Monotonicity of quantumness of ensembles under commutativity-preserving channels

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A quantum channel (operation) is commutativity-preserving if it maps commuting states to commuting states. We show that commutativity-preserving channels cannot increase quantumness of ensembles, as quantified in terms of the commutator of square roots of constituent states in a quantum ensemble recently introduced [Phys. Rev. A **96**, 022132 (2017)].

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I. INTRODUCTION

Noncommutativity, between quantum states (as represented by density operators) and/or quantum observables (as represented by self-adjoint operators), is one of the most important characteristics of quantum mechanics which signifies the radical departure of the quantum from the classical [1–5]. Although a *single* quantum state or observable, being a self-adjoint operator without any coupling or interaction with other objects, can always be regarded as classical in the sense that it can always be diagonalized, the situation changes fundamentally when at least two quantum states or observables are involved: the representing operators may be not commutative (from the mathematical perspective), and thus quantumness arises (from the physical perspective) regarding the relations between these states or observables.

In particular, consider a quantum ensemble $\mathcal{E} = \{(p_i, \rho_i): i \in I\}$ describing a quantum system whose states are represented by density operators ρ_i indexed by $i \in I$ with non-negative probabilities p_i adding up to 1. Whenever any two constituent states in the ensemble are not commutative and therefore cannot be diagonalized simultaneously, this ensemble exhibits quantumness from various physical perspectives, such as no cloning [6,7], no broadcasting [8–10], unattainability to the Holevo bound for the accessible information [11], etc. Based on each of the above features, one may introduce a quantifier to synthesize quantumness contained in the ensemble [12–22]. In general, these quantifiers are not consistent with each other in the sense that they may yield different quantumness orderings for the same family of ensembles [18], and each of them could capture only a certain aspect of the complex nature of ensembles.

Although we should not expect a universal quantifier for quantumness of ensembles, just as we should not expect a universal measure of correlations or entanglement, it is still desirable to quantify quantumness of ensembles from both physical and mathematical perspectives. Since the physical quantumness is mathematically characterized by noncommutativity, it is natural to regard the degree of noncommutativity between the constituent states as a quantumness quantifier of ensembles. Based on this intuitive idea, and in terms of the commutator between operators, a quantumness quantifier for ensemble $\mathcal{E} = \{(p_i, \rho_i) : i \in I\}$ is introduced as [22]

$$Q(\mathcal{E}) = -\sum_{i,j} \sqrt{p_i p_j} \mathrm{tr}[\sqrt{\rho_i}, \sqrt{\rho_j}]^2, \tag{1}$$

where tr denotes trace of operators (matrices) and $[X, Y] = XY - YX$ denotes the commutator between two operators X and Y. This measure enjoys several desirable and intuitive properties required naturally for a quantumness quantifier, such as positivity, unitary invariance, subadditivity, concavity under probabilistic union, convexity under state decomposition, decreasing under coarse graining, and increasing under fine graining, as established in Ref. [22]. In particular, an ensemble $\mathcal{E}$ is classical (without quantumness) if its constituent states are commutative, which is equivalent to $Q(\mathcal{E}) = 0$.

Now consider a quantum channel Λ which preserves commutativity in the sense that whenever two states ρ and σ are commutative, i.e., $[\rho, \sigma] = 0$, then $[\Lambda(\rho), \Lambda(\sigma)] = 0$. Since commutativity-preserving channel Λ preserves the commutativity between any pair of states, it maps any classical ensemble into a classical ensemble in the sense that $Q(\mathcal{E}) = 0$ implies $Q(\Lambda(\mathcal{E})) = 0$. Here $\Lambda(\mathcal{E}) = \{(p_i, \Lambda(\rho_i)), i \in I\}$ is the image (i.e., output) ensemble under the commutativity-preserving channel Λ. A natural question arises: Can a commutativity-preserving channel increase quantumness of ensembles? The answer is no. More precisely, we will show that for any quantum ensemble $\mathcal{E}$ and any commutativity-preserving channel Λ, it holds true that

$$Q(\Lambda(\mathcal{E})) \leqslant Q(\mathcal{E}). \tag{2}$$

This monotonicity of quantumness of ensembles is the main result of this work.

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This paper is organized as follows. In Sec. II we review commutativity-preserving channels and their characterization, which will be used in the proof of inequality (2) in Sec. III. The proofs are quite different for qubit systems and for higher-dimensional systems, and thus we treat them separately in two subsections. We conclude with discussions in Sec. IV. In this paper we consider only ensembles with a finite number of constituent states for finite-dimensional systems, and the input and output of the channels are of the same dimension.

II. COMMUTATIVITY-PRESERVING CHANNELS

In order to establish our main result, inequality (2), we first review basic properties and classification of commutativity-preserving channels. Recall that a quantum channel (operation) Λ is a linear, trace-preserving, completely positive map between sets of quantum states (density operators). A commutativity-preserving channel Λ is just a channel that preserves the commutativity between any two input states ρ and σ; i.e., $[\rho,\sigma]=0$ implies that $[\Lambda(\rho),\Lambda(\sigma)]=0$. Such channels are of physical significance. For example, it is proved that commutativity-preserving channels are the only kind of channels that cannot create any quantum correlations from any quantum-classical state $\rho^{ab}=\sum_i q_i\rho_i^a\otimes|i^b\rangle\langle i^b|$, when the channel acts on subsystem b [23–26]. Here $\{\rho_i^a\}$ is a set of quantum states on subsystem a, $\{|i^b\rangle\}$ is an orthonormal basis for subsystem b, and $\{q_i\}$ is a probability distribution.

A channel Λ on a d-dimensional system space is a commutativity-preserving channel if and only if its image constitutes commutative states (e.g., completely decohering channel whose output states are all diagonal in a certain base) or (1) it is a unital channel when $d=2$ [24,25] or (2) it is an isotropic channel when $d\geqslant 3$ [25,26].

Recall that a unital channel refers to the channel satisfying $\Lambda(\mathbf{1})=\mathbf{1}$ with $\mathbf{1}$ being the identity operator, and an isotropic channel is defined as one of the following two forms [25,26]:

$$\Lambda(\rho)=tU\rho U^\dagger+(1-t)\frac{\mathbf{1}}{d},\quad \frac{-1}{d^2-1}\leqslant t\leqslant 1, \tag{3}$$

$$\Lambda(\rho)=tU\rho^T U^\dagger+(1-t)\frac{\mathbf{1}}{d},\quad \frac{-1}{d-1}\leqslant t\leqslant \frac{1}{d+1}, \tag{4}$$

which are called depolarizing channel and transpose-depolarizing channel, respectively [27]. Here U is any unitary operator, ρ^T denotes the transpose of ρ, and the parameter t is chosen to ensure that Λ is completely positive. As an isotropic channel must be unital, we know that the set of isotropic channels is contained in the set of unitial channels.

The issue of the characterization of commutativity-preserving mapping on operator algebras has been studied intensively from a mathematical perspective [28–35]. It originated from the linear preserver problems, which concern the characterization of linear operators on matrix spaces that leave certain functions, subsets, relations, etc., invariant [29]. A prototypical problem is the characterization of those linear operators preserving certain relations such as commutativity. The issue of the linear preserver problems is of fundamental theoretical interest in matrix theory [29], because it is related to the structure of matrix space. Its significance in quantum mechanics is apparent and deserves further investigations. With these preparations, we move to the detailed proof of inequality (2).

III. MONOTONICITY OF QUANTUMNESS

This section is devoted to proving inequality (2). We start by introducing the quantity

$$Q(\rho_1,\rho_2)=-\mathrm{tr}[\sqrt{\rho_1},\sqrt{\rho_2}]^2 \tag{5}$$

to quantify the degree of noncommutativity between two states ρ_1 and ρ_2. Actually, $Q(\rho_1,\rho_2)=2I(\rho_1,\sqrt{\rho_2})=2I(\rho_2,\sqrt{\rho_1})$, where

$$I(\rho_1,\sqrt{\rho_2})=-\tfrac{1}{2}\mathrm{tr}[\sqrt{\rho_1},\sqrt{\rho_2}]^2 \tag{6}$$

is the Wigner-Yanase skew information of the state ρ_1 with respect to $\sqrt{\rho_2}$ (formally considered as an observable) [36–39]. Furthermore, $I(\rho_1,\sqrt{\rho_2})$ can be regarded as a quantifier for coherence of ρ_1 with respect to $\sqrt{\rho_2}$ [40,41].

Performing orthogonal spectral decomposition of ρ_1 yields $\rho_1=\sum_i\lambda_i|i\rangle\langle i|$ with $\{\lambda_i\}$ the eigenvalues of ρ_1 and $\{|i\rangle\}$ the corresponding eigenvectors, then by the calculations in Ref. [38], we know that

$$\begin{aligned} Q(\rho_1,\rho_2)&=2I(\rho_1,\sqrt{\rho_2})\\ &=2(\mathrm{tr}\rho_1\rho_2-\mathrm{tr}\sqrt{\rho_1}\sqrt{\rho_2}\sqrt{\rho_1}\sqrt{\rho_2}) \qquad (7)\\ &=\sum_{i,j}(\sqrt{\lambda_i}-\sqrt{\lambda_j})^2|\langle i|\sqrt{\rho_2}|j\rangle|^2. \qquad (8)\end{aligned}$$

Noting that the roles of ρ_1 and ρ_2 can be interchanged in view of the symmetry of $Q(\rho_1,\rho_2)$ defined by Eq. (5) with respect to ρ_1 and ρ_2, and

$$Q(\mathcal{E})=\sum_{i,j}\sqrt{p_ip_j}Q(\rho_i,\rho_j),$$

thus in order to prove inequality (2), it suffices to establish the following inequality

$$Q(\Lambda(\rho_1),\Lambda(\rho_2))\leqslant Q(\rho_1,\rho_2) \tag{9}$$

for any commutativity-preserving channel Λ and any pair of states ρ_1,ρ_2. To prove this inequality, we treat the cases $d=2$ (qubit systems) and $d\geqslant 3$ (higher-dimensional systems) separately. When the image of the channel Λ consists of commutating states, the above result is trivially true since $Q(\Lambda(\rho_1),\Lambda(\rho_2))=0$; thus we consider only nontrivial cases in the following.

A. Proof for qubit systems

For qubit systems, a channel Λ preserves commutativity (except for the case when the image of the channel is commutative) if and only if it is a unital channel [24,25]. Note that every unital qubit channel can be represented as a random unitary channel (a convex combination of unitary channels) [42–44]

$$\Lambda(\rho)=\sum_i q_iU_i\rho U_i^\dagger,$$

where U_i are all unitary operators and $\{q_i\}$ is a probability distribution. Therefore, by use of Eq. (7), together with

Lieb's inequality [45] (which implies the joint convexity of $-\mathrm{tr}\sqrt{\rho_1}\sqrt{\rho_2}\sqrt{\rho_1}\sqrt{\rho_2}$ with respect to ρ_1 and ρ_2), and the unitary invariance of $Q(\cdot,\cdot)$, we have

$$\begin{aligned}Q(\Lambda(\rho_1),\Lambda(\rho_2)) &= Q\bigg(\sum_i q_i U_i\rho_1 U_i^\dagger, \sum_i q_i U_i\rho_2 U_i^\dagger\bigg)\\ &\leqslant \sum_i q_i Q(U_i\rho_1U_i^\dagger, U_i\rho_2 U_i^\dagger)\\ &= \sum_i q_i Q(\rho_1,\rho_2)\\ &= Q(\rho_1,\rho_2).\end{aligned}$$

This completes the proof of inequality (9) for qubit systems.

In this context, we note that the statement "every unital qubit channel can be represented as a random unitary channel" is a quantum extension (only for two-dimensional systems) of the celebrated Birkhoff theorem for representing doubly stochastic matrices as convex combinations of permutations [42], and it is definitely not true for higher-dimensional systems (with dimension $\geqslant 3$) [42].

B. Proof for higher-dimensional systems

For higher-dimensional systems with $d \geqslant 3$, a channel Λ is a commutativity-preserving channel if and only if (1) its image is a family of commuting states or (2) it is an isotropic channel [24,25], which are of the forms as in Eqs. (3) and (4). Case (1) is trivial, and we need to treat only case (2).

For any two states ρ_1 and ρ_2, let

$$\eta_i = t\rho_i + (1-t)\frac{\mathbf{1}}{d}, \quad i = 1,2,$$

then by the unitary invariance of $Q(\cdot,\cdot)$ for the depolarizing channel

$$\Lambda(\rho) = tU\rho U^\dagger + (1-t)\frac{\mathbf{1}}{d} = U\eta_i U^\dagger, \quad \frac{-1}{d^2-1} \leqslant t \leqslant 1,$$

we have

$$Q(\Lambda(\rho_1),\Lambda(\rho_2)) = Q\big(U\eta_1 U^\dagger, U\eta_2 U^\dagger\big) = Q(\eta_1,\eta_2).$$

Next we prove that

$$Q(\eta_1,\eta_2) \leqslant Q(\rho_1,\rho_2),$$

for the cases $0 \leqslant t \leqslant 1$ and $-\frac{1}{d^2-1} \leqslant t < 0$, respectively.

When $0 \leqslant t \leqslant 1$, η_i is just the convex combination of ρ_i and $\frac{\mathbf{1}}{d}$. By use of the convexity under state decomposition of the quantumness quantifier $Q(\cdot)$ [22], we have

$$\begin{aligned}Q(\eta_1,\eta_2) &\leqslant t^2 Q(\rho_1,\rho_2) + (1-t)^2 Q\bigg(\frac{\mathbf{1}}{d},\frac{\mathbf{1}}{d}\bigg)\\ &\quad + t(1-t)Q\bigg(\frac{\mathbf{1}}{d},\rho_2\bigg) + t(1-t)Q\bigg(\rho_1,\frac{\mathbf{1}}{d}\bigg)\\ &= t^2 Q(\rho_1,\rho_2)\\ &\leqslant Q(\rho_1,\rho_2).\end{aligned}$$

When $-\frac{1}{d^2-1} \leqslant t < 0$, η_i can be rewritten as

$$\eta_i = v_1\frac{\mathbf{1}-\rho_i}{d-1} + v_2\frac{\mathbf{1}}{d}, \quad i = 1,2,$$

which is a convex combination of two states $\frac{\mathbf{1}-\rho_i}{d-1}$ and $\frac{\mathbf{1}}{d}$. Here

$$v_1 = -t(d-1), \quad v_2 = 1 + (d-1)t$$

satisfy $v_1 \geqslant 0$, $v_2 \geqslant 0$ and $v_1 + v_2 = 1$. By use of the convexity under state decomposition of the quantumness quantifier $Q(\cdot)$ [22], we have

$$Q(\eta_1,\eta_2) \leqslant v_1^2 Q\bigg(\frac{\mathbf{1}-\rho_1}{d-1},\frac{\mathbf{1}-\rho_2}{d-1}\bigg).$$

Now we proceed to prove the following inequality

$$Q\bigg(\frac{\mathbf{1}-\rho_1}{d-1},\rho_2\bigg) \leqslant \frac{1}{d-1}Q(\rho_1,\rho_2), \tag{10}$$

which is equivalent to

$$-\mathrm{tr}[\sqrt{\mathbf{1}-\rho_1},\sqrt{\rho_2}]^2 \leqslant -\mathrm{tr}[\sqrt{\rho_1},\sqrt{\rho_2}]^2. \tag{11}$$

Actually, by use of Eq. (8), inequality (11) is equivalent to

$$\begin{aligned}&\sum_{i,j}(\sqrt{1-\lambda_i}-\sqrt{1-\lambda_j})^2|\langle i|\sqrt{\rho_2}|j\rangle|^2\\ &\quad\leqslant \sum_{i,j}(\sqrt{\lambda_i}-\sqrt{\lambda_j})^2|\langle i|\sqrt{\rho_2}|j\rangle|^2.\end{aligned}$$

The above inequality is true and follows from the fact that

$$(\sqrt{1-\lambda_i}-\sqrt{1-\lambda_j})^2 \leqslant (\sqrt{\lambda_i}-\sqrt{\lambda_j})^2, \tag{12}$$

where $\{\lambda_i\}$ are the eigenvalues of ρ_1 and $\{|i\rangle\}$ are the corresponding eigenvectors, and thus $\sum_i \lambda_i = 1$, $\lambda_i \geqslant 0$, $i = 1,2,\ldots$. In particular, $\lambda_i + \lambda_j \leqslant 1$.

To prove inequality (12), let $\lambda_i + \lambda_j = 1 - s$, $0 \leqslant s \leqslant 1$, then inequality (12) is equivalent to

$$(\sqrt{\lambda_i + s}-\sqrt{\lambda_j + s})^2 \leqslant (\sqrt{\lambda_i}-\sqrt{\lambda_j})^2, \quad 0 \leqslant s \leqslant 1,$$

which is apparently true by noting that $f(s) = (\sqrt{\lambda_i + s}-\sqrt{\lambda_j + s})^2$ is a decreasing function of $s \in [0,1]$.

Now invoking inequality (11) twice yields [noting that inequality (11) actually holds true for any non-negative ρ_2]

$$\begin{aligned}-\mathrm{tr}[\sqrt{\mathbf{1}-\rho_1},\sqrt{\mathbf{1}-\rho_2}]^2 &\leqslant -\mathrm{tr}[\sqrt{\rho_1},\sqrt{\mathbf{1}-\rho_2}]^2\\ &\leqslant -\mathrm{tr}[\sqrt{\rho_1},\sqrt{\rho_2}]^2.\end{aligned}$$

Consequently,

$$Q\bigg(\frac{\mathbf{1}-\rho_1}{d-1},\frac{\mathbf{1}-\rho_2}{d-1}\bigg) \leqslant \frac{1}{(d-1)^2}Q(\rho_1,\rho_2)$$

and

$$Q(\eta_1,\eta_2) \leqslant \frac{v_1^2}{(d-1)^2}Q(\rho_1,\rho_2) \leqslant Q(\rho_1,\rho_2).$$

In summary, we have established that any depolarizing channel Λ cannot increase quantumness between ρ_1 and ρ_2 in the sense that

$$Q(\Lambda(\rho_1),\Lambda(\rho_2)) \leqslant Q(\rho_1,\rho_2).$$

We proceed to prove inequality (9) for the transpose depolarizing channel:

$$\Lambda(\rho) = tU\rho^T U^\dagger + (1-t)\frac{\mathbf{1}}{d}, \quad \frac{-1}{d-1} \leqslant t \leqslant \frac{1}{d+1}.$$

For any two states ρ_1 and ρ_2, let

$$\tau_i = t\rho_i^T + (1-t)\frac{\mathbf{1}}{d}, \quad i = 1, 2,$$

then

$$\Lambda(\rho_i) = U\tau_i U^\dagger, \qquad i = 1, 2,$$

and by the unitary invariance of $Q(\cdot,\cdot)$ [22], we have

$$Q(\Lambda(\rho_1), \Lambda(\rho_2)) = Q\big(U\tau_1U^\dagger, U\tau_2U^\dagger\big) = Q(\tau_1, \tau_2).$$

Repeating the processes for the depolarizing channel, we get that when $0 \leqslant t \leqslant \frac{1}{d+1}$,

$$Q(\tau_1, \tau_2) \leqslant Q\big(\rho_1^T, \rho_2^T\big),$$

since τ_i is just a convex combination of ρ_i^T and $\frac{1}{d}$, and that when $\frac{-1}{d-1} \leqslant t < 0$,

$$Q(\tau_1, \tau_2) \leqslant Q\big(\rho_1^T, \rho_2^T\big),$$

since τ_i can be rewritten as a convex composition of $\frac{\mathbf{1}-\rho_i^T}{d-1}$ and $\frac{\mathbf{1}}{d}$. Due to the fact that $\sqrt{\rho^T} = \sqrt{\rho}^T$ for any self-adjoint operator ρ, we know that

$$\begin{aligned} Q\big(\rho_1^T, \rho_2^T\big) &= 2\mathrm{tr}\rho_1^T\rho_2^T - 2\mathrm{tr}\sqrt{\rho_1^T}\sqrt{\rho_2^T}\sqrt{\rho_1^T}\sqrt{\rho_2^T} \\ &= 2\mathrm{tr}\rho_1\rho_2 - 2\mathrm{tr}\sqrt{\rho_1}^T\sqrt{\rho_2}^T\sqrt{\rho_1}^T\sqrt{\rho_2}^T \\ &= 2\mathrm{tr}\rho_1\rho_2 - 2\mathrm{tr}\sqrt{\rho_1}\sqrt{\rho_2}\sqrt{\rho_1}\sqrt{\rho_2} \\ &= Q(\rho_1, \rho_2). \end{aligned}$$

Consequently,

$$Q(\Lambda(\rho_1), \Lambda(\rho_2)) = Q(\tau_1, \tau_2) \leqslant Q\big(\rho_1^T, \rho_2^T\big) = Q(\rho_1, \rho_2).$$

Combining the cases $d = 2$ and $d \geqslant 3$, we conclude that inequality (9), or equivalently inequality (2), holds for any commutativity-preserving channel on any finite-dimensional systems.

IV. DISCUSSION

For the quantumness of quantum ensembles quantified in terms of commutator of square roots of quantum states, any commutativity-preserving channel Λ preserves null quantumness in the sense that $Q(\mathcal{E}) = 0$ implies that $Q(\Lambda(\mathcal{E})) = 0$. To strengthen this intuitive relation considerably we have established that $Q(\Lambda(\mathcal{E})) \leqslant Q(\mathcal{E})$, which is tantamount to the monotonicity of quantumness under commutativity-preserving channels. This desirable and intuitive property corroborates the information-theoretic significance of the measure of quantumness defined by Eq. (1). We expect that this quantity will be useful in addressing quantumness in physical systems.

We emphasize that the square root in the construction of quantumness measure plays a crucial role, and the Wigner-Yanase skew information manifests itself here. The results do not hold if one employs the density operators themselves rather than their square roots in the commutator. In this context, a generalization of the Wigner-Yanase skew information, i.e., the Wigner-Yanase-Dyson skew information $I_s(\rho, H) = -\frac{1}{2}\mathrm{tr}[\rho^s, H][\rho^{1-s}, H]$, which involves a general exponent s in the interval (0,1), also has many desirable properties as the original skew information [45]. It will be interesting to investigate related issues involving other exponents of the density operators.

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