

Two-time correlators for propagating squeezed microwave fields in transients

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We analyze two-time correlators as the most natural characteristic of a propagating quadrature-squeezed field in the transient regime. The considered system is a parametrically driven resonator with a time-dependent drive. Using a semiclassical approach derived from the input-output theory, we develop a technique for calculation of the two-time correlators, which are directly related to fluctuations of the measured integrated signal. While in the steady state the correlators are determined by three parameters (as for the phase-space ellipse describing a squeezed state), four parameters are necessary in the transient regime. The formalism can be generalized to weakly nonlinear resonators with additional coherent drive. We focus on squeezed microwave fields relevant to the measurement of superconducting qubits; however, our formalism is also applicable to optical systems. The results can be readily verified experimentally.

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I. INTRODUCTION

Squeezed microwave fields (SMFs) [1] have recently become the focus of extensive research efforts related to superconducting quantum computing. This was enabled by a rapid progress in the development of practical superconducting parametric amplifiers [2–6], which have become versatile sources as well as detectors of SMFs. Applications of intracavity and propagating (itinerant) SMFs include qubit readout [7,8], metrology [9–11], continuous-variable entanglement [12,13], control of artificial-atom fluorescence [14], etc. Among other experimental achievements are demonstrations of the dynamical Casimir effect [15–17], tomography of an itinerant SMF [18], and detection of SMF radiation pressure [19].

Besides generation in phase-sensitive parametric amplification, SMFs are also self-generated in the process of circuit QED measurement of superconducting qubits [20,21] due to effective nonlinearity of the resonator induced by coupling with the qubit. Since squeezing affects the qubit measurement error, and for fast readout the steady-state regime is not reached, analysis of squeezing in transients is very important. The corresponding dynamics of the intracavity squeezing has been recently analyzed [22]; however, there is still no theory for transient squeezing of the propagating SMF, which determines the qubit measurement accuracy. Moreover, our extensive search for any papers discussing transient evolution for a resonator-produced propagating squeezed field resulted in only a few remotely related references [23–26], which cannot serve as a starting point in developing a theory to

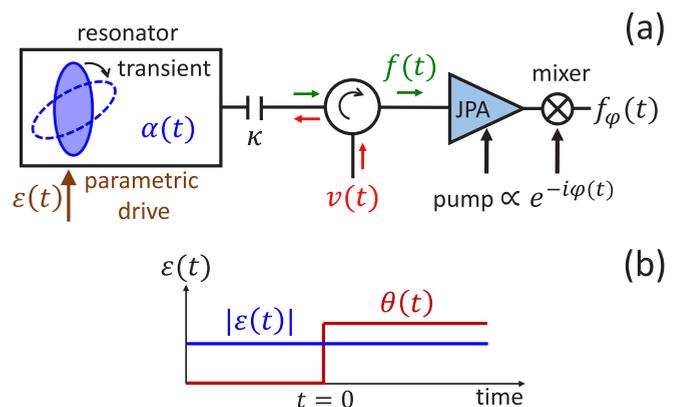


FIG. 1. (a) Analyzed system. Propagating microwave field [described by operator $F(t)$ or complex stochastic variable $f(t)$] is squeezed due to parametric drive of the resonator with changing in time amplitude $\varepsilon(t) = |\varepsilon(t)| e^{i\theta(t)}$. The amplified quadrature phase $\varphi(t)$ also changes in time, producing the noisy output signal $f_\varphi(t)$. The resonator damping rate is κ , and the incoming vacuum noise is described by $v(t)$. (b) An example of the parametric drive change, producing transient evolution of the resonator field [depicted in panel (a)] and of the propagating field.

answer this physically interesting and practically important question. Therefore, while our motivation is to describe fast measurement of superconducting qubits, we need to start with development of the basic theory of propagating SMFs in transients.

In this work, we analyze the *transient regime* of the propagating SMF, generated by a parametrically driven linear resonator [27], as shown in Fig. 1(a). The case of a weakly nonlinear resonator with a coherent drive (as in the qubit measurement) is slightly more complicated but equivalent—see Appendix B. As needed in practical applications (e.g., for optimized qubit readout), we focus on two-time correlators [28] for the quadrature (homodyne) measurement [29,30],

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with quadrature angle φ changing in time (similar to the variational measurement [31,32]). In particular, we find that in transients the dependence of the correlator on two angles φ_1 and φ_2 is characterized by four parameters, in contrast to only three parameters needed in a steady state, as for the ellipse in phase space, which is traditionally used to describe squeezing. Our results can be readily checked experimentally.

II. SYSTEM AND HAMILTONIAN

Let us consider a parametrically modulated resonator [Fig. 1(a)] described in the rotating-wave approximation by the Hamiltonian ($\hbar = 1$)

$$H = \Omega(t) a^\dagger a + \frac{i}{4} [\varepsilon^*(t) a^2 - \varepsilon(t) a^{\dagger 2}], \quad (1)$$

where the resonator detuning $\Omega(t) = \omega_r(t) - \omega_d$ and the parametric drive amplitude $\varepsilon(t) = |\varepsilon(t)| e^{i\theta(t)}$ can depend on time [17,23,24,33] (slowly in comparison with the rotating-frame frequency ω_d , but arbitrarily fast in comparison with evolution in the rotating frame). In the laboratory frame, this Hamiltonian corresponds to the resonator frequency modulation at the double frequency, $\omega_r - |\varepsilon| \sin(2\omega_d t - \theta)$. The more general case of a nonlinear resonator and added coherent drive is discussed in Appendix B.

The propagating microwave field leaking from the resonator, described by operator $F(t)$, is amplified by a *phase-sensitive* amplifier, which amplifies the quadrature phase φ , so that the measured operator is $F_\varphi(t) = [F(t) e^{-i\varphi} + F^\dagger(t) e^{i\varphi}]/2$. In contrast to most previous works, we assume a time-dependent phase $\varphi(t)$. After the mixer, the φ -quadrature measurement produces a *classical* (normalized) fluctuating output signal $f_\varphi(t)$, which in a typical experiment is integrated with a weight function $w(t)$ to produce [31] the measurement result $R = \int w(t) f_\varphi(t) dt$. To analyze fluctuations of R , we need $\langle R^2 \rangle = \iint w(t_1) w(t_2) \langle f_{\varphi_1}(t_1) f_{\varphi_2}(t_2) \rangle dt_1 dt_2$, where $\varphi_k \equiv \varphi(t_k)$. Therefore, in experiments it is important to know the correlator

$$K_{\varphi_1 \varphi_2}(t_1, t_2) \equiv \langle f_{\varphi_1}(t_1) f_{\varphi_2}(t_2) \rangle, \quad (2)$$

which will be the main object analyzed in this paper. Note that in our model, $f_\varphi(t)$ is only noise (amplified and measured propagating squeezed vacuum), i.e., $\langle f_\varphi(t) \rangle = 0$; it is simple to add a nonzero signal by adding a coherent drive into Eq. (1) [cf. Eq. (A18)], but this does not affect fluctuations because of linearity. For simplicity, we assume that the resonator energy decay rate κ is only due to coupling κ_{out} with the transmission line, $\kappa = \kappa_{\text{out}}$ (generalization to the case $\kappa > \kappa_{\text{out}}$ is trivial in the same-temperature case, see below).

In the simplest case of zero detuning ($\Omega = 0$), zero temperature, and time-independent φ and ε , the propagating squeezed vacuum produces the steady-state correlator

$$K_{\varphi\varphi}(0, \tau) = \frac{\delta(\tau)}{4} - \frac{\kappa|\varepsilon|}{4\kappa_+} e^{-\kappa+|\tau|/2} \cos^2(\varphi - \theta/2) + \frac{\kappa|\varepsilon|}{4\kappa_-} e^{-\kappa-|\tau|/2} \sin^2(\varphi - \theta/2), \quad \kappa_\pm = \kappa \pm |\varepsilon|, \quad (3)$$

as can be obtained via the conventional input-output formalism [34,35], assuming $|\varepsilon| < \kappa$. Correspondingly, the integrated correlator for $\varphi = \theta/2$ is $\int_{-\infty}^{\infty} K_{\varphi\varphi}(0, \tau) d\tau = (1/4)(\kappa_-/\kappa_+)^2$, so it is squeezed compared with the vacuum value of $1/4$, while for $\varphi = (\theta + \pi)/2$ it is antisqueezed: $\int_{-\infty}^{\infty} K_{\varphi\varphi}(0, \tau) d\tau = (1/4)(\kappa_+/\kappa_-)^2$.

Note that dependence of the correlator $K_{\varphi\varphi}(0, \tau)$ on φ is described by three real parameters. Also note that since in the steady state $K_{\varphi\varphi}(0, \tau)$ depends only on the time difference $\tau \equiv t_2 - t_1$, it is natural to use the Fourier transform, so the squeezing is usually analyzed in terms of the squeezing spectrum [29,36] $S_\varphi(\omega) \equiv 4 \int_{-\infty}^{\infty} e^{-i\omega\tau} K_{\varphi\varphi}(0, \tau) d\tau$. However, during transients the two-time correlator $K_{\varphi_1 \varphi_2}(t_1, t_2)$ depends on both times, so introducing a time-dependent spectrum $S_{\varphi_1 \varphi_2}(t_1, \omega)$ via a similar Fourier transform is not natural. This is why we focus on $K_{\varphi_1 \varphi_2}(t_1, t_2)$.

III. SEMICLASSICAL MODEL FOR MEASURED FLUCTUATIONS

Instead of using the conventional input-output formalism [34], we will use a simpler semiclassical stochastic model [37] to analyze the temporal correlations of the output signal $f_\varphi(t)$. As shown in Appendix A, the correlators obtained using this model are *exact* for our linear system (1); the model is still a good approximation for a weakly nonlinear resonator.

In this semiclassical model, the fluctuation of the (quantum) propagating output field $F(t)$ is treated as a complex-valued stochastic variable,

$$f(t) = -v(t) + \sqrt{\kappa} \alpha(t), \quad (4)$$

where the complex-valued stochastic variable $\alpha(t)$ describes fluctuations of the intracavity field, while the incoming vacuum noise [Fig. 1(a)] is described by a complex-valued Gaussian noise $v(t)$ with two-time correlators

$$\langle v(t) v^*(t') \rangle = (\bar{n}_b + 1/2) \delta(t - t'), \quad \langle v(t) v(t') \rangle = 0, \quad (5)$$

where $\langle \dots \rangle$ denotes ensemble average and $\bar{n}_b = [\exp(\omega_r/T) - 1]^{-1}$ is the average number of bath thermal photons. For brevity of formulas, we will assume the temperature T to be zero (so $\bar{n}_b = 0$); however, for $T \neq 0$ all correlators in this paper can be simply multiplied by the factor $1 + 2\bar{n}_b$.

The intracavity field fluctuation $\alpha(t)$ for a parametrically modulated resonator (1) evolves as

$$\dot{\alpha}(t) = -\left(\frac{\kappa}{2} + i\Omega\right)\alpha(t) - \frac{\varepsilon(t)}{2}\alpha^*(t) + \sqrt{\kappa} v(t). \quad (6)$$

Note that in our normalization, $|\alpha|^2$ corresponds to the number of photons in the resonator, while $|f|^2$ corresponds to the propagating number of photons per second. The decay rate κ is frequency independent, i.e., we use the Markovian approximation [28]. The term $-\varepsilon\alpha^*/2$ describes effective increase of κ by $|\varepsilon|$ for the quadrature phase $\varphi = \theta/2$ and its decrease by $|\varepsilon|$ for $\varphi = (\theta + \pi)/2$.

The output signal $f_\varphi(t)$ from the quadrature measurement is given by the real-valued stochastic variable

$$f_\varphi(t) = \text{Re}[e^{-i\varphi(t)} f(t)], \quad (7)$$

so the correlator of interest (2) can be calculated as

$$K_{\varphi_1\varphi_2}(t_1, t_2) = \frac{1}{2} \text{Re}[K_{ff}(t_1, t_2) e^{-i(\varphi_1+\varphi_2)}] + \frac{1}{2} \text{Re}[K_{ff^*}(t_1, t_2) e^{-i(\varphi_1-\varphi_2)}], \quad (8)$$

$$K_{ff}(t_1, t_2) \equiv \langle f(t_1) f(t_2) \rangle, \quad (9)$$

$$K_{ff^*}(t_1, t_2) \equiv \langle f(t_1) f^*(t_2) \rangle. \quad (10)$$

We see that for given t_1 and t_2 , the dependence of $K_{\varphi_1\varphi_2}(t_1, t_2)$ on φ_1 and φ_2 is described by *four real parameters* [e.g., $\text{Re}(K_{ff})$, $\text{Im}(K_{ff})$, $\text{Re}(K_{ff^*})$, and $\text{Im}(K_{ff^*})$]. As will be discussed later, in the steady state there are only three independent real parameters because K_{ff^*} in this case is real. Note that K_{ff} and K_{ff^*} obviously satisfy the symmetry relations [38],

$$K_{ff}(t, t') = K_{ff}(t', t), \quad K_{ff^*}(t, t') = [K_{ff^*}(t', t)]^*. \quad (11)$$

Now let us calculate the correlators $K_{ff}(t_1, t_2)$ and $K_{ff^*}(t_1, t_2)$ using the semiclassical model (4)–(6). Because of the symmetry, it is sufficient to assume $t_2 > t_1$ (the δ -function contribution to K_{ff^*} at $t_1 = t_2$ is discussed below). Let us introduce the column vector containing both correlators, $\mathbf{K}(t_1, t_2) = [K_{ff}(t_1, t_2), K_{ff^*}(t_1, t_2)]^T$. From Eq. (4) we obtain

$$\mathbf{K}(t_1, t_2) = \kappa \begin{bmatrix} \langle \alpha(t_2) \alpha(t_1) \rangle \\ \langle \alpha^*(t_2) \alpha(t_1) \rangle \end{bmatrix} - \sqrt{\kappa} \begin{bmatrix} \langle \alpha(t_2) v(t_1) \rangle \\ \langle \alpha^*(t_2) v(t_1) \rangle \end{bmatrix}, \quad (12)$$

since $\langle v(t_2) \alpha(t_1) \rangle = \langle v^*(t_2) \alpha(t_1) \rangle = 0$ because of causality. Now using Eq. (6), we find the evolution of $\mathbf{K}(t_1, t_2)$ as a function of t_2 ,

$$\partial \mathbf{K}(t_1, t_2) / \partial t_2 = M(t_2) \mathbf{K}(t_1, t_2), \quad (13)$$

where the matrix $M(t)$ describes the ensemble-averaged evolution of the vector $(\alpha, \alpha^*)^T$ following from Eq. (6) without the noise term (contribution from the noise v averages to zero because of linearity),

$$M(t) = \begin{bmatrix} -\kappa/2 - i\Omega & -\varepsilon(t)/2 \\ -\varepsilon^*(t)/2 & -\kappa/2 + i\Omega \end{bmatrix}. \quad (14)$$

Note that $M(t)$ is Hermitian only if $\Omega = 0$.

To find the initial condition for Eq. (13) at $t_2 = t_1 + 0$, we use Eq. (12) with $\langle \alpha(t_1 + 0) v(t_1) \rangle = 0$ and $\langle \alpha^*(t_1 + 0) v(t_1) \rangle = \sqrt{\kappa}/2$, where the last equation follows from Eq. (6): $\alpha^*(t_1 + dt) \approx \alpha^*(t_1) + \sqrt{\kappa} v^*(t_1) dt$, while $\langle |v(t_1)|^2 \rangle = 1/(2 dt)$ from Eq. (5). Therefore,

$$\mathbf{K}(t_1, t_1 + 0) = \kappa \begin{bmatrix} \langle \alpha^2(t_1) \rangle \\ \langle |\alpha^2(t_1)| \rangle - 1/2 \end{bmatrix}. \quad (15)$$

The solution of Eq. (13) with the initial condition (15) can be expressed via the Green's function 2×2 matrix $G(t|t_{\text{in}})$, defined as

$$\partial G(t|t_{\text{in}}) / \partial t = M(t) G(t|t_{\text{in}}), \quad G(t_{\text{in}}|t_{\text{in}}) = \mathbb{1}. \quad (16)$$

Thus, for \mathbf{K} (now expressed via K_{ff} and K_{ff^*}) we obtain

$$\begin{bmatrix} K_{ff}(t_1, t_2) \\ K_{ff^*}(t_1, t_2) \end{bmatrix} = \kappa G(t_2|t_1) \begin{bmatrix} \langle \alpha^2(t_1) \rangle \\ \langle |\alpha^2(t_1)| \rangle - 1/2 \end{bmatrix}. \quad (17)$$

To complete the calculation of K_{ff} and K_{ff^*} , we need the second moments of the intracavity field fluctuations, $\langle \alpha^2(t_1) \rangle$ and $\langle |\alpha^2(t_1)| \rangle$. Following the result of Ref. [22], they can be obtained as a solution of a system of four first-order differential equations. Alternatively, they can be obtained from Eq. (6) as (see [39,40])

$$\begin{bmatrix} \langle |\alpha^2(t_1)| \rangle & \langle \alpha^2(t_1) \rangle \\ \langle \alpha^{*2}(t_1) \rangle & \langle |\alpha^2(t_1)| \rangle \end{bmatrix} = \frac{\kappa}{2} \int_{t_0}^{t_1} G(t_1|t') G^\dagger(t_1|t') dt' + G(t_1|t_0) \begin{bmatrix} \langle |\alpha^2(t_0)| \rangle & \langle \alpha^2(t_0) \rangle \\ \langle \alpha^{*2}(t_0) \rangle & \langle |\alpha^2(t_0)| \rangle \end{bmatrix} G^\dagger(t_1|t_0), \quad (18)$$

where $\langle \alpha^2(t_0) \rangle = \text{Tr}[a^2 \rho(t_0)]$, $\langle |\alpha^2(t_0)| \rangle = \text{Tr}[a^\dagger a \rho(t_0)] + 1/2$, and $\rho(t_0)$ is a given intracavity state at an initial time t_0 , for which we assume a positive Wigner function (for $t_0 \rightarrow -\infty$, the initial state is irrelevant). This assumption allows us to regard $\alpha(t)$ as a classical complex-valued stochastic variable—see Appendix A.

Equations (16)–(18) are the main result of this paper. Using these equations with $M(t)$ defined in Eq. (14), we can find the correlators K_{ff} and K_{ff^*} , which can then be used to obtain the main correlator of interest $K_{\varphi_1\varphi_2}(t_1, t_2)$ via Eq. (8). As mentioned above, in the case of a nonzero bath temperature, the correlators should be multiplied by $1 + 2\bar{n}_b$.

At $t_2 = t_1$, the correlator K_{ff^*} contains the singular contribution $(\bar{n}_b + 1/2) \delta(t_2 - t_1)$, as follows from Eqs. (4) and (5), while K_{ff} does not have a singularity. Since in this case $\varphi_1 = \varphi_2$, the correlator $K_{\varphi_1\varphi_2}(t_1, t_2)$ has the singular contribution $(1/4)(1 + 2\bar{n}_b) \delta(t_2 - t_1)$. In a real experiment, at $t_2 \approx t_1$ there is also a contribution from the additional noise of a not-quantum-limited amplifier.

In the derivation we assumed that energy decay in the resonator is due only to coupling with the outgoing transmission line, i.e., $\kappa = \kappa_{\text{out}}$. If this is not the case, the correlators $K_{\varphi_1\varphi_2}$, K_{ff} , and K_{ff^*} for $t_1 \neq t_2$ should be simply multiplied by the factor $\kappa_{\text{out}}/\kappa$. This can be shown by repeating the derivation with Eq. (4) replaced by $f = -v + \sqrt{\kappa_{\text{out}}}\alpha$ and Eq. (6) replaced by $\dot{\alpha} = -(\kappa/2 + i\Omega)\alpha - (\varepsilon/2)\alpha^* + \sqrt{\kappa_{\text{out}}}v + \sqrt{\kappa - \kappa_{\text{out}}}v_{\text{add}}$, where the additional uncorrelated noise $v_{\text{add}}(t)$ satisfies Eq. (5) with the same temperature. Alternatively, the multiplication of the correlators by $\kappa_{\text{out}}/\kappa$ is rather obvious because the system is then equivalent to adding a beamsplitter with transmission amplitude $\sqrt{\kappa_{\text{out}}/\kappa}$ to the outgoing transmission line (after the circulator) in Fig. 1(a). Note that the singularity of correlators at $t_2 = t_1$ does not change when $\kappa_{\text{out}} \neq \kappa$, because of the additional noise.

Even though our results have been derived for the case of a linear parametrically driven resonator (1), we emphasize that they remain practically the same if a weak nonlinearity is added to the resonator, as well as a coherent drive (see

Appendix B). In this case the evolution of fluctuations should be linearized in the vicinity of the classical evolution (this modifies the matrix M) and we need to use the Gaussian approximation.

IV. STEADY-STATE REGIME

In the steady state we can assume that the parametric drive amplitude ε does not depend on time (as well as parameters Ω and κ). This is the case considered in the literature (e.g., [29,34,37]). Using our formalism (with $\bar{n}_b = 0$), we can easily find the Green's function $G(t|t_{in})$ by finding eigenvalues and eigenvectors of the matrix M . Then from Eqs. (17) and (18) we obtain

$$K_{ff}(0, \tau) = -\frac{\kappa\varepsilon}{4} \left[\left(1 - \frac{2i\Omega}{\varepsilon}\right) \frac{e^{-\kappa-|\tau|/2}}{\kappa_-} + \left(1 + \frac{2i\Omega}{\varepsilon}\right) \frac{e^{-\kappa+|\tau|/2}}{\kappa_+} \right], \quad (19)$$

$$K_{ff^*}(0, \tau) = \frac{\delta(\tau)}{2} + \frac{\kappa|\varepsilon|^2}{4\varepsilon} \left(\frac{e^{-\kappa-|\tau|/2}}{\kappa_-} - \frac{e^{-\kappa+|\tau|/2}}{\kappa_+} \right), \quad (20)$$

where $\kappa_{\pm} = \kappa \pm \varepsilon$ and $\varepsilon = \sqrt{|\varepsilon|^2 - 4\Omega^2}$ if $|\Omega| < |\varepsilon|/2$ (overdamped case) or $\varepsilon = i\sqrt{4\Omega^2 - |\varepsilon|^2}$ if $|\Omega| > |\varepsilon|/2$ (underdamped case). The condition of stability is obviously $|\varepsilon|^2 < \kappa^2 + 4\Omega^2$. The singular contribution $\delta(\tau)/2$ added into Eq. (20) follows from Eqs. (4) and (5).

We see that in the steady state, $K_{ff^*}(0, \tau)$ is always real. Therefore, the squeezing is determined by three real parameters (which depend on τ), in contrast to four parameters in the general (transient) case.

V. GENERAL FORM OF TWO-TIME CORRELATOR

A convenient way of introducing the four real parameters (A , B , ϕ , and ψ) is by rewriting Eq. (8) as

$$K_{\varphi_1\varphi_2}(t_1, t_2) = A \cos(\varphi_1 - \phi) \cos(\varphi_2 - \psi) + B \sin(\varphi_1 - \phi) \sin(\varphi_2 - \psi) + \delta(t_2 - t_1)/4, \quad (21)$$

where we explicitly added the singular term (note that $\varphi_1 = \varphi_2$ when $t_1 = t_2$), and the parameters A , B , ϕ , and ψ (all depending on t_1 and t_2) can be obtained from the equations $(A - B)e^{i(\phi+\psi)} = K_{ff}(t_1, t_2)$ and $(A + B)e^{i(\phi-\psi)} = K_{ff^*}(t_1, t_2) - \delta(t_2 - t_1)/2$. As discussed above, in the steady state K_{ff^*} is real, and therefore $\phi = \psi$, thus again leaving only three independent real parameters.

Note that for $\varphi_1 = \varphi_2$, the correlator $K_{\varphi\varphi}(t_1, t_2)$ as a function of φ is always (even in the transient regime) parameterized by only three parameters. In the steady state, measurement of these three parameters fully defines Eq. (21) since $\phi = \psi$, thus predicting the correlator for $\varphi_1 \neq \varphi_2$ as well. However, in the general (transient) case this is impossible because of one extra parameter. Comparing Eqs. (3) and (21), it is tempting to interpret the angles ϕ and ψ in Eq. (21) as squeezing directions at t_1 and t_2 , respectively. However, this interpretation is incorrect, because in general both ϕ and ψ depend on both t_1 and t_2 , not only as $\phi(t_1)$ and $\psi(t_2)$.

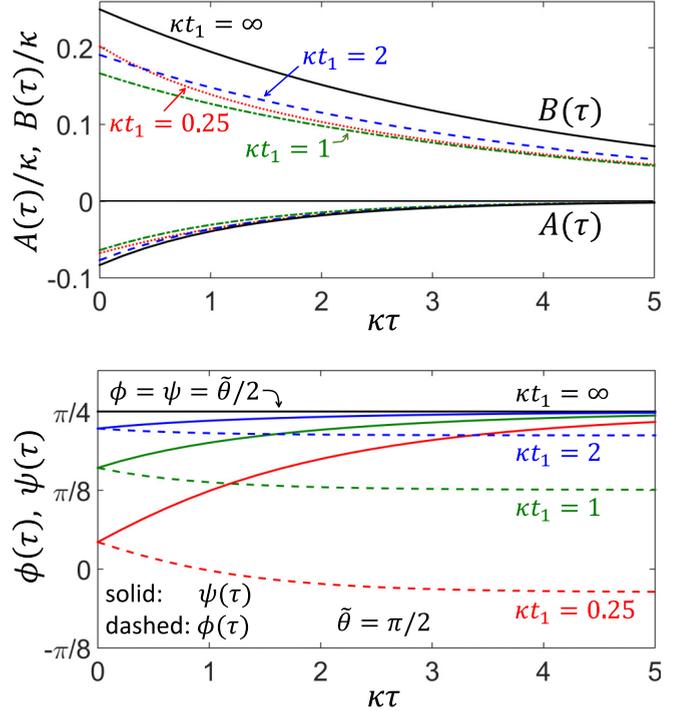


FIG. 2. Parameters A and B (top panel) and ϕ and ψ (bottom panel) as functions of $\tau = t_2 - t_1$ for several values of time t_1 passed after the abrupt change of the parametric drive shown in Fig. 1(b), $\kappa t_1 = 0.25, 1, 2, \infty$. In the steady state, $\phi = \psi$. We use $\tilde{\theta} = \pi/2$, $|\varepsilon|/\kappa = 0.5$, and $\Omega = 0$.

VI. EXAMPLE OF TRANSIENT EVOLUTION WITH $\phi \neq \psi$

To observe experimentally the discussed features of the squeezing in transients, the simplest case is to use no detuning ($\Omega = 0$) and to change abruptly the parametric drive amplitude $|\varepsilon(t)| e^{i\theta(t)}$ (with a reasonably long cycle to accumulate ensemble statistics). If only $|\varepsilon(t)|$ is changing [23], then the dynamics is still not very interesting (squeezing is still characterized by only three parameters—see Appendix C3). Therefore, the natural choice is to keep $|\varepsilon|$ constant but to abruptly change the phase $\theta(t)$, as shown in Fig. 1(b). Let us assume that $\theta(t) = 0$ for $t < 0$ and $\theta(t) = \tilde{\theta}$ for $t > 0$. Then solving Eqs. (16)–(18) we obtain

$$K_{ff}(t_1, t_1 + \tau) = \kappa [P_- + P_+] e^{i\tilde{\theta}}, \quad (22)$$

$$K_{ff^*}(t_1, t_1 + \tau) = \kappa [P_+ - P_-], \quad (23)$$

$$P_{\pm} = \left\{ \frac{\kappa|\varepsilon|}{4(\kappa^2 - |\varepsilon|^2)} [(1 - \cos \tilde{\theta}) e^{-\kappa_{\pm} t_1} + i \sin \tilde{\theta} e^{-\kappa t_1}] - \frac{|\varepsilon|}{4\kappa_{\pm}} \right\} \times e^{-\kappa_{\pm} \tau/2}, \quad (24)$$

where $\kappa_{\pm} = \kappa \pm |\varepsilon|$ and $\tau > 0$. Figure 2 shows the corresponding parameters A , B , ϕ , and ψ in Eq. (21) as functions of τ for several values of t_1 . As expected, we see that $\phi \neq \psi$, except in the steady state ($t_1 \rightarrow \infty$).

Thus, in this example the steady-state squeezing is described by three parameters: $A(\tau)$, $B(\tau)$, and ϕ (not depending on τ), while the transient squeezing is described by four parameters: A , B , ϕ , and ψ , which all depend on both τ and t_1 . The same conclusion of three versus four parameters

remains true if the correlator $K_{\varphi_1\varphi_2}(t_1, t_1 + \tau)$ is integrated over τ or if we apply the Fourier transform over τ (to obtain a time-dependent squeezing spectrum).

To check our results experimentally (especially the presence of the fourth parameter), it is natural to use a narrower-band flux-pumped degenerate Josephson parametric amplifier (JPA) as the parametrically driven resonator and another broader-band degenerate JPA as a phase-sensitive amplifier. For $\kappa/2\pi \sim 1\text{--}3$ MHz (and $|\varepsilon|/\kappa \sim 0.5$), it should be easy to change the pump phase abruptly on the scale of κ^{-1} , while the amplifier still can resolve the transients. The amplified phase in this case should also change in time, so that the correlator $K_{\varphi_1\varphi_2}(t_1, t_2)$ is obtained via ensemble averaging for all phases φ_1 and φ_2 . Note that experimentally it is easier to use a phase-preserving (nondegenerate) amplifier in Fig. 1(a) instead of the phase-sensitive amplifier with time-varying amplified quadrature. As discussed in Appendix C 1, all our results remain the same for a phase-preserving amplifier, except the singular contribution to K_{ff^*} becomes twice as large. (In a real experiment the singular contribution broadens because of the finite bandwidth of the amplifier.)

VII. CONCLUSIONS

We have developed the theory for analyzing the squeezing of a propagating microwave field in the transient regime. The most natural way to characterize squeezing in this case is via the two-time correlators $K_{\varphi_1\varphi_2}(t_1, t_2)$ of the detector output with different quadrature angles φ_1 and φ_2 , since in experiments these correlators are directly related to the fluctuations of the integrated signal. In our theory the correlators $K_{\varphi_1\varphi_2}$ are expressed via the field fluctuation correlators K_{ff} and K_{ff^*} , for which the differential equations have been derived using the semiclassical model. For practical applications, the amplified phase $\varphi(t)$ will need to be optimized using the ideas of the variational measurement. Our theory is equally applicable to squeezing in optics (then phase-sensitive amplification is simply replaced with homodyne detection), though it is more challenging to realize transients of optical squeezing experimentally.

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APPENDIX A: CORRELATORS FOR PROPAGATING SQUEEZED FIELD—SEMICLASSICAL MODEL

In this section we discuss a semiclassical description of the quantum fluctuations of the propagating field leaking out of the cavity. The main advantage of this approach is that it enables us to calculate the temporal correlators of the measured quadrature signals using classical stochastic equations. The semiclassical model is derived from the conventional input-output formalism [34,35]. We show that for a linear resonator, the correlators calculated within the semiclassical model are *exactly* equal to the correlators calculated in the standard quantum way.

1. Correlators in the input-output formalism

In the standard quantum approach, the correlator $K_{\varphi_1\varphi_2}(t_1, t_2)$ for the measured normalized quadrature signal $f_\varphi(t)$ with a time-varying quadrature phase $\varphi(t)$ is given by the symmetrized combination

$$K_{\varphi_1\varphi_2}^q(t_1, t_2) = \frac{\langle F_{\varphi_1}(t_1) F_{\varphi_2}(t_2) \rangle + \langle F_{\varphi_2}(t_2) F_{\varphi_1}(t_1) \rangle}{2}, \quad (\text{A1})$$

where $F_\varphi(t)$ is the quadrature operator for the propagating field. It is related to the field operator $F(t)$ as

$$F_\varphi(t) = \frac{1}{2}[e^{-i\varphi}F(t) + e^{i\varphi}F^\dagger(t)], \quad (\text{A2})$$

and the Heisenberg picture is used for all operators.

In the input-output theory [34], the field leaked from the cavity is written as

$$F(t) = -V(t) + \sqrt{\kappa}a(t), \quad (\text{A3})$$

where $a(t)$ is the annihilation operator for the intracavity mode, for simplicity we assume $\kappa = \kappa_{\text{out}}$, and the operator $V(t)$ of the incoming vacuum noise satisfies the commutation relations

$$[V(t), V^\dagger(t')] = \delta(t - t'), \quad [V(t), V(t')] = 0, \quad (\text{A4})$$

while the average values of the products are

$$\langle V^\dagger(t)V(t') \rangle = \bar{n}_b \delta(t - t'), \quad \langle V(t)V(t') \rangle = 0, \quad (\text{A5})$$

where $\bar{n}_b = [\exp(\omega_r/T) - 1]^{-1}$ depends on the bath temperature T . In this section, we will assume $T = 0$ ($\bar{n}_b = 0$), but generalization to a nonzero temperature is rather straightforward. The evolution of the operator $a(t)$ is [34,35]

$$\dot{a}(t) = -\frac{\kappa}{2}a(t) + i[H(t), a(t)] + \sqrt{\kappa}V(t). \quad (\text{A6})$$

It is possible to show [37] that the propagating field $F(t)$ satisfies the same commutation relations as $V(t)$,

$$[F(t), F^\dagger(t')] = \delta(t - t'), \quad [F(t), F(t')] = 0, \quad (\text{A7})$$

and, therefore, the correlator (A1) for quadratures can be written without symmetrization,

$$K_{\varphi_1\varphi_2}^q(t_1, t_2) = \langle F_{\varphi_1}(t_1) F_{\varphi_2}(t_2) \rangle. \quad (\text{A8})$$

Using Eq. (A2), we can write the correlator as

$$\begin{aligned} \langle F_{\varphi_1}(t_1) F_{\varphi_2}(t_2) \rangle &= \frac{1}{4}[\langle F(t_1) F(t_2) \rangle e^{-i(\varphi_1 + \varphi_2)} \\ &\quad + \langle F(t_1) F^\dagger(t_2) \rangle e^{i(\varphi_2 - \varphi_1)} \\ &\quad + \langle F^\dagger(t_1) F(t_2) \rangle e^{i(\varphi_1 - \varphi_2)} \\ &\quad + \langle F(t_2) F(t_1) \rangle^* e^{i(\varphi_1 + \varphi_2)}], \end{aligned} \quad (\text{A9})$$

where the two-time averages are [34]

$$\langle F(t_1) F(t_2) \rangle = \kappa \langle \mathcal{T}[a(t_2) a(t_1)] \rangle, \quad (\text{A10a})$$

$$\langle F(t_1) F^\dagger(t_2) \rangle = \delta(t_1 - t_2) + \kappa \langle a^\dagger(t_2) a(t_1) \rangle, \quad (\text{A10b})$$

$$\langle F^\dagger(t_2) F(t_1) \rangle = \kappa \langle a^\dagger(t_2) a(t_1) \rangle. \quad (\text{A10c})$$

The relations (A10a)–(A10c) are the standard results of the input-output theory; they are valid for arbitrary t_1 and t_2 . In Eq. (A10a), the time-ordering operator \mathcal{T} is defined

in the usual way: $\mathcal{T}[A(t_1)B(t_2)] = B(t_2)A(t_1)$ if $t_1 < t_2$ and $\mathcal{T}[A(t_1)B(t_2)] = A(t_1)B(t_2)$ if $t_1 > t_2$.

Without loss of generality we assume $t_1 < t_2$. Then, using Eqs. (A8)–(A10), the correlator (A1) for the measured quadrature signal can be written as

$$K_{\varphi_1\varphi_2}^q(t_1, t_2) = \frac{\kappa}{2} \operatorname{Re}[\langle a(t_2)a(t_1) \rangle e^{-i(\varphi_1+\varphi_2)} + \langle a^\dagger(t_2)a(t_1) \rangle e^{-i(\varphi_1-\varphi_2)}]. \quad (\text{A11})$$

2. Intracavity correlators via Wigner representation

The two-time averages $\langle a(t_2)a(t_1) \rangle$ and $\langle a^\dagger(t_2)a(t_1) \rangle$ in Eq. (A11) can be calculated using the standard quantum-regression formulas [34]

$$\langle a(t_2)a(t_1) \rangle = \operatorname{Tr}[a\tilde{\rho}(t_2)], \quad (\text{A12a})$$

$$\langle a^\dagger(t_2)a(t_1) \rangle = \operatorname{Tr}[a^\dagger\tilde{\rho}(t_2)], \quad (\text{A12b})$$

where $\tilde{\rho}(t)$ is an unphysical (in particular, non-Hermitian) density matrix, which has the initial condition

$$\tilde{\rho}(t_1) = a\rho(t_1), \quad (\text{A13})$$

relating it to the physical density matrix $\rho(t)$ of the resonator at the moment t_1 , while in between t_1 and t_2 it evolves in the same way as ρ [cf. Eq. (A6)],

$$\dot{\rho} = -i[H, \rho] + \kappa(\rho a^\dagger - a^\dagger \rho/2 - \rho a^\dagger/2), \quad (\text{A14})$$

so that the equation for $\tilde{\rho}$ is Eq. (A14) with ρ replaced by $\tilde{\rho}$. Note that in Eqs. (A12) the left-hand sides assume the Heisenberg picture, while the right-hand sides use the Schrödinger picture.

The justification of our semiclassical model for the output field can be based on the Wigner representation of the resonator density matrix $\rho(t)$. Instead of the standard Wigner function, depending on x and p [corresponding to $(a + a^\dagger)/2$

and $(a - a^\dagger)/2i$], we will use a slight modification, as in Refs. [34,37], in which the Wigner function depends on $\alpha = x + ip$ and $\alpha^* = x - ip$. So we will use the Wigner transformation \mathcal{W} , defined as

$$\begin{aligned} \mathcal{W}[\rho(t)] &= W(\alpha, \alpha^*, t) \\ &= \int \operatorname{Tr}[\rho(t) \exp(z a^\dagger - z^* a)] e^{z^* \alpha - z \alpha^*} \frac{d^2 z}{\pi^2}, \end{aligned} \quad (\text{A15})$$

where $d^2 z \equiv d(\operatorname{Re} z) d(\operatorname{Im} z)$ corresponds to the integration over the complex phase space. Note that W is real if ρ is Hermitian (then complex conjugation of W reduces to the transformation $z \rightarrow -z$). However, W is complex if ρ is non-Hermitian. The normalization following from Eq. (A15) is $\int W(\alpha, \alpha^*, t) d^2 \alpha = \operatorname{Tr}[\rho(t)]$. Note that the definition (A15) allows us to think of α and α^* as independent variables which are not necessarily conjugate to each other (even though the final expressions are evaluated for conjugate values). For example, the partial derivative $\partial_\alpha W \equiv \partial W / \partial \alpha$ is given by Eq. (A15) with extra factor z^* inside the integral, while $\partial_{\alpha^*} W$ produces the factor $-z$ inside the integral.

To use the Wigner representation in Eq. (A12), we need to apply the Wigner transformation (A15) to the nonphysical density matrix $\tilde{\rho}$; for that we will need the relations [34] (which are straightforward to derive)

$$\begin{aligned} \mathcal{W}[a\rho] &= \left(\alpha + \frac{1}{2}\partial_{\alpha^*}\right) W, & \mathcal{W}[\rho a^\dagger] &= \left(\alpha^* + \frac{1}{2}\partial_\alpha\right) W, \\ \mathcal{W}[a^\dagger\rho] &= \left(\alpha^* - \frac{1}{2}\partial_\alpha\right) W, & \mathcal{W}[\rho a] &= \left(\alpha - \frac{1}{2}\partial_{\alpha^*}\right) W. \end{aligned} \quad (\text{A16})$$

Then using Eqs. (A12), (A13), and (A16), we can express the two-time averages in Eq. (A11) as [34]

$$\begin{aligned} \langle a(t_2)a(t_1) \rangle &= \int d^2\alpha_1 d^2\alpha_2 \left(\alpha_2 + \frac{1}{2}\partial_{\alpha_2^*}\right) W(2|1) \left(\alpha_1 + \frac{1}{2}\partial_{\alpha_1^*}\right) W(1) \\ &= \int \alpha_2 W(2|1) \left(\alpha_1 + \frac{1}{2}\partial_{\alpha_1^*}\right) W(1) d^2\alpha_1 d^2\alpha_2, \end{aligned} \quad (\text{A17a})$$

$$\langle a^\dagger(t_2)a(t_1) \rangle = \int \alpha_2^* W(2|1) \left(\alpha_1 + \frac{1}{2}\partial_{\alpha_1^*}\right) W(1) d^2\alpha_1 d^2\alpha_2, \quad (\text{A17b})$$

where $W(1) \equiv W(\alpha_1, \alpha_1^*, t_1)$ is the Wigner function at time t_1 and $W(2|1) \equiv W(\alpha_2, \alpha_2^*, t_2 | \alpha_1, \alpha_1^*, t_1)$ is the propagator for the Wigner function from time t_1 to time t_2 , which can be obtained from Eq. (A14). Note that the first and second forms of Eq. (A17a) differ only by the term $(1/2)\partial_{\alpha_2^*}$, which gives zero contribution after integration by parts over the whole space of α_2 ; the same cancellation is used in Eq. (A17b).

To find the propagator $W(2|1)$, we need to convert the evolution equation (A14) into the Wigner representation. The conversion is relatively simple for a linear resonator. Let us consider the rotating-frame Hamiltonian

$$H = \Omega a^\dagger a + \frac{i}{4}[\varepsilon^* a^2 - \varepsilon a^{\dagger 2}] + \varepsilon_c^* a + \varepsilon_c a^\dagger, \quad (\text{A18})$$

which is more general than the Hamiltonian (1) in the main text due to addition of a coherent drive with amplitude ε_c , while the parametric drive still has the amplitude ε . For example, ε_c can represent the input signal in a parametric amplifier. Note that in the case $\varepsilon_c \neq 0$, the intracavity and outgoing fields include the signal component, in contrast to the main text, where we considered only noise. Also note that all parameters in the Hamiltonian (Ω , ε , ε_c) as well as the decay rate κ can depend on time. (This dependence should be slow in comparison with the resonator frequency ω_r but can be arbitrarily fast compared with evolution in the rotating frame.)

Applying the Wigner transformation (A15) to Eq. (A14) and using Eqs. (A16) and (A18), we obtain the following

evolution equation for the Wigner function [34,37]:

$$\partial_t W = -\partial_\alpha(\Lambda W) - \partial_{\alpha^*}(\Lambda^* W) + \frac{\kappa}{2} \partial_{\alpha\alpha^*}^2 W, \quad (\text{A19})$$

$$\Lambda(\alpha, \alpha^*, t) = -\left(\frac{\kappa}{2} + i\Omega\right)\alpha - \frac{\varepsilon}{2}\alpha^* - i\varepsilon_c. \quad (\text{A20})$$

Note that in this derivation, the relations (A16) should be applied several times, e.g.,

$$\begin{aligned} \mathcal{W}[a^2\rho] &= \left(\alpha + \frac{1}{2}\partial_{\alpha^*}\right)^2 W, \quad \mathcal{W}[\rho a^2] = \left(\alpha - \frac{1}{2}\partial_{\alpha^*}\right)^2 W, \\ \mathcal{W}[a^\dagger a\rho] &= \left(\alpha^* - \frac{1}{2}\partial_\alpha\right)\left(\alpha + \frac{1}{2}\partial_{\alpha^*}\right) W. \end{aligned} \quad (\text{A21})$$

Most importantly, Eq. (A19) for the Wigner function evolution has the same form as the Fokker-Planck equation [34,41] for the evolution of a probability distribution, in which $\Lambda(\alpha, \alpha^*, t)$ has the physical meaning of a drift velocity. Moreover, Eq. (A20) for Λ has the same form as for the evolution of a classical field in the cavity. This similarity between the Wigner function and the classical probability distribution will be the basis of the proof that the correlator (A1) can be calculated within the semiclassical model.

Note that for the averages (A12) we need to consider the evolution of an unphysical (non-Hermitian) density matrix $\tilde{\rho}$ and therefore unphysical (complex) Wigner function W . Nevertheless, for the propagator $W(2|1)$ in Eqs. (A17), it is sufficient to consider physical (real) W in Eq. (A19). This is because of the linearity of Eq. (A14) (linearity of quantum mechanics), so that $W(2|1)$ is just the Green's function of Eq. (A19) with initial condition $W(\alpha_2, \alpha_2^*, t_1|\alpha_1, \alpha_1^*, t_1) = \delta^2(\alpha_1 - \alpha_2) \equiv \delta[\text{Re}(\alpha_2 - \alpha_1)]\delta[\text{Im}(\alpha_2 - \alpha_1)]$.

Moreover, in the calculation of the propagator $W(2|1)$ using Eq. (A19), the Wigner function W remains positive, since the initial condition $\delta^2(\alpha_1 - \alpha_2)$ is positive and Eq. (A19) is a second-order partial differential equation. (This fact follows from Pawula's theorem [41]; normalization for the Wigner function is preserved automatically.) Therefore, in this case the Wigner function W can be interpreted as a classical probability distribution in phase space, which evolves due to the drift Λ and diffusion (see below). Similarly, $W(1)$ in Eq. (A17) is also positive (and therefore can be interpreted as a classical probability distribution) if the Wigner function at some earlier time $t_0 < t_1$ is positive. This is what we assume below, for example, assuming that in a distant past the evolution started from vacuum (which has positive Wigner function).

Note that if at some time moment t_0 the Wigner function is Gaussian (for example, the resonator state is vacuum in a distant past), then it will remain Gaussian at any later time $t > t_0$ [this can be shown explicitly using Eqs. (A19) and (A20)]. However, in our proof of the semiclassical model for the linear case (S18), we *do not need Gaussianity* of the resonator state; we only need positivity of the Wigner function at the initial time t_0 . [Actually, as can be shown from Eqs. (A11), (A17), (A19), and (A20), even the Wigner function positivity is not needed for our main result, Eqs. (16)–(18) of the main text; however, for the proof presented here, we need to assume the positivity.]

3. Semiclassical model

As discussed in the main text, in the semiclassical model we consider a stochastic evolution of the classical field $\alpha(t)$ in the cavity, which is caused by the Hamiltonian, dissipation, and classical complex-valued noise $v(t)$, which imitates the vacuum noise $V(t)$ incident on the cavity from the transmission line. This noise has correlators

$$\langle v(t)v^*(t') \rangle = (\bar{n}_b + 1/2)\delta(t - t'), \quad \langle v(t)v(t') \rangle = 0, \quad (\text{A22})$$

which are classical counterparts of the quantum relations (A4) and (A5). For simplicity we assume zero temperature, so that $\bar{n}_b = 0$, though generalization to a nonzero temperature is simple.

The evolution of the intracavity field [counterpart of Eq. (A6)] is

$$\dot{\alpha} = -\frac{\kappa}{2}\alpha - i\partial_{\alpha^*}h(\alpha, \alpha^*) + \sqrt{\kappa}v(t), \quad (\text{A23})$$

where $h(\alpha, \alpha^*)$ is the classical Hamiltonian, corresponding to the quantum Hamiltonian H . For the Hamiltonian (A18) of a driven *linear* resonator, the classical Hamiltonian h can be obtained from H by simply replacing a with α and a^\dagger with α^* , so that

$$h(\alpha, \alpha^*) = \Omega|\alpha|^2 + \frac{i}{4}[\varepsilon^*\alpha^2 - \varepsilon(\alpha^*)^2] + \varepsilon_c^*\alpha + \varepsilon_c\alpha^*, \quad (\text{A24})$$

and therefore the field evolution is

$$\dot{\alpha} = -\left(\frac{\kappa}{2} + i\Omega\right)\alpha - \frac{\varepsilon}{2}\alpha^* - i\varepsilon_c + \sqrt{\kappa}v(t). \quad (\text{A25})$$

In general, the classical Hamiltonian h should be chosen so that $\dot{\alpha}$ correctly describes the evolution of the field α in the classical case. An initial condition for $\alpha(t)$ is usually not needed, because if the evolution starts at $t = -\infty$, then the initial condition does not matter. However, if we want to start evolution from $t = t_0$, then $\alpha(t_0)$ in the semiclassical model should be treated as a random complex number, with two-dimensional probability distribution equal to the Wigner function $W(\alpha, \alpha^*, t_0)$, which is positive (and normalized) by the above-discussed assumption.

The outgoing field [counterpart of Eq. (A3)] is

$$f(t) = -v(t) + \sqrt{\kappa}\alpha(t), \quad (\text{A26})$$

and the measured quadrature signal $f_\varphi(t)$ for the quadrature phase φ is

$$f_\varphi(t) = \frac{1}{2}[e^{-i\varphi}f(t) + e^{i\varphi}f^*(t)]. \quad (\text{A27})$$

Our goal is to prove that the two-time correlator $K_{\varphi_1\varphi_2}^q(t_1, t_2)$ for the signal $f_\varphi(t)$ calculated in the quantum way (A8) is *exactly* equal to the correlator

$$K_{\varphi_1\varphi_2}(t_1, t_2) = \langle f_{\varphi_1}(t_1)f_{\varphi_2}(t_2) \rangle, \quad (\text{A28})$$

calculated in the semiclassical model. Note that here $\alpha(t)$ and $f_\varphi(t)$ contain contributions due to the “signal” $\alpha_c(t)$, in contrast to the main text, where we considered only noise (in the main text $\alpha_c = 0$).

Let us first check the equivalence for the singular contribution to the correlator at $t_1 = t_2$ (then $\varphi_1 = \varphi_2$ as well). From Eqs. (A2)–(A5) we see that at zero temperature the singular

part of the quantum correlator is $(1/4)\delta(t_1 - t_2)$, and from Eqs. (A22) and (A26)–(A28) we obtain the same result for the semiclassical correlator. Next, for the equivalence in the case $t_1 \neq t_2$, it is sufficient to consider $t_1 < t_2$ (because of the symmetry). In this case the quantum correlator is given by Eq. (A11), while the semiclassical correlator is

$$K_{\varphi_1\varphi_2}(t_1, t_2) = \frac{1}{2} \operatorname{Re}[\langle f(t_2)f(t_1) \rangle e^{-i(\varphi_1+\varphi_2)} + \langle f^*(t_2)f(t_1) \rangle e^{-i(\varphi_1-\varphi_2)}]. \quad (\text{A29})$$

Therefore, we only need to prove two relations for $t_1 < t_2$:

$$\langle f(t_2)f(t_1) \rangle = \kappa \langle a(t_2)a(t_1) \rangle, \quad (\text{A30})$$

$$\langle f^*(t_2)f(t_1) \rangle = \kappa \langle a^\dagger(t_2)a(t_1) \rangle. \quad (\text{A31})$$

Let us prove Eq. (A30) first [the proof of Eq. (A31) is similar]. Using Eq. (A26) for $f(t)$, we can write the left-hand side of Eq. (A30) as

$$\langle f(t_2)f(t_1) \rangle = \kappa \langle \alpha(t_2)\alpha(t_1) \rangle - \sqrt{\kappa} \langle \alpha(t_2)v(t_1) \rangle, \quad (\text{A32})$$

since $\langle v(t_2)\alpha(t_1) \rangle = 0$ for $t_1 < t_2$. Comparing this equation with Eq. (A17a), we see that we can prove Eq. (A30) by proving the following two relations:

$$\langle \alpha(t_2)\alpha(t_1) \rangle = \int \alpha_2 W(2|1)\alpha_1 W(1)d^2\alpha_1 d^2\alpha_2, \quad (\text{A33})$$

$$\frac{-2}{\sqrt{\kappa}} \langle \alpha(t_2)v(t_1) \rangle = \int \alpha_2 W(2|1)\partial_{\alpha_1^*}W(1)d^2\alpha_1 d^2\alpha_2. \quad (\text{A34})$$

Note that in these relations, $\alpha(t)$ in the left-hand side is the semiclassical random process, while α_1 and α_2 in the right-hand side are the integration variables.

To prove Eq. (A33), let us show that the Wigner function $W(\alpha, \alpha^*, t)$ is equal to the probability distribution of $\alpha(t)$ in the semiclassical model. Introducing the probability distribution $P(x, p, t)$ on the two-dimensional plane with real coordinates $x = \operatorname{Re}(\alpha)$ and $p = \operatorname{Im}(\alpha)$, from the Langevin equation (A25) with noise given by Eq. (A22), we can write the standard Fokker-Planck equation

$$\begin{aligned} \partial_t P = & -\partial_x \left\{ \left[-\left(\frac{\kappa}{2} + \operatorname{Re}\frac{\varepsilon}{2}\right)x + \left(\Omega - \operatorname{Im}\frac{\varepsilon}{2}\right)p + \operatorname{Im}\varepsilon_c \right] P \right\} \\ & -\partial_p \left\{ \left[-\left(\frac{\kappa}{2} - \operatorname{Re}\frac{\varepsilon}{2}\right)p - \left(\Omega + \operatorname{Im}\frac{\varepsilon}{2}\right)x - \operatorname{Re}\varepsilon_c \right] P \right\} \\ & + \frac{\kappa}{8} (\partial_x^2 + \partial_p^2) P. \end{aligned} \quad (\text{A35})$$

It is easy to check that if we formally introduce the same probability distribution as a function of α and α^* , i.e., $P(x, p, t) = \tilde{P}(\alpha, \alpha^*, t)$, then Eq. (A35) can be rewritten as

$$\begin{aligned} \partial_t \tilde{P} = & -\partial_\alpha \{ [-(\kappa/2 + i\Omega)\alpha - (\varepsilon/2)\alpha^* - i\varepsilon_c] \tilde{P} \} \\ & -\partial_{\alpha^*} \{ [-(\kappa/2 - i\Omega)\alpha - (\varepsilon^*/2)\alpha + i\varepsilon_c^*] \tilde{P} \} \\ & + (\kappa/2) \partial_{\alpha\alpha^*}^2 \tilde{P}. \end{aligned} \quad (\text{A36})$$

This is exactly the same equation as Eq. (A19) for the Wigner function. Therefore, if $W(\alpha, \alpha^*, t_0) = P(x, p, t_0)$ at some initial time t_0 (as we assumed above), then the Wigner function will be equal to the probability distribution of $\alpha(t)$ in the semiclassical model at any later time, $W(\alpha, \alpha^*, t) = P(x, p, t)$.

Thus, we have shown that $W(1)$ in Eq. (A33) is equal to the probability distribution of the semiclassical intracavity field α at time t_1 . Similarly, the propagator $W(2|1)$ in Eq. (A33) is equal to the probability distribution of the field $\alpha(t_2)$ at time t_2 in the semiclassical model if at time $t_1 < t_2$ the field is α_1 . Therefore, Eq. (A33) is obviously valid.

It is a little more difficult to prove Eq. (A34). Let us introduce discrete time with very small time steps $\Delta t \rightarrow 0$. Then the average $\langle \alpha(t_2)v(t_1) \rangle$ in the left-hand side of Eq. (A34) is replaced with $\langle \alpha(t_2)\tilde{v}(t_1) \rangle$, where $\tilde{v}(t_1) = (1/\Delta t) \int_{t_1}^{t_1+\Delta t} v(t) dt$. Now $\tilde{v}(t_1)$ is a (large) complex number, which is Gaussian-distributed with $\langle |\tilde{v}|^2 \rangle = 1/(2\Delta t)$ and $\langle \tilde{v} \rangle = 0$. Because of the linearity of the Fokker-Planck equation,

$$\langle \alpha(t_2)\tilde{v}(t_1) \rangle = \int \alpha_2 P(2|1) \langle \delta P(1)\tilde{v}(t_1) \rangle d^2\alpha_1 d^2\alpha_2, \quad (\text{A37})$$

where $P(2|1)$ is the propagator for probabilities [the same as $W(2|1)$, we have used a different notation only to emphasize that we consider the semiclassical model], and $\delta P(1) \equiv P(x_1 + \delta x_1, p_1 + \delta p_1, t_1 + \Delta t) - P(x_1, p_1, t_1)$ is the change of the probability distribution between time moments t_1 and $t_1 + \Delta t$ due to the kick to $\alpha(t)$ produced by $\tilde{v}(t_1)$. Note that the averaging in the right-hand side of Eq. (A37) is only over $\tilde{v}(t_1)$, while in the left-hand side it also includes averaging over random trajectories. As seen from Eq. (A25), the noise $\tilde{v}(t_1)$ shifts $\alpha(t_1)$ by $\delta\alpha(t_1) = \sqrt{\kappa} \tilde{v}(t_1) \Delta t$, and therefore the leading-order change of the probability distribution is $\delta P(1) = -\sqrt{\kappa} [\partial_x P \operatorname{Re} \tilde{v}(t_1) + \partial_p P \operatorname{Im} \tilde{v}(t_1)] \Delta t$. Now using averages $\langle \tilde{v}(t_1) \operatorname{Re} \tilde{v}(t_1) \rangle \Delta t = 1/4$ and $\langle \tilde{v}(t_1) \operatorname{Im} \tilde{v}(t_1) \rangle \Delta t = i/4$, we obtain

$$\begin{aligned} \langle \alpha(t_2)\tilde{v}(t_1) \rangle = & -\frac{\sqrt{\kappa}}{4} \int \alpha_2 P(2|1) [\partial_{x_1} P(1) + i\partial_{p_1} P(1)] \\ & \times d^2\alpha_1 d^2\alpha_2. \end{aligned} \quad (\text{A38})$$

This equation is the same as Eq. (A34), since $W(2|1) = P(2|1)$ and $\partial_{\alpha_1^*} W(1) = (1/2) \partial_{x_1} P(1) + (i/2) \partial_{p_1} P(1)$ [as follows from the change of variables: $x = (\alpha + \alpha^*)/2$ and $p = (\alpha - \alpha^*)/2i$]. Thus, we have proved Eq. (A34).

By proving Eqs. (A33) and (A34), we have proved Eq. (A30). The proof of Eq. (A31) is very similar. Instead of Eqs. (A33) and (A34), we need to prove the relations

$$\langle \alpha^*(t_2)\alpha(t_1) \rangle = \int \alpha_2^* W(2|1)\alpha_1 W(1)d^2\alpha_1 d^2\alpha_2, \quad (\text{A39})$$

$$\frac{-2}{\sqrt{\kappa}} \langle \alpha^*(t_2)v(t_1) \rangle = \int \alpha_2^* W(2|1)\partial_{\alpha_1^*}W(1)d^2\alpha_1 d^2\alpha_2, \quad (\text{A40})$$

which can be done in the same way as above.

Thus, we have shown that for a system with the Hamiltonian (A18), the quantum correlators (A1) are *exactly* equal to the correlators (A28) calculated within the semiclassical model.

APPENDIX B: TRANSIENT SQUEEZING IN A WEAKLY NONLINEAR RESONATOR

In the main text we considered the case of a linear resonator, with squeezing produced by a parametric drive. However, the main initial motivation for this work was to develop a theory capable of calculating fluctuations of the integrated signal in circuit QED measurement of a superconducting qubit (these fluctuations are directly related to the probability of error in qubit measurement). In the process of qubit measurement, the squeezing is self-generated due to resonator nonlinearity induced by interaction with the qubit [21,22]. Therefore, it is important to consider the nonlinear case as well.

As shown in Ref. [22], in the case of a weakly nonlinear resonator (typical in qubit measurement), the intracavity state remains approximately Gaussian during the transient squeezing caused by abruptly applied coherent drive. This is why the theory developed in the main text is directly applicable to this case as well. However, while for a linear resonator our semiclassical theory is exact, for a weakly nonlinear resonator it is only approximate, being based on the Gaussian ansatz. (We believe it is a very good approximation for typical parameters of a qubit measurement; however, the accuracy still has to be analyzed.)

In a weakly nonlinear case, the Hamiltonian can be written as

$$H = \sum_n E(n)|n\rangle\langle n| + \frac{i}{4}[\varepsilon^* a^2 - \varepsilon a^{\dagger 2}] + \varepsilon_c^* a + \varepsilon_c a^\dagger, \quad (\text{B1})$$

where $|n\rangle$ is n th eigenstate of the resonator and the rotating-frame energy $E(n)$ is related to the experimentally measurable (laboratory-frame) weakly changing resonator frequency $\omega_r(n)$ as

$$E(n) = \sum_{k=0}^{n-1} [\omega_r(k) - \omega_{\text{rf}}]. \quad (\text{B2})$$

Here ω_{rf} is an (arbitrary) rotating-frame frequency, for which it is most natural to choose the frequency of the coherent drive. A small difference between the frequencies of the rotating frame, coherent drive, and (halved) parametric drive can be taken into account by slowly changing the complex amplitudes ε_c and ε . Note that for the qubit measurement we do not need parametric drive, $\varepsilon = 0$, but we keep this term for generality. Also note that the Hamiltonian (B1) uses the basis of actual eigenstates, so the creation and annihilation operators (which are constructed from \hat{x} and \hat{p} operators) satisfy the relations $\langle n|a^\dagger|k\rangle = \sqrt{n}\delta_{n-1,k}$, $\langle k|a|n\rangle = \sqrt{n}\delta_{n-1,k}$, and $a^\dagger a|n\rangle = n|n\rangle$ only approximately, and therefore we need the assumption of weak nonlinearity to use these standard relations (see [22]).

The main idea of using the semiclassical model in this case is to separate evolution of the average field α_c (corresponding to the maximum of the Wigner function, which is assumed to be Gaussian) and fluctuations $\delta\alpha$,

$$\alpha(t) = \alpha_c(t) + \delta\alpha(t), \quad (\text{B3})$$

so that the ‘‘center’’ evolves as

$$\dot{\alpha}_c = -i[\omega_r(|\alpha_c|^2) - \omega_{\text{rf}}]\alpha_c - \frac{\kappa}{2}\alpha_c - \frac{\varepsilon}{2}\alpha_c^* - i\varepsilon_c, \quad (\text{B4})$$

while for fluctuations we use linearization [37,39] near the center,

$$\begin{aligned} \frac{d}{dt}\delta\alpha = & -i\left[\omega_r(|\alpha_c|^2) - \omega_{\text{rf}} + \frac{d\omega_r}{dn}|\alpha_c|^2\right]\delta\alpha - \frac{\kappa}{2}\delta\alpha \\ & - \left(\frac{\varepsilon}{2} + i\frac{d\omega_r}{dn}\alpha_c^2\right)(\delta\alpha)^* + \sqrt{\kappa}v(t), \end{aligned} \quad (\text{B5})$$

where $d\omega_r(n)/dn$ is evaluated at $n = |\alpha_c|^2$. Note that the terms with $d\omega_r/dn$ come from the contribution $-i\alpha_c(d\omega_r/dn)2\text{Re}(\alpha_c^*\delta\alpha)$ describing the resonator frequency change due to fluctuations.

Equation (B5) has the same form as Eq. (6) in the main text, with α replaced by $\delta\alpha$ (in the main text we considered only fluctuations) and also Ω and ε replaced by the corresponding terms in Eq. (B5). Therefore, squeezing of fluctuations in the case of a weakly nonlinear resonator is still described by Eqs. (16)–(18) of the main text ($\alpha \rightarrow \delta\alpha$, $f \rightarrow \delta f \equiv f - \sqrt{\kappa}\alpha_c$), with the following replacements in Eq. (14) of the main text:

$$\Omega \rightarrow \omega_r(|\alpha_c|^2) - \omega_{\text{rf}} + \frac{d\omega_r(n)}{dn}\Big|_{n=|\alpha_c|^2}|\alpha_c|^2, \quad (\text{B6})$$

$$\varepsilon \rightarrow \varepsilon + 2i\frac{d\omega_r(n)}{dn}\Big|_{n=|\alpha_c|^2}\alpha_c^2, \quad (\text{B7})$$

where α_c depends on time via Eq. (B4).

As an example, Fig. 3 illustrates the self-developing squeezing of the field leaked from the resonator in the process of circuit QED measurement of a superconducting transmon qubit (for the qubit in the ground state $|0\rangle$). We assume that the readout resonator with frequency $\omega_r/2\pi = 6$ GHz is coupled to a superconducting transmon qubit with frequency $\omega_q/2\pi = 5$ GHz and anharmonicity of $\eta/2\pi = 250$ MHz (see, e.g., Ref. [21] for details and notations). The assumed coupling of $g/2\pi = 104$ MHz produces the dispersive shift of the resonator frequency $2\chi/2\pi = 5$ MHz; the resonator damping is $\kappa/2\pi = 5$ MHz. The resonator is coherently driven (starting at $t = 0$) with the amplitude $\varepsilon/2\pi = 19.1$ MHz and detuning $\Omega = \chi = 2\pi \times 2.5$ MHz (as in the standard circuit QED measurement); in the steady state this drive produces on average 46.2 photons in the resonator, which is equal to $2n_{\text{crit}}$, where $n_{\text{crit}} = (\omega_r - \omega_q)^2/4g^2$ is the so-called critical number of photons. The assumed parameters are typical for the state-of-the-art fast measurement of transmons (e.g., Ref. [42]).

Nonlinearity of the transmon induces a weak nonlinearity $\omega_r(n)$ of the resonator, which is computed numerically [21] (in particular, in the steady state the resonator frequency decreases by 1.22 MHz). This weak nonlinearity leads to the self-developing squeezing illustrated in Fig. 3. The lines in the upper and lower panels show parameters A , B , ϕ , and ψ defined in Eq. (21) of the main text (similar to Fig. 2 of the main text). The figure shows dependence of these parameters on $\tau = t_2 - t_1$ for several values of t_1 ($\kappa t_1 = 2, 4, 6$, and ∞). We see that in the steady state (i.e., for $\kappa t_1 = \infty$) $\phi(\tau) = \psi(\tau)$, so that the correlator $K_{\varphi_1\varphi_2}(t_1, t_1 + \tau)$ is characterized by three parameters (all depending on τ), while in transients four parameters are needed. These four parameters (depending on τ and t_1) are sufficient to find the noise of the integrated

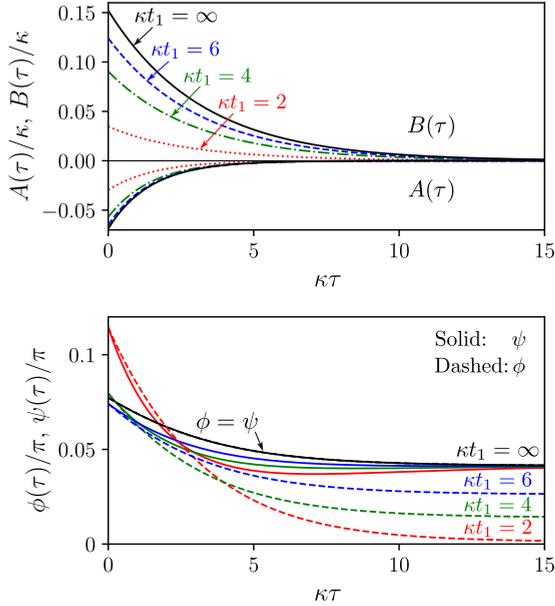


FIG. 3. Transient evolution of the self-developing squeezing in the process of circuit QED measurement of a transmon qubit. The lines show the correlator parameters A , B , ϕ , and ψ , defined in Eq. (21) of the main text, as functions of $\tau = t_2 - t_1$ for several values of t_1 ($\kappa t_1 = 2, 4, 6$, and ∞ , corresponding to red, green, blue, and black lines). In the upper panel, $A(\tau)$ are negative, $B(\tau)$ are positive. In the lower panel, solid lines show $\psi(\tau)$, dashed lines show $\phi(\tau)$. The assumed circuit QED parameters are described in the text; they are typical for the state-of-the-art fast measurement of transmon qubits ($\bar{n} = 2n_{\text{crit}}$). The resulting squeezing in the steady state is 6.8 dB.

signal $\int w(t)f_\varphi(t)dt$ for an arbitrary time-dependent phase $\varphi(t)$ and an arbitrary weight function $w(t)$, as discussed in the main text.

In particular, in the steady state the squeezing is 6.8 dB, i.e., the noise variance of the integrated signal for the properly chosen quadrature ($\varphi = 0.19$) is a factor of 4.8 less than for vacuum (the antisqueezing is also 6.8 dB). This large magnitude of self-developing squeezing clearly shows the importance of the analysis for circuit QED measurement of superconducting qubits.

APPENDIX C: SOME TECHNICAL DETAILS

1. Phase-sensitive vs phase-preserving amplifier

In the main text, we have assumed a phase-sensitive amplifier with a time-dependent amplified quadrature phase $\varphi(t)$. In this case only the homodyne output signal $f_\varphi(t)$ is available from an experiment [see Fig. 1(a) of the main text], while the orthogonal quadrature is deamplified and therefore is not measurable. Correspondingly, even though the experimentally measurable correlator $K_{\varphi_1\varphi_2}(t_1, t_2) \equiv \langle f_{\varphi_1}(t_1)f_{\varphi_2}(t_2) \rangle$ can be formally expressed via $K_{ff}(t_1, t_2) \equiv \langle f(t_1)f(t_2) \rangle$ and $K_{ff^*}(t_1, t_2) \equiv \langle f(t_1)f^*(t_2) \rangle$ [see Eq. (8) of the main text], the field correlators K_{ff} and K_{ff^*} are not directly measurable because $f(t)$ is not available from an experiment.

The situation is different for a setup based on a phase-preserving amplifier (heterodyne measurement). Then all

quadratures are equally amplified, and from the two experimental normalized outputs $I(t)$ and $Q(t)$ we can construct the standard complex combination

$$f_{\text{pp}}(t) = I(t) + iQ(t), \quad (\text{C1})$$

so that for any phase $\varphi(t)$, the quadrature signal

$$f_{\varphi, \text{pp}}(t) = \text{Re}[e^{-i\varphi(t)}f_{\text{pp}}(t)] \quad (\text{C2})$$

can be obtained from the same experiment (here the subscript “pp” indicates the phase-preserving case).

However, a phase-preserving amplifier necessarily has an “additional noise” [35], which increases the total noise by at least a factor of 2 compared with the ideal phase-sensitive amplifier. (The additional noise is essentially the price for simultaneous measurement of noncommuting quadratures.) Therefore, the output $f_{\text{pp}}(t)$ in the phase-preserving case can be related to the leaked field $f(t)$ considered in the main text as

$$f_{\text{pp}}(t) = f(t) + v'(t), \quad (\text{C3})$$

where the complex-valued additional noise $v'(t)$ is uncorrelated with the noise $v(t)$ introduced in the semiclassical formalism, and

$$\langle v'(t_1)v'(t_2) \rangle = 0, \quad \langle v'(t_1)v'^*(t_2) \rangle = (1/2 + n_a)\delta(t_1 - t_2). \quad (\text{C4})$$

Here $n_a = 0$ for a quantum-limited phase-preserving amplifier, while $n_a > 0$ characterizes an extra noise compared with the quantum-limited case. Note that the additional noise v' does not drive the intracavity field fluctuations $\alpha(t)$, affecting only the output signal.

In the phase-preserving case, the correlators

$$K_{ff}^{\text{pp}}(t_1, t_2) \equiv \langle f_{\text{pp}}(t_1)f_{\text{pp}}(t_2) \rangle, \quad (\text{C5})$$

$$K_{ff^*}^{\text{pp}}(t_1, t_2) \equiv \langle f_{\text{pp}}(t_1)f_{\text{pp}}^*(t_2) \rangle \quad (\text{C6})$$

are measurable directly. Their relation to the correlator $K_{\varphi_1\varphi_2}^{\text{pp}}(t_1, t_2) \equiv \langle f_{\varphi_1, \text{pp}}(t_1)f_{\varphi_2, \text{pp}}(t_2) \rangle$ is the same as in Eq. (8) of the main text [this trivially follows from Eq. (C2)]. The important difference, however, is that $K_{\varphi_1\varphi_2}^{\text{pp}}(t_1, t_2)$ is well defined for $\varphi_1 \neq \varphi_2$ even when $t_1 = t_2$, in contrast to $K_{\varphi_1\varphi_2}(t_1, t_2)$ in the phase-sensitive case.

From Eqs. (C3) and (C4) it follows that for different time moments the correlators in the phase-preserving and phase-sensitive cases coincide,

$$K_{\varphi_1\varphi_2}^{\text{pp}}(t_1, t_2) = K_{\varphi_1\varphi_2}(t_1, t_2), \quad t_1 \neq t_2. \quad (\text{C7})$$

However, the singular contributions are different: in the phase-preserving case

$$K_{\varphi_1\varphi_2}^{\text{pp}}(t_1, t_2 \approx t_1) = \frac{1 + \bar{n}_b + n_a}{2} \cos(\varphi_1 - \varphi_2) \times \delta(t_2 - t_1), \quad (\text{C8})$$

while in the phase-sensitive case

$$K_{\varphi_1\varphi_1}(t_1, t_2 \approx t_1) = \frac{1/2 + \bar{n}_b + n_a}{2} \delta(t_2 - t_1), \quad (\text{C9})$$

where we have also included nonideality of a phase-sensitive amplifier by similarly adding n_a in the numerator.

To verify experimentally the theory developed in this paper, it is easier to use a phase-preserving amplifier (which gives direct access to K_{ff} and K_{ff^*}) instead of a phase-sensitive amplifier, while the correlators at $t_1 \neq t_2$ coincide—see Eq. (C7). However, a phase-sensitive amplifier is needed in real applications for measurements well beyond the standard quantum limit. For example, in the simplest phase-sensitive steady-state case discussed in Eq. (3) of the main text, squeezing can in principle provide a very large (unlimited) suppression of the integrated noise variance. In contrast, in the phase-preserving case the noise suppression can be only up to the factor of 2, because of the additional noise. We expect a similar relation for measurement in transients; however, this is still a subject of future research.

2. Effect of energy loss

The signal from the leaked field can be partially lost in the transmission line because of the energy decay in the transmission line. Moreover, insertion of a circulator leads to an extra decay (typically around 0.5 dB). If we characterize the total energy decay between the resonator and the amplifier by the decrease of the efficiency η_{loss} from the perfect value of 1, then the field reaching the amplifier decreases as $f \rightarrow \sqrt{\eta_{\text{loss}}} f$. Correspondingly, the nonsingular part of the correlators decreases as

$$K_{\varphi_1\varphi_2}(t_1, t_2) \rightarrow \eta_{\text{loss}} K_{\varphi_1\varphi_2}(t_1, t_2), \quad t_1 \neq t_2, \quad (\text{C10})$$

and similarly for K_{ff} and K_{ff^*} . However, the singular part of the correlator (at $t_2 \approx t_1$) is not affected by the energy loss because it physically originates from the noise of the amplifier. An alternative explanation of the fact that $K_{\varphi_1\varphi_1}(t_1, t_2 \approx t_1)$ does not change with η_{loss} can be based on inserting a fictitious beam splitter with transmission coefficient η_{loss} between the resonator and the amplifier. Then the decrease of the propagating vacuum fluctuations (at $t_2 \approx t_1$) due to imperfect transmission η_{loss} is exactly compensated by extra vacuum fluctuations with relative strength $1 - \eta_{\text{loss}}$, coming from the other arm of the beam splitter.

A similar effect on the correlators is produced by the extra decay in the resonator, leading to $\kappa > \kappa_{\text{out}}$, where κ_{out} is the resonator damping due to coupling with the outgoing transmission line, while κ is the total energy decay rate for the resonator (which includes coupling to other transmission lines, energy loss in the surrounding dielectric materials, etc.). The effect can also be modelled by a fictitious beam splitter with transmission coefficient $\kappa_{\text{out}}/\kappa$ inserted into the transmission line [between the circulator and amplifier in Fig. 1(a) of the main text]. Therefore, the correlators derived in this paper (for the same κ) are decreased as

$$K_{\varphi_1\varphi_2}(t_1, t_2) \rightarrow (\kappa_{\text{out}}/\kappa) K_{\varphi_1\varphi_2}(t_1, t_2), \quad t_1 \neq t_2, \quad (\text{C11})$$

with the same relation for K_{ff} and K_{ff^*} . The singular part of the correlator is again not affected (because it physically originates from the amplifier or, alternatively, because of the extra vacuum noise coming from the beam splitter).

Combining the effects of extra energy decay in the resonator and transmission line, we obtain

$$K_{\varphi_1\varphi_2}(t_1, t_2) \rightarrow \eta_{\text{col}} K_{\varphi_1\varphi_2}(t_1, t_2), \quad t_1 \neq t_2, \quad (\text{C12})$$

$$K_{\varphi_1\varphi_1}(t_1, t_2 \approx t_1) \rightarrow K_{\varphi_1\varphi_1}(t_1, t_2 \approx t_1), \quad (\text{C13})$$

where $\eta_{\text{col}} = (\kappa_{\text{out}}/\kappa)\eta_{\text{loss}}$ is the total collection efficiency.

As an example, for 1 dB loss in the collection efficiency, all correlators at $t_1 \neq t_2$ are multiplied by the factor 0.794. Since the collection efficiency loss between 1 and 2 dB is typical in circuit QED experiments (e.g., Ref. [43]), the squeezing correlators discussed in this paper should be easily measurable experimentally (with the decrease by less than a factor of 2 due to losses).

Note that while the correlators discussed in this paper are affected only by the collection efficiency η_{col} , the actual applications of the transient squeezing will depend on the total quantum efficiency $\eta = \eta_{\text{col}}\eta_{\text{amp}}$, which includes the quantum efficiency η_{amp} of the amplifier.

3. Three vs four parameters for squeezing in transients

As discussed in the main text, dependence of the correlator $K_{\varphi_1\varphi_2}(t_1, t_2)$ on quadratures φ_1 and φ_2 can always be described by three real parameters in the steady state, while in transients four parameters are generally needed. However, there are at least two special cases discussed here, when three parameters are sufficient even in transients.

The first special case is when there is no detuning, $\Omega = 0$, and the parametric pump phase θ is time independent, so that the drive amplitude $\varepsilon(t) = |\varepsilon(t)| e^{i\theta}$ changes in time only because of (an arbitrary) time dependence $|\varepsilon(t)|$. In this case we can redefine the quadratures to set $\theta = 0$; then the matrix $M(t)$ [given by Eq. (14) of the main text] is real, and therefore the Green's function $G(t|t_{\text{in}})$ in Eq. (16) of the main text is also real. As a consequence, the matrix given by Eq. (18) of the main text is also real (assuming that the evolution starts from $t_0 = -\infty$ or assuming that $\langle \alpha^2(t_0) \rangle$ is real), and therefore all terms in Eq. (17) of the main text are real. Thus, $K_{ff}(t_1, t_2)$ and $K_{ff^*}(t_1, t_2)$ are both real, giving us two real parameters. Adding the phase θ (which was zeroed in the rotated basis), we obtain three parameters.

The second special case is when θ is time-independent, the detuning Ω is arbitrary, but time-independent, and $|\varepsilon(t)|$ is a step function: $|\varepsilon(t < 0)| = 0$ and $|\varepsilon(t \geq 0)| = \text{const}$. The derivation in this case is lengthier, but it is also based on Eqs. (14)–(18) of the main text (assuming vacuum state at $t_0 = 0$). The analytical result shows that $K_{ff^*}(t_1, t_2)$ is real for arbitrary t_1 and t_2 , and numerical simulations confirm this result. Therefore, in this case only three real parameters are sufficient again.

4. Derivation of Eq. (18) of the main text

From Eq. (6) of the main text for the intracavity field fluctuation $\alpha(t)$ and Eq. (14) of the main text for the evolution matrix M , we obtain

$$\frac{d}{dt} \begin{bmatrix} \alpha(t) \\ \alpha^*(t) \end{bmatrix} = M(t) \begin{bmatrix} \alpha(t) \\ \alpha^*(t) \end{bmatrix} + \sqrt{\kappa} \begin{bmatrix} v(t) \\ v^*(t) \end{bmatrix}. \quad (\text{C14})$$

The solution of this equation can be written in terms of the Green's function $G(t|t')$ [2×2 matrix defined in Eq. (16) of the main text], which gives

$$\begin{bmatrix} \alpha(t_1) \\ \alpha^*(t_1) \end{bmatrix} = G(t_1|t_0) \begin{bmatrix} \alpha(t_0) \\ \alpha^*(t_0) \end{bmatrix} + \sqrt{\kappa} \int_{t_0}^{t_1} G(t_1|t') \begin{bmatrix} v(t') \\ v^*(t') \end{bmatrix} dt'. \quad (\text{C15})$$

From this equation and Eq. (5) of the main text we obtain

$$\begin{bmatrix} \langle |\alpha^2(t_1)| \rangle & \langle \alpha^2(t_1) \rangle \\ \langle \alpha^{*2}(t_1) \rangle & \langle |\alpha^2(t_1)| \rangle \end{bmatrix} = \frac{\kappa(1 + 2\bar{n}_b)}{2} \int_{t_0}^{t_1} G(t_1|t') G^\dagger(t_1|t') dt' + G(t_1|t_0) \begin{bmatrix} \langle |\alpha^2(t_0)| \rangle & \langle \alpha^2(t_0) \rangle \\ \langle \alpha^{*2}(t_0) \rangle & \langle |\alpha^2(t_0)| \rangle \end{bmatrix} G^\dagger(t_1|t_0), \quad (\text{C16})$$

which becomes Eq. (18) of the main text at zero temperature, $\bar{n}_b = 0$.

Note that instead of using Eq. (18) of the main text, the evolution of the intracavity squeezing can be obtained from differential equations (similar to Ref. [22]),

$$\frac{d}{dt} \langle \alpha^2 \rangle = (-2i\Omega - \kappa) \langle \alpha^2 \rangle - \varepsilon \langle |\alpha|^2 \rangle, \quad (\text{C17})$$

$$\frac{d}{dt} \langle |\alpha|^2 \rangle = -\kappa \langle |\alpha|^2 \rangle - \text{Re}(\varepsilon^* \langle \alpha^2 \rangle) + \kappa(\bar{n}_b + 1/2), \quad (\text{C18})$$

starting with an initial condition at $t = t_0$.

5. Derivation of Eqs. (22)–(23) of the main text

We consider the special case when there is no detuning, $\Omega = 0$, and the parametric drive $\varepsilon = |\varepsilon| e^{i\theta}$ has a time-independent amplitude $|\varepsilon| = \text{const}$, while the phase $\theta(t)$ abruptly changes from $\theta(t) = 0$ for $t < 0$ to $\theta(t) = \tilde{\theta} = \text{const}$ for $t > 0$. The temperature is assumed to be zero. We are interested in the correlators after the phase jump, so we

assume $t_1 > 0$ and $\tau = t_2 - t_1 > 0$. These correlators can be obtained from Eqs. (16)–(18) of the main text. Let us first calculate the propagator G from Eq. (16):

$$\begin{aligned} G(t_1 + \tau|t_1) &= \exp(M\tau) \\ &= \frac{1}{2} \begin{bmatrix} e^{-\frac{\kappa-\tau}{2}} + e^{-\frac{\kappa+\tau}{2}} & (e^{-\frac{\kappa+\tau}{2}} - e^{-\frac{\kappa-\tau}{2}}) e^{i\tilde{\theta}} \\ (e^{-\frac{\kappa+\tau}{2}} - e^{-\frac{\kappa-\tau}{2}}) e^{-i\tilde{\theta}} & e^{-\frac{\kappa-\tau}{2}} + e^{-\frac{\kappa+\tau}{2}} \end{bmatrix}, \end{aligned} \quad (\text{C19})$$

where $\kappa_{\pm} = \kappa \pm |\varepsilon|$ and the matrix M is given by Eq. (14) of the main text with $\Omega = 0$ and $\varepsilon = |\varepsilon| e^{i\tilde{\theta}}$. As the next step, to find $\langle \alpha^2(t_1) \rangle$ and $\langle |\alpha(t_1)|^2 \rangle$, we use Eq. (18) of the main text with the initial condition at $t_0 = 0$ (which is the steady state before the jump),

$$\begin{bmatrix} \langle \alpha^2(0) \rangle \\ \langle |\alpha^2(0)| \rangle \end{bmatrix} = \frac{\kappa}{4} \begin{bmatrix} 1/\kappa_+ - 1/\kappa_- \\ 1/\kappa_+ + 1/\kappa_- \end{bmatrix}, \quad (\text{C20})$$

which gives

$$\begin{aligned} \langle \alpha^2(t_1) \rangle &= \left[\frac{1}{4} \left(\frac{\kappa}{\kappa_+} - \frac{\kappa}{\kappa_-} \right) + i \frac{\sin \theta}{4} \left(\frac{\kappa}{\kappa_-} - \frac{\kappa}{\kappa_+} \right) e^{-\kappa t_1} \right. \\ &\quad \left. + \frac{1 - \cos \theta}{8} \left(\frac{\kappa}{\kappa_-} - \frac{\kappa}{\kappa_+} \right) (e^{-\kappa t_1} + e^{-\kappa_+ t_1}) \right] e^{i\tilde{\theta}}, \end{aligned} \quad (\text{C21})$$

$$\begin{aligned} \langle |\alpha^2(t_1)| \rangle &= \frac{1}{4} \left(\frac{\kappa}{\kappa_+} + \frac{\kappa}{\kappa_-} \right) \\ &\quad - \frac{1 - \cos \theta}{8} \left(\frac{\kappa}{\kappa_-} - \frac{\kappa}{\kappa_+} \right) (e^{-\kappa t_1} - e^{-\kappa_+ t_1}). \end{aligned} \quad (\text{C22})$$

Note that (C21)–(C22) reduce to Eq. (C20) when $\tilde{\theta} = 0$. Finally, using Eq. (17) of the main text with the propagator G given by Eq. (C19) and the initial condition given by Eqs. (C21)–(C22), we obtain Eqs. (22)–(23) of the main text for the correlators $K_{ff}(t_1, t_1 + \tau)$ and $K_{ff^*}(t_1, t_1 + \tau)$.

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