,\pm}(\text{in})\big[1 - N^{(i)}_{n,\beta,\pm}(\text{in})\big]^{-1}. \tag{71}$$

Using then Eqs. (A41)–(A43) and (A50), it is possible to present weights $W^{(i)}_l$ via differential mean numbers of initial particles, reflection $|R_n|^2$ and transition $|T_n|^2$ probabilities, and the number of particles created from vacuum N^{cr}_n.

C. Entropy corresponding to spatial reduction (left and right)

We can also calculate von Neumann entropy for the left and right reduced density operators, found in Eqs. (55) and (56). For the reduced generating functionals from Ω_1 this entropy has the form

$$\begin{aligned} S\big(R^{(1)}_{\text{left}}\big) &= -\text{tr}R^{(1)}_{\text{left}}\ln R^{(1)}_{\text{left}} = \sum_{n\in\Omega_1}\big[\ln Z^{(1)}_n - \ln(1+\tilde{C}_{++}) - N^{(1)}_{n,-}(\text{out})\ln(1+L_-)\big], \\ S\big(R^{(1)}_{\text{right}}\big) &= -\text{tr}R^{(1)}_{\text{right}}\ln R^{(1)}_{\text{right}} = \sum_{n\in\Omega_1}\big[\ln Z^{(1)}_n - \ln(1+\tilde{C}_{--}) - N^{(1)}_{n,+}(\text{out})\ln(1+L_+)\big], \end{aligned} \tag{72}$$

where $N^{(1)}_{n,-}(\text{out})$ and $N^{(1)}_{n,+}(\text{out})$ are the differential mean numbers of left and right final electrons in Ω_1,

$$\begin{aligned} N^{(1)}_{n,-}(\text{out}) &= \text{tr}R^{(1)}_{\text{left}}\,{}_-a^{\dagger}_n\,{}_-a_n = \big[Z^{(1)}_n\big]^{-1}[(1+\tilde{C}_{++})\tilde{C}_{--} - C_{+-}C_{-+}], \\ N^{(1)}_{n,+}(\text{out}) &= \text{tr}R^{(1)}_{\text{right}}\,{}_+a^{\dagger}_n\,{}_+a_n = \big[Z^{(1)}_n\big]^{-1}[(1+\tilde{C}_{--})\tilde{C}_{++} - C_{+-}C_{-+}]. \end{aligned} \tag{73}$$

Using the fact that reduced generating functionals $R^{(1)}_{\text{left}}$ are normalized, $\text{tr}R^{(1)}_{\text{left}} = 1$, one can show that the following relations hold true:

$$\begin{aligned} 1 + L_- &= \frac{N^{(1)}_{n,-}(\text{out})}{1 - N^{(1)}_{n,-}(\text{out})}, \\ 1 + L_+ &= \frac{N^{(1)}_{n,+}(\text{out})}{1 - N^{(1)}_{n,+}(\text{out})}, \\ \frac{1+\tilde{C}_{\pm\pm}}{Z^{(1)}_n} &= (2+L_{\mp})^{-1}. \end{aligned} \tag{74}$$

Using these expressions, we can represent Eq. (72) as

$$\begin{aligned} S\big(R^{(1)}_{\text{left}}\big) &= -\textstyle\sum_{n\in\Omega_1}\big\{\big[1 - N^{(1)}_{n,-}(\text{out})\big]\ln\big[1 - N^{(1)}_{n,-}(\text{out})\big] + N^{(1)}_{n,-}(\text{out})\ln N^{(1)}_{n,-}(\text{out})\big\}, \\ S\big(R^{(1)}_{\text{right}}\big) &= -\textstyle\sum_{n\in\Omega_1}\big\{\big[1 - N^{(1)}_{n,+}(\text{out})\big]\ln\big[1 - N^{(1)}_{n,+}(\text{out})\big] + N^{(1)}_{n,+}(\text{out})\ln N^{(1)}_{n,+}(\text{out})\big\}. \end{aligned} \tag{75}$$

For the reduced generating functionals from Ω_5 the result reads

$$\begin{aligned} S\big(R^{(5)}_{\text{left}}\big) &= -\text{tr}R^{(5)}_{\text{left}}\ln R^{(5)}_{\text{left}} = \sum_{n\in\Omega_5}\big[\ln Z^{(5)}_n - \ln(1+\tilde{D}_{++}) - N^{(5)}_{n,+}(\text{out})\ln(1+K_+)\big], \\ S\big(R^{(5)}_{\text{right}}\big) &= -\text{tr}R^{(5)}_{\text{right}}\ln R^{(5)}_{\text{right}} = \sum_{n\in\Omega_5}\big[\ln Z^{(5)}_n - \ln(1+\tilde{D}_{--}) - N^{(5)}_{n,-}(\text{out})\ln(1+K_-)\big], \end{aligned} \tag{76}$$

where $N^{(5)}_{n,+}(\text{out})$ and $N^{(5)}_{n,-}(\text{out})$ are the differential mean numbers of final positrons,

$$\begin{aligned} N^{(5)}_{n,+}(\text{out}) &= \text{tr}R^{(5)}_{\text{left}}\,{}_+b^{\dagger}_n\,{}_+b_n = \big[Z^{(5)}_n\big]^{-1}[(1+\tilde{D}_{++})\tilde{D}_{--} - D_{+-}D_{-+}], \\ N^{(5)}_{n,-}(\text{out}) &= \text{tr}R^{(5)}_{\text{right}}\,{}_-b^{\dagger}_n\,{}_-b_n = \big[Z^{(5)}_n\big]^{-1}[(1+\tilde{D}_{--})\tilde{D}_{++} - D_{+-}D_{-+}]. \end{aligned} \tag{77}$$

The entropies (76) in terms of mean differential numbers (77) take the form

$$\begin{aligned} S\big(R^{(5)}_{\text{left}}\big) &= -\sum_{n\in\Omega_5}\big\{\big[1 - N^{(5)}_{n,+}(\text{out})\big]\ln\big[1 - N^{(5)}_{n,+}(\text{out})\big] + N^{(5)}_{n,+}(\text{out})\ln N^{(5)}_{n,+}(\text{out})\big\}, \\ S\big(R^{(5)}_{\text{right}}\big) &= -\sum_{n\in\Omega_5}\big\{\big[1 - N^{(5)}_{n,-}(\text{out})\big]\ln\big[1 - N^{(5)}_{n,-}(\text{out})\big] + N^{(5)}_{n,-}(\text{out})\ln N^{(5)}_{n,-}(\text{out})\big\}. \end{aligned} \tag{78}$$

D. Loss of information due to electron-positron reduction in the L-constant field

Here we illustrate some of the obtained formulas by considering the deformation of the quantum vacuum between two infinite capacitor plates separated by a finite distance L. Several aspects of particle creation by the constant electric field between such plates (this field is also called the L-constant electric field) were studied in Ref. [40]. The latter field is a particular case of the x-electric potential step. Thus, we consider the L-constant electric field in $d = D + 1$ dimensions. We choose $\mathbf{E}(x) = E^i$ $(i = 1, \ldots, D)$, $E^1 = E_x(x)$, $E^{2,\ldots,D} = 0$, and

$$E_x(x) = \begin{cases} 0, & x \in (-\infty, -L/2] \\ E = \text{const} > 0, & x \in (-L/2, L/2) \\ 0, & x \in [L/2, \infty). \end{cases}$$

We consider a particular case with a sufficiently large length L between the capacitor plates,

$$\sqrt{eE}L \gg \max\{1, E_c/E\}. \tag{79}$$

Here $E_c = m^2/e$ is the critical Schwinger field. We conditionally call this approximation as large work approximation when $\Delta U = eEL \gg 2m$. Such an x-electric step represents a regularization for a constant uniform electric field and is suitable for imitating a small-gradient field.

1. Initial vacuum state

Let us calculate von Neumann entropy corresponding to subsystems of electrons and positrons created from vacuum in Ω_3. The leading asymptotic contributions to the differential and total number of those created from the vacuum particles in the large work approximation have the form [40]

$$N_n^{\text{cr}} \approx \exp\left[-\pi \frac{\pi_\perp^2}{eE}\right], \quad N^{\text{cr}} \approx \frac{J_{(d)} TV (eE)^{d/2}}{(2\pi)^{d-1}} \exp\left(-\pi \frac{E_c}{E}\right), \tag{80}$$

where $V = LV_\perp$ is the volume inside the capacitor (the volume occupied by the electric field, L is the distance between capacitor plates, and $V_\perp$ is the transversal volume of capacitor), $J_{(d)} = 2^{[d/2]-1}$ is a spin summation factor,[3] and $e > 0$ is an absolute value of electron charge.

Let us estimate the information loss of the reduced electron and positron subsystems, which can be calculated as entropies (63) of these states. Performing summation over quantum modes n (for the details of this operation see Refs. [35,40]), we obtain the expression

$$S\big(\hat{\rho}_\pm^{(3)}\big) \approx \frac{J_{(d)} TV (eE)^{d/2}}{(2\pi)^{d-1}} \exp\left(-\pi \frac{E_c}{E}\right) A(d, E_c/E) \quad \text{if } d > 2, \tag{81}$$

where the factor $A(d, E_c/E)$ has the form

$$A(d, E_c/E) = \left[(\pi E_c/E + d/2 - 1) + \sum_{l=1}^{\infty} [l^{-d/2} - l^{-1}(l+1)^{(2-d)/2} \exp(-\pi E_c/E)] \exp[-\pi(l-1)E_c/E]\right]. \tag{82}$$

Comparing Eqs. (81) and (80), one can see that the entropy is proportional to the total number of particles created, i.e.,

$$S\big(\hat{\rho}_\pm^{(3)}\big) \approx N^{\text{cr}} A(d, E_c/E). \tag{83}$$

The result coincides with that obtained for the T-constant electric field in Ref. [20], i.e., we reproduce exactly the same expression for the entropy of the electron-positron subsystem for the vacuum initial state. This result shows that despite the fact that L-constant and T-constant electric fields are physically distinct, they can be considered as two different regularizations of the uniform constant electric field in the limit $T, L \to \infty$.

[3] Here $[\cdots]$ denotes the integer part of the expression.

2. Initial thermal state

Here we only consider the Klein zone Ω_3 as well, as for the case of the electron-positron subsystem reduction the density operators of the other quantum ranges Ω_i either are completely traced out and do not contribute to the von Neumann entropy, or are undisturbed by the reduction and therefore their initial entropy does not change after the reduction.

We take the Fermi distributions as those of the initial particles. They depend on particle energy and are given by Eq. (25). In the Klein zone these distributions have the form

$$N_{n,\beta,\pm}^{(3)}(\text{in}) = \{\exp[\beta(\varepsilon_n^\pm - \mu^\pm)] + 1\}^{-1}. \tag{84}$$

At any given $p_\perp$ the available quantum numbers p_0 in the Klein zone for the L-constant field are restricted by the definition of the Klein zone [34]

$$U_{\text{L}} + \pi_\perp \leqslant p_0 \leqslant U_{\text{R}} - \pi_\perp, \quad U_{\text{R}} = -U_{\text{L}} = \Delta U/2 = eEL/2 \tag{85}$$

such that

$$\varepsilon_n^{\pm} = \pm p_0 + \frac{\Delta U}{2}\big[1 - N_n^{\rm cr}\big]. \tag{86}$$

Here $U_{\rm L} = -eA_0(x \to -\infty)$ and $U_{\rm R} = -eA_0(x \to +\infty)$ are the left and right asymptotic potential energies, respectively.

Let us analyze Eq. (84) for initial electrons. The number $N_n^{\rm cr}$ is even with respect to the change $p_0 \to -p_0$ and has the form (80) for the large range if $|p_0|$, $\pi_\perp \ll \Delta U/2$. At the left (right) edge of the Klein zone asymptotic longitudinal momenta $|p^{\rm L}|$ ($|p^{\rm R}|$),

$$|p^{\rm L\,(R)}| = \sqrt{[\pm p_0 + \Delta U/2]^2 - \pi_\perp^2},$$

tends to zero and one of the following limits holds true: $N_n^{\rm cr} \sim |p^{\rm L}|/\sqrt{eE} \to 0$ or $N_n^{\rm cr} \sim |p^{\rm R}|/\sqrt{eE} \to 0$, respectively. We see that kinetic energies $\varepsilon_n^{\pm}$ tend to the minimum, given by transversal energy $\varepsilon_n^{\pm} \to \pi_\perp$. Therefore, it is more likely to find a particle with a lower kinetic energy $\sim\pi_\perp$, just as one would expect.

For further analysis it is convenient to rewrite the expressions (66) for the final differential number of electrons and positrons as

$$\begin{aligned} N^{(3)}_{n,\beta,+}({\rm out}) &= N^{(3)}_{n,\beta,+}({\rm in}) + N_n^{\rm cr}\big[1 - N^{(3)}_{n,\beta,-}({\rm in}) - N^{(3)}_{n,\beta,+}({\rm in})\big], \\ N^{(3)}_{n,\beta,-}({\rm out}) &= N^{(3)}_{n,\beta,-}({\rm in}) + N_n^{\rm cr}\big[1 - N^{(3)}_{n,\beta,-}({\rm in}) - N^{(3)}_{n,\beta,+}({\rm in})\big]. \end{aligned} \tag{87}$$

Note that if $\mu^+ = \mu^- = \mu$, the sum $N^{(3)}_{n,\beta,-}({\rm in}) + N^{(3)}_{n,\beta,+}({\rm in})$ is even with respect to the change $p_0 \to -p_0$. Further consideration can be easily extended to the case when, for example, $N^{(3)}_{n,\beta,+}({\rm in}) = 0$ or $N^{(3)}_{n,\beta,-}({\rm in}) = 0$, i.e., when only one type of initial particle is present. We can sum these expression over quantum numbers $n \in \Omega_3$ as

$$\begin{aligned} N^{(3)}_{\beta,\pm}({\rm out}) &= \sum_{n\in\Omega_3} N^{(3)}_{n,\beta,\pm}({\rm out}) \\ &= \frac{J_{(d)}TV_\perp}{(2\pi)^{d-1}} \int_{p_\perp, p_0 \in \Omega_3} d^{d-2}p_\perp dp_0 N^{(3)}_{n,\beta,\pm}({\rm out}). \end{aligned} \tag{88}$$

It was shown in Ref. [40] that a leading contribution to $N_n^{\rm cr}$, given by Eq. (80), comes from the inner subrange D, defined as

$$\begin{aligned} &\frac{\pi_\perp}{\sqrt{eE}} < K_\perp, \quad |p_0|/\sqrt{eE} < \sqrt{eE}L/2 - K, \\ &\sqrt{eE}L/2 \gg K \gg K_\perp^2 \gg \max\{1, m^2/eE\}. \end{aligned} \tag{89}$$

For the second terms of Eqs. (87) $N_n^{\rm cr}$ acts as a cutoff factor, so we can integrate over subrange D only. Note that for quantum modes n' where $N_{n'}^{\rm cr}$ is small enough, i.e., the number of particles created is small enough, distributions $N^{(3)}_{n,\beta,\pm}({\rm out})$ are only slightly differ from initial distributions $N^{(3)}_{n,\beta,\pm}({\rm in})$. In this situation, the corresponding entropy will almost coincide with the initial entropy of each subsystem,

$$\begin{aligned} S\big(\hat\rho^{(3)}_{\pm,n'}\big) \approx -\big\{&\big[1 - N^{(3)}_{n',\beta,\pm}({\rm in})\big]\ln\big[1 - N^{(3)}_{n',\beta,\pm}({\rm in})\big] \\ &+ N^{(3)}_{n',\beta,\pm}({\rm in})\ln N^{(3)}_{n',\beta,\pm}({\rm in})\big\}. \end{aligned} \tag{90}$$

To calculate the impact of a pair creation we can rewrite Eq. (88) as

$$N^{(3)}_{\beta,\pm}({\rm out}) \approx \frac{J_{(d)}TV_\perp}{(2\pi)^{d-1}} \int_{p_\perp, p_0 \in D} d^{d-2}p_\perp dp_0 N^{(3)}_{n,\beta,\pm}({\rm out}). \tag{91}$$

Let us consider, for example, the case $N_n^{\rm cr} \ll 1$. Taking the relation (86) and integrating $N^{(3)}_{n,\beta,\pm}({\rm in})$ over p_0, we obtain that the leading term is

$$\begin{aligned} &\int_D dp_0 N^{(3)}_{n,\beta,+}({\rm in}) = N^{(3)}_{\perp,\beta,\pm}({\rm in}), \\ &N^{(3)}_{\perp,\beta,\pm}({\rm in}) \approx \frac{1}{\beta}\ln\frac{1 + \exp[-\beta(\sqrt{eE}K - \mu)]}{1 + \exp[-\beta(eEL - \mu)]}. \end{aligned} \tag{92}$$

In particular, for low temperature and not very large μ, $\sqrt{eE}K \gg \mu$, we have $\beta(eEL - \mu) \gg \beta(\sqrt{eE}K - \mu) \gg 1$ and then

$$\begin{aligned} N^{(3)}_{\perp,\beta,\pm}({\rm in}) &\approx \frac{1}{\beta}\ln|1 + \exp[-\beta(\sqrt{eE}K - \mu)]| \\ &\approx \frac{1}{\beta}\exp[-\beta(\sqrt{eE}K - \mu)]. \end{aligned} \tag{93}$$

For high temperature, $1 \gg \beta(eEL - \mu) \gg \beta(\sqrt{eE}K - \mu)$,

$$N^{(3)}_{\perp,\beta,\pm}({\rm in}) \approx \tfrac{1}{2}(eEL - \sqrt{eE}K). \tag{94}$$

Integrating it over the transversal momentum, we get

$$\begin{aligned} N^{(3)}_{\beta,\pm}({\rm in}) &\approx \frac{J_{(d)}TV_\perp}{(2\pi)^{d-1}}(2K_\perp)^{d-2}(eE)^{d/2-1}N^{(3)}_{\perp,\beta,\pm}({\rm in}) \\ &\approx \frac{J_{(d)}TV(eE)^{d/2}}{2(2\pi)^{d-1}}(2K_\perp)^{d-2}. \end{aligned} \tag{95}$$

The second terms of (87) can be integrated in a similar way,

$$\begin{aligned} &\int_D d^{d-2}p_\perp dp_0 N_n^{\rm cr}\big[1 - N^{(3)}_{n,\beta,-}({\rm in}) - N^{(3)}_{n,\beta,+}({\rm in})\big] \\ &\quad = \int_D d^{d-2}p_\perp N_n^{\rm cr}\big[eEL - 2\sqrt{eE}K - N^{(3)}_{\perp,\beta,+}({\rm in}) \\ &\qquad - N^{(3)}_{\perp,\beta,-}({\rm in})\big]. \end{aligned} \tag{96}$$

Note that for high temperature, the second terms of (87) vanish. This means that for this particular case $N^{(3)}_{\beta,\pm}({\rm out}) \approx N^{(3)}_{\beta,\pm}({\rm in})$, and the starting entropy of the system does not change significantly due to pair creation and subsequent reduction over one of the subsystems.

The corresponding entropy for the general case is not difficult to write,

$$\begin{aligned} S\big(\hat\rho^{(3)}_{\pm}\big) = -\sum_{n\in D}\big\{&\big[1 - N^{(3)}_{n,\beta,\pm}({\rm out})\big]\ln\big[1 - N^{(3)}_{n,\beta,\pm}({\rm out})\big] \\ &+ N^{(3)}_{n,\beta,\pm}({\rm out})\ln N^{(3)}_{n,\beta,\pm}({\rm out})\big\}, \end{aligned} \tag{97}$$

where the summation over the quantum numbers can be done in the same manner as in Eq. (88). However, unlike the case of the vacuum initial state, the expression (97) is complicated. To obtain further results from it one must utilize numerical calculations with definite parameters of a particular system configuration: temperature $\Theta = \beta^{-1}$, field strength E, and capacitor length L.

We can see that for very low temperatures, $\beta \to \infty$, and $N^{(3)}_{n,\beta,\pm}(\text{in}) \to 0$, we reproduce the result obtained for the vacuum initial condition. Similar to the case of a vacuum initial state, Eq. (97) reproduces the one that can be obtained for the case of a T-constant electric field in the first order of magnitude, supporting the conclusion that both fields can be considering the two different regularizations of the uniform constant electric field in the limit $T, L \to \infty$.

We also note that there is the following difference when considering the thermal initial state for an x-electric potential step. Unlike the case of time-dependent electric fields, the initial left and right subsystems are spatially separated and may in principle have different temperatures.

VI. CONCLUSION

In this work, we have considered the deformation of different initial states by constant nonuniform electric fields and statistical properties of the resulting states. We introduced a special generating functional that allows us to construct density operators for different initial conditions. In graphene and similar materials any electric field can be considered as critical due to the fact that charge carriers are massless. Because of this, a significant number of carrier pairs is produced. Possible dissipative processes lead to a loss of coherence of the states arising from vacuum and it becomes necessary to study the statistical properties of the state generated by the field. For this reason, we considered two cases of initial states of the system other than vacuum: the case when the system was initially in thermodynamic equilibrium at an absolute temperature $\Theta = \beta^{-1}$ and the case of a pure state with a certain number of particles with fixed quantum numbers. In the framework of QED with x-electric potential steps, we have to introduce five partial generating functionals for each range of quantum numbers Ω. To simplify further calculations, we construct the normal form of these generating functionals in terms of creation and annihilation operators corresponding to final particles. Setting appropriate sources in these generating functionals, we obtain density operators for different initial states of the system: the vacuum state, pure states with a definite number of particles with fixed quantum numbers, and the thermal initial state. We also note that it is formally possible to construct the generating functional for a system with different initial conditions in different areas of quantum numbers Ω. For example, choosing $J^{(3)}_{\pm,n} = J^{(2)}_{n} = J^{(4)}_{n} = 0$ and $J^{(1)}_{\pm,n} = J^{(5)}_{\pm,n} = e^{-E^{\pm}_{n}}$, we can construct the following density operator:

$$\hat{\rho}_{\text{mix}} = \prod_{n\in\Omega_1} \hat{\rho}^{(1)}_{\beta,n} \otimes \prod_{n\in\Omega_2} \hat{\rho}^{(2)}_{v,n} \otimes \prod_{n\in\Omega_3} \hat{\rho}^{(3)}_{v,n} \otimes \prod_{n\in\Omega_4} \hat{\rho}^{(4)}_{v,n} \otimes \prod_{n\in\Omega_5} \hat{\rho}^{(5)}_{\beta,n}. \tag{98}$$

This density operator corresponds to the case when there are particles with quantum numbers n from ranges Ω_1 and Ω_5 in thermal equilibrium at the initial time instant, but there are no particles that belong to the Klein zone Ω_3 (i.e., the initial state in this range was the vacuum state) as well as in the ranges Ω_2 and Ω_4. Moreover, the functionals $R^{(i)}(J)$ permit factorization in quantum modes n and each of those modes evolve separately. This fact allows one to assemble the general density operator as a product of partial operators $R^{(i)}_n(J)$, setting their initial conditions individually for each mode n. Sometimes there are situations when only part of the system is available for observation; in this case we need to construct reduced density operators that describe this available part only. Another possible scenario for reduction is a measurement with the classical tool, which causes decoherence and deforms the general density operator of the system. We considered three types of reduction: the reduction over electron and positron subsystems, the reduction due to measurement of the number of final particles, and the spatial reduction over left or right final particles. We note that the latter kind of reduction is of interest when considering the types of fields that are concentrated in restricted space areas. We can compare the situation at hand to the case of QED with t-electric potential steps [15]: For time-dependent uniform fields spatial reduction always coincides with reduction over electron or positron subsystems, as formulation of the problem suggests that the field occupies the entire space. Therefore, for a field acting for a sufficiently long time period all the final electrons, regardless of their initial state, move in the direction of the field (and all the final positrons move in the opposite direction). The same can be said about electron-positron pairs created from vacuum by an external field. For this reason the electron subsystem always coincides with the left spatial subsystem and the positron subsystem coincides with the right spatial subsystem in uniform time-dependent electric fields. However, for x-potential electric steps [34] the electric field is restricted in a finite area of space. Thus, there exist initial particles in ranges Ω_1 and/or Ω_5, which can go through the potential barrier and end up at the opposite side of the potential barrier as free final particles. Taking that into account, we can conclude that the spatial reduction is different from electron-positron subsystem reduction in the general case. However, this difference exists only when there are initial particles in ranges Ω_1 and Ω_5. When there are no initial particles in these ranges, e.g., for the case of the initial vacuum state, spatial reduction coincides with electron-positron subsystem reduction. We have constructed reduced density operators corresponding to each of the three types of reduction. We have calculated von Neumann entropy for the reduced density operators. Using the so-called L-constant field as an example, we have shown that for the reduced density operators of electron and positron subsystems this entropy is proportional to the total number of pairs created. Comparing the result obtained for the L-constant field to that obtained for the T-constant electric field in Ref. [20], we reproduced exactly the same expressions for the entropy of the electron-positron subsystem. This result shows that, despite the fact that L-constant and T-constant electric fields are physically distinct, they can be considered as two different regularizations of the uniform constant electric field in the limit $T, L \to \infty$.

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APPENDIX A

In this Appendix we present some more results regarding QED with x-electric potential steps, which may be useful for the reader. The operators $\hat{\Psi}_i(X)$ for each particular range Ω_i can be decomposed using the specific sets of solutions of the Dirac equation with quantum numbers $n \in \Omega_i$. These decompositions have the form

$$\begin{aligned}\hat{\Psi}_1(X) &= \sum_{n\in\Omega_1} \mathcal{M}_n^{-1/2}[{}_+a_n(\mathrm{in})\,{}_+\psi_n(X) + {}^-a_n(\mathrm{in})\,{}^-\psi_n(X)]\\ &= \sum_{n\in\Omega_1} \mathcal{M}_n^{-1/2}[{}^+a_n(\mathrm{out})\,{}^+\psi_n(X) + {}_-a_n(\mathrm{out})\,{}_-\psi_n(X)],\\ \hat{\Psi}_3(X) &= \sum_{n\in\Omega_3} \mathcal{M}_n^{-1/2}[{}^-a_n(\mathrm{in})\,{}^-\psi_n(X) + {}_-b_n^\dagger(\mathrm{in})\,{}_-\psi_n(X)]\\ &= \sum_{n\in\Omega_3} \mathcal{M}_n^{-1/2}[{}^+a_n(\mathrm{out})\,{}^+\psi_n(X) + {}_+b^\dagger(\mathrm{out})\,{}_+\psi_n(X)],\\ \hat{\Psi}_5(X) &= \sum_{n\in\Omega_5} \mathcal{M}_n^{-1/2}[{}^+b_n^\dagger(\mathrm{in})\,{}^+\psi_n(X) + {}_-b_n^\dagger(\mathrm{in})\,{}_-\psi_n(X)]\\ &= \sum_{n\in\Omega_5} \mathcal{M}_n^{-1/2}[{}_+b_n^\dagger(\mathrm{out})\,{}_+\psi_n(X) + {}^-b_n^\dagger(\mathrm{out})\,{}^-\psi_n(X)]\end{aligned} \tag{A1}$$

in the ranges Ω_i, $i = 1, 3, 5$, and

$$\begin{aligned}\hat{\Psi}_2(X) &= \sum_{n\in\Omega_2} \mathcal{M}_n^{-1/2} a_n \psi_n(X),\\ \hat{\Psi}_4(X) &= \sum_{n\in\Omega_4} \mathcal{M}_n^{-1/2} b_n^\dagger \psi_n(X)\end{aligned} \tag{A2}$$

in the ranges Ω_i, $i = 2, 4$.

Operators (17) and (18) obey the following anticommutation relations. All operators with different sets of quantum numbers n anticommute. This implies that all operators from different ranges Ω_i anticommute. Existing inside each range Ω_i are the nonzero anticommutation relations

$$\begin{aligned}&[{}_+a_n, {}_+a_{n'}^\dagger]_+ = [{}^-a_n, {}^-a_{n'}^\dagger]_+ = [{}_-a_n, {}_-a_{n'}^\dagger]_+ = [{}^+a_n, {}^+a_{n'}^\dagger]_+ = \delta'_{nn'}, \quad n \in \Omega_1\\ &[{}^-a_n, {}^-a_{n'}^\dagger]_+ = [{}_-b_n, {}_-b_{n'}^\dagger]_+ = [{}^+a_n, {}^+a_{n'}^\dagger]_+ = [{}_+b_n, {}_+b_{n'}^\dagger]_+ = \delta_{nn'}, \quad n \in \Omega_3\\ &[{}^+b_n, {}^+b_{n'}^\dagger]_+ = [{}_-b_n, {}_-b_{n'}^\dagger]_+ = [{}_+b_n, {}_+b_{n'}^\dagger]_+ = [{}^-b_n, {}^-b_{n'}^\dagger]_+ = \delta_{nn'}, \quad n \in \Omega_5\end{aligned} \tag{A3}$$

in the ranges Ω_i, $i = 1, 3, 5$, and

$$\begin{aligned}n \in \Omega_2 :\ [a_n, a_{n'}^\dagger]_+ &= \delta_{nn'},\\ n \in \Omega_4 :\ [b_n, b_{n'}^\dagger]_+ &= \delta_{nn'},\end{aligned} \tag{A4}$$

in the ranges Ω_i, $i = 2, 4$. Initial and final vacuum vectors are defined as state vectors annihilated by operators of initial and final particles

$$\begin{aligned}{}_+a_n|0,\mathrm{in}\rangle = {}^-a_n|0,\mathrm{in}\rangle = {}_-b_n|0,\mathrm{in}\rangle = {}^+b_n|0,\mathrm{in}\rangle = 0,\\ {}_-a_n|0,\mathrm{out}\rangle = {}^+a_n|0,\mathrm{out}\rangle = {}_+b_n|0,\mathrm{out}\rangle = {}^-b_n|0,\mathrm{out}\rangle = 0\end{aligned} \tag{A5}$$

for quantum numbers from ranges Ω_i, $i = 1, 3, 5$, and

$$\begin{aligned}a_n|0,\mathrm{in}\rangle = a_n|0,\mathrm{out}\rangle = 0, \quad n \in \Omega_2\\ b_n|0,\mathrm{in}\rangle = b_n|0,\mathrm{out}\rangle = 0, \quad n \in \Omega_4\end{aligned} \tag{A6}$$

in ranges Ω_2 and Ω_4. Since all operators from different Ω_i anticommute, the total initial and final vacua vectors can be represented as the tensor product

$$\begin{aligned}|0,\mathrm{in}\rangle &= \otimes \prod_{1,3,5} |0,\mathrm{in}\rangle^{(i)} \otimes |0\rangle^{(2)} \otimes |0\rangle^{(4)},\\ |0,\mathrm{out}\rangle &= \otimes \prod_{1,3,5} |0,\mathrm{out}\rangle^{(i)} \otimes |0\rangle^{(2)} \otimes |0\rangle^{(4)},\end{aligned} \tag{A7}$$

where $|0,\mathrm{in}\rangle^{(i)}$ and $|0,\mathrm{out}\rangle^{(i)}$ denote partial in and out vacua in ranges Ω_i, $i = 1, 3, 5$, and $|0\rangle^{(2)}$ and $|0\rangle^{(4)}$ partial vacua in ranges Ω_2 and Ω_4, respectively,

$$|0\rangle^{(2)} = |0,\mathrm{in}\rangle^{(2)} = |0,\mathrm{out}\rangle^{(2)}, \quad |0\rangle^{(4)} = |0,\mathrm{in}\rangle^{(4)} = |0,\mathrm{out}\rangle^{(4)}. \tag{A8}$$

In addition, inside each range Ω_i the partial vacua can be presented in turn as the tensor products in quantum modes:

$$\begin{aligned}|0,\mathrm{in}\rangle^{(i)} &= \prod_{n\in\Omega_i} |0,\mathrm{in}\rangle_n^{(i)},\\ |0,\mathrm{out}\rangle^{(i)} &= \prod_{n\in\Omega_i} |0,\mathrm{out}\rangle_n^{(i)},\\ |0\rangle^{(2,4)} &= \prod_{n\in\Omega_{2,4}} |0\rangle_n^{(2,4)}.\end{aligned} \tag{A9}$$

Each of these partial vacuum vectors is destroyed only by annihilation operators with the corresponding quantum numbers n. The in and out sets of operators of creation and annihilation of electrons and positrons as well as in and out vacua are connected via the special unitary evolution operators V ($VV^\dagger = I$), $|0,\mathrm{in}\rangle = V|0,\mathrm{out}\rangle$,

$$\begin{aligned}&\{a(\mathrm{in}), a^\dagger(\mathrm{in}), b(\mathrm{in}), b^\dagger(\mathrm{in})\}\\ &\quad = V\{a(\mathrm{out}), a^\dagger(\mathrm{out}), b(\mathrm{out}), b^\dagger(\mathrm{out})\}V^\dagger.\end{aligned} \tag{A10}$$

This in particular implies that

$$\hat{F}(\mathrm{in}) = V\hat{F}(\mathrm{out})V^\dagger, \tag{A11}$$

where $\hat{F}(\mathrm{in})$ is an operator-valued function written in terms of the in set of the operators of creation and annihilation

operators while $\hat{F}(\text{out})$ is the same function written in terms of the out set. The explicit form of the operator V is given in Ref. [34]. The initial partial vacuum states remain vacua [34] in ranges $\Omega_{1,2,4,5}$ (i.e., the vacuum is stable in these ranges). In other words,

$$|0,\text{in}\rangle^{(i)} = |0,\text{out}\rangle^{(i)}, \quad i = 1,2,4,5. \tag{A12}$$

We can also define the vacuum-to-vacuum transition amplitude as

$$c_{\text{v}} = \langle 0,\text{out}|0,\text{in}\rangle = {}^{(3)}\langle 0,\text{out}|0,\text{in}\rangle^{(3)}. \tag{A13}$$

Taking into account relations (A9), we can also introduce partial transition amplitudes for each quantum mode n,

$$c_{\text{v},n} = {}^{(3)}_{n}\langle 0,\text{out}|0,\text{in}\rangle^{(3)}_{n}, \quad c_{\text{v}} = \prod_{n\in\Omega_3} c_{\text{v},n}. \tag{A14}$$

The connection between in and out operators can be presented also via the linear canonical transformation (also called Bogoliubov transformation), which has the following form in different ranges of quantum numbers Ω_i. In the range Ω_1 for electrons the transformation reads

$$\begin{aligned}
{}^{+}a_n &= \eta_{\text{R}} g({}_{+}|{}^{+})^{-1}\,{}_{+}a_n + g({}^{-}|{}_{-})^{-1} g({}^{+}|{}_{-})\,{}^{-}a_n,\\
{}_{-}a_n &= g({}_{+}|{}^{+})^{-1} g({}_{-}|{}^{+})\,{}_{+}a_n - \eta_{\text{L}} g({}^{-}|{}_{-})^{-1}\,{}^{-}a_n,\\
{}_{+}a_n &= g({}_{-}|{}^{-})^{-1} g({}_{+}|{}^{-})\,{}_{-}a_n + \eta_{\text{L}} g({}^{+}|{}_{+})^{-1}\,{}^{+}a_n,\\
{}^{-}a_n &= -\eta_{\text{R}} g({}_{-}|{}^{-})^{-1}\,{}_{-}a_n + g({}^{+}|{}_{+})^{-1} g({}^{-}|{}_{+})\,{}^{+}a_n.
\end{aligned} \tag{A15}$$

The parameters $\eta_{\text{L (R)}} = \text{sgn}(p_0 - U_{\text{L (R)}})$ denote the signs of asymptotic particle kinetic energy. Canonical transformations between the initial and final pairs of creation operators of positrons in Ω_5 can be derived from the expression (A15) by replacing ${}^{\pm}a_n \to {}^{\pm}b^{\dagger}_n$, ${}_{\pm}a_n \to {}_{\pm}b^{\dagger}_n$, and $\eta_{\text{L}} \rightleftarrows \eta_{\text{R}}$. In the Klein zone Ω_3, the canonical transformation takes the form

$$\begin{aligned}
{}^{+}a_n &= -g({}_{-}|{}^{+})^{-1}\,{}_{-}b^{\dagger}_n + g({}^{-}|{}_{+})^{-1} g({}^{+}|{}_{+})\,{}^{-}a_n,\\
{}_{+}b^{\dagger}_n &= g({}_{-}|{}^{+}) g({}_{+}|{}^{+})^{-1}\,{}_{-}b^{\dagger}_n + g({}^{-}|{}_{+})^{-1}\,{}^{-}a_n,\\
{}_{-}b^{\dagger}_n &= g({}_{+}|{}^{-})^{-1} g({}_{-}|{}^{-})\,{}_{+}b^{\dagger}_n - g({}^{+}|{}_{-})^{-1}\,{}^{+}a_n,\\
{}^{-}a_n &= g({}_{+}|{}^{-})^{-1}\,{}_{+}b^{\dagger}_n + g({}^{+}|{}_{-})^{-1} g({}^{-}|{}_{-})\,{}^{+}a_n.
\end{aligned} \tag{A16}$$

The functions g are mutual decomposition coefficients of Dirac equation solutions,

$$\begin{aligned}
\eta_{\text{L}}\,{}^{\zeta}\psi_n(X) &= {}_{+}\psi_n(X) g({}_{+}|{}^{\zeta}) - {}_{-}\psi_n(X) g({}_{-}|{}^{\zeta}),\\
\eta_{\text{R}}\,{}_{\zeta}\psi_n(X) &= {}^{+}\psi_n(X) g({}^{+}|{}_{\zeta}) - {}^{-}\psi_n(X) g({}^{-}|{}_{\zeta}),
\end{aligned} \tag{A17}$$

with respect to the inner product on the x-constant hyperplane (see Ref. [34] for details), and have the following properties:

$$\begin{aligned}
&({}_{\zeta}\psi_n, {}^{\zeta'}\psi_{n'})_x = \delta_{n,n'} g({}_{\zeta}|{}^{\zeta'}), \quad g({}_{\zeta}|{}^{\zeta'}) = g({}^{\zeta'}|{}_{\zeta})^{*},\\
&|g({}_{-}|{}^{+})|^2 = |g({}_{+}|{}^{-})|^2, \quad |g({}_{+}|{}^{+})|^2 = |g({}_{-}|{}^{-})|^2,\\
&\frac{g({}_{+}|{}^{-})}{g({}_{-}|{}^{-})} = \frac{g({}^{+}|{}_{-})}{g({}^{+}|{}_{+})}.
\end{aligned} \tag{A18}$$

1. Generating functionals for density operators

We introduce special generating functionals that allow us to obtain the explicit forms of density operators (matrices) for different initial states by choosing an appropriate set of sources. Note that the results of this section are valid for any x-electric potential step. As we mentioned in the preceding section, all the creation and annihilation operators (17) from different ranges Ω_i anticommute. The density operator $\hat{\rho}$ of the system under consideration is a function of quadratic combinations of these creation and annihilation operators. This fact allows us to present the density operator $\hat{\rho}$ as a tensor product of partial density operators $\hat{\rho}^{(i)}$ for each range Ω_i,

$$\hat{\rho} = \otimes \prod_{i=1}^{5} \hat{\rho}^{(i)}. \tag{A19}$$

One can see that due to Eqs. (A3) and (A4), the operators $\hat{\rho}^{(i)}$ anticommute and can be considered separately. Thus, it is convenient to introduce the separate partial generating functional for each range of quantum numbers Ω_i. We will refer to each of these generating functionals as $R^{(i)}(J)$, and $J = \{J_n\}_{n\in\Omega_i}$ is a complete set of sources in each range which fully describes (parametrizes) the initial state of the system in each range Ω_i. The total generating functional can be obtained as a direct tensor product of functionals $R^{(i)}(J)$,

$$R(J) = \otimes \prod_{i=1}^{5} R^{(i)}(J). \tag{A20}$$

a. Generating functionals in Ω_1 and Ω_5

In Ω_1 the generating functional $R^{(1)}(J)$ has the form

$$\begin{aligned}
&R^{(1)}(J) = \prod_{n\in\Omega_1} R^{(1)}_n, \quad R^{(1)}_n = \left[Z^{(1)}_n\right]^{-1} \underline{R}^{(1)}_n, \quad \text{tr} R^{(1)}_n = 1,\\
&\underline{R}^{(1)}_n = {:}\exp\left[{}_{+}a^{\dagger}_n \left(J^{(1)}_{+,n} - 1\right) {}_{+}a_n + {}^{-}a^{\dagger}_n \left(J^{(1)}_{-,n} - 1\right) {}^{-}a_n\right]{:}.
\end{aligned} \tag{A21}$$

In Ω_5 the generating functional $R^{(5)}(J)$ has the form

$$\begin{aligned}
&R^{(5)}(J) = \prod_{n\in\Omega_5} R^{(5)}_n, \quad R^{(5)}_n = \left[Z^{(5)}_n\right]^{-1} \underline{R}^{(5)}_n, \quad \text{tr} R^{(5)}_n = 1,\\
&\underline{R}^{(5)}_n = {:}\exp\left[{}^{+}b^{\dagger}_n \left(J^{(5)}_{+,n} - 1\right) {}^{+}b_n + {}_{-}b^{\dagger}_n \left(J^{(5)}_{-,n} - 1\right) {}_{-}b_n\right]{:}.
\end{aligned} \tag{A22}$$

Here $Z^{(1)}_n$ and $Z^{(5)}_n$ are normalization factors (statistical sums); colons $:\cdots:$ always denote the normal form with respect to creation and annihilation operators inside them. Using Eq. (C6), one can calculate each of them as

$$Z^{(1,5)}_n = \left(J^{(1,5)}_{+,n} + 1\right)\left(J^{(1,5)}_{-,n} + 1\right). \tag{A23}$$

b. Generating functionals in Ω_2 and Ω_4

In these ranges the corresponding generating functionals $R^{(2,4)}(J)$ have the following structure:

$$\begin{aligned}
&R^{(2)}(J) = \prod_{n\in\Omega_2} R^{(2)}_n, \quad R^{(2)}_n = \left[Z^{(2)}_n\right]^{-1} {:}\exp\left[a^{\dagger}_n \left(J^{(2)}_n - 1\right) a_n\right]{:},\\
&R^{(4)}(J) = \prod_{n\in\Omega_4} R^{(4)}_n, \quad R^{(4)}_n = \left[Z^{(4)}_n\right]^{-1} {:}\exp\left[b^{\dagger}_n \left(J^{(4)}_n - 1\right) b_n\right]{:}.
\end{aligned} \tag{A24}$$

Here $J_n^{(2)}$ and $J_n^{(4)}$ are the corresponding sources in Ω_2 and Ω_4, and the corresponding normalization factors are

$$Z_n^{(2,4)} = \left(J_n^{(2,4)} + 1\right). \tag{A25}$$

The structure of operators $R^{(2,4)}(J)$ are trivial as there is no particle production in these ranges and all initial particles are subjected to total reflection [34]. For this reason we often omit the consideration of ranges Ω_2 and Ω_4 throughout the article.

c. Generating functional in the Klein zone

In the Klein zone Ω_3, the corresponding generating functional $R^{(3)}(J)$ has the form

$$R^{(3)}(J) = \prod_{n\in\Omega_3} R_n^{(3)}, \quad R_n^{(3)} = \left[Z_n^{(3)}\right]^{-1} \underline{R}_n^{(3)}, \quad \mathrm{tr} R_n^{(3)} = 1,$$
$$\underline{R}_n^{(3)} = :\exp\left[{}^{-}a_n^{\dagger}\left(J_{+,n}^{(3)} - 1\right){}^{-}a_n + {}_{-}b_n^{\dagger}\left(J_{-,n}^{(3)} - 1\right){}_{-}b_n\right]:, \tag{A26}$$

where the normalization factor $Z_n^{(3)}$ has the form

$$Z_n^{(3)} = \left(J_{+,n}^{(3)} + 1\right)\left(J_{-,n}^{(3)} + 1\right). \tag{A27}$$

2. Normal form of generating functional

The problem of calculating the mean value $F(\mathrm{out})$ of an operator $\hat{F}(\mathrm{out})$ at the final state of the system is related to the problem of calculating the quantity $\mathrm{tr}[\hat{F}(\mathrm{out})\hat{\rho}]$, which is

$$\mathrm{tr}[\hat{F}(\mathrm{out})\hat{\rho}] = \sum_{M,N=0}^{\infty} \sum_{M!N!} \langle \Psi_{\{m\}_M;\{n\}_N}(\mathrm{out})|\hat{F}(\mathrm{out})\hat{\rho}|\Psi_{\{m\}_M;\{n\}_N}(\mathrm{out})\rangle,$$
$$|\Psi_{\{m\}_M;\{n\}_N}(\mathrm{out})\rangle = a_{m_1}^{\dagger}\cdots a_{m_M}^{\dagger} b_{n_1}^{\dagger}\cdots b_{n_N}^{\dagger}|0,\mathrm{out}\rangle. \tag{A28}$$

For this reason, it is convenient to have the expression for generating functionals $R^{(i)}(J)$ (and subsequently for density operators $\hat{\rho}$) in terms of the out set of creation and annihilation operators. According to (A11),

$$\underline{R}(J) = VU(J)V^{\dagger},$$

where $U(J)$ are operators $\underline{R}(J)$ with creation and annihilation operators from the in set replaced by corresponding operators from the out set. Taking into account Eq. (A20) and the fact that the evolution operator V can also be factorized as [34]

$$V = \otimes \prod_{i=1}^{5} V^{(i)},$$

we can write that

$$\underline{R}^{(i)}(J) = V^{(i)}U^{(i)}(J)V^{(i)\dagger}, \tag{A29}$$

where $U^{(i)}(J)$ are operators $\underline{R}^{(i)}(J)$ with creation and annihilation operators from the in set replaced by the corresponding operators from the out set for each range Ω_i. Utilizing the explicit forms of operators $V^{(i)}$, $i = 1, 3, 5$, found in Ref. [34], we can construct the expression for generating functionals $R^{(i)}(J)$ in terms of the out set. It should be noted that the unitary evolution operators $V^{(i)}$ have the same functional form in terms of the in and out sets of operators of particle creation and annihilation due to the properties (A10) and (A11).

a. Ranges Ω_1 and Ω_5

In Ω_1 the partial evolution operator $V^{(1)} = \prod_{n\in\Omega_1} V_n^{(1)}$ has the form

$$V_n^{(1)} = \exp[{}^{+}a_n^{\dagger}S_4\,{}_{-}a_n]\exp[{}_{-}a_n^{\dagger}S_3\,{}_{-}a_n]\exp[{}^{+}a_n^{\dagger}S_2\,{}^{+}a_n]\exp[{}_{-}a_n^{\dagger}S_1\,{}^{+}a_n],$$
$$S_4 = g({}_{-}|{}^{+})^{-1}, \quad S_3 = \ln[g({}^{-}|{}_{-})^{-1}g({}^{-}|{}_{+})], \quad S_2 = \ln[g({}^{+}|{}_{+})g({}^{-}|{}_{+})^{-1}], \quad S_1 = -g({}^{-}|{}_{+})^{-1}. \tag{A30}$$

In Ω_5 the operator $V^{(5)} = \prod_{n\in\Omega_5} V_n^{(5)}$ is

$$V_n^{(5)} = \exp[{}_{+}b_n^{\dagger}S_4'\,{}^{-}b_n]\exp[{}^{-}b_n^{\dagger}S_3'\,{}^{-}b_n]\exp[{}_{+}b_n^{\dagger}S_2'\,{}_{+}b_n]\exp[{}^{-}b_n^{\dagger}S_1'\,{}_{+}b_n],$$
$$S_4' = -g({}_{+}|{}^{-})^{-1}, \quad S_3' = \ln[g({}^{-}|{}_{-})^{-1}g({}^{+}|{}_{-})], \quad S_2' = \ln[g({}^{+}|{}_{+})g({}^{+}|{}_{-})^{-1}], \quad S_1' = g({}^{+}|{}_{-})^{-1}. \tag{A31}$$

Then one can write that

$$\underline{R}_n^{(1)} = V_n^{(1)}U_n^{(1)}V_n^{(1)\dagger}, \tag{A32}$$

where $U_n^{(1)}$ is the operator that can be obtained from $\underline{R}_n^{(1)}$ by simultaneous replacements ${}_+a_n \to {}^+a_n$ and ${}^-a_n \to {}_-a_n$. Similarly, we have that

$$\underline{R}_n^{(5)} = V_n^{(5)} U_n^{(5)} V_n^{(5)\dagger}, \tag{A33}$$

where $U_n^{(5)}$ is the operator that can be obtained from $\underline{R}_n^{(5)}$ by simultaneous replacements ${}_-b_n \to {}^-b_n$ and ${}^+b_n \to {}_+b_n$. Let us calculate the normal form of the operator $\underline{R}_n^{(1)}$. This can be done using the relation (C3). Utilizing the anticommutation relations for the creation and annihilation operators (A3), one can find that

$$\begin{aligned}
\exp[{}^+a_n^\dagger S_2\, {}^+a_n]\exp[{}_-a_n^\dagger S_1\, {}^+a_n] &= \exp[{}_-a_n^\dagger S_1 e^{-S_2}\, {}^+a_n]\exp[{}^+a_n^\dagger S_2\, {}^+a_n],\\
\exp[{}_-a_n^\dagger S_3\, {}_-a_n]\exp[{}_-a_n^\dagger S_1 e^{-S_2}\, {}^+a_n] &= \exp[{}_-a_n^\dagger e^{S_3} S_1 e^{-S_2}\, {}^+a_n]\exp[{}_-a_n^\dagger S_3\, {}_-a_n].
\end{aligned} \tag{A34}$$

Then partial operators $V_n^{(1)}$ with the help of relation (C2) can be presented as

$$\begin{aligned}
V_n^{(1)} &= Y_n \tilde{V}_n, \quad Y_n = \exp[{}^+a_n^\dagger S_4\, {}_-a_n]\exp[{}_-a_n^\dagger e^{S_3}\, S_1 e^{-S_2}\, {}^+a_n],\\
\tilde{V}_n &= :\exp[{}^+a_n^\dagger (e^{S_2}-1)\, {}^+a_n + {}_-a_n^\dagger (e^{S_3}-1)\, {}_-a_n]:.
\end{aligned} \tag{A35}$$

Using the relation (C4), one can present the product $\tilde{V}_n U_n^{(1)} \tilde{V}_n^\dagger$ as follows:

$$\begin{aligned}
\tilde{V}_n U_n^{(1)} \tilde{V}_n^\dagger &= :\exp[{}^+a_n^\dagger A_{++}\, {}^+a_n + {}_-a_n^\dagger A_{--}\, {}_-a_n]:,\\
A_{++} &= J_{n,+}^{(1)} |e^{S_2}|^2 - 1,\ A_{--} = J_{n,-}^{(1)} |e^{S_3}|^2 - 1.
\end{aligned} \tag{A36}$$

On the other hand, with the help of the relation (C5), the operator Y_n can be presented as

$$Y_n = :\exp[{}^+a_n^\dagger S_4\, {}_-a_n + {}_-a_n^\dagger \tilde{S}_1\, {}^+a_n + {}^+a_n^\dagger S_4 \tilde{S}_1\, {}^+a_n]:, \quad \tilde{S}_1 = e^{S_3} S_1 e^{-S_2}. \tag{A37}$$

Then one can calculate $Y_n \tilde{V}_n U_n^{(1)} \tilde{V}_n^\dagger$ to be

$$\begin{aligned}
Y_n \tilde{V}_n U_n^{(1)} \tilde{V}_n^\dagger &= :\exp[{}^+a_n^\dagger B_{+-}\, {}_-a_n + {}_-a_n^\dagger B_{-+}\, {}^+a_n + {}^+a_n^\dagger B_{++}\, {}^+a_n + {}_-a_n^\dagger B_{--}\, {}_-a_n]:,\\
B_{++} &= A_{++} + (1 + A_{++}) S_4 \tilde{S}_1,\ B_{--} = A_{--},\ B_{+-} = S_4 (1 + A_{--}),\ B_{-+} = (1 + A_{++}) \tilde{S}_1.
\end{aligned} \tag{A38}$$

Finally, we can attach the last remaining operator $Y_n^\dagger$ from the right side to obtain

$$\begin{aligned}
\underline{R}_n^{(1)} &= :\exp[{}^+a_n^\dagger C_{+-}\, {}_-a_n + {}_-a_n^\dagger C_{-+}\, {}^+a_n + {}^+a_n^\dagger C_{++}\, {}^+a_n + {}_-a_n^\dagger C_{--}\, {}_-a_n]:,\\
C_{++} &= S_4^* \tilde{S}_1^* + B_{+-} S_4^* + B_{++} (1 + S_4^* \tilde{S}_1^*),\ C_{--} = B_{--} + B_{-+} \tilde{S}_1^*,\\
C_{-+} &= S_4^* + B_{-+} + B_{-+} S_4^* \tilde{S}_1^* + B_{--} S_4^*,\ C_{+-} = \tilde{S}_1^* + B_{+-} + B_{++} \tilde{S}_1^*.
\end{aligned} \tag{A39}$$

Substituting $B_{\zeta\zeta'}$ and $A_{\pm\pm}$ into the expression (A39), we find the explicit form of the operator $R^{(1)}$ in terms of out operators to be

$$\begin{aligned}
C_{++} &= -1 + J_{n,-}^{(1)} |S_4|^2 |e^{S_3}|^2 + J_{n,+}^{(1)} (1 + S_4 \tilde{S}_1)(1 + S_4^* \tilde{S}_1^*) |e^{S_2}|^2,\\
C_{--} &= -1 + J_{n,-}^{(1)} |e^{S_3}|^2 + J_{n,+}^{(1)} |\tilde{S}_1|^2 |e^{S_2}|^2,\\
C_{-+} &= J_{n,+}^{(1)} \tilde{S}_1 (1 + S_4^* \tilde{S}_1^*) |e^{S_2}|^2 + J_{n,-}^{(1)} S_4^* |e^{S_3}|^2,\\
C_{+-} &= J_{n,-}^{(1)} S_4 |e^{S_3}|^2 + J_{n,+}^{(1)} \tilde{S}_1^* (1 + S_4 \tilde{S}_1) |e^{S_2}|^2.
\end{aligned} \tag{A40}$$

In ranges Ω_1 and Ω_5 the matrices g are connected [34] to the relative amplitudes of an electron and positron reflection $R_\pm$ and transmission $T_\pm$ as

$$\begin{aligned}
R_{+,n} &= g({}_+|^+)^{-1} g({}_-|^+), \quad T_{+,n} = \eta_L g({}_+|^+)^{-1},\\
R_{-,n} &= g({}^-|_-)^{-1} g({}^+|_-), \quad T_{-,n} = -\eta_R g({}^-|_-)^{-1}.
\end{aligned} \tag{A41}$$

One can use these definitions and the properties of matrices g given by (A18) to present the coefficients from Eq. (A40) as

$$\begin{aligned}
1 + S_4 \tilde{S}_1 &= 1 - |g({}^-|_-)|^{-2} \frac{|g({}^-|_+)|^2}{|g({}^+|_-)|^2} = 1 - |g({}^-|_-)|^{-2} = |R_{+,n}|^2,\\
|S_4|^2 &= |g({}^-|_+)|^{-2} = |T_{+,n}|^2 / |R_{+,n}|^2,\\
|e^{S_2}|^2 &= |g({}^+|_+)|^2 |g({}^-|_+)|^2 = |R_{+,n}|^{-2},\\
|e^{S_3}|^2 &= |g({}^-|_-)|^{-2} |g({}^+|_-)|^2 = |R_{+,n}|^2.
\end{aligned} \tag{A42}$$

Thus coefficients C in Eq. (A40) take the compact form

$$\begin{aligned} C_{++} &= -1 + J^{(1)}_{n,-}|T_n|^2 + J^{(1)}_{n,+}|R_n|^2, \\ C_{--} &= -1 + J^{(1)}_{n,-}|R_n|^2 + J^{(1)}_{n,+}|T_n|^2, \\ C_{-+} &= J^{(1)}_{n,-} g(^+|_-)^{-1}|R_n|^2 - J^{(1)}_{n,+} g(_-|^+)|T_n|^2, \\ C_{+-} &= J^{(1)}_{n,-} g(_-|^+)^{-1}|R_n|^2 - J^{(1)}_{n,+} g(^+|_-)|T_n|^2, \end{aligned} \tag{A43}$$

where we introduced the notation

$$|T_n|^2 = |T_{-,n}|^2 = |T_{+,n}|^2, \quad |R_n|^2 = |R_{-,n}|^2 = |R_{+,n}|^2, \tag{A44}$$

where $|T_n|^2$ and $|R_n|^2$ are the absolute probability of electron transmission and the absolute probability of electron reflection, respectively, so $|T_n|^2 + |R_n|^2 = 1$. The normal form of the operator $\underline{R}^{(5)}_n$ can be constructed in the same manner and has the form

$$\begin{aligned} \underline{R}^{(5)}_n &= :\exp[{}_+b^\dagger_n D_{++}\, {}_+b_n + {}_+b^\dagger_n D_{+-}\, {}^-b_n + {}^-b^\dagger_n D_{-+}\, {}_+b_n + {}^-b^\dagger_n D_{--}\, {}^-b_n]:, \\ D_{++} &= -1 + J^{(1)}_{n,-}|T_n|^2 + J^{(1)}_{n,+}|R_n|^2, \quad D_{-+} = -J^{(1)}_{n,-} g(^-|_+)^{-1}|R_n|^2 + J^{(1)}_{n,+} g(_+|^-)|T_n|^2, \\ D_{--} &= -1 + J^{(1)}_{n,-}|R_n|^2 + J^{(1)}_{n,+}|T_n|^2, \quad D_{+-} = -J^{(1)}_{n,-} g(_+|^-)^{-1}|R_n|^2 + J^{(1)}_{n,+} g(^-|_+)|T_n|^2. \end{aligned} \tag{A45}$$

b. Range Ω_3

The unitary evolution operator $V^{(3)}$ has the form

$$\begin{aligned} V^{(3)} &= \prod_{n\in\Omega_3} V^{(3)}_n, \quad V^{(3)}_n = w_n(-|-)^{-1} v^{(3)}_4 v^{(3)}_3 v^{(3)}_2 v^{(3)}_1, \\ v^{(3)}_4 &= \exp\{-{}^+a^\dagger_n w_n(+-|0)\, {}_+b^\dagger_n\}, \quad v^{(3)}_3 = \exp\{{}_+b^\dagger_n \ln w_n(-|-)\, {}_+b_n\}, \\ v^{(3)}_2 &= \exp\{{}^+a_n \ln w_n(+|+)\, {}^+a_n\}, \quad v^{(3)}_1 = \exp\{-{}_+b_n w_n(0|-+)\, {}^+a_n\}, \end{aligned} \tag{A46}$$

where $w_n(\zeta|\zeta')$ are elementary amplitudes of scattering and pair creation processes, defined as

$$\begin{aligned} w_n(-|-) &= c_{\rm v}^{-1}\langle 0, \text{out}|\, {}_+b_n\, {}_-b^\dagger_n|0, \text{in}\rangle, \quad w_n(+|+) = c_{\rm v}^{-1}\langle 0, \text{out}|\, {}^+a_n\, {}^-a^\dagger_n|0, \text{in}\rangle, \\ w_n(+-|0) &= c_{\rm v}^{-1}\langle 0, \text{out}|\, {}^+a_n\, {}_+b_n|0, \text{in}\rangle, \quad w_n(0|-+) = c_{\rm v}^{-1}\langle 0, \text{out}|\, {}_-b^\dagger_n\, {}^-a^\dagger_n|0, \text{in}\rangle. \end{aligned} \tag{A47}$$

All these amplitudes are diagonal in quantum numbers due to Eq. (A3) and can be expressed in terms of the coefficients $g(^\zeta|_{\zeta'})$ as follows:

$$\begin{aligned} w_n(-|-) &= g(^-|_+)g(^-|_-)^{-1} = g(_-|^+)g(_+|^+)^{-1}, \\ w_n(+|+) &= g(^+|_-)g(^-|_-)^{-1} = g(_+|^-)g(_+|^+)^{-1}, \\ w_n(+-|0) &= g(_+|^+)^{-1}, \quad w_n(0|-+) = -g(^-|_-)^{-1}. \end{aligned} \tag{A48}$$

The relative amplitude of pair creation $w_n(+-|0)$ is also connected with the differential number of pairs created from vacuum,

$$N^{\rm cr}_n = |g(_-|^+)|^{-2} = \frac{|w_n(+-|0)|^2}{1+|w_n(+-|0)|^2}, \quad |w_n(+-|0)|^2 = \frac{N^{\rm cr}_n}{1-N^{\rm cr}_n}, \quad |c_{{\rm v},n}|^2 = 1 - N^{\rm cr}_n. \tag{A49}$$

We note that the structure of the operator $V^{(3)}$ can be formally identified with the structure of the unitary evolution operator V for QED with time-dependent uniform electric potential steps [17] with the formal replacements $a_n \to {}^+a^\dagger_n$ and $b_n \to {}_+b^\dagger_n$. Thus, the normal form of the operator $\underline{R}^{(3)}_n$ can be obtained in the exact same way as in Ref. [17] and has the form

$$\begin{aligned} \underline{R}^{(3)}_n &= |w_n(-|-)|^{-2}(1+AB){:}\exp[-{}^+a^\dagger_n(1-D_+)\, {}^+a_n - {}_+b^\dagger_n(1-D_-)\, {}_+b_n - {}^+a^\dagger_n C^\dagger\, {}_+b^\dagger_n - {}_+b_n C\, {}^+a_n]:, \\ D_+ &= |w_n(+|+)|^2(1+AB)^{-1}J^{(3)}_{+,n}, \quad B = w_n(0|-+), \\ D_- &= |w_n(-|-)|^2 J^{(3)}_{-,n}(1+AB)^{-1}, \quad A = J^{(3)}_{+,n} B^* J^{(3)}_{-,n}, \\ C &= w_n(-|-)^* A^* (1+AB)^{-1} w_n(+|+)^* + w_n(+-|0)^*. \end{aligned} \tag{A50}$$

APPENDIX B

In this Appendix, we consider the case of pure initial states other than vacuum, namely, initial states with a definite number of particles.

1. Generating functionals

The generating functionals $R^{(i)}$ also allow us to construct the partial density operators $\hat{\rho}^{(i)}_{\{m\}_M;\{n\}_N}$ of the system which is initially found in a pure state with a definite number of particles with fixed sets of quantum numbers $\{m\}_M = \{m_1, m_2, \ldots, m_M\}$ and $\{n\}_N = \{n_1, n_2, \ldots, n_N\}$ as follows. In the ranges Ω_1, Ω_3, and Ω_5,

$$\hat{\rho}^{(i)}_{\{m\}_M;\{n\}_N} = \frac{\partial^{M+N}\underline{R}^{(i)}(J)}{\partial\left(J^{(i)}_{+,m_1}\cdots J^{(i)}_{+,m_M}J^{(i)}_{-,n_1}\cdots J^{(i)}_{-,n_N}\right)}\bigg|_{J=0} = |\Psi_{\{m\}_M;\{n\}_N}(\mathrm{in})\rangle^{(i)\,(i)}\langle\Psi_{\{m\}_M;\{n\}_N}(\mathrm{in})|, \quad m,n\in\Omega_i, \tag{B1}$$

where the states $|\Psi_{\{m\}_M;\{n\}_N}(\mathrm{in})\rangle^{(i)}$ are defined as

$$\begin{aligned}
|\Psi_{\{m\}_M;\{n\}_N}(\mathrm{in})\rangle^{(1)} &= {}_+a^\dagger_{m_1}\cdots{}_+a^\dagger_{m_M}\,{}^-a^\dagger_{n_1}\cdots{}^-a^\dagger_{n_N}|0,\mathrm{in}\rangle^{(1)},\\
|\Psi_{\{m\}_M;\{n\}_N}(\mathrm{in})\rangle^{(3)} &= {}^-a^\dagger_{m_1}\cdots{}^-a^\dagger_{m_M}\,{}_-b^\dagger_{n_1}\cdots{}_-b^\dagger_{n_N}|0,\mathrm{in}\rangle^{(3)},\\
|\Psi_{\{m\}_M;\{n\}_N}(\mathrm{in})\rangle^{(5)} &= {}^+b^\dagger_{m_1}\cdots{}^+b^\dagger_{m_M}\,{}_-b^\dagger_{n_1}\cdots{}_-b^\dagger_{n_N}|0,\mathrm{in}\rangle^{(5)}.
\end{aligned} \tag{B2}$$

In the ranges Ω_2 and Ω_4,

$$\hat{\rho}^{(i)}_{\{n\}_N} = \frac{\partial^{N}\underline{R}^{(i)}(J)}{\partial\left(J^{(i)}_{n_1}\cdots J^{(i)}_{n_N}\right)}\bigg|_{J=0} = |\Psi_{\{n\}_N}(\mathrm{in})\rangle^{(i)\,(i)}\langle\Psi_{\{n\}_N}(\mathrm{in})|, \quad n\in\Omega_i, \tag{B3}$$

with the states $|\Psi_{\{n\}_N}(\mathrm{in})\rangle^{(i)}$ having the form

$$|\Psi_{\{n\}_N}(\mathrm{in})\rangle^{(2)} = a^\dagger_{n_1}\cdots a^\dagger_{n_N}|0\rangle^{(2)}, \quad |\Psi_{\{n\}_N}(\mathrm{in})\rangle^{(4)} = b^\dagger_{n_1}\cdots b^\dagger_{n_N}|0\rangle^{(4)}. \tag{B4}$$

2. Reduced density operators and entropy production

First, we consider the reduction due to the measurement of the number of final particles. It should be stressed that the fact that quantum modes evolve separately substantially simplifies the technical side of the consideration. Suppose that the initial particles are present in only one quantum mode $m \in \Omega_i$. In this case, the partial density operator $\hat{\rho}^{(i)}$ for the range Ω_i can be presented as

$$\hat{\rho}^{(i)} = \hat{\rho}^{(i)}_m \otimes \prod_{n\neq m\in\Omega_i}\hat{\rho}^{(i)}_{v,n}, \tag{B5}$$

where $\hat{\rho}^{(i)}_m$ is the partial density operator for the quantum mode m corresponding to the initial state with a definite number of particles in question. Due to the structure of the operator $\hat{\rho}^{(i)}$ given by Eq. (B5), the operator $\hat{\rho}^{(i)}_N$ takes the form

$$\hat{\rho}^{(i)}_N = \hat{\rho}^{(i)}_{N,m} \otimes \prod_{n\neq m\in\Omega_i}\hat{\rho}^{(i)}_{N,n}. \tag{B6}$$

One can see that the procedure of reduction for the case under consideration differs from the deformation of the vacuum initial state only in the quantum mode m, where initial particles are present. One can also see that it is not difficult to generalize the consideration for the case when initial particles are present in more than one quantum mode. Let us first consider the deformations for the range Ω_1. It is easy to verify that the only nonzero weights $W^{(1)}_{s,n}$, $n \neq m$, in Eq. (B6) are those where $|s_i, \mathrm{out}\rangle^{(1)}_n = |0, \mathrm{out}\rangle^{(1)}_n$, i.e., partial density operators for vacuum quantum modes do not change due to measurement of the number of particles; it is possible to write that

$$\hat{\rho}^{(1)}_N = \hat{\rho}^{(1)}_{N,m} \otimes \prod_{n\neq m\in\Omega_i}\hat{\rho}^{(1)}_{v,n}. \tag{B7}$$

Then, all that is left is to deal with is the quantum mode m where initial particles are present. Constructing the pure states with (a) a single right initial electron ${}^-a^\dagger_m$ in mode m, (b) a single left initial electron ${}_+a^\dagger_m$, and (c) with both left and right electrons in the initial state, it is easy to obtain the measurement-reduced density operators

$$\begin{aligned}
\hat{\rho}^{(1)}_{N,m} &= |T_{-,m}|^2\,{}_-a^\dagger_m\hat{\rho}^{(1)}_{v,m}\,{}_-a_m + |R_{-,m}|^2\,{}_+a^\dagger_m\hat{\rho}^{(1)}_{v,m}\,{}_+a_m \quad \text{for (a)},\\
\hat{\rho}^{(1)}_{N,m} &= |R_{+,m}|^2\,{}_-a^\dagger_m\hat{\rho}^{(1)}_{v,m}\,{}_-a_m + |T_{+,m}|^2\,{}_+a^\dagger_m\hat{\rho}^{(1)}_{v,m}\,{}_+a_m \quad \text{for (b)},\\
\hat{\rho}^{(1)}_{N,m} &= [|R_m|^2 + |T_m|^2]^2\,{}_+a^\dagger_m\,{}_-a^\dagger_m\hat{\rho}^{(1)}_{v,m}\,{}_-a_m\,{}_+a_m\\
&= {}_+a^\dagger_m\,{}_-a^\dagger_m\hat{\rho}^{(1)}_{v,m}\,{}_-a_m\,{}_+a_m \quad \text{for (c)}.
\end{aligned} \tag{B8}$$

From the expression (B8) one can see that for the case of a single initial electron we have two terms: the first term corresponds to the reflection of the initial electron, while the second term corresponds to the transition of the electron through the barrier. When there are two electrons in the initial state, we see that they can be either simultaneously reflected or simultaneously transmitted through the barrier. One can show that

$$S\big(\hat{\rho}^{(1)}_{N,n}\big) = -[|R_n|^2 \ln |R_n|^2 + |T_n|^2 \ln |T_n|^2] \quad \text{for (a) and (b)}, \tag{B9}$$

where the notation in (A44) for the reflection and transmission coefficients has been used. Entropy for case (c) vanishes, i.e., $S(\hat{\rho}^{(1)}_{N,n}) = 0$, as $\hat{\rho}^{(1)}_{N,n}$ corresponding to case (c) describes the pure state, despite the fact that a measurement has been performed in the system. Taking into account that for this case the differential numbers of final particles are

$$\begin{aligned} N_{n,-}(\text{out}) &= \text{tr}\hat{\rho}^{(1)}_{N,n}\, {}_-a^\dagger_n\, {}_-a_n = |T_n|^2, \quad |R_n|^2 = 1 - N_{n,-}(\text{out}), \\ N_{n,+}(\text{out}) &= \text{tr}\hat{\rho}^{(1)}_{N,n}\, {}^+a^\dagger_n\, {}^+a_n = |T_n|^2, \quad |R_n|^2 = 1 - N_{n,+}(\text{out}), \end{aligned} \tag{B10}$$

we can represent entropies (B9) as

$$\begin{aligned} S\big(\hat{\rho}^{(1)}_{N,n}\big) &= -\{[1 - N_{n,-}(\text{out})]\ln[1 - N_{n,-}(\text{out})] + N_{n,-}(\text{out})\ln N_{n,-}(\text{out})\} \quad \text{for (a)}, \\ S\big(\hat{\rho}^{(1)}_{N,n}\big) &= -\{[1 - N_{n,+}(\text{out})]\ln[1 - N_{n,+}(\text{out})] + N_{n,+}(\text{out})\ln N_{n,+}(\text{out})\} \quad \text{for (b)}. \end{aligned} \tag{B11}$$

The entropies for the range Ω_5 have the same form

$$\begin{aligned} S\big(\hat{\rho}^{(5)}_{N,n}\big) &= -\{[1 - N_{n,+}(\text{out})]\ln[1 - N_{n,+}(\text{out})] + N_{n,+}(\text{out})\ln N_{n,+}(\text{out})\}, \\ S\big(\hat{\rho}^{(5)}_{N,n}\big) &= -\{[1 - N_{n,-}(\text{out})]\ln[1 - N_{n,-}(\text{out})] + N_{n,-}(\text{out})\ln N_{n,-}(\text{out})\} \end{aligned} \tag{B12}$$

for the cases of single left and single right initial positrons. In a similar way and with the same result one can consider the range Ω_5.

In ranges Ω_2, Ω_3, and Ω_4 all initial particles are subjected to total reflection; for this reason the consideration of modes with only one initial particle (electron or positron) is trivial. The only exception is the case when we have an initial electron-positron pair in mode $m \in \Omega_3$. In this situation one can show, using Eqs. (A10), (A16), and (A46) and relations (A48) and (A18), that the initial state evolves as follows:

$${}^-a^\dagger_m\, {}_-b^\dagger_m|0, \text{in}\rangle^{(3)}_m = w^*_m(+|+)^{-1}[{}^+a^\dagger_m\, {}_+b^\dagger_m - w^*_m(+-|0)]|0, \text{out}\rangle^{(3)}_m. \tag{B13}$$

In this expression, the first term on the right-hand side is the state vector corresponding to the situation when both initial particles are reflected from the potential step and the second is the vacuum state vector corresponding to the situation when the initial pair is annihilated. The partial density operator reduced by measurement of the number of particles with an initial pair in mode $m \in \Omega_3$ has the form

$$\hat{\rho}^{(3)}_{N,m} = |c_{\text{v},m}|^2 |w_m(+-|0)|^2 P^{(3)}_{v,n}(\text{out}) + |c_{\text{v},m}|^2\, {}^+a^\dagger_m\, {}_+b^\dagger_m P^{(3)}_{v,n}(\text{out})\, {}_+b_m\, {}^+a_m, \tag{B14}$$

where $|c_{\text{v},m}|^2|w_m(+-|0)|^2$ and $|c_{\text{v},m}|^2$ are the probabilities of pair annihilation and pair scattering, respectively. The von Neumann entropy for the density operator $\hat{\rho}^{(3)}_{N,m}$ is

$$\begin{aligned} S\big(\hat{\rho}^{(3)}_{N,m}\big) &= -[|c_{\text{v},m}|^2 \ln |c_{\text{v},m}|^2 + |c_{\text{v},m}|^2 |w_m(+-|0)|^2 \ln |c_{\text{v},m}|^2 |w_m(+-|0)|^2] \\ &= -\big[\big(1 - N^{\text{cr}}_n\big)\ln\big(1 - N^{\text{cr}}_n\big) + N^{\text{cr}}_n \ln N^{\text{cr}}_n\big], \end{aligned} \tag{B15}$$

the same as for the case of the density operator with the vacuum initial condition, reduced by the measurement of final particles.

Now let us consider a reduction over the subsystems of electrons or positrons. Due to the nature of this reduction, the partial density operators $\hat{\rho}^{(i)}_n$, $i = 1, 2, 4, 5$, either are not affected by the reduction or are completely traced out. The states with a definite number of initial particles with fixed quantum numbers n are pure states. This means that the corresponding entropies vanish, $S(\hat{\rho}^{(1,2,4,5)}_n) = 0$. In the range Ω_3 initial electrons and positrons are subjected to total reflection [34], i.e., states with a single initial electron or a single initial positron remain pure states. Then, by applying the procedure given by Eq. (B1) to the normal form of the generating functional (A50) it is a simple matter to show that the corresponding entropies $S(\hat{\rho}^{(3)}_{\pm,n})$ vanish. The only interesting result arises when we consider a mode with an initial electron-positron pair in Ω_3. The partial density operator $\hat{\rho}^{(3)}_n$ then takes the form

$$\hat{\rho}^{(3)}_n = |\Psi\rangle_{n\,n}\langle\Psi|, \quad |\Psi\rangle_n = w^*_n(+|+)^{-1}[{}^+a^\dagger_n\, {}_+b^\dagger_n - w^*_n(+-|0)]|0, \text{out}\rangle^{(3)}_n \tag{B16}$$

and the reduced partial density operators $\hat{\rho}^{(3)}_{\pm,n}$ can be calculated as

$$\begin{aligned} \hat{\rho}^{(3)}_{+,n} &= |w_n(+|+)|^{-2}\big[{}^+a^\dagger_n P^{(3)}_{+,n}(\text{out})\, {}^+a_n + |w_n(+-|0)|^2 P^{(3)}_{+,n}(\text{out})\big], \\ \hat{\rho}^{(3)}_{-,n} &= |w_n(+|+)|^{-2}\big[{}_+b^\dagger_n P^{(3)}_{-,n}(\text{out})\, {}_+b_n + |w_n(+-|0)|^2\, P^{(3)}_{-,n}(\text{out})\big], \end{aligned} \tag{B17}$$

where $|w_n(+|+)|^{-2} = |c_{\mathrm{v},n}|^2$ and projectors $P^{(3)}_{\pm,n}(\mathrm{out})$ are

$$P^{(3)}_{+,n}(\mathrm{out}) = |0,\mathrm{out}\rangle^{(3)}_{a,n}\,{}^{(3)}_{a,n}\langle 0,\mathrm{out}|,$$
$$P^{(3)}_{-,n}(\mathrm{out}) = |0,\mathrm{out}\rangle^{(3)}_{b,n}\,{}^{(3)}_{b,n}\langle 0,\mathrm{out}|. \tag{B18}$$

The corresponding von Neumann entropies can be calculated,

$$S\big(\hat\rho^{(3)}_{\pm,n}\big) = -[|c_{\mathrm{v},n}|^2 \ln|c_{\mathrm{v},n}|^2 + |c_{\mathrm{v},n}|^2|w_n(+-|0)|^2 \times \ln|c_{\mathrm{v},n}|^2|w_n(+-|0)|^2]. \tag{B19}$$

Using the relations (A49), we can present it as

$$S\big(\hat\rho^{(3)}_{\pm,n}\big) = -\big[\big(1-N^{\mathrm{cr}}_n\big)\ln\big(1-N^{\mathrm{cr}}_n\big) + N^{\mathrm{cr}}_n \ln N^{\mathrm{cr}}_n\big]. \tag{B20}$$

One can see that this result coincides with the one obtained for the case of the initial vacuum state, Eq. (62), i.e., with the case when an electron-positron pair is produced from the vacuum, and with the entropy for the density operator $\hat\rho^{(3)}_{N,m}$ given by Eq. (B15).

APPENDIX C

Here we provide several relations that we have used during the calculations. For both the Fermi and Bose cases the relations [17]

$$ae^{a^\dagger Da} = e^{a^\dagger Da}e^{D}a, \quad a^\dagger e^{a^\dagger Da} = e^{a^\dagger Da}a^\dagger e^{-D}, \tag{C1}$$
$$e^{a^\dagger Da} = :\exp\{a^\dagger(e^D-1)a\}: \tag{C2}$$

hold, where in the general case D is an arbitrary matrix. Note that in the case under consideration all products are diagonal in quantum numbers n, $a^\dagger Da = a^\dagger_n D_{nn} a_n$, and the matrices D_{nn} are diagonal and single rank, i.e., are just c-numbers. One can also easily see that the following generalization of Eq. (C1) holds:

$$e^{a^\dagger Da}a = e^{-D}ae^{a^\dagger Da}, \quad e^{a^\dagger Da}a^\dagger = e^{D}a^\dagger e^{a^\dagger Da}. \tag{C3}$$

For the product of two normal-form operators the relation

$$:e^{a^\dagger Da}::e^{a^\dagger \tilde D a}: = :e^{a^\dagger(D+\tilde D+D\tilde D)a}: \tag{C4}$$

is useful, where D and $\tilde D$ are matrices, and its simple generalization

$$:e^{b^\dagger Da}::e^{a^\dagger \tilde D c}: = :\exp[b^\dagger Da + a^\dagger\tilde D c + b^\dagger D\tilde D c]:, \tag{C5}$$

where for the case of Fermi operators the decomposition of the exponent is finite and has the form

$$:\exp[b^\dagger Da + a^\dagger\tilde D c + b^\dagger D\tilde D c]: = 1 + b^\dagger Da + a^\dagger\tilde D c + b^\dagger D\tilde D c - b^\dagger a^\dagger Dac.$$

The trace of a normal product of creation and annihilation operators can be calculated by using the following path-integral representation. Suppose that $X(a^\dagger, a)$ is an operator expression of creation and annihilation operators. Then the trace of its normal form can be expressed as [17]

$$\mathrm{tr}\{:X(a^\dagger,a):\} = \langle 0|\int \exp\{\lambda^*\lambda + \lambda^* a\}:X(a^\dagger,a):\exp\{a^\dagger\lambda\}\prod d\lambda^* d\lambda|0\rangle. \tag{C6}$$

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