

Flag additivity in quantum resource theoriesC. L. Liu,^{1,2} Xiao-Dong Yu,^{1,3,*} and D. M. Tong^{1,†}¹*Department of Physics, Shandong University, Jinan 250100, China*²*Institute of Physics, Beijing National Laboratory for Condensed Matter Physics, Chinese Academy of Sciences, Beijing 100190, China*³*Naturwissenschaftlich-Technische Fakultät, Universität Siegen, Walter-Flex-Strasse 3, D-57068 Siegen, Germany*

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Quantum resource theories offer a powerful framework for studying various phenomena in quantum physics. Although considerable effort has been devoted to developing a unified framework of resource theories, there are few common properties that hold for all quantum resources. In this paper, we fill this gap by introducing the flag additivity based on the tensor product structure and the flag basis for the general quantum resources. To illustrate the usefulness of flag additivity, we show that flag additivity can be used to derive other nontrivial properties in quantum resource theories, e.g., strong monotonicity, convexity, and full additivity.

DOI: [10.1103/PhysRevA.99.042322](https://doi.org/10.1103/PhysRevA.99.042322)**I. INTRODUCTION**

From quantum entanglement to quantum coherence, quantum resource theories have been used to quantify desirable quantum effects, develop new protocols for resource detection, and identify processes that optimize the resource utilization for a given application [1–6].

All quantum resource theories have two common fundamental ingredients: free states and free operations [7–10]. For a specific quantum resource, free states are quantum states that do not contain this kind of resource. Correspondingly, free operations cannot generate this kind of resource from free states. Based on the definition of free states and free operations, the resource measures can be introduced. In general, a resource measure must satisfy the non-negativity and the monotonicity. Other useful properties, such as strong monotonicity, convexity, and additivity, may also be introduced in different physical and mathematical contexts [11–22].

This common structure of quantum resource theories suggests the existence of common properties that can be applied to the general quantum resources [8,23]. For example, in Ref. [8], the authors showed that under a few physically motivated assumptions, a resource theory is asymptotically reversible if the set of allowed operations is maximal. However, although considerable effort has been devoted to developing a unified framework of resource theories, few common properties that hold for general resource measures have been found.

In this paper, we fill this gap by introducing the notions of flag bases and flag additivity for general quantum resources. To illustrate the usefulness of these general properties, we show that the flag additivity implies other nontrivial properties of resource measures in quantum resource theories, e.g., strong monotonicity, convexity, and additivity. We find that the flag additivity holds if and only if both the strong monotonicity and the convexity hold; the flag additivity implies the equivalence between the additivity and the full additivity,

and for regularized resource measures the flag additivity is equivalent to the full additivity.

The paper is organized as follows. In Sec. II, we recall the basic framework of quantum resource theories and introduce the notions of flag bases and flag additivity. In Sec. III, we prove that the flag additivity holds if and only if the additivity and the strong monotonicity hold. In Sec. IV, we show that the flag additivity implies the equivalence between the additivity and the full additivity. In Sec. V, we discuss the flag additivity for the regularized resource measures and prove that the flag additivity is equivalent to the full additivity in this special case. We conclude in Sec. VI.

II. FLAG ADDITIVITY IN QUANTUM RESOURCE THEORIES

For a specific quantum resource, the set of free states, denoted by \mathcal{F} , contains all the quantum states that do not contain this kind of resource, and the set of free operations, denoted by \mathcal{O} , contains the quantum operations that cannot create this kind of resource. We will use the 2-tuple $(\mathcal{F}, \mathcal{O})$ to denote this specific quantum resource theory. For example, in the resource theory of entanglement, the free states \mathcal{F} are all separable states and the free operations \mathcal{O} can be chosen as all local operations and classical communication (LOCC) [11,24]. In the resource theory of coherence, the free states \mathcal{F} are all incoherent states, and the free operations \mathcal{O} can be chosen as all incoherent operations [16].

With the definitions of the free states and free operations, resource measures can be introduced. The basic requirements for a functional M being a measure for $(\mathcal{F}, \mathcal{O})$ are as follows:

(M1) (Non-negativity): $M(\rho) \geq 0$ for any quantum state ρ , and $M(\rho) = 0$ if (and only if) $\rho \in \mathcal{F}$.

(M2) (Monotonicity): $M(\rho) \geq M(\Lambda(\rho))$ for any quantum state ρ and any $\Lambda \in \mathcal{O}$.

To study the flag additivity of the quantum resource $(\mathcal{F}, \mathcal{O})$, we need to consider the situation of appending or discarding an auxiliary system. For this, we recall the following tensor product structure of $(\mathcal{F}, \mathcal{O})$ (see, e.g., Refs. [8,10]):

*Xiao-Dong.Yu@uni-siegen.de

†tdm@sdu.edu.cn

(T1) Appending a free state is a free operation: For any given free state $\delta^B \in \mathcal{H}_B$, the operation $\Phi_\delta(\rho^A) = \rho^A \otimes \delta^B$ is a free operation from \mathcal{H}_A to \mathcal{H}_{AB} .

(T2) Discarding a system is a free operation: The partial trace $\text{Tr}_B(\rho^{AB}) = \rho^A$ is a free operation \mathcal{H}_{AB} to \mathcal{H}_A .

(T3) A free operation is completely free: If Λ_A is a free operation on \mathcal{H}_A , then $\Lambda_A \otimes \text{id}_B$ is a free operation on \mathcal{H}_{AB} .

For example, both the resource theory of entanglement and the resource theory of coherence satisfy this tensor product structure [10].

Equipped with these notions in quantum resource theories, we can define the flag basis and the flag additivity. A basis $\{|\varphi_i\rangle\}_{i=1}^n$ of quantum system \mathcal{H} is called a flag basis if it satisfies (i) $|\varphi_i\rangle$ are free states for all $i = 1, 2, \dots, n$; and (ii) the projective measurement $P = \{|\varphi_i\rangle\langle\varphi_i|\}_{i=1}^n$ is a free operation. For example, in the resource theory of entanglement, any separable basis is a flag basis, and in the resource theory of coherence, any incoherent basis is a flag basis. Hereafter, we will assume that the quantum resource $(\mathcal{F}, \mathcal{O})$ always satisfies the tensor product structure and the flag basis always exists.

Consider a flagged state $\sum_{i=1}^n p_i \rho_i^A \otimes |\varphi_i\rangle\langle\varphi_i|^B$, where $\{|\varphi_i\rangle\}_{i=1}^n$ is a flag basis. As all $|\varphi_i\rangle$ are free states, the resource is only contained in the system \mathcal{H}^A , which is an ensemble $\{p_i, \rho_i\}_{i=1}^n$. A reasonable assumption is that the resource measure M satisfies the following additivity condition:

$$M\left(\sum_{i=1}^n p_i \rho_i^A \otimes |\varphi_i\rangle\langle\varphi_i|^B\right) = \sum_{i=1}^n p_i M(\rho_i^A). \quad (1)$$

If Eq. (1) holds for any state ensemble $\{p_i, \rho_i\}_{i=1}^n$ and any flag basis $\{|\varphi_i\rangle\}_{i=1}^n$, we say that the resource measure M is flag additive.

III. STRONG MONOTONICITY AND CONVEXITY

To show that the flag additivity is of fundamental importance in quantum resource theories, we first prove that for any quantum resource $(\mathcal{F}, \mathcal{O})$, the flag additivity holds if and only if both the strong monotonicity and the convexity hold.

The strong monotonicity is introduced in the situation that the experimenter is able to postselect the multiple outcomes of a quantum measurement. Suppose that the free operation $\Lambda(\rho^A) = \sum_{i=1}^n K_n \rho^A K_n^\dagger$ is a general quantum measurement; then the measurement gives the outcome ρ_i^A with the probability p_i , where $p_i \rho_i^A = K_n \rho^A K_n^\dagger$. If the postselection is possible, we can further perform different free operations to different outcomes and the final operation will also be free. As a special case, we can add different flags $|\varphi_i\rangle$ to different measurement outcomes. According to (T1), the final operation,

$$\begin{aligned} \tilde{\Lambda}(\rho^A) &= \sum_{i=1}^n K_n \rho^A K_n^\dagger \otimes |\varphi_i\rangle\langle\varphi_i|^B \\ &= \sum_{i=1}^n p_i \rho_i^A \otimes |\varphi_i\rangle\langle\varphi_i|^B, \end{aligned} \quad (2)$$

is still a free operation. Then the monotonicity condition (M2) implies that

$$M(\rho^A) \geq M(\tilde{\Lambda}(\rho^A)) = M\left(\sum_{i=1}^n p_i \rho_i^A \otimes |\varphi_i\rangle\langle\varphi_i|^B\right). \quad (3)$$

If the flag additivity in Eq. (1) is satisfied, then we can get that

$$M(\rho^A) \geq \sum_{i=1}^n p_i M(\rho_i^A). \quad (4)$$

Equation (4) is usually referred to as the strong monotonicity of the resource measure M , i.e., nonincreasing of M under the selective measurement on average.

The convexity is related to the mixing of quantum states. For any ensemble of quantum states $\{p_i, \rho_i^A\}_{i=1}^n$ in \mathcal{H}_A , let us consider the auxiliary state $\sum_{i=1}^n p_i \rho_i^A \otimes |\varphi_i\rangle\langle\varphi_i|^B$, where $\{|\varphi_i\rangle\}_{i=1}^n$ is a flag basis in \mathcal{H}_B . Now, we discard the quantum system \mathcal{H}_B , i.e.,

$$\text{Tr}_B\left(\sum_{i=1}^n p_i \rho_i^A \otimes |\varphi_i\rangle\langle\varphi_i|^B\right) = \sum_{i=1}^n p_i \rho_i^A, \quad (5)$$

which is a free operation according to (T2). Thus, the monotonicity condition (M2) implies that

$$M\left(\sum_{i=1}^n p_i \rho_i^A \otimes |\varphi_i\rangle\langle\varphi_i|^B\right) \geq M\left(\sum_{i=1}^n p_i \rho_i^A\right). \quad (6)$$

If the flag additivity in Eq. (1) is satisfied, then we can get that

$$\sum_{i=1}^n p_i M(\rho_i^A) \geq M\left(\sum_{i=1}^n p_i \rho_i^A\right), \quad (7)$$

which is usually referred to as the convexity of the resource measure M , i.e., nonincreasing of M under the mixing of quantum states.

Up to now, we have shown that the flag additivity in Eq. (1) is sufficient for the strong monotonicity in Eq. (4) and the convexity in Eq. (7). In the following, we will show that it is also necessary.

Consider that the quantum system in \mathcal{H}_{AB} is the flagged state $\sum_{i=1}^n p_i \rho_i^A \otimes |\varphi_i\rangle\langle\varphi_i|^B$, where $\{|\varphi_i\rangle\}_{i=1}^n$ is a flag basis in \mathcal{H}_B . On the one hand, if we perform the measurement $\{|\varphi_i\rangle\langle\varphi_i|^B\}_{i=1}^n$ on \mathcal{H}_B , which is a free operation by the definition of flag bases, then we will get the measurement outcomes $\rho_i^A \otimes |\varphi_i\rangle\langle\varphi_i|^B$ with the probability p_i . By using the strong monotonicity in Eq. (4), we can obtain that

$$\begin{aligned} M\left(\sum_{i=1}^n p_i \rho_i^A \otimes |\varphi_i\rangle\langle\varphi_i|^B\right) &\geq \sum_{i=1}^n p_i M(\rho_i^A \otimes |\varphi_i\rangle\langle\varphi_i|^B) \\ &\geq \sum_{i=1}^n p_i M(\rho_i^A), \end{aligned} \quad (8)$$

where the second inequality follows from condition (T2). On the other hand, by using the convexity, we can get that

$$\begin{aligned} M\left(\sum_{i=1}^n p_i \rho_i^A \otimes |\varphi_i\rangle\langle\varphi_i|^B\right) &\leq \sum_{i=1}^n p_i M(\rho_i^A \otimes |\varphi_i\rangle\langle\varphi_i|^B) \\ &\leq \sum_{i=1}^n p_i M(\rho_i^A), \end{aligned} \quad (9)$$

where the second inequality follows from condition (T1). Combining Eqs. (8) and (9), we immediately obtain the flag additivity in Eq. (1).

The preceding results can be summarized as the following theorem:

Theorem 1. For any resource measure, the strong monotonicity and the convexity is equivalent to the flag additivity.

We should note that in some contexts, the strong monotonicity and the convexity are only desired but not mandatory requirements of resource measures. Sometimes one or both of them may not hold. In this situation, we can still consider two conditions that are weaker than the flag additivity: the flag supadditivity and the flag subadditivity. The flag supadditivity is defined as

$$M\left(\sum_{i=1}^n p_i \rho_i^A \otimes |\varphi_i\rangle\langle\varphi_i|^B\right) \geq \sum_{i=1}^n p_i M(\rho_i^A), \quad (10)$$

and the flag subadditivity is defined as

$$M\left(\sum_{i=1}^n p_i \rho_i^A \otimes |\varphi_i\rangle\langle\varphi_i|^B\right) \leq \sum_{i=1}^n p_i M(\rho_i^A), \quad (11)$$

for any state ensemble $\{p_i, \rho_i\}_{i=1}^n$ and any flag basis $\{|\varphi_i\rangle\}_{i=1}^n$. By slightly modifying the proof in Theorem 1, we can get the following corollary, which can be viewed as a refinement of Theorem 1.

Corollary 1. For any resource measure, the strong monotonicity is equivalent to the flag supadditivity and the convexity is equivalent to the flag subadditivity.

It is worth noting that Theorem 1 and Corollary 1 can be directly applied to entanglement measures and coherence measures. When they are applied to coherence measures, they give the results in Ref. [17] as a special case. Compared with the strong monotonicity and the convexity, the flag supadditivity and flag subadditivity are much easier to prove or disprove because they do not involve the Kraus operators and the structure of flagged states is much simpler than the mixing of ensembles. We can see this simplification from an example, the trace norm of coherence C_{tr} . It is difficult to prove whether or not C_{tr} satisfies the strong monotonicity by examining it directly [25,26]. However, one can easily prove that C_{tr} violates the strong monotonicity by examining the flag additivity [17].

IV. ADDITIVITY AND FULL ADDITIVITY

As another application of the flag additivity, we show that it implies the equivalence between the additivity and the full additivity. In quantum resource theories, a resource measure M is said to be additive if it satisfies that

$$M(\rho^{\otimes N}) = NM(\rho), \quad (12)$$

for any $\rho^{\otimes N} \in \mathcal{H}_A^{\otimes N}$, where N is any positive integer. The full additivity is a stronger condition, which is defined as

$$M(\rho \otimes \sigma) = M(\rho) + M(\sigma), \quad (13)$$

for any $\rho \in \mathcal{H}_{A_1}$ and $\sigma \in \mathcal{H}_{A_2}$.

In the following, we will prove that both the additivity and the full additivity of M are equivalent to the simpler condition

$$M(\rho \otimes \rho) = 2M(\rho), \quad (14)$$

for any $\rho \in \mathcal{H}_A$. It is obvious that Eq. (13) \Rightarrow Eq. (12) \Rightarrow Eq. (14). In order to show that they are equivalent, we only need to prove that Eq. (14) \Rightarrow Eq. (13).

We first consider the case that both ρ and σ are states in the identical quantum systems \mathcal{H}_A , i.e., $\rho \otimes \sigma \in \mathcal{H}_A \otimes \mathcal{H}_A$. Consider an auxiliary state,

$$\omega = \frac{1}{2}\rho \otimes |\varphi_1\rangle\langle\varphi_1|^B + \frac{1}{2}\sigma \otimes |\varphi_2\rangle\langle\varphi_2|^B, \quad (15)$$

where $\{|\varphi_1\rangle^B, |\varphi_2\rangle^B\}$ is a flag basis in an auxiliary system \mathcal{H}_B ; then the flag additivity implies that

$$M(\omega) = \frac{1}{2}M(\rho) + \frac{1}{2}M(\sigma). \quad (16)$$

Further, there is

$$\begin{aligned} \omega \otimes \omega &= \frac{1}{4}\rho \otimes \rho \otimes |\varphi_1\rangle\langle\varphi_1|^B \otimes |\varphi_1\rangle\langle\varphi_1|^B \\ &\quad + \frac{1}{4}\rho \otimes \sigma \otimes |\varphi_1\rangle\langle\varphi_1|^B \otimes |\varphi_2\rangle\langle\varphi_2|^B \\ &\quad + \frac{1}{4}\sigma \otimes \rho \otimes |\varphi_2\rangle\langle\varphi_2|^B \otimes |\varphi_1\rangle\langle\varphi_1|^B \\ &\quad + \frac{1}{4}\sigma \otimes \sigma \otimes |\varphi_2\rangle\langle\varphi_2|^B \otimes |\varphi_2\rangle\langle\varphi_2|^B. \end{aligned} \quad (17)$$

From the definition of flag bases, we can easily see that the tensor product of flag bases is still a flag basis. Hence, $\{|\varphi_i\rangle^B|\varphi_j\rangle^B\}_{i,j=1}^2$ is still a flag basis. Applying the flag additivity to Eq. (17), we get that

$$\begin{aligned} M(\omega \otimes \omega) &= \frac{1}{4}M(\rho \otimes \rho) + \frac{1}{4}M(\rho \otimes \sigma) \\ &\quad + \frac{1}{4}M(\sigma \otimes \rho) + \frac{1}{4}M(\sigma \otimes \sigma) \\ &= \frac{1}{4}M(\rho \otimes \rho) + \frac{1}{2}M(\rho \otimes \sigma) + \frac{1}{4}M(\sigma \otimes \sigma), \end{aligned} \quad (18)$$

where we have used the relation that $M(\rho \otimes \sigma) = M(\sigma \otimes \rho)$, as they are the same state under reordering $\mathcal{H}_A \otimes \mathcal{H}_A$. Combining Eqs. (14), (16), and (18), we can obtain that

$$\begin{aligned} M(\rho \otimes \sigma) &= 4M(\omega) - M(\rho) - M(\sigma) \\ &= M(\rho) + M(\sigma), \end{aligned} \quad (19)$$

which is the full additivity for $\rho, \sigma \in \mathcal{H}_A$.

We then consider the general case, $\rho \in \mathcal{H}_{A_1}$ and $\sigma \in \mathcal{H}_{A_2}$. Suppose that δ_1 and δ_2 are two free states in \mathcal{H}_{A_1} and \mathcal{H}_{A_2} , respectively. Consider the states

$$\tilde{\rho} = \rho \otimes \delta_2, \quad \tilde{\sigma} = \delta_1 \otimes \sigma;$$

then both $\tilde{\rho}$ and $\tilde{\sigma}$ are states in the quantum system $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$. Then, applying Eq. (19) to $\tilde{\rho}$ and $\tilde{\sigma}$, we get that

$$M(\rho \otimes \delta_2 \otimes \delta_1 \otimes \sigma) = M(\rho \otimes \delta_2) + M(\delta_1 \otimes \sigma). \quad (20)$$

By using conditions (T1) and (T2), we can get that appending and discarding a free state are both free operations. We immediately obtain that the resource measure M does not change for appending or discarding a free state according to condition (M2). Hence, Eq. (20) implies that

$$M(\rho \otimes \sigma) = M(\rho) + M(\sigma), \quad (21)$$

which is the full additivity for any $\rho \in \mathcal{H}_{A_1}$ and $\sigma \in \mathcal{H}_{A_2}$. Thus, we prove that Eq. (14) \Rightarrow Eq. (13). In summary, we get the following theorem:

Theorem 2. For any resource measure, the flag additivity implies the equivalence between the additivity and the full additivity.

V. FLAG ADDITIVITY FOR REGULARIZED RESOURCE MEASURES

In general, the converse of Theorem 2 is not true. The equivalence of additivity and full additivity (i.e., the full additivity itself) does not imply the flag additivity, while it holds for an important class of resource measures, the regularized resource measures.

Suppose M is a resource measure; then the regularization of M is defined as

$$M^\infty(\rho) = \lim_{N \rightarrow \infty} \frac{M(\rho^{\otimes N})}{N}. \quad (22)$$

The most important example of regularized resource measures is the entanglement cost, which is the regularization of the entanglement of formation [27]. To study the asymptotic property of $M(\rho^{\otimes N})/N$, as $N \rightarrow \infty$, we need to assume a special kind of continuity called the asymptotic continuity [14,28–31]. A resource measure M is said to be asymptotically continuous if it satisfies that for all states ρ and σ in the Hilbert space \mathcal{H} ,

$$\|\rho - \sigma\|_{\text{tr}} \rightarrow 0 \Rightarrow \frac{|M(\rho) - M(\sigma)|}{\log_2 \dim(\mathcal{H})} \rightarrow 0, \quad (23)$$

where $\dim(\mathcal{H})$ is the dimension of the Hilbert space \mathcal{H} , and $\|\rho - \sigma\|_{\text{tr}}$ is the trace distance between ρ and σ [32].

For regularized resource measures, we have the following theorem:

Theorem 3. Suppose M is a resource measure satisfying the flag additivity and the asymptotic continuity. Then the regularized measure M^∞ is flag additive if and only if it is fully additive.

From the definition of the regularized resource measure M^∞ , we can easily see that it automatically satisfies the additivity condition defined as Eq. (12). Then, the necessity follows directly from Theorem 2.

To prove the sufficiency, we only need to consider the special flagged state,

$$\rho = p_1 \rho_1 \otimes |\varphi_1\rangle\langle\varphi_1| + p_2 \rho_2 \otimes |\varphi_2\rangle\langle\varphi_2|, \quad (24)$$

where ρ_1 and ρ_2 are any two states in a Hilbert space \mathcal{H} , $p_1 + p_2 = 1$, and $\{|\varphi_1\rangle, |\varphi_2\rangle\}$ is a flag basis. The proof can be generalized to the general flagged state.

Without loss of generality, we may assume that $p_1 \leq 1/2$. According to the information theory, for any $\delta > 0$ and $\varepsilon > 0$, $\rho^{\otimes N}$ can be written as

$$\rho^{\otimes N} = (1 - \varepsilon)\rho_{\text{typ}} + \varepsilon\rho_{\text{atyp}}, \quad (25)$$

when N is large enough [33,34]. In Eq. (25), the typical part ρ_{typ} is defined as

$$\rho_{\text{typ}} = \frac{1}{T} \sum_{k=\lfloor Np_1(1-\delta) \rfloor}^{\lfloor Np_1(1+\delta) \rfloor} \mathcal{S}[(\rho_1 \otimes |\varphi_1\rangle\langle\varphi_1|)^{\otimes k} \otimes (\rho_2 \otimes |\varphi_2\rangle\langle\varphi_2|)^{\otimes (N-k)}], \quad (26)$$

where T is a normalization factor and \mathcal{S} is the symmetrized tensor product. For example, $\mathcal{S}(\tilde{\rho}_1 \otimes \tilde{\rho}_1 \otimes \tilde{\rho}_2) = \tilde{\rho}_1 \otimes \tilde{\rho}_1 \otimes \tilde{\rho}_2 + \tilde{\rho}_1 \otimes \tilde{\rho}_2 \otimes \tilde{\rho}_1 + \tilde{\rho}_2 \otimes \tilde{\rho}_1 \otimes \tilde{\rho}_1$, where $\tilde{\rho}_i = \rho_i \otimes |\varphi_i\rangle\langle\varphi_i|$. As $k \geq \lfloor Np_1(1-\delta) \rfloor$, we can get that $N - k \leq N - \lfloor Np_1(1-\delta) \rfloor = \lceil N - Np_1(1-\delta) \rceil \leq \lceil Np_2(1+\delta) \rceil$, where

we have used the condition that $p_1 \leq 1/2$. Similarly, we can obtain that $N - k \geq N - \lceil Np_1(1+\delta) \rceil = \lfloor N - Np_1(1+\delta) \rfloor \geq \lfloor Np_2(1-\delta) \rfloor$. Thus, there is $\lfloor Np_2(1-\delta) \rfloor \leq N - k \leq \lceil Np_2(1+\delta) \rceil$. Then, conditions (T2) and (M2) imply that

$$M(\rho_1^{\otimes \lfloor Np_1(1-\delta) \rfloor} \otimes \rho_2^{\otimes \lfloor Np_2(1-\delta) \rfloor}) \leq M(\rho_1^{\otimes k} \otimes \rho_2^{\otimes (N-k)}) \leq M(\rho_1^{\otimes \lceil Np_1(1+\delta) \rceil} \otimes \rho_2^{\otimes \lceil Np_2(1+\delta) \rceil}). \quad (27)$$

From the definition of flag bases, we can easily see that the tensor product of flag bases is still a flag basis. Thus, by combining the flag additivity of M and Eq. (27), we can get that

$$M(\rho_1^{\otimes \lfloor Np_1(1-\delta) \rfloor} \otimes \rho_2^{\otimes \lfloor Np_2(1-\delta) \rfloor}) \leq M(\rho_{\text{typ}}) \leq M(\rho_1^{\otimes \lceil Np_1(1+\delta) \rceil} \otimes \rho_2^{\otimes \lceil Np_2(1+\delta) \rceil}). \quad (28)$$

Let m_1, m_2 , and m be positive integers such that

$$p_1(1-\delta) = \frac{m_1}{m} + \varepsilon_1, \quad p_2(1-\delta) = \frac{m_2}{m} + \varepsilon_2, \quad (29)$$

where the positive parameters ε_1 and ε_2 converge to zero when we choose suitable and large enough integers, m_1, m_2 , and m . Choose N to be $N'm$, where N' are also large enough integers. Then, the first inequality in Eq. (28) implies that

$$M(\rho_{\text{typ}}) \geq M(\rho_1^{\otimes N'm_1} \otimes \rho_2^{\otimes N'm_2}). \quad (30)$$

When $\varepsilon \rightarrow 0$ ($N \rightarrow \infty$), $\|\rho^{\otimes N} - \rho_{\text{typ}}\| \rightarrow 0$. Hence, the asymptotic continuity implies that

$$\lim_{N \rightarrow \infty} \frac{1}{N} M(\rho^{\otimes N}) = \lim_{N' \rightarrow \infty} \frac{1}{N'm} M(\rho^{\otimes N'm}) = \lim_{N' \rightarrow \infty} \frac{1}{N'm} M(\rho_{\text{typ}}). \quad (31)$$

Combining Eqs. (30) and (31), we have

$$M^\infty(\rho) \geq \frac{1}{m} M^\infty(\rho_1^{m_1} \otimes \rho_2^{m_2}). \quad (32)$$

Then if M^∞ is fully additive, we have

$$M^\infty(\rho) \geq \frac{m_1}{m} M^\infty(\rho_1) + \frac{m_2}{m} M^\infty(\rho_2). \quad (33)$$

Let $\varepsilon_1, \varepsilon_2 \rightarrow 0$ ($m_1, m_2, m \rightarrow \infty$), we get the flag supadditivity of M^∞ ,

$$M^\infty(\rho) \geq p_1 M^\infty(\rho_1) + p_2 M^\infty(\rho_2). \quad (34)$$

Similarly, we can get the flag subadditivity of M^∞ from the second inequality in Eq. (28),

$$M^\infty(\rho) \leq p_1 M^\infty(\rho_1) + p_2 M^\infty(\rho_2), \quad (35)$$

when M^∞ is fully additive. Combining Eqs. (34) and (35), we get the flag additivity,

$$M^\infty(\rho) = p_1 M^\infty(\rho_1) + p_2 M^\infty(\rho_2). \quad (36)$$

Thus, this completes the proof of the Theorem 3.

When Theorem 3 is applied to the entanglement measures, we immediately get the equivalence between flag additivity and full additivity for regularized entanglement measures.

This is a generalization of the result in Ref. [34], where an additional condition (the subadditivity) is assumed.

VI. CONCLUSIONS

In this paper, we have introduced the notion of the flag basis and defined the flag additivity for general quantum resources. To illustrate the usefulness of the flag additivity, we have shown that it can be used to derive other nontrivial properties in quantum resource theories. As examples, we have proved that the flag additivity holds if and only if both the strong monotonicity and the convexity hold at the same time, the flag additivity implies the equivalence between the additivity and the full additivity, and for regularized resource

measures the flag additivity is equivalent to the full additivity. We think the technique of flag additivity together with its refinements, flag subadditivity and flag supadditivity, will become a fundamental tool for studying the resource measures.

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