Cross-spectral purity of nonstationary light

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The concept of cross-spectral purity of light, established by Mandel for stationary optical fields, is extended to nonstationary fields. Separability conditions that ensure the cross-spectral purity of nonstationary light are derived in the space-frequency and space-time domains. It is also shown that the property of cross-spectral purity is not strictly preserved on propagation of fields with any appreciable spectral bandwidth. Further, we introduce a method for generating cross-spectrally pure (stationary and nonstationary) fields from spatially incoherent light sources by means of achromatic Fourier-transform systems.

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I. INTRODUCTION

The concept of cross-spectral purity, introduced by Mandel in 1961 [1], is one of the most fundamental yet least widely recognized concepts in optical coherence theory. Physically, in the case of stationary fields, cross-spectral purity amounts to two-beam interference with the superposition field having the same normalized spectrum as the input waves. Mathematically it is manifested in a reduction formula which expresses the complex degree of coherence as a product of spatial and temporal coherence factors [2]. The concept of cross-spectral purity has more recently been extended to electromagnetic fields [3–7] and examined in the context of spectral modulation [8], diffusers [9], ghost imaging [10], statistical similarity [11], and scattering [12]. All of the studies on cross-spectral purity thus far have dealt with stationary optical fields.

In this paper we extend the concept of cross-spectral purity into the domain of nonstationary fields, such as trains of short optical pulses. We begin, in Sec. II, by briefly recalling the concept of cross-spectral purity of stationary light and establishing the relevant terminology. The extension to nonstationary fields is presented in Sec. III, where the criteria for cross-spectral purity of nonstationary light fields are established. We proceed to show that, unlike with stationary fields, cross-spectral purity of nonstationary wave fields requires that the path difference involved in the two-beam interference is zero. Reduction formulas for cross-spectrally pure nonstationary light in the space-frequency and space-time domains are then derived. In Sec. IV we show that the property of cross-spectral purity is, in general, not preserved on propagation. We further introduce, in Sec. V, methods for generating cross-spectrally pure optical fields from incoherent sources with the aid of achromatic Fourier-transform systems. Conclusions and some final remarks are provided in Sec. VI.

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II. CROSS-SPECTRALLY PURE STATIONARY LIGHT

Let us consider a superposition of light from two points $\mathbf{r}_1$ and $\mathbf{r}_2$ of a wave field with a spectral representation $E(\mathbf{r};\omega)$, formed at some observation point $\mathbf{R}$. Denoting by τ the temporal delay between the fields originating from $\mathbf{r}_1$ and $\mathbf{r}_2$, the superposition field has the spectral representation

$$E(\mathbf{R};\omega) = E(\mathbf{r}_1;\omega) + E(\mathbf{r}_2;\omega)\exp(i\omega\tau). \tag{1}$$

Superpositions of this kind can be implemented with a variety of interferometric schemes, but we do not specify any particular approach. One of the possible methods is the classic Young's two-pinhole interferometer [13], which, however, involves somewhat inconvenient (weakly frequency-dependent) proportionality factors. These can be avoided by forming the superposition with wavefront folding or shearing interferometers.

The spectral density (or the spectrum) of the field at point $\mathbf{r}$ is defined as $S(\mathbf{r};\omega) = \langle |E(\mathbf{r};\omega)|^2\rangle$, where the angle brackets denote ensemble averaging. A stationary light field is called cross-spectrally pure if the normalized spectral density

$$s(\mathbf{r};\omega) = \frac{S(\mathbf{r};\omega)}{\int_0^\infty S(\mathbf{r};\omega)d\omega} \tag{2}$$

of the superposition is the same as the normalized spectra of the fields at the two points $\mathbf{r}_1$ and $\mathbf{r}_2$ for some value $\tau = \tau_0$ of the time delay, i.e., if

$$s(\mathbf{R};\omega) = s(\mathbf{r}_1;\omega) = s(\mathbf{r}_2;\omega), \tag{3}$$

when $\tau = \tau_0$. It is then possible to show (see Sec. 4.5.1 of [13]) that the condition for cross-spectrally pure stationary light is

$$\mu(\mathbf{r}_1,\mathbf{r}_2;\omega) = \gamma(\mathbf{r}_1,\mathbf{r}_2;\tau_0)\exp(i\omega\tau_0), \tag{4}$$

where $\mu(\mathbf{r}_1,\mathbf{r}_2;\omega)$ is the complex degree of spectral (spatial) coherence of the light field at points $\mathbf{r}_1$ and $\mathbf{r}_2$ and $\gamma(\mathbf{r}_1,\mathbf{r}_2;\tau_0)$ is the corresponding complex degree of (temporal) coherence at $\tau = \tau_0$. Moreover, the so-called reduction formula [13]

$$\gamma(\mathbf{r}_1, \mathbf{r}_2; \tau) = \gamma(\mathbf{r}_1, \mathbf{r}_2; \tau_0)\gamma(\mathbf{r}, \mathbf{r}; \tau - \tau_0), \tag{5}$$

where $\mathbf{r} = \mathbf{r}_1$ or $\mathbf{r} = \mathbf{r}_2$, holds. This formula expresses the complex degree of coherence as a product of a term that characterizes the spatial coherence (at points $\mathbf{r}_1$ and $\mathbf{r}_2$) and a term that characterizes the temporal coherence of the field at a single point ($\mathbf{r}_1$ or $\mathbf{r}_2$).

III. CROSS-SPECTRALLY PURE NONSTATIONARY LIGHT

In the case of nonstationary light, coherence in the space-frequency domain is characterized by the two-frequency cross-spectral density (CSD) function, introduced at points $\mathbf{r}_1$ and $\mathbf{r}_2$ as the ensemble average:

$$W(\mathbf{r}_1, \mathbf{r}_2; \omega_1, \omega_2) = \langle E^*(\mathbf{r}_1; \omega_1)E(\mathbf{r}_2; \omega_2)\rangle. \tag{6}$$

We define the cross-spectral purity of a nonstationary field by demanding that not only the normalized form (2) of the spectral density $S(\mathbf{r}; \omega) = W(\mathbf{r}, \mathbf{r}; \omega, \omega)$ but also the two-frequency complex degree of spectral (spatial) coherence

$$\mu(\mathbf{r}_1, \mathbf{r}_2; \omega_1, \omega_2) = \frac{W(\mathbf{r}_1, \mathbf{r}_2; \omega_1, \omega_2)}{\sqrt{S(\mathbf{r}_1; \omega_1)S(\mathbf{r}_2; \omega_2)}}, \tag{7}$$

evaluated at a single point $\mathbf{r}_1 = \mathbf{r}_2$, is the same at points $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{R}$:

$$\mu(\mathbf{R}, \mathbf{R}; \omega_1, \omega_2) = \mu(\mathbf{r}_1, \mathbf{r}_1; \omega_1, \omega_2) = \mu(\mathbf{r}_2, \mathbf{r}_2; \omega_1, \omega_2). \tag{8}$$

On inserting Eq. (1) into the definition (6), we find that the two-frequency CSD at point $\mathbf{R}$ is given by

$$\begin{aligned} W(\mathbf{R}, \mathbf{R}; \omega_1, \omega_2) = {} & W(\mathbf{r}_2, \mathbf{r}_2; \omega_1, \omega_2)\exp[-i(\omega_1 - \omega_2)\tau] \\ & + W(\mathbf{r}_2, \mathbf{r}_1; \omega_1, \omega_2)\exp(-i\omega_1\tau) \\ & + W(\mathbf{r}_1, \mathbf{r}_2; \omega_1, \omega_2)\exp(i\omega_2\tau) \\ & + W(\mathbf{r}_1, \mathbf{r}_1; \omega_1, \omega_2). \end{aligned} \tag{9}$$

In particular, the spectral density at point $\mathbf{R}$ takes on the form

$$\begin{aligned} S(\mathbf{R}; \omega) = {} & S(\mathbf{r}_1; \omega) + S(\mathbf{r}_2; \omega) \\ & + 2\mathrm{Re}[W(\mathbf{r}_1, \mathbf{r}_2; \omega, \omega)\exp(i\omega\tau)], \end{aligned} \tag{10}$$

where the CSD Hermiticity property $W(\mathbf{r}_2, \mathbf{r}_1; \omega, \omega) = W^*(\mathbf{r}_1, \mathbf{r}_2; \omega, \omega)$ has been used and Re denotes the real part. Equation (10) constitutes a spectral interference law for nonstationary light.

Let us first consider the necessary condition of Eq. (3) for cross-spectrally pure nonstationary light and denote $S(\mathbf{r}_2; \omega) = CS(\mathbf{r}_1; \omega)$, where $C = C(\mathbf{r}_1, \mathbf{r}_2)$ is a spatial proportionality factor. Making use of Eq. (7), we may cast Eq. (10) into the form

$$\begin{aligned} S(\mathbf{R}; \omega) = {} & S(\mathbf{r}_1; \omega)\{1 + C + 2\sqrt{C}|\mu(\mathbf{r}_1, \mathbf{r}_2; \omega, \omega)| \\ & \times \cos[\alpha(\mathbf{r}_1, \mathbf{r}_2; \omega, \omega) + \omega\tau]\}, \end{aligned} \tag{11}$$

where $\alpha(\mathbf{r}_1, \mathbf{r}_2; \omega, \omega)$ is the phase of $\mu(\mathbf{r}_1, \mathbf{r}_2; \omega, \omega)$. This result shows that we must demand $\tau = 0$ for the equalities in Eq. (3) to hold; otherwise spectral interference fringes would appear. Hence nonstationary light can be cross-spectrally pure only in the zero-time-delay region. This is in stark contrast to stationary light, which never exhibits spectral interference effects due to the inherent spectral incoherence.

On the basis of the result obtained above, Eqs. (9) and (10) can be simplified into the forms

$$W(\mathbf{R}, \mathbf{R}; \omega_1, \omega_2) = W(\mathbf{r}_1, \mathbf{r}_1; \omega_1, \omega_2) + W(\mathbf{r}_2, \mathbf{r}_2; \omega_1, \omega_2) + W(\mathbf{r}_1, \mathbf{r}_2; \omega_1, \omega_2) + W(\mathbf{r}_2, \mathbf{r}_1; \omega_1, \omega_2) \tag{12}$$

and

$$S(\mathbf{R}; \omega) = S(\mathbf{r}_1; \omega)\{1 + C + 2\sqrt{C}\mathrm{Re}[\mu(\mathbf{r}_1, \mathbf{r}_2; \omega, \omega)]\}, \tag{13}$$

respectively. Again, the factor in the braces in Eq. (13) must not depend on frequency ω for Eq. (3) to hold. Making use of Eq. (7) and the latter equality of Eq. (8), we find that $W(\mathbf{r}_2, \mathbf{r}_2; \omega_1, \omega_2) = CW(\mathbf{r}_1, \mathbf{r}_1; \omega_1, \omega_2)$. We may then—after some manipulations—express the complex degree of spectral coherence at point $\mathbf{R}$ in the form

$$\mu(\mathbf{R}, \mathbf{R}; \omega_1, \omega_2) = \mu(\mathbf{r}_1, \mathbf{r}_1; \omega_1, \omega_2)\frac{1 + C + [W_{12}(\omega_1, \omega_2) + W_{21}(\omega_1, \omega_2)]/W_{11}(\omega_1, \omega_2)}{1 + C + 2\sqrt{C}\mathrm{Re}[\mu(\mathbf{r}_1, \mathbf{r}_2)]}, \tag{14}$$

where in the numerator we have denoted the spatial dependence with subscripts for brevity and have written $\mathrm{Re}[\mu(\mathbf{r}_1, \mathbf{r}_2; \omega, \omega)] = \mathrm{Re}[\mu(\mathbf{r}_1, \mathbf{r}_2)]$ in the denominator. For the first equality in Eq. (8) to hold it is required that

$$\frac{W(\mathbf{r}_1, \mathbf{r}_2; \omega_1, \omega_2) + W(\mathbf{r}_2, \mathbf{r}_1; \omega_1, \omega_2)}{W(\mathbf{r}_1, \mathbf{r}_1; \omega_1, \omega_2)} = 2\sqrt{C}\mathrm{Re}[\mu(\mathbf{r}_1, \mathbf{r}_2)]. \tag{15}$$

The left-hand side of this equation, which is generally complex valued, depends on two frequency coordinates ω_1 and ω_2, whereas the real-valued right-hand side is independent of frequency.

When $\mathbf{r}_2 = \mathbf{r}_1$, whereby both C and $\mu(\mathbf{r}_1, \mathbf{r}_1)$ are unity, Eq. (15) is identically satisfied. It is conceivable that there exist correlation functions which satisfy Eq. (15) at some special points. However, we demand this equality to hold for all values of $\mathbf{r}_2$ within the wave field; when $\mathbf{r}_2$ differs from $\mathbf{r}_1$, this condition is fulfilled given that the space and frequency dependencies of the CSD separate

$$W(\mathbf{r}_1, \mathbf{r}_2; \omega_1, \omega_2) = W_{\mathrm{s}}(\mathbf{r}_1, \mathbf{r}_2)W_{\mathrm{f}}(\omega_1, \omega_2), \tag{16}$$

where $W_s(\mathbf{r}_1, \mathbf{r}_2)$ and $W_f(\omega_1, \omega_2)$ are correlation functions. In this case the left-hand side of Eq. (15) is also independent of frequency and reduces precisely to the right-hand side, which is a function of the spatial coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ only.

The result expressed by Eq. (16) shows that nonstationary light is cross-spectrally pure over the whole field, if the two-point two-frequency CSD separates into a product of space- and frequency-domain correlation functions. The spectral density is thus expressible as

$$S(\mathbf{r};\omega) = W_s(\mathbf{r}, \mathbf{r})W_f(\omega, \omega) = S_s(\mathbf{r})S_f(\omega). \tag{17}$$

Further, the complex degree of spectral coherence can be cast into the form

$$\mu(\mathbf{r}_1, \mathbf{r}_2;\omega_1, \omega_2) = \mu_s(\mathbf{r}_1, \mathbf{r}_2)\mu_f(\omega_1, \omega_2). \tag{18}$$

Here the spatial correlation factor

$$\mu_s(\mathbf{r}_1, \mathbf{r}_2) = \frac{W_s(\mathbf{r}_1, \mathbf{r}_2)}{\sqrt{S_s(\mathbf{r}_1)S_s(\mathbf{r}_2)}} \tag{19}$$

is independent of frequency, and the spectral correlation factor is given by

$$\mu_f(\omega_1, \omega_2) = \frac{W_f(\omega_1, \omega_2)}{\sqrt{S_f(\omega_1)S_f(\omega_2)}}. \tag{20}$$

We refer to Eq. (18) above as the space-frequency domain reduction formula for cross-spectrally pure nonstationary light fields.

Considering correlations in the space-time domain, we make use of the two-time mutual coherence function (MCF) of nonstationary fields, defined as

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2;t_1, t_2) = \langle E^*(\mathbf{r}_1;t_1)E(\mathbf{r}_2;t_2)\rangle, \tag{21}$$

where

$$E(\mathbf{r};t) = \int_0^\infty E(\mathbf{r};\omega)\exp(-i\omega t)d\omega \tag{22}$$

are the space-time domain realizations of the electric field. In view of Eqs. (6), (21), and (22), the CSD and MCF are connected by the Fourier-type relationship

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2;t_1, t_2) = \iint_0^\infty W(\mathbf{r}_1, \mathbf{r}_2;\omega_1, \omega_2) \times \exp[i(\omega_1 t_1 - \omega_2 t_2)]d\omega_1 d\omega_2. \tag{23}$$

Hence the separability condition (16) for the CSD immediately implies a separability condition

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2;t_1, t_2) = W_s(\mathbf{r}_1, \mathbf{r}_2)\Gamma_t(t_1, t_2) \tag{24}$$

for the MCF, where

$$\Gamma_t(t_1, t_2) = \iint_0^\infty W_f(\omega_1, \omega_2) \times \exp[i(\omega_1 t_1 - \omega_2 t_2)]d\omega_1 d\omega_2. \tag{25}$$

The temporal intensity of a cross-spectrally pure field is therefore

$$I_t(\mathbf{r};t) = W_s(\mathbf{r}, \mathbf{r})\Gamma_t(t, t) = S_s(\mathbf{r})I(t), \tag{26}$$

i.e., the spatial intensity distribution is the same in both domains. The normalized form of the MCF, namely, the two-point two-time complex degree of coherence, defined as

$$\gamma(\mathbf{r}_1, \mathbf{r}_2;t_1, t_2) = \frac{\Gamma(\mathbf{r}_1, \mathbf{r}_2;t_1, t_2)}{\sqrt{I(\mathbf{r}_1, t_1)I(\mathbf{r}_2, t_2)}}, \tag{27}$$

therefore factors into the product

$$\gamma(\mathbf{r}_1, \mathbf{r}_2;t_1, t_2) = \gamma_s(\mathbf{r}_1, \mathbf{r}_2)\gamma_t(t_1, t_2). \tag{28}$$

Here the spatial correlation factor is time independent, and fulfills

$$\gamma_s(\mathbf{r}_1, \mathbf{r}_2) = \mu_s(\mathbf{r}_1, \mathbf{r}_2), \tag{29}$$

whereas the temporal correlation factor is

$$\gamma_t(t_1, t_2) = \frac{\Gamma_t(t_1, t_2)}{\sqrt{I_t(t_1)I_t(t_2)}}. \tag{30}$$

Equation (28), which we call the space-time domain reduction formula for cross-spectrally pure nonstationary fields, is the counterpart of the reduction formula (5) for stationary fields. On the other hand, Eq. (29) is the nonstationary equivalent of formula (4).

IV. PROPAGATION OF LIGHT FROM CROSS-SPECTRALLY PURE SOURCES

Let us assume that the spectral electric field is known at the plane $z = 0$, where the transverse coordinate is denoted by $\boldsymbol{\rho} = (x, y)$, and consider its propagation into the positive half space $z > 0$. The free-space propagation is governed rigorously by the well-known angular spectrum representation (see [13], Sec. 3.2), which states that the field at an arbitrary point $\mathbf{r} = (x, y, z)$ is given by

$$E(\mathbf{r};\omega) = \int_{-\infty}^\infty A(\boldsymbol{\kappa};\omega)\exp[ik_z(\omega)z]\exp(i\boldsymbol{\kappa}\cdot\boldsymbol{\rho})d^2\kappa. \tag{31}$$

Here $\boldsymbol{\kappa} = (k_x, k_y)$ is a two-dimensional spatial-frequency vector,

$$A(\boldsymbol{\kappa};\omega) = \frac{1}{(2\pi)^2}\int_{-\infty}^\infty E(\boldsymbol{\rho};\omega)\exp(-i\boldsymbol{\kappa}\cdot\boldsymbol{\rho})d^2\rho \tag{32}$$

is the angular spectrum, and

$$k_z(\omega) = \sqrt{(\omega/c)^2 - |\boldsymbol{\kappa}|^2} \tag{33}$$

is the longitudinal component of the wave vector $\mathbf{k}$ (which is real for homogeneous waves and purely imaginary for evanescent waves). The propagation law for the CSD of nonstationary fields is obtained by directly inserting Eq. (31) into Eq. (6). This yields

$$W(\mathbf{r}_1, \mathbf{r}_2;\omega_1, \omega_2) = \iint_{-\infty}^\infty T(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2;\omega_1, \omega_2)\exp\{-i[k_{z1}^*(\omega_1)z_1 - k_{z2}(\omega_2)z_2]\}\exp\big[-i(\boldsymbol{\kappa}_1\cdot\boldsymbol{\rho}_1 - \boldsymbol{\kappa}_2\cdot\boldsymbol{\rho}_2)\big]d^2\kappa_1 d^2\kappa_2, \tag{34}$$

where

$$T(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2; \omega_1, \omega_2) = \frac{1}{(2\pi)^4} \iint_{-\infty}^{\infty} W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_1, \omega_2) \exp[i(\boldsymbol{\kappa}_1 \cdot \boldsymbol{\rho}_1 - \boldsymbol{\kappa}_2 \cdot \boldsymbol{\rho}_2)] d^2\rho_1 d^2\rho_2 \tag{35}$$

is known as the (spectral) angular correlation function and

$$k_{zj}(\omega_j) = \sqrt{(\omega_j/c)^2 - |\boldsymbol{\kappa}_j|^2}, \tag{36}$$

with $j = 1, 2$.

Let us now assume that the field at $z = 0$ is cross-spectrally pure in the frequency domain, i.e., in view of Eq. (16) the CSD can be expressed in the product form

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_1, \omega_2) = W_{\mathrm{s}}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) W_{\mathrm{f}}(\omega_1, \omega_2). \tag{37}$$

Then, it follows from Eq. (35) that

$$T(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2; \omega_1, \omega_2) = T(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) W_{\mathrm{f}}(\omega_1, \omega_2), \tag{38}$$

where

$$T(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) = \frac{1}{(2\pi)^4} \iint_{\infty}^{\infty} W_{\mathrm{s}}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \times \exp[i(\boldsymbol{\kappa}_1 \cdot \boldsymbol{\rho}_1 - \boldsymbol{\kappa}_2 \cdot \boldsymbol{\rho}_2)] d^2\rho_1 d^2\rho_2. \tag{39}$$

Hence the angular correlation function is cross-spectrally pure. However, if we insert Eq. (35) into Eq. (34), we immediately see that the cross-spectral purity of the CSD is generally broken at points $\mathbf{r}_1$ and $\mathbf{r}_2$ beyond the plane $z = 0$. The same conclusion is true also if paraxial propagation is considered by approximating

$$k_{zj}(\omega_j) \approx \frac{\omega_j}{c} - \frac{c}{2\omega_j} |\boldsymbol{\kappa}_j|^2, \tag{40}$$

which leads to the Fresnel propagation formula for nonstationary fields.

Since cross-spectral purity is not preserved on propagation in the spectral domain, it is not preserved in the temporal domain either. However, if a narrow-band field with a spectrum concentrated around some frequency $\omega = \omega_0$ is considered, we may approximate

$$k_{zj}(\omega_j) \approx \sqrt{(\omega_0/c)^2 - |\boldsymbol{\kappa}_j|^2}. \tag{41}$$

In this case, considered by Mandel (for stationary fields) in his original paper [1], cross-spectral purity is approximately maintained throughout the half space $z > 0$. Of course, this condition is not fulfilled for short optical pulses.

V. GENERATION OF CROSS-SPECTRALLY PURE LIGHT

Mandel and Wolf state in their classic text on optical coherence (see the first paragraph of Sec. 4.5.1 of [13]) that "We will now show that, under certain circumstances which are often encountered in practice, the spectral composition of the superposed light depends on the spectral composition of the interfering beams in a relatively simple matter." Such commonly encountered circumstances have, to our knowledge, not been elaborated in the literature. In fact, it is difficult to identify primary (or secondary) nontrivial natural light fields that would fulfill the conditions for cross-spectral purity even in the stationary case, let alone the nonstationary case. Apart from narrow-band fields, polychromatic plane waves satisfy these conditions [8], though in a trivial sense since they are spatially fully coherent. In this section we introduce a scheme, based on achromatic Fourier transformation [14–21], for generating cross-spectrally pure fields from spatially incoherent light sources.

A. Stationary fields

Let us denote the transverse coordinates in the input and output planes of an optical system by $\mathbf{v}$ and $\boldsymbol{\rho}$, respectively, and the impulse response of the system by $K(\boldsymbol{\rho}, \mathbf{v}; \omega)$. Then, with stationary light, the CSD functions at the input and output planes are related by

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega) = \int_{-\infty}^{\infty} W(\mathbf{v}_1, \mathbf{v}_2; \omega) K^*(\boldsymbol{\rho}_1, \mathbf{v}_1; \omega) K(\boldsymbol{\rho}_2, \mathbf{v}_2; \omega) d^2v_1 d^2v_2. \tag{42}$$

It is of interest to consider two particular types of systems that perform a spatial Fourier transform of the field in the input plane. For a conventional Fourier-transforming system realized with an achromatic lens of focal length F the impulse response is given by

$$K(\boldsymbol{\rho}, \mathbf{v}; \omega) = \frac{\omega}{i2\pi cF} \exp\left(-\frac{i\omega}{cF} \boldsymbol{\rho} \cdot \mathbf{v}\right). \tag{43}$$

On the other hand, the impulse response of an achromatic Fourier-transform system of focal length F, designed to operate around some wavelength $\lambda_0 = 2\pi c/\omega_0$, is

$$K(\boldsymbol{\rho}, \mathbf{v}; \omega) = \frac{\omega_0}{i2\pi cF} \exp\left(-\frac{i\omega_0}{cF} \boldsymbol{\rho} \cdot \mathbf{v}\right). \tag{44}$$

We note that achromatic Fourier-transforming systems can be implemented in a number of ways with the aid of purely refractive or hybrid refractive-diffractive optical systems [16–21]

Let us first assume that the field in the input plane is spatially incoherent and that its normalized spectrum is independent of position. Then the CSD in the input plane of the system is of the form

$$W(\mathbf{v}_1, \mathbf{v}_2; \omega) = s(\omega) S_{\mathrm{s}}(\mathbf{v}_1) \delta(\mathbf{v}_1 - \mathbf{v}_2). \tag{45}$$

This field is cross-spectrally pure, but in a somewhat trivial sense since the field is spatially delta correlated.

If a conventional Fourier-transform (Köhler illumination) system is used, the CSD at the output plane is well known to be

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega) = \left(\frac{\omega}{cF}\right)^2 s(\omega) \tilde{S}_{\mathrm{s}}\left(\frac{\omega}{cF} \Delta\boldsymbol{\rho}\right), \tag{46}$$

where $\Delta\boldsymbol{\rho} = \boldsymbol{\rho}_2 - \boldsymbol{\rho}_1$ and

$$\tilde{S}_{\mathrm{s}}(\mathbf{f}) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} S_{\mathrm{s}}(\mathbf{v}) \exp(-i\mathbf{f} \cdot \mathbf{v}) d^2v. \tag{47}$$

The complex degree of spectral (spatial) coherence

$$\mu(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega) = \frac{\tilde{S}_s(\omega \Delta\boldsymbol{\rho}/cF)}{\tilde{S}_s(0)} \tag{48}$$

thus satisfies Wolf's scaling law [22]. With an achromatic Fourier-transform system, the output-plane CSD has the form

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega) = \left(\frac{\omega_0}{cF}\right)^2 s(\omega) \tilde{S}_s\left(\frac{\omega_0}{cF}\Delta\boldsymbol{\rho}\right), \tag{49}$$

and the complex degree of coherence is

$$\mu(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega) = \frac{\tilde{S}_s(\omega_0 \Delta\boldsymbol{\rho}/cF)}{\tilde{S}_s(0)} = \mu(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_0). \tag{50}$$

It is also readily seen that

$$\gamma(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; 0) = \mu(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_0). \tag{51}$$

In view of Eqs. (4), (15), and (51), the resulting field is cross-spectrally pure around the time delay $\tau = \tau_0 = 0$. Hence an achromatic Fourier-transform system provides a simple means to generate cross-spectrally pure stationary fields from incoherent light sources.

B. Nonstationary fields

Proceeding to consider the nonstationary case, the relationship between the CSD functions in the input and output planes is

$$\begin{aligned} W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2 \omega_1, \omega_2) &= \iint_{-\infty}^{\infty} W(\mathbf{v}_1, \mathbf{v}_2; \omega_1, \omega_2) K^*(\boldsymbol{\rho}_1, \mathbf{v}_1; \omega_1) \\ &\quad \times K(\boldsymbol{\rho}_2, \mathbf{v}_2; \omega_2) d^2 v_1 d^2 v_2. \end{aligned} \tag{52}$$

Let us assume that the CSD in the input plane is of the form

$$\begin{aligned} W(\mathbf{v}_1, \mathbf{v}_2; \omega_1, \omega_2) &= \sqrt{s(\omega_1) s(\omega_2)} c(\omega_1, \omega_2) \\ &\quad \times S_s(\mathbf{v}_1) \delta(\mathbf{v}_1 - \mathbf{v}_2), \end{aligned} \tag{53}$$

where $c(\omega, \omega) = 1$. Here again the normalized spectrum $s(\omega)$ of the field is taken to be independent of position. In addition, the spectral correlation function $c(\omega_1, \omega_2)$ is assumed to be position independent and, as in Sec. V A, the field is spatially incoherent. These assumptions hold, at least to a good approximation, if a spectrally partially coherent pulse train passes through a rotating diffuser, so that each individual pulse experiences a different roughness distribution.

The CSD at the output plane of a conventional Fourier-transform system is obtained by inserting Eqs. (7) and (53) into Eq. (52), with the result

$$\begin{aligned} W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_1, \omega_2) &= \frac{\omega_1 \omega_2}{(cF)^2} \sqrt{s(\omega_1) s(\omega_2)} c(\omega_1, \omega_2) \\ &\quad \times \tilde{S}_s[(\omega_2 \boldsymbol{\rho}_2 - \omega_1 \boldsymbol{\rho}_1)/cF]. \end{aligned} \tag{54}$$

Hence the complex degree of coherence has the form

$$\mu(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_1, \omega_2) = c(\omega_1, \omega_2) \frac{\tilde{S}_s[(\omega_1 \boldsymbol{\rho}_1 - \omega_2 \boldsymbol{\rho}_2)/cF]}{\tilde{S}_s(0)}. \tag{55}$$

If an achromatic Fourier-transform lens is used, the CSD at the output plane becomes

$$\begin{aligned} W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_1, \omega_2) &= \left(\frac{\omega_0}{cF}\right)^2 \sqrt{s(\omega_1) s(\omega_2)} c(\omega_1, \omega_2) \\ &\quad \times \tilde{S}_s(\omega_0 \Delta\boldsymbol{\rho}/cF), \end{aligned} \tag{56}$$

which is of the separable form of Eq. (16). The normalized spectrum at the output plane is therefore equal to $s(\omega)$ and the complex degree of spectral coherence is of the form of Eq. (18), with

$$\mu_s(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \frac{\tilde{S}_s(\omega_0 \Delta\boldsymbol{\rho}/cF)}{\tilde{S}_s(0)} \tag{57}$$

and

$$\mu_f(\omega_1, \omega_2) = c(\omega_1, \omega_2). \tag{58}$$

Hence the output field is cross-spectrally pure. Its MCF is of the separable form of Eq. (24), with

$$\begin{aligned} \Gamma_t(t_1, t_2) &= \iint_0^{\infty} \sqrt{s(\omega_1) s(\omega_2)} \mu_f(\omega_1, \omega_2) \\ &\quad \times \exp\left[i(\omega_1 t_1 - \omega_2 t_2)\right] d\omega_1 d\omega_2, \end{aligned} \tag{59}$$

and the complex degree of coherence follows Eq. (28).

VI. FINAL REMARKS

We have derived conditions under which nonstationary light is cross-spectrally pure by demanding that, apart from the spectrum, also the spatial self-correlation function in the frequency domain (and thereby also in the temporal domain) must have the same form as it has at any two selected points on the incident wave front. Our conditions for cross-spectral purity lead to strict separability criteria for the two-frequency cross-spectral density function and the two-time mutual correlation function. In particular, we have shown that cross-spectral purity of nonstationary fields is possible only in the zero-time delay region of a superposition of two light beams. Spectral correlations, however minor, enforce this conclusion. We have also shown that cross-spectrally pure light cannot, in general, retain its purity upon propagation unless the field is nearly monochromatic. It therefore follows that fields associated with trains of short optical pulses with broad spectra can be cross-spectrally pure across at most one plane.

Thus far we have not been able to identify any nontrivial naturally occurring fields (stationary or nonstationary) that would intrinsically be cross-spectrally pure. However, we have shown that transformation of fields from spatially incoherent sources into a cross-spectrally pure form is possible by means of achromatic Fourier-transform systems. In this paper we have considered only scalar wave fields, but an electromagnetic extension of the results given here is possible along the lines discussed in [3,4].

It is finally worth noting that, in the existing literature on nonstationary fields, cross-spectral purity in the sense discussed here is sometimes assumed implicitly. This is the case, for instance, in the works that deal with partially spatially and spectrally coherent pulsed beams [23–25], where the spatial beam parameters (beam width and coherence width) are typically taken to be frequency independent. Such an assumption is common also in the studies of stationary partially coherent beams. To our knowledge, only Christov [23] mentions the concept of cross-spectral purity explicitly in this context, but in passing and without further justification or elucidation.

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