

## Evaluating the performance of photon-number-resolving detectors

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(Received 20 December 2018; published 16 April 2019)

We analyze the performance of photon-number-resolving (PNR) detectors and introduce a figure of merit for the accuracy of such detectors. This figure of merit is the (worst-case) probability that the photon-number-resolving detector correctly predicts the input photon number. Simulations of various PNR detectors based on multiplexed single-photon “click detectors” is performed. We conclude that the required quantum efficiency is very high in order to achieve even moderate (up to a handful) photon resolution, we derive the required quantum efficiency as a function of the the maximal photon number one wants to resolve, and we show that the number of click detectors required grows quadratically with the maximal number of photons resolvable.

DOI: [10.1103/PhysRevA.99.043822](https://doi.org/10.1103/PhysRevA.99.043822)

### I. INTRODUCTION

Photon-number-resolving (PNR) detectors have been shown to be useful in various optical applications, such as linear optical quantum computing [1], quantum key exchange [2], entanglement distribution [3], photon-counting laser radars [4], x-ray astronomy [5], evaluation of single-photon sources [6], and elementary particle detection [7]. Due to the wide applicability of these detectors many different schemes have been proposed, both inherent PNR detectors [8–16] and various schemes for multiplexing single-photon detectors to construct what is called a multiplexed PNR detector [17–28]. However, so far only limited work has been dedicated to systematically and realistically evaluate the actual performance of multiplexed PNR detectors [18,19,26,28].

Figures of merit used to evaluate the performance in previous work has generally been input signal dependent, which makes it difficult to determine how the detectors will perform in applications with varying input signals. Furthermore, the figures of merit used for single-photon detectors, such as quantum efficiency, cannot be directly applied to PNR detectors since they can have elaborate internal detection mechanisms and therefore behave differently for different input signals.

In this paper we investigate the efficiency of PNR detectors in applications where it is important that every input event is labeled with the correct number of photons, where the input state is unknown, but where it is likely that the input photon number exceeds unity. We are therefore interested in characterizing how well the output is reproducing the input. To do this we introduce a figure of merit, the PNR quality, which we show is a natural generalization of the quantum efficiency. The figure is input signal independent for a selected set of input signals and any uncertainty within this set. However, generalizing the set to any input, the figure becomes completely input independent. We also show how the PNR

quality can be used to set an upper, resolvable input photon-number for PNR detectors such that the number prediction for any input not exceeding this number can be trusted to within a specified “confidence” level.

We simulate three different multiplexed PNR detector schemes, a spatial array, a temporal array constructed using fiber couplers, and a loop-multiplexed detector, and evaluate their PNR performance. Our analysis is much in the spirit of Refs. [28,29] and qualitatively our results agree but the assumptions and details differ. We show quantitatively how the quantum efficiency of the single-photon detector(s) limits the number of photons resolvable with each PNR-detector type, and we show that the needed number of detector elements grows quadratically with the number of photons one desires to resolve. We also analyze quantitatively how the dark-count probability affects the multiplexed PNR-detector quality.

### II. PNR QUALITY

An ideal PNR detector gives an output signal indicating the input number of photons independent of the input signal used. However, in practice all detectors have nonideal characteristics, so there exists some upper input number  $n$  which is the largest number of photons that the detector is capable of resolving with reasonable certainty. To incorporate such nonideal properties in a model, we introduce the  $n$  detector which is a PNR detector capable of detecting up to  $n$  photons, with probabilities  $P_{k,m}$  that the output is  $S_O = k$ , given that the input was  $S_I = m$  photons. The probability to receive an output larger than  $n$  is zero and thereby the  $n$  detector is effectively classifying the input signal into the  $n + 1$  output classes  $0, 1, \dots, n - 1$  and  $\geq n$  photons. It is possible to transform any PNR detector into an  $n$  detector by mapping the output  $S_O \mapsto \min\{S_O, n\}$ .

Quite naturally we say that the output from a PNR detector was desired if the input was classified into the correct class, which for an  $n$  detector corresponds to

$$S_{O,\text{desired}} = \begin{cases} m & \text{if } S_I = m \leq n, \\ n & \text{otherwise.} \end{cases} \quad (1)$$

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Given the desired output, we want to define a figure of merit. Two possibilities are to use Bayesian inference or, more simply, use the probability that the  $n$  detector's output is desired. Both these figures of merit capture the detector's performance, yet they measure different properties. In the former case we get a figure of merit that depends on how well the input signal is known. In general such a figure will be of the form

$$B[p] = f\left(\left\{\frac{P_{m,m}p(m)}{\sum_{l \in \mathbb{N}} P_{m,l}p(l)}\right\}_{m \in \mathbb{N}}\right), \quad (2)$$

where  $f$  is a function that maps the sequence of Bayesian probabilities on  $\mathbb{R}$  and  $p$  is the input photon-number distribution. Because we like to define a measure that is input signal independent, we need to reduce the measure  $B$  from a functional of  $p$  to a detector-dependent constant.

Finding a reduction that generates a measure that is both representative for all input signals and that captures the behavior of the detector is nontrivial. One possibility would, e.g., be to minimize with respect to  $p$ , i.e.,  $B = \inf_p B[p]$ . However, this is problematic since the Bayesian quotient is independent of  $P_{m,m}$  when  $p(l) = \delta_{l,m}$ , that is, we have a Fock-state input. Another possible reduction is to use the expected value, i.e.,  $B = \mathbb{E}_p[B[p]]$ . However this is also problematic since the measure is only representative if the input signal distribution is known or if its relative variance is small.

Due to the problems with generating an input-independent Bayesian figure of merit we instead suggest using a figure that is derived from the probability that the detector classifies the input correctly. Minimizing this functional on the set of all allowed input photon-number distributions  $\mathcal{F}$  we get the PNR quality. This is thus a “worst case” figure of merit which is input signal independent for all allowed inputs. Formally we define the PNR quality as

$$Q_n(\mathcal{F}) \equiv \inf_{p \in \mathcal{F}} \left( \sum_{m \leq n} P_{m,m}p(m) + \sum_{m > n} P_{n,m}p(m) \right), \quad (3)$$

where the set  $\mathcal{F}$  is a subset of all probability distributions on  $\mathbb{N}$ . Consequently, it holds that if the input signal distribution is in  $\mathcal{F}$  then the probability for the output to be the desired one is at least  $Q_n(\mathcal{F})$ . The set  $\mathcal{F}$  should therefore be selected so it includes all input signals that the detector is expected to handle, for example, any Poissonian photon-number distribution.

One should note that our chosen figure of merit depends only on the detector's ability to correctly map the obtained signal into the “correct” corresponding input photon number. One can imagine detectors where  $P_{m,m}$  is very low so that for one or several  $k < m$  one has  $P_{k,m} > P_{m,m}$ . Our measure would give a low figure of merit for such detectors. However, given that one knows that the input is limited to a maximum of  $m$  photons and that for all  $m' < m$  the probabilities  $P_{m,m'} \ll P_{m,m}$ , the (rare) output signal  $k = m$  would signal (with high confidence) that an  $m$ -photon event was detected. Such a detector would have certain applications, although it would not be rated high by our figure of merit.

The choice of input photon-number distribution set that the detector is expected to handle and the maximal number of photons that can be resolved by the detector both affect the PNR quality of the detector. For the former it holds trivially that a reduction of the distribution set from  $\mathcal{F}$  to  $\mathcal{F}' \subset \mathcal{F}$  can

help to improve the PNR quality of the detector. Hence, quite naturally, there is a trade-off between allowing a large input set and having a high guaranteed probability to get the correct output. For the latter it holds generally that (see the Appendix)

$$Q_n(\mathcal{F}) \leq Q_{n-1}(\mathcal{F}) \quad \forall \mathcal{F}, n > 0, \quad (4)$$

so by reducing the maximal number of photons that can be resolved we may improve the PNR quality of the detector.

Finding the desired output probability infimum on a function set can be difficult if the number of free parameters is large or if there is uncertainty about which signal is used as the input. For example, assume that there are two possible input signals,  $p_1$  and  $p_2$ , but that there is an uncertainty as to which of the two will be sent. The set of possible input distributions then is not  $\mathcal{F} = \{p_1, p_2\}$ , but rather

$$\mathcal{F}' = \{ap_1 + bp_2 \mid a + b = 1 \wedge a, b \geq 0\}. \quad (5)$$

However, the uncertainty does not increase the complexity of the optimization problem since it holds that

$$Q_n(\mathcal{F}') = Q_n(\mathcal{F}), \quad (6)$$

so it is therefore possible to simplify the minimization problem to only consider  $\mathcal{F}$ . Generally it holds that (see the Appendix)

$$Q_n(\mathcal{A}) = Q_n(\mathcal{B}) \quad (7)$$

if it is possible to write all elements in  $\mathcal{A}$  as a linear combination of elements in  $\mathcal{B}$  where the coefficients are non-negative.

Let us consider a special case of the PNR quality when  $\mathcal{F}$  is the set of all probability distributions on  $\mathbb{N}$ . We denote this special case with  $Q_n$  and we can show that (see the Appendix)

$$Q_n = \min \left\{ \min_{m \leq n} P_{m,m}, \inf_{m > n} P_{n,m} \right\}. \quad (8)$$

Hence, the most difficult input signals to resolve are the distributions with 100% probability for some photon number, i.e., any Fock state. Furthermore, for a single-photon detector with quantum efficiency  $\eta$  and dark-count rate  $r_d = 0$ , the quality of the detector is  $Q_1 = \eta$ , which makes the signal  $p(k) = \delta_{k,1}$  the most difficult signal to resolve. The quality for the full probability set could be thought of as a generalization of the quantum efficiency for single-photon detectors.

A reasonable requirement of a PNR detector is that it should outperform guessing the outcome. If a detector has  $Q_n \geq 0.5$  then it has better-than-guessing quality for any signal with unknown probabilities consisting of two or more outcomes. Therefore in the following we often use this value to define for what photon number  $n$  specific PNR detectors can reasonably be said to resolve 0, 1,  $\dots$ ,  $n - 1$  or  $n$  or more input photons.

### III. DETECTOR SIMULATION

In this section we present the PNR quality obtained from simulations of three different “multiplexed” detectors: a spatial array exemplified in Refs. [21–24,27], a temporal array exemplified in Refs. [18,19,26,29], and a loop-multiplexed detector such as in Refs. [17,30]. We limit ourselves to two different function sets, the set of all probability distributions

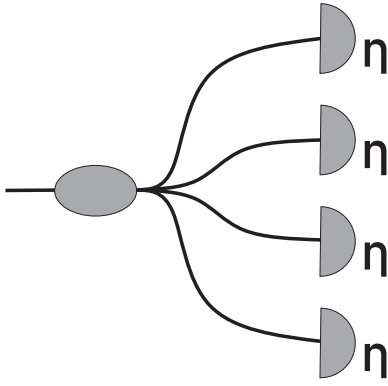


FIG. 1. Schematic image of a spatial array consisting of four detector elements with quantum efficiency  $\eta$ . An input signal equally distributed over the four detector elements, e.g., with a 1-to-4 fiber coupler.

on  $\mathbb{N}$  and the set of all Poisson distributions

$$\mathcal{F} = \left\{ f : k \in \mathbb{N} \mapsto \frac{\mu^k e^{-\mu}}{k!} \mid \mu \in [0, n] \right\}, \quad (9)$$

where we have limited the mean  $\mu$  to be at most  $n$  for an  $n$  detector. We run this simulation for different values of  $n$  to determine what quantum efficiency is required to reliably ( $Q_n \geq 0.5$ ) detect  $n$  photons.

In the simulations we model the three multiplexed detectors as devices that distribute photons on click elements. Thus the individual detector is assumed to only distinguish between zero and one or more photons, while the multiplexed detector is able to use the combined result from the individual detector elements to detect more than one photon. To get the probabilities  $P_{k,m}$  for different outcomes of the multiplexed detector we sum over all possible distributions on the detector elements and all possible detection outcomes when the photons hit the single-photon detector elements. The exact distribution of photons over the segments is implementation dependent and is presented in the subsections below; however, the detector element model is shared among the considered PNR detectors. We assume that if a click-detector has quantum efficiency  $\eta$ , dark-count probability  $p_d$ , and  $m$  photons hit the detector then the probability for the detector to click is

$$\Pr(\text{click} | m) = 1 - (1 - p_d)(1 - \eta)^m. \quad (10)$$

That is, we have assumed that the dark-count probability is independent of the photon-detection probability.

### A. Spatial array

The spatial array consists of  $M$  click detector elements and the photons are distributed equiprobably over the elements as seen in Fig. 1. A prototype implementation is a uniformly illuminated array of single-photon avalanche photodiodes [22,24] or an array of superconducting nanowire detectors [23,27]. The outcomes when  $m$  photons are used as the input can be represented as the matrix  $\vec{x} \in \mathbb{N}^M$ , where element  $k$  is the number of photons that hit detector  $k$  and  $\|\vec{x}\|_1 = m$ . The corresponding probability for each outcome is given by

the multinomial distribution

$$\Pr(\{x_1, x_2, \dots\}) = \frac{M!}{M^m \prod_{i=1}^M x_i!}. \quad (11)$$

The probabilities  $P_{k,m}$  for the spatial array are in general difficult to compute analytically; however, it is possible to show that

$$P_{m,m} = \frac{M!}{M^m (M-m)!} \eta^m. \quad (12)$$

This gives us an upper bound for the PNR quality  $Q_n$  and it holds that  $Q_n \leq \eta^n$ . Consequently, for a given PNR quality the maximal number of photons that can be resolved is bounded by  $n \leq \ln Q_n / \ln \eta$ , independent of the number of detector elements in the array. Furthermore, if we assume that  $M \gg n$  we can show, using Eq. (12) and Stirling's formula, that

$$M \geq \frac{1}{2} \frac{n^2}{n \ln \eta - \ln Q_n} + O\left(\frac{n}{M} + M \ln M\right), \quad (13)$$

if  $n < \ln Q_n / \ln \eta$ . In Fig. 3 it is shown that the number of detector elements  $M$  needed is well approximated by Eq. (13) for sufficiently large  $M$ . This implies that for a fixed PNR quality the number of detector elements grows quadratically with the number of photons that the PNR detector can resolve according to our quality criterion. It was noted already in Ref. [31] that the number of detector elements must be much larger the number of photons one wants to resolve. In Fig. 2 of Ref. [31] one sees that for nine photons the minimum  $M$  is somewhat less than 100, which is in agreement with our Fig. 3. To resolve five photons  $\geq 20$  elements are needed, and to resolve ten photons one needs  $\geq 72$  detector elements. Hence, to resolve many photons, many detector elements, all of them having a high quantum efficiency, are required. As we shall see, it is very challenging to build PNR detectors using the spatial array scheme.

In Fig. 2 the simulated PNR quality is presented for spatial arrays consisting of 8 and 32 detector elements where dark counts have been neglected. As expected there is a significant improvement for both arrays when the distribution set is restricted to Poisson distributions, although the requirement on the quantum efficiency is still very high in order to resolve many photons. Comparing an 8-segment detector in Fig. 2(a) with a 32-segment detector in Fig. 2(c) we notice that the input photon number for which  $Q \geq 0.8$  doubles for  $\eta = 1$ , which is what was predicted by Eq. (13).

Using the  $Q_n \geq 0.5$  requirement to evaluate the spatial arrays yields the result that an ideal 8-detector element array ([as seen in Fig. 2(a)] is only capable of resolving three photons if all possible input signals are allowed. If restricted to Poisson distributions [as seen in Fig. 2(b)] such a detector is able to detect up to five photons, although in both cases the requirement on the quantum efficiency is close to unity ( $\eta \geq 0.92$  and  $\eta \geq 0.96$ , respectively).

In Fig. 4 it is shown how the PNR quality is affected by the dark-count probability for a fixed quantum efficiency. As expected, the general trend is that the PNR quality quickly decreases as the dark-count probability increases, yet  $Q_6$  violates this trend in the region  $p_d \in [0, 0.06]$  where it is increasing. Hence, the probability to resolve the signal that minimized the desired output must have been increased by the added

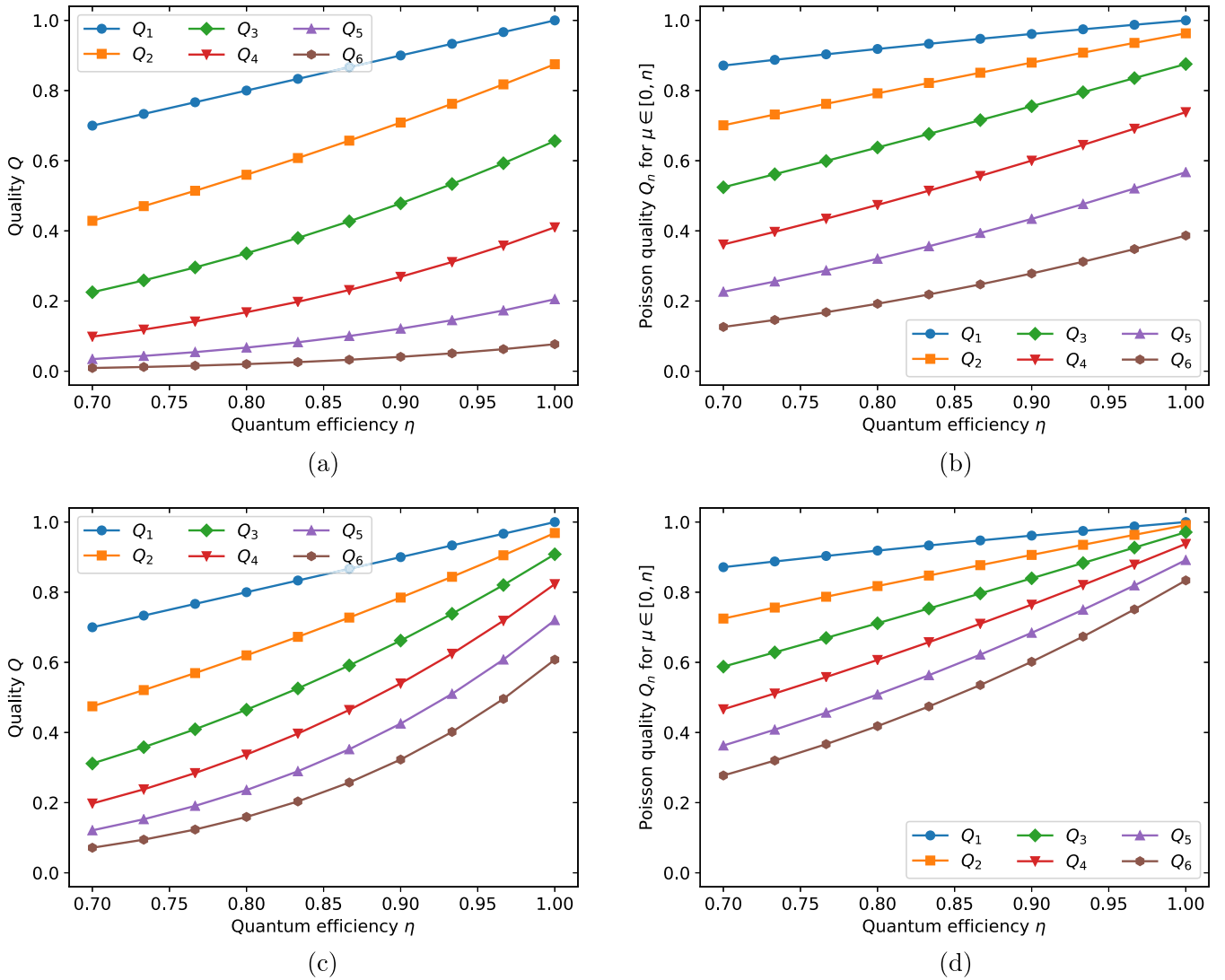


FIG. 2. Simulation result of the PNR quality for spatial arrays with negligible dark count rates. In panels (a) and (c), the PNR quality using the set of all probability distributions is presented for an array consisting of 8 and 32 elements, respectively. In panels (b) and (d), the PNR quality using the set of Poisson distributions with mean  $\mu \leq n$  is presented for an array consisting of 8 and 32 elements, respectively. Both array sizes show improvement in the quality when restricted to the set of Poisson distributions. However, reaching a quality larger than 0.5 for many photons still requires a very high quantum efficiency.

dark counts. Loosely speaking, the dark counts compensate for the nondetected incident photons for high photon numbers. However, at the same time the probability to resolve other signals decreases ( $Q_1$  to  $Q_4$  all decrease with  $p_d$ ), so quite intuitively it is therefore not beneficial for the overall PNR detector performance to add dark counts for the purpose of increasing the quality for large  $n$ .

### B. Temporal array

The temporal array consists of two single-photon detectors and a series of fiber couplers that split the signal equally between different fiber paths. Such setups, each with three couplers, have been reported in Refs. [18,19,26]. The lengths of the paths should be chosen both so that the detectors have time to recover between photons taking different paths and also so that photons taking different paths never combine and

interfere in a subsequent coupler. One scheme that can fulfill these two criteria is presented in Fig. 5, where  $l$  and  $L$  are introduced. The length  $l$  is chosen to be some arbitrary length, while  $L$  is chosen to be the shortest distance between light pulses needed for the click detectors to recover. The lengths of the top fiber and the bottom fiber after coupler  $k$  are selected to be  $2^{k-1}L + l$  and  $l$ , respectively [18]. In Ref. [29] it was shown that for the number of detector elements we are considering ( $< 100$ ), fiber dispersion effects can be neglected.

The setup in Fig. 5 produces eight equidistantly spaced pulses with equal amplitude at each detector from a single input pulse. Thus the scheme mimics, by temporal splitting, a 16-detector element spatial array. This scheme thus results in a detector with an accuracy that is mathematically equivalent to the accuracy of a uniformly illuminated spatial array and the PNR quality is therefore described by Fig. 2. However, by introducing the couplers in front of the detector we increase



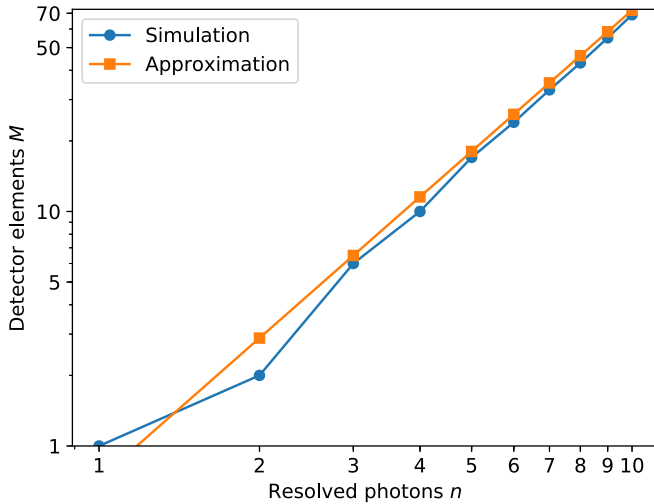


FIG. 3. The number of detector elements  $M$  needed get a PNR quality of  $Q_n \geq 0.5$  in a spatial array with quantum efficiency  $\eta = 1$  and negligible dark counts as a function of the number of resolved photons. The simulated result is in agreement with the approximation in Eq. (13) for sufficiently large  $M$ .

the effective recovery time of the temporally multiplexed detector by a factor  $M/2$ , where  $M$  is the number of effective detector segments. Moreover, the quantum efficiency of the multiplexed detector is lowered by the linear losses in the fiber couplers. If the fiber couplers have an efficiency  $\eta_c$  and the number of effective detector segments is  $M$  then

$$\eta_{\text{eff}} = \eta_c^{\log_2 M} \eta. \quad (14)$$

Hence, in this configuration there exists a trade-off between having a large number of detector elements and having a high quantum efficiency. At some point, by increasing the number of couplers one will therefore decrease the PNR quality due to a drop in quantum efficiency.

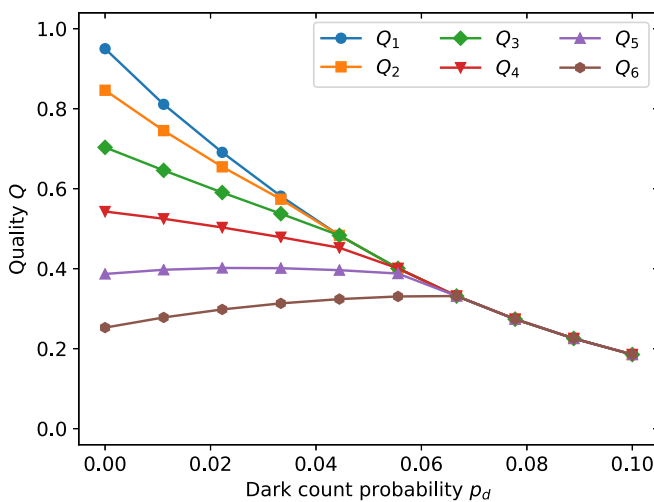


FIG. 4. PNR quality for a 16-element array with a quantum efficiency of  $\eta = 0.95$  as a function of the dark-count probability of each detector element. The general trend shows that the PNR quality decreases with increased dark counts; however, in some instances it is possible to have an increasing PNR quality.

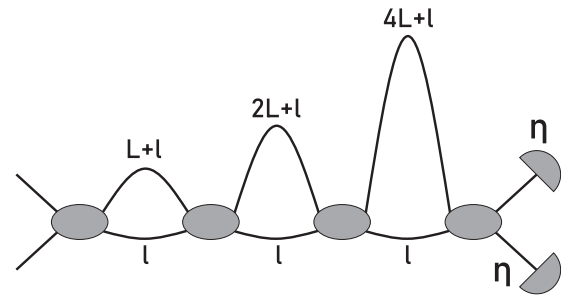


FIG. 5. Schematic image of a temporally multiplexed detector. The fibers between the 50:50 couplers are chosen so the detectors have time to recover between pulses and so a split signal can not recombine. Between coupler  $k$  and  $k + 1$  the lower fiber has a short (but arbitrary) length  $l$  and the upper fiber has length  $2^{k-1}L + l$ , where  $L$  corresponds to the smallest length rendering two subsequent pulses that the detector can resolve.

### C. Loop-multiplexed detector

The loop-multiplexed detector consists of a multimode fiber loop with inherent mode scrambling, a multimode to single-mode coupler circuit and a single-photon detector as seen in Fig. 6. Properly designed, when an input enters the circuit each photon has a probability  $T$  to exit the loop and hit the detector. The remaining photons loop back into the circuit and the process repeats from the beginning up to  $l$  times. Hence, during each revolution in the loop a photon has probability  $T$  to exit. It is not possible to build such a splitting circuit with a single-mode fiber coupler since it would result in an exit probability  $T$  at the first coupler passage and a probability  $(1 - T)^2$  at the second passage. Consequently, using a single-mode fiber coupler results in every photon having at least a 75% probability to exit the loop during the first two passes of the coupler [17,30]. This makes it difficult to resolve multiple ( $>2$ ) photons with any confidence using a click detector in such a setup.

The probabilities for different outcomes in the loop-multiplexed detector can be described with a recursion

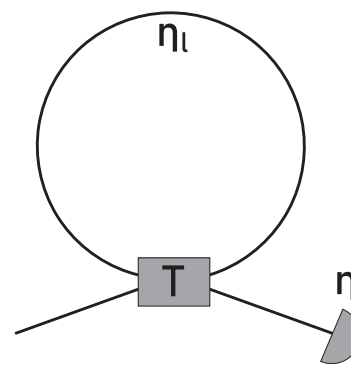


FIG. 6. Schematic image of a loop-multiplexed detector. Incoming light enters the multimode fiber end to the left and enters a splitting circuit in the middle. The probability for photons to exit to the detector is  $T$  independent of which of the two input fibers was used by the photon. The remaining probability is that the photon enters the loop, where it has an  $\eta_l$  probability to survive per loop.

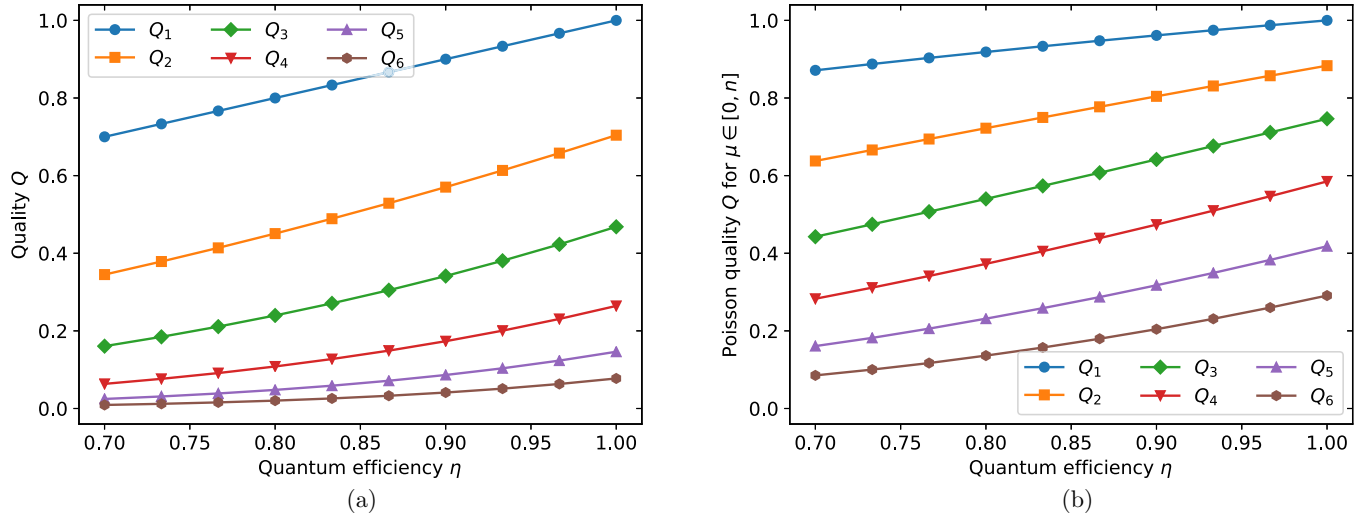


FIG. 7. PNR quality of a loop-multiplexed detector without dark counts. The probability that a photon survives the loop is  $\eta_l = 0.97$  and at most  $l = 32$  loops are allowed. (a) The PNR quality for all distributions is presented. Only two photons can be detected with a quality higher than 0.5. (b) The PNR quality for all Poisson distributions with mean  $\mu \leq n$ . Compared to panel (a) the quality has improved significantly, yet it is still only possible to detect up to four photons even with a perfect detector.

relation:

$$P_{k,m}^{(l)} = \sum_{a,b} f_{\text{bin}}(t, a, m) f_{\text{bin}}(\eta_l, b, m - a) (\text{Pr}(\text{click} | a) P_{k-1,b}^{(l-1)} + \text{Pr}(\text{not click} | a) P_{k,b}^{(l-1)}), \quad (15)$$

where  $\eta_l$  is the probability that a photon survives the loop and can re-enter the circuit,  $P_{k,m}^{(l)}$  is the probability to get an output  $k$  given that the input is  $m$  and at most  $l$  loops are allowed after which the detector is turned off (or its output is ignored) and  $f_{\text{bin}}(p, a, b)$  is the binomial probability to get  $a$  successful outcomes given  $b$  tries and a probability  $p$ .

Figure 7 shows the simulated result from the loop-multiplexed detector where we have assumed a maximal number of allowed loops,  $l = 32$ . As seen in Fig. 7(a), for arbitrary inputs only two photons can be resolved with a PNR quality larger than 0.5 and in Fig. 7(b) only four photons can be resolved with a PNR quality of 0.5 or more if the input photon distribution is limited to Poisson distributions. Consequently, this detector is outperformed by a temporal array with 32 detector elements if the losses in the couplers are assumed to be equal to  $\eta_l$ .

#### IV. SUMMARY

In this paper we introduced a figure of merit that is useful to assess the resolution accuracy of photon-number-resolving (PNR) detectors. The figure can be made input-signal independent and is equal to the smallest probability that the detector gives the correct output.

Simulations of three different PNR detectors, implemented as multiplexed click detectors show that the requirements on the quantum efficiency are very high in order to resolve a handful of, or more, photons. With an eight-segment detector, one cannot resolve more than three photons with better-than-guessing quality even with ideal click detectors. Furthermore, the needed number of detector segments in an array grows

quadratically with the number of photons resolved, so large arrays consisting of high quantum efficiency detector elements are imperative to resolve more than a few photons.

In the literature one encounters claims of PNR detectors that appear to contradict the limits we derive. Here it is important to differentiate between the resolution of the input photon number and the resolution between the output signals corresponding to different numbers of absorbed photons. In Ref. [23] array-PNR detectors are presented containing four to six segments. Even at unit quantum efficiency the input photon-number resolution ability of such detectors is limited to  $n = 3$ , but at the reported quantum efficiency of 2%, not even single-photon input resolution is reached. In contrast, in the (few) events where one, two, or three photons were absorbed by the array, the three different output signals were well resolved. Thus, the authors' [23] claim about output signal resolution is correct. However, the output signals do not allow one to conclude much about the input photon number. In Ref. [27], arrays with 4, 5, 12, and 24 elements were reported. Again, a 4-element array can only resolve, with accuracy, two photons, even at unit quantum efficiency. At the reported quantum efficiency of 0.17% (for the 12-element detector) none of the arrays have even single-photon resolution, although, again, the output signals corresponding to the absorption of 0, 1, ..., 24 photons are shown to be resolvable. Finally in Ref. [21], a 132-element detector is reported with a 16% efficiency. The authors make no claims about the input photon number resolution but note that the signals corresponding to different number of absorbed photons are well resolved. According to our resolution criterion the detector is not even able to resolve single input photons from none.

With the technology available today only a few detector types, such as transition-edge detectors [13,15] and superconducting nanowire detectors [32,33], have quantum efficiencies high enough to give better-than-guessing quality for more than a few photons. Even so, most realizations of PNR detectors with spatial arrays of click detectors have

difficulty increasing the number of detector elements to what is required for better-than-guessing PNR capability. Therefore, temporal arrays seem to be the most reasonable option to implement multiplexed PNR detectors if the losses in the fiber couplers can be made small and if some temporal resolution can be sacrificed.

### ACKNOWLEDGMENTS

This work was supported by the Knut and Alice Wallenberg Foundation grant “Quantum Sensing” and by the Swedish Research Council (VR) through Grant No. 621-2014-5410 and through its support of the Linnæus Excellence Center ADOPT. The simulations were performed on resources provided by the Swedish National Infrastructure for Computing (SNIC) at PDC.

### APPENDIX: PROOFS

*Theorem 1.* Let  $\mathcal{F}$  be a subset of all probability distributions on  $\mathbb{N}$  and let  $Q_n(\mathcal{F})$  be the PNR quality on the set for an  $n$  detector. It then holds that

$$Q_n(\mathcal{F}) \leq Q_{n-1}(\mathcal{F}) \quad \forall \mathcal{F}, n > 0. \quad (\text{A1})$$

*Proof.* Assume that  $P_{k,m}$  and  $P'_{k,m}$  are the conditional probabilities for the  $n$  detector and the  $(n-1)$  detector, respectively. An  $n$  detector is made into an  $(n-1)$  detector by mapping the output  $S_O \mapsto \min\{S_O, n-1\}$  and hence  $P'_{n-1,m} = P_{n-1,m} + P_{n,m}$ . Furthermore, the desired output of the  $n$  detector is

$$\begin{aligned} & \sum_{m \leq n} P_{m,m} p(m) + \sum_{m > n} P_{n,m} p(m) \\ &= P_{n,n} p(n) + \sum_{m \leq n-1} P_{m,m} p(m) + \sum_{m > n} P_{n,m} p(m) \\ &\leq (P_{n-1,n} + P_{n,n}) p(n) + \sum_{m \leq n-1} P_{m,m} p(m) \\ &\quad + \sum_{m > n} (P_{n-1,m} + P_{n,m}) p(m) \\ &= \sum_{m \leq n-1} P'_{m,m} p(m) + \sum_{m > n-1} P'_{n,m} p(m), \end{aligned} \quad (\text{A2})$$

where the expression in the last equality is the desired output of an  $(n-1)$  detector. It follows directly that  $Q_n(\mathcal{F}) \leq Q_{n-1}(\mathcal{F})$  if  $n > 0$ . ■

*Definition 1.* Let  $\mathcal{F}$  be a subset of all probability distributions on  $\mathbb{N}$ . The extended distribution set contains  $\mathcal{F}$  and all countable linear combinations of the elements in  $\mathcal{F}$ , so

$$\bar{\mathcal{F}} = \mathcal{F} \cup \left\{ \sum_k a_k p_k \mid p_k \in \mathcal{F}, \sum_k a_k = 1, a_k > 0 \right\}. \quad (\text{A3})$$

*Theorem 2.* For any set  $\mathcal{F}$  it holds that

$$Q_n(\mathcal{F}) = Q_n(\bar{\mathcal{F}}). \quad (\text{A4})$$

*Proof.* Introduce the function

$$f : p \in \bar{\mathcal{F}} \mapsto \sum_{m \in \mathbb{N}} R_{m,m} p(m), \quad (\text{A5})$$

where the coefficients

$$R_{m,m} = \begin{cases} P_{m,m} & \text{if } m \leq n, \\ P_{n,m} & \text{otherwise.} \end{cases} \quad (\text{A6})$$

It then holds by definition that  $Q_n(\mathcal{A}) = \inf_{p \in \mathcal{A}} f(p)$  for  $\mathcal{A} \subseteq \bar{\mathcal{F}}$ .

We can assume that  $\bar{\mathcal{F}} \setminus \mathcal{F} \neq \emptyset$ , otherwise the theorem holds trivially. Take  $p \in \bar{\mathcal{F}} \setminus \mathcal{F}$  and compute

$$f(p) = \sum_{k \in \mathbb{N}} a_k f(p_k) \geq \inf_{l \in \mathbb{N}} f(p_l) \sum_{k \in \mathbb{N}} a_k = \inf_{l \in \mathbb{N}} f(p_l). \quad (\text{A7})$$

By definition  $p_l \in \mathcal{F} \forall l \Rightarrow \inf_{l \in \mathbb{N}} f(p_l) \geq Q_n(\mathcal{F})$ , so  $Q_n(\mathcal{F})$  is a lower bound to  $f$  on  $\bar{\mathcal{F}} \setminus \mathcal{F}$ . It therefore follows that  $Q_n(\bar{\mathcal{F}} \setminus \mathcal{F}) \geq Q_n(\mathcal{F})$  and therefore it must be true that  $Q_n(\bar{\mathcal{F}}) = Q_n(\mathcal{F})$ . ■

*Corollary 1.* Let  $\mathcal{A}$  and  $\mathcal{B}$  be subsets of all probability distributions on  $\mathbb{N}$ . If  $\bar{\mathcal{A}} = \bar{\mathcal{B}}$  then it holds that

$$Q_n(\mathcal{A}) = Q_n(\mathcal{B}). \quad (\text{A8})$$

*Proof.* Using Theorem 2 yields

$$Q_n(\mathcal{A}) = Q_n(\bar{\mathcal{A}}) = Q_n(\bar{\mathcal{B}}) = Q_n(\mathcal{B}). \quad (\text{A9})$$

*Theorem 3.* The PNR quality for the set of all probability distributions on  $\mathbb{N}$  is

$$Q_n = \min \left\{ \min_{m \leq n} P_{m,m}, \inf_{m > n} P_{n,m} \right\}. \quad (\text{A10})$$

*Proof.* Introduce the set  $\mathcal{A}$  as the set of all probability distributions on  $\mathbb{N}$  and let

$$\mathcal{F} = \{f_k : m \in \mathbb{N} \mapsto \delta_{k,m} \mid k \in \mathbb{N}\}, \quad (\text{A11})$$

where  $\delta_{k,m}$  is the Kronecker delta. Any  $p \in \mathcal{A}$  can be written as a linear combination of elements in  $\mathcal{F}$  so it holds that  $\bar{\mathcal{F}} = \mathcal{A}$ . Using Corollary 1 yields

$$Q_n = Q_n(\mathcal{A}) = Q_n(\mathcal{F}). \quad (\text{A12})$$

Noticing that the infimum on the set of Kronecker deltas can be written as

$$Q_n(\mathcal{F}) = \min \left\{ \min_{m \leq n} P_{m,m}, \inf_{m > n} P_{n,m} \right\} \quad (\text{A13})$$

completes the proof. ■

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