Complete and incomplete state estimation via the simultaneous unsharp measurement of two incompatible qubit operators

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We consider the simultaneous coupling of a qubit system to two qubit probes, designed to measure noncommuting qubit observables when working in isolation. While the single meter model usually corresponds to an unsharp measurement of a qubit operator, the double meter model usually describes an informationally complete measurement. The double meter positive operator measurement can be characterized by three vectors associated with the space of Hermitian traceless observables of the system. The dimension of the linear space generated by these vectors (Bloch rank) is equal to the number of independent operators (or the number of independent Bloch-vector components) which can be estimated. $\mathbb{S}$, the subspace of parameters associated with informationally incomplete measurements (IICs), is characterized either analytically or numerically, and their Bloch rank is used to classify the IICs. Bloch rank of the "simultaneous measurement" of two Pauli matrices is shown to be 2 if the isolated measurements are projective and 3 if they are weak; however, two-meter weak measurements are expected to converge more slowly than their single-meter counterparts.

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I. INTRODUCTION

The historic axiomatic description of quantum measurements, based on the Born rule [1] and the wave function collapse [2], renounces the understanding of the mensuration process. However, von Neumann showed that it is possible to model the measuring instrument and the system actually observed using the quantum formalism [3]. Subsequent refinements to the von Neumann model of projective measurements include the Lüder's rule to update the state [4] and the demonstration that decoherence can explain the transition from an initial pure state to a mixture of eigenstates of the observable being measured [5]. It has been shown that the dynamics of the system alone, after the degrees of freedom describing the probe and its environment have been traced out, can be described by a positive operator valued measure (POVM) [6,7]. Mensuration schemes more general than projective measurements, such as imperfect measurement setups employing nonideal detectors, informationally incomplete tomographic protocols [8], and weak measurements [9–11], can all be represented by POVMs. Despite recent proposals based on nonlinear stochastic variants of the Schrödinger equation [12], the description of the transition to a single eigenstate is still considered an open problem, which will not be addressed in this work.

The theoretical discussion of the simultaneously measurement of noncompatible observables has a long history in quantum mechanics, which can be traced back to Heisenberg's microscope [2]. After the first discussion of this problem using the full quantum formalism [13], many papers have been published on the subject, highlighting its different aspects. For example, Ref. [14] discusses the joint unbiasedness condition (jointly measured expectation values proportional to those separately measured), and Ref. [15] analyzes how quantum cloning can be exploited to measure incompatible observables. Continuous monitoring of incompatible observables, closely related to the use of weak measurements to estimate quantum states [16], was recently proposed [17] and tested [18].

The objective of this work is to describe the attempt to simultaneously measure two noncommuting qubit observables by introducing two qubit meters, an approach similar to that of Ref. [13]. Because our treatment is simpler than that of Ref. [19], which considers a complete microscopic model to describe the simultaneous measurement of two Pauli matrices, we are able to perform a more systematic exploration of the parameter space of our model. (However, we are not able to describe inhibition of registration of one probe due to a strong coupling of the system to the other probe.) Once the meters's states are ignored, our approach becomes similar to that of Ref. [20], where a simple form of a universal observable [21] was found. Universal observables are those which permit complete tomography.

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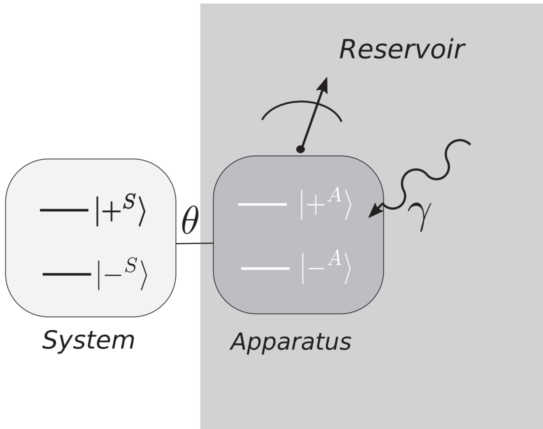

FIG. 1. von Neumann model of the measurement of a qubit observable. The system-probe interaction is characterized by the parameter θ. The probe also interacts with a reservoir and loses coherence at the rate γ.

The next section of this paper shows that the von Neumann model employed here describes a family of asymmetric unsharp measurements [22]. When two meters, each of them designed to perform an unsharp measurement of a single qubit (in absence of the other), are simultaneously coupled to the system, generally two observables can be estimated (Sec. III); moreover, if coincidences between the outcomes of both apparatuses are taken into account, the initial state of the system can be estimated (Sec. IV). We give a formula to estimate the system's Bloch vector in terms of the probabilities of the four possible outcomes valid almost everywhere in the space of parameters. Bloch rank, the number of components of the system's Bloch vector which can be estimated by a measurement or the number of independent observables that can be measured, is used to classify IICs (informationally incomplete measurements) in Sec. V. Special consideration is given to the possibility to have simultaneous projective or weak measurements (Sec. VI). Finally, some conclusions are drawn.

II. UNSHARP ASYMMETRIC MEASUREMENT OF A QUBIT OPERATOR

Figure 1 illustrates a setup in which a qubit probe measures an observable of a two-level system. We employ a von Neumann model of the mensuration process, where the system interacts with a meter. The probe is assumed to lose coherence as a consequence of its interaction with a reservoir. We take the stance that when a measurement of a quantum system is performed in the laboratory, a meter interacts with the system and leaves an objective, indelible record; this process is described by decoherence only in a statistical sense. Since it is necessary to consider an ensemble of identically prepared systems, it is natural to assume that an observable is measured when its probability distribution (PD) can be estimated. Qubit observables can be written as $O = \sum_{\mu=0}^{3} c_\mu \sigma_\mu = c_0 I + \boldsymbol{c}\cdot\boldsymbol{\sigma}$, where c_μ are real coefficients, $\sigma_0 = I$ is the identity matrix, and σ_i, $i = 1, 2, 3$, are the well-known Pauli's matrices. However, it is enough to consider operators of the form $\tilde{O}(\boldsymbol{n}) = \boldsymbol{n}\cdot\boldsymbol{\sigma}$, where $n_i = c_i/\sqrt{c_1^2 + c_2^2 + c_3^2}$, because the PD of O can be obtained from the PD of $\tilde{O}(\boldsymbol{n})$. In this paper emphasis is made on the vectors $\boldsymbol{c} = (c_1, c_2, c_3)^T$ used to defined the observables $\boldsymbol{c}\cdot\boldsymbol{\sigma}$. However, we could have employed the operators themselves, just by using the identities $\boldsymbol{a}\cdot\boldsymbol{b} = \frac{1}{2}\,\mathrm{Tr}[(\boldsymbol{a}\cdot\boldsymbol{\sigma})^\dagger \boldsymbol{b}\cdot\boldsymbol{\sigma}]$ and $\boldsymbol{a}\times\boldsymbol{b} = \frac{1}{4i}\,\mathrm{Tr}\,([\boldsymbol{a}\cdot\boldsymbol{\sigma}, \boldsymbol{b}\cdot\boldsymbol{\sigma}]\boldsymbol{\sigma})$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ were assumed to be real.

To measure the observable $\tilde{O}(\boldsymbol{n})$ one usually considers a system-probe Hamiltonian $g(t)\boldsymbol{n}\cdot\boldsymbol{\sigma}^S \otimes \sigma_z^A$. Superscripts S and A are used to denote states and operators belonging, respectively, to the observed system and to the apparatus, and $\otimes$ represents the tensor product. Following Peres [23,24], we consider a slightly different interaction Hamiltonian

$$H(t) = \frac{g(t)}{4}(I^S + \boldsymbol{n}\cdot\boldsymbol{\sigma}^S) \otimes \left(I^A + \sigma_z^A\right). \tag{1}$$

The interaction Hamiltonian can be recast as a product of projectors over the states $|+_{\boldsymbol{n}}^S\rangle$ and $|+^A\rangle$, the eigenstates of $\boldsymbol{n}\cdot\boldsymbol{\sigma}^S$ and σ_z^A with unity eigenvalue,

$$H(t) = g(t)\Pi_{+\boldsymbol{n}}^S \otimes \Pi_+^A = g(t)\big|+_{\boldsymbol{n}}^S\big\rangle\big\langle+_{\boldsymbol{n}}^S\big| \otimes |+^A\rangle\langle+^A|. \tag{2}$$

For simplicity, the tensor products and the superscripts will be omitted when no confusion arises.

The interaction is assumed to be turned on at the initial time $t = 0$ and turned off at some time $t_f = T$. Hence, the function $g(t)$ vanishes outside this time interval, considered to be very small as compared to all other relevant timescales. The probe is assumed to interact with a reservoir which causes the system-probe state to lose coherence. The dynamics of the global state ρ is given by

$$\begin{aligned} \frac{d\rho}{dt} &= \frac{1}{i\hbar}[H(t), \rho] + \gamma\left(\sigma_{\boldsymbol{m}}^A \rho \sigma_{\boldsymbol{m}}^A - \rho\right) \\ &= \left(\frac{1}{i\hbar}[H(t), \cdot] + \mathcal{L}_D\right)\rho, \end{aligned} \tag{3}$$

with γ and $\hbar$ denoting, respectively, the decoherence rate and Planck's constant. Here $\sigma_{\boldsymbol{m}}^A$ is shorthand for $I \otimes \boldsymbol{m}\cdot\boldsymbol{\sigma}^A$, and $\mathcal{L}_D = \gamma(\sigma_{\boldsymbol{m}}^A \cdot \sigma_{\boldsymbol{m}}^A - 1)$, where we have employed the dot $(\cdot)$ convention. The place of the dot is occupied by the operator to its right. For example, $(A \cdot B)\rho = A\rho B$. We consider a factorizing initial state

$$\rho(0) = \rho^S(0) \otimes \big|+_{\boldsymbol{m}'}^A\big\rangle\big\langle+_{\boldsymbol{m}'}^A\big|,$$

where the initial system state $\rho^S(0)$ is arbitrary. The initial state of the probe $|+_{\boldsymbol{m}'}^A\rangle$ is the eigenstate of $\boldsymbol{m}'\cdot\boldsymbol{\sigma}^A$ with unity eigenvalue.

In the remainder of the section we will focus on the measurement of the operator $\sigma_z = \sigma_3$. Moreover, we choose $\boldsymbol{m} = \boldsymbol{e}_x = \boldsymbol{m}'$, so that $\boldsymbol{m}'\cdot\boldsymbol{\sigma}^A = \sigma_x$, and $|+_{\boldsymbol{m}'}^A\rangle = \frac{1}{\sqrt{2}}(|+^A\rangle + |-^A\rangle)$.

Although the interval $[0, T]$ is considered to be very small, the average value of the coupling function $g(t)$ is considered to be much larger than all other relevant energy scales in $[0, T]$. Under this assumption, the state at a time $t > T$ can be approximated by

$$\begin{aligned} \rho(t) &\approx e^{\mathcal{L}_D t} e^{\frac{1}{i\hbar}\int_0^T [H(\tau), \cdot]d\tau}\rho(0) \\ &= e^{-\gamma t} e^{\gamma\sigma_x^A \cdot \sigma_x^A t} e^{-i\theta\Pi_+^S\Pi_+^A}\rho(0)e^{i\theta\Pi_+^S\Pi_+^A} \\ &= e^{-\gamma t}\sum_{n=0}^{\infty}\frac{(\gamma t)^n}{n!}\left(\sigma_x^A\right)^n\rho(T)\left(\sigma_x^A\right)^n, \end{aligned} \tag{4}$$

where $\rho(T) = (e^{-i\theta\Pi_+^S\Pi_+^A}\rho(0)e^{i\theta\Pi_+^S\Pi_+^A})$ and $\theta = \frac{1}{\hbar}\int_0^T g(t)dt$.

To simplify the intermediate algebra, we consider an initial pure system state, where $\rho^S(0) = |\psi_0\rangle\langle\psi_0|$ and $|\psi_0\rangle = \alpha|+\rangle + \beta|-\rangle$. The joint state is also pure at time T, $\rho(T) = |\Psi(T)\rangle\langle\Psi(T)|$. After the system-probe interaction is turned on and then off, $|\Psi\rangle$, the joint system-probe state becomes entangled,

$$|\Psi(T)\rangle = e^{-i\theta\Pi_+^S\Pi_+^A}|\psi_0\rangle \otimes |+\rangle_x = |s_+\rangle|+\rangle + |s_-\rangle|-\rangle, \quad (5)$$

where $|s_+\rangle = \frac{1}{\sqrt{2}}(e^{-i\theta}\alpha|+\rangle + \beta|-\rangle)$ and $|s_-\rangle = \frac{1}{\sqrt{2}}(\alpha|+\rangle + \beta|-\rangle)$. We have omitted the S and A superscripts in the understanding that the first ket label refers to the system and the second to the apparatus. The interaction with the environment produces decoherence: the off-diagonal elements of the density state, written in the basis of the eigenvalues of σ_x^A, are suppressed. For times long compared to $1/\gamma$, where γ is the decoherence rate, the joint statistical operator becomes

$$\rho_\infty = |d_+\rangle\langle d_+| \otimes |+_x\rangle\langle +_x| + |d_-\rangle\langle d_-| \otimes |-_x\rangle\langle -_x|, \quad (6)$$

where the states $|d_+\rangle$ and $|d_-\rangle$ are the unnormalized symmetric and antisymmetric superpositions of the states $|s_+\rangle$ and $|s_-\rangle$,

$$|d_+\rangle = (|s_+\rangle + |s_-\rangle)/\sqrt{2} = e^{-i\theta/2}\cos(\theta/2)\alpha|+\rangle + \beta|-\rangle,$$
$$|d_-\rangle = (|s_+\rangle - |s_-\rangle)/\sqrt{2} = -ie^{-i\theta/2}\sin(\theta/2)\alpha|+\rangle.$$

The final state of the apparatus is $|+_x\rangle(|-_x\rangle)$ with probability P_+ (P_-), where

$$P_+ = \langle d_+|d_+\rangle = \cos^2(\theta/2)|\alpha|^2 + |\beta|^2 \quad (7)$$

and

$$P_- = \langle d_-|d_-\rangle = 1 - P_+ = \sin^2(\theta/2)|\alpha|^2. \quad (8)$$

When the apparatus ends up in the state $|-_x\rangle$, the final state of the system is the state $|+\rangle$, an eigenstate of the third component of the Bloch vector. However, when the final state of the apparatus is $|+_x\rangle$, the system ends up in a superposition of both eigenstates of the third component of the Bloch vector, unless θ is an odd multiple of π (when the system's final state is $|-\rangle$). Except when $\theta = 2n\pi$, $n = 0, 1, 2, \dots$, probabilities P_+ and P_- allow the estimation of the probability distribution of σ_z^S, for the initial state of the system, that is, they allow us to infer the probabilities of $|+^S\rangle$ and $|-^S\rangle$, as well as the third component of the Bloch vector,

$$P(|+^S\rangle) = |\alpha|^2 = \frac{P_-}{\sin^2(\theta/2)}, \quad (9)$$

$$P(|-^S\rangle) = |\beta|^2 = P_+ - P_-\cot^2(\theta/2), \quad (10)$$

$$\langle\sigma_z^S\rangle = |\alpha|^2 - |\beta|^2 = c_1P_+ + c_2(\theta)P_-$$
$$= -P_+ + \frac{3+\cos\theta}{1-\cos\theta}P_-. \quad (11)$$

The function $c_2(\theta)$, of period 2π, becomes very large for θ near even multiples of π, and attains its minimum, unity, when θ is an odd multiple of π (see Fig. 2). The former case describes a weak measurement, understood as a one in which, with large probability, the system's state barely changes. Of

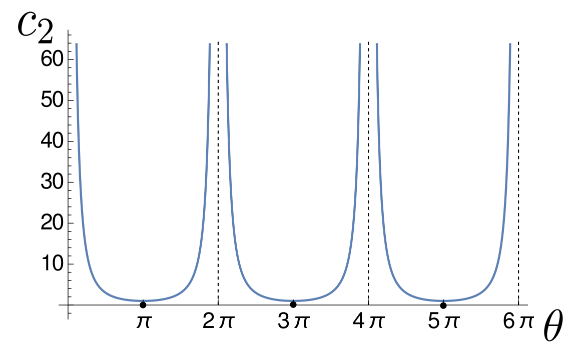

FIG. 2. Coefficient $c_2(\theta)$ as a function of the parameter θ.

course, when θ is an exact even multiple of π, the final state of the meter is always $|+\rangle$, and no information about the system state is obtained. The latter case corresponds to a projective measurement.

Up to now we have taken into account both the measuring apparatus and the measured system. However, we can make a description from the point of view of the observed system alone. In this case, the initial state of the system ρ_0^S is transformed into a final state ρ_f^S,

$$\rho_f^S = |d_+\rangle\langle d_+| + |d_-\rangle\langle d_-|. \quad (12)$$

The unnormalized states $|d_+\rangle$ and $|d_-\rangle$ are linear transformations of the initial pure state of the system

$$|d_+\rangle = e^{-i\theta/2}\cos(\theta/2)\alpha|+\rangle + \beta|-\rangle$$
$$= \underbrace{\left(e^{-i\frac{\theta}{2}}\cos\frac{\theta}{2}|+\rangle\langle+| + |-\rangle\langle-|\right)}_{E_+}\underbrace{(\alpha|+\rangle + \beta|-\rangle)}_{|\psi_0\rangle}, \quad (13)$$

$$|d_-\rangle = -ie^{-i\frac{\theta}{2}}\sin\frac{\theta}{2}|+\rangle\langle+||\psi_0\rangle = E_-|\psi_0\rangle. \quad (14)$$

The Kraus operators E_i, $i = \pm$, allow us to write the final system state in the form of an operator sum representation

$$\rho_f^S = \sum_{i=\pm} E_i|\psi_0\rangle\langle\psi_0|E_i^\dagger. \quad (15)$$

Since any initial system state can be written as a statistical superposition of pure states, $\rho_0^S = \sum_m p_m|\psi_{0m}\rangle\langle\psi_{0m}|$, and given that the transformation (15) is linear, the final system state can be written in terms of the initial state as

$$\rho_f^S = \sum_m p_m \sum_i E_i|\psi_{0m}\rangle\langle\psi_{0m}|E_i^\dagger = \sum_i E_i\rho_0^S E_i^\dagger. \quad (16)$$

It is usual to define also the measurement operators $M_i = E_i^\dagger E_i$,

$$M_+ = \frac{1}{2}\left(1 + \cos^2\frac{\theta}{2}\right)I - \frac{1}{2}\sin^2\frac{\theta}{2}\sigma_z, \quad (17)$$

$$M_- = \frac{1}{2}\sin^2\frac{\theta}{2}I + \frac{1}{2}\sin^2\frac{\theta}{2}\sigma_z. \quad (18)$$

They satisfy the sum rule $\sum_i M_i = \sum_i E_i^\dagger E_i = I^S$, a nonorthogonal resolution of the identity [25]. Transformations of the form of Eq. (16) are known as positive operator-valued measures (POVMs), positive operator measures [26],

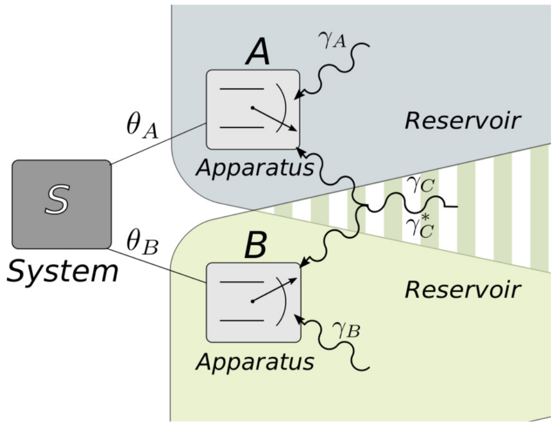

FIG. 3. The simultaneous coupling of the system to two meters attempts to measure two noncommutative qubit observables.

or probability operator measures (POMs) [27]. These POVM elements only give information about the observable σ_z as can be seen from the expressions (17) and (18). Using the terminology of Konrad [22], the scheme presented in this section is an unsharp measurement, i.e., the POVM elements M_i mutually commute and not all of them are projection operators. This measurement presented in this work is not symmetric in the exchange of system's states $|+\rangle$ and $|-\rangle$.

The probabilities P_i, $i=\pm$, that the meter ends up in the state $|i_x\rangle$, can be given in terms of the measurement operators M_i. Indeed, for a pure system's initial state $|\psi_0\rangle$, this probability is $P_i = \langle d_i|d_i\rangle = \mathrm{Tr}(E_i^\dagger E_i|\psi_0\rangle\langle\psi_0|)$, where the form of the post-measurement joint statistical operator (6) was taken into account. The probability P_i can be generalized to initial mixed states ρ_0^S,

$$P_i = \mathrm{Tr}\big(E_i^\dagger E_i \rho_0^S\big) = \mathrm{Tr}\big(M_i \rho_0^S\big), \quad i=\pm.$$

III. SEPARATED SIMULTANEOUS MEASUREMENTS

In the previous section we saw how Peres' interaction Hamiltonian leads to an unsharp measurement of a qubit observable. Would we obtain twice as much data using the same number of copies of a system, if we try to simultaneously perform unsharp measurements of two observables. In the context of this paper, we model the attempt to measure the operators σ_z and σ_x, using meters A and B, respectively. Our results show that each apparatus measures an operator, usually not the intended one. There are instances in which σ_z or σ_x can be measured, and instances in which both meters measure exactly the same operator.

Figure 3 sketches the situation we consider in this section. The effective dynamics of the global state ρ, which includes the system and both meters, is given by

$$\frac{d\rho}{dt} = \frac{i}{\hbar}[K(t),\rho] + \mathcal{L}_D^{AB}\rho, \tag{19}$$

where $K(t)$ is the system-probes Hamiltonian and $\mathcal{L}_D^{AB}$ describes the decoherence process. If each apparatus is assumed to interact with its own reservoir, we can write $\mathcal{L}_D^{AB}\rho = \mathcal{L}_D^{A}\rho + \mathcal{L}_D^{B}\rho = \gamma_A(\sigma_x^A\rho\sigma_x^A - \rho) + \gamma_B(\sigma_x^B\rho\sigma_x^B - \rho)$, where γ_A and γ_B are the decoherence rates associated with the probes A and B. If the reservoirs of both meters are not completely independent, but it is still possible to describe the decoherence process of the system by a Markovian master equation with constant coefficients, we can write [28]

$$\mathcal{L}_D^{AB} = \mathcal{L}_D^{A} + \mathcal{L}_D^{B} + \frac{\gamma_C}{2}\big(2\sigma_x^A\cdot\sigma_x^B - \sigma_x^B\sigma_x^A\cdot - \cdot\,\sigma_x^B\sigma_x^A\big) + \frac{\gamma_C^*}{2}\big(2\sigma_x^B\cdot\sigma_x^A - \sigma_x^A\sigma_x^B\cdot - \cdot\,\sigma_x^A\sigma_x^B\big), \tag{20}$$

where we have employed the dot convention. The complex coefficient γ_C satisfies the inequality $|\gamma_C|^2 \leqslant \gamma_A\gamma_B$ [28].

The simultaneous interaction of the observed system with the measuring apparatuses is described by the Hamiltonian

$$\begin{aligned} K(t) &= \tfrac{1}{4}g_A(t)\big(I^S+\sigma_z^S\big)\big(I^A+\sigma_z^A\big) \\ &\quad + \tfrac{1}{4}g_B(t)\big(I^S+\sigma_x^S\big)\big(I^B+\sigma_z^B\big) \\ &= g_A(t)\Pi_+^S\Pi_+^A + g_B(t)\Pi_{+x}^S\Pi_+^B, \end{aligned} \tag{21}$$

where g_A and g_B are much larger than any other energies (in the common interval $[0,T]$ where they do not vanish). Our conventions for the projectors are similar to those of the previous section. For example, Π_{+x}^S designates the projection over the system's state $|+_x^S\rangle$, the eigenstate of σ_x^S with positive eigenvalue. The initial state of the probes and the observed system is assumed to be $\rho(0) = \rho_S(0)|+_x^A,+_x^B\rangle\langle+_x^A,+_x^B|$.

The basic ingredients of the model described so far are the decoherence basis (eigenstates of $\sigma_x^A\otimes\sigma_x^B$), the system-meters Hamiltonian $[K(t)]$, and the initial states of the meters $(|+_x^A,+_x^B\rangle\langle+_x^A,+_x^B|)$. This is a particular but typical model, in the sense that its qualitative features hold for most choices of decoherence basis, Hamiltonian, and initial meters's state, provided each meter in isolation is able to measure a system's observable.

As in the previous section, without loss of generality and in order to simplify the calculations, we consider the initial state of the system to be pure; thus, the joint initial state is also pure, $|\psi_0\rangle = (\alpha|+^S\rangle + \beta|-^S\rangle)\otimes|+_x^A\rangle|+_x^B\rangle$. Since $g_A, g_B \gg \gamma_A, \gamma_B$ we can approximate the statistical operator, for $t>T$, by

$$\rho(t) \approx e^{\mathcal{L}_D^{AB}t}\mathcal{T}\big(e^{\frac{1}{i\hbar}\int_0^T d\tau[K(\tau),\cdot]}\big)\rho(0) = e^{\mathcal{L}_D^{AB}t}\rho(T). \tag{22}$$

We have used Dyson's time-ordering operator $\mathcal{T}$, which puts operators at earliest times to the right of operators at later times,

$$\mathcal{T}(K(t_1)K(t_2)) = \begin{cases} K(t_1)K(t_2) & \text{if } t_1 \geqslant t_2, \\ K(t_2)K(t_1) & \text{if } t_2 > t_1. \end{cases} \tag{23}$$

If the initial state is pure, the statistical operator $\rho(T) = |\Psi(T)\rangle\langle\Psi(T)|$ is also pure. The state $|\Psi(T)\rangle$ can be written as

$$|\Psi(T)\rangle = \frac{1}{2}\sum_{m,n=\pm} U_{mn}\big|\psi_0^S\big\rangle|m^A,n^B\rangle, \tag{24}$$

where $U_{--} = I$,

$$U_{+-} = e^{-i\theta_A(I^S+\sigma_z^S)/2} = e^{-i\theta_A\Pi_+^S},$$

$$U_{-+} = e^{-i\theta_B(I^S+\sigma_x^S)/2} = e^{-i\theta_B/2}\big[\cos(\theta_B/2)I^S - i\sin(\theta_B/2)\sigma_x^S\big],$$

where $\theta_A = \frac{1}{\hbar}\int_0^T d\tau\, g_A(\tau)$ and $\theta_B = \frac{1}{\hbar}\int_0^T d\tau\, g_B(\tau)$. Finally,

$$U_{++} = \mathcal{T}\big(e^{\frac{1}{i\hbar}\int_0^T d\tau[g_A(\tau)(I^S+\sigma_z^S)/2 + g_B(\tau)(I^S+\sigma_x^S)/2]}\big).$$

The precise form of U_{++} depends on the details of the functions $g_A(t)$ and $g_B(t)$. We focus in the case defined by $g_\alpha(t) = \lambda_\alpha g(t)$, where λ_α, $\alpha = A, B$, are constants. In this case we can give a more explicit expression for U_{++},

$$\begin{aligned} U_{++} &= e^{-i\theta_A(I^S+\sigma_z^S)/2 - i\theta_B(I^S+\sigma_x^S)/2} \\ &= e^{-i\frac{\theta_A+\theta_B}{2}}\left(\cos\frac{\theta_C}{2}I^S - i\frac{\sin\frac{\theta_C}{2}}{\theta_C}\left(\theta_A\sigma_z^S + \theta_B\sigma_x^S\right)\right), \end{aligned}$$

where θ_A and θ_B have already been defined and $\theta_C = \sqrt{\theta_A^2 + \theta_B^2}$. Besides the expected contributions in which only one apparatus couples with the observed system U_{+-} and U_{-+}, there is a contribution U_{++} in which both measuring devices simultaneously interact with the system.

At times much larger than the inverses of the decoherence rates, the global statistical operator is given by

$$\rho_\infty = \sum_{k,l=\pm} |d_{kl}\rangle\langle d_{kl}| \otimes \left|k_x^A l_x^B\right\rangle\!\left\langle k_x^A l_x^B\right|.$$

Each unnormalized system state $|d_{kl}\rangle$ can be written as a linear transformation of the initial pure state of the system

$$|d_{kl}\rangle = \frac{1}{2}\sum_{m,n=\pm}\left\langle k_x^A l_x^B\middle|m^A n^B\right\rangle U_{mn}\left|\psi_0^S\right\rangle = E_{kl}\left|\psi_0^S\right\rangle.$$

Here the Kraus operators E_{kl} are given by the following linear combinations of the unitary operators U_{mn},

$$E_{kl} = \tfrac{1}{4}(U_{++} + lU_{+-} + kU_{-+} + klI), \quad k, l = \pm 1,$$

where the overlaps $\langle \pm_x|+\rangle = \frac{1}{\sqrt{2}}$ and $\langle \pm_x|-\rangle = \pm\frac{1}{\sqrt{2}}$ were used.

In terms of Kraus operators, the long time statistical operator reads

$$\rho_\infty = \sum_{k,l=\pm} E_{kl}\rho_S(0)E_{kl}^\dagger \otimes \left|k_x^A l_x^B\right\rangle\!\left\langle k_x^A l_x^B\right|.$$

The final state of meter A is $|k_x\rangle$ with probability

$$\begin{aligned} p_k^A &= \mathrm{Tr}\left(\left\langle k_x^A\middle|\rho_\infty\middle|k_x^A\right\rangle\right) = \mathrm{Tr}\left(\sum_l M_{kl}\rho_S(0)\right) \\ &= \mathrm{Tr}\left[M_k^A\rho_S(0)\right], \end{aligned} \tag{25}$$

where $M_{kl} = E_{kl}^\dagger E_{kl}$. A similar expression holds for the probability that the final state of meter B is $|l_x\rangle$, $p_l^B = \mathrm{Tr}\left[\sum_k M_{kl}\rho_S(0)\right] = \mathrm{Tr}\left[M_l^B\rho_S(0)\right]$. Though the previous expressions were found assuming a system's pure initial state, it is not difficult to show they are also valid for initial mixed states.

The measurement operators M_{kl} can be written as

$$\begin{aligned} M_{kl} = (1/4)I &+ \left(\sum_{\mu=0}^{3} a_\mu\sigma_\mu\right)k + \left(\sum_{\mu=0}^{3} b_\mu\sigma_\mu\right)l \\ &+ \left(\sum_{\mu=0}^{3} c_\mu\sigma_\mu\right)kl, \end{aligned} \tag{26}$$

where σ_0 is the identity and σ_i, $i = 1, 2, 3$, designate the Pauli matrices, as usual. The coefficients a_μ, b_μ, c_μ are real functions of θ_A and θ_B, explicitly given in the Appendix. The measurement operators M_k^A and M_l^B can be also written in terms of these coefficients,

$$M_k^A = \frac{1}{2}I + 2k\left(\sum_{\mu=0}^{3} a_\mu\sigma_\mu\right), \tag{27}$$

$$M_l^B = \frac{1}{2}I + 2l\left(\sum_{\mu=0}^{3} b_\mu\sigma_\mu\right). \tag{28}$$

Measurement operators (27) are similar to (17) and (18), discussed in the previous section. Equations (27) and (28) show that we can associate an unsharp measurement to each apparatus ($[M_+^A, M_-^A] = 0 = [M_+^B, M_-^B]$). Meters A and B measure observables $\boldsymbol{a}\cdot\boldsymbol{\sigma} = \sum_{\mu=1}^{3} a_\mu\sigma_\mu$ and $\boldsymbol{b}\cdot\boldsymbol{\sigma}$, respectively. Consider apparatus A; the difference between the probability of obtaining the plus outcome and that of obtaining the minus outcome is $p_+^A - p_-^A = 4(a_0 + \boldsymbol{a}\cdot\boldsymbol{s})$. The expectation value of the operator $\boldsymbol{a}\cdot\boldsymbol{\sigma}$ is $\langle\boldsymbol{a}\cdot\boldsymbol{\sigma}\rangle = \mathrm{Tr}[\boldsymbol{a}\cdot\boldsymbol{\sigma}(\frac{1}{2}I + \boldsymbol{s}\cdot\boldsymbol{\sigma})] = \boldsymbol{a}\cdot\boldsymbol{s}$. We finally have

$$\langle\boldsymbol{a}\cdot\boldsymbol{\sigma}\rangle = \boldsymbol{a}\cdot\boldsymbol{s} = \frac{p_+^A - p_-^A}{4} - a_0. \tag{29}$$

A similar equation holds for the observable $\boldsymbol{b}\cdot\boldsymbol{\sigma}$ measured by apparatus B.

A numeric search shows that any observable (up to a positive or negative scale factor) can be measured by either meter. However, there are isolated points in the (θ_A, θ_B) parameter space, where no information is acquired about the system. In effect, when θ_A and θ_B are integer multiples of 2π, the operators U_{ij}, $i, j = \pm$ are proportional to the identity, except U_{++}. In contrast to operators E_{kl} $k, l = \pm$, which have a nontrivial contribution U_{++}, the four measurement operators M_k^A and M_l^B $k, l = \pm$, are all proportional to the identity.

When only θ_A is an integer multiple of 2π, $\theta_A = 2\pi m$, $m \in \mathbb{Z}$, only meter B obtains information on the system: it measures any operator which is a linear combination of σ_x and σ_z. In particular, it measures observable σ_z when $\theta_B = \pm 2\pi\sqrt{n^2 - m^2}$ for integers n such that $|n| > |m|$ (clearly $\sqrt{n^2 - m^2}$ must not be an integer), and measures σ_x when θ_A satisfies $\mathrm{sinc}(\theta_B/2) = (-1)^{m+1}\,\mathrm{sinc}[\sqrt{\theta_B^2 + (2\pi n)^2}/2]$. Similar results hold for meter A (obtained by exchange of θ_A and θ_B, on one hand, and of σ_x and σ_z, on the other).

The squared magnitude of the vector $v = \boldsymbol{a}\times\boldsymbol{b}$, v^2, vanishes when both meters measure the same observable, that is, when $\boldsymbol{a}(\theta_A, \theta_B) = \lambda\boldsymbol{b}(\theta_A, \theta_B)$ for some nonzero constant λ. After eliminating factors that either are always positive or that correspond to previous cases (θ_A or θ_B even multiple of π), v^2 turns out to be proportional to $\sin(\sqrt{\theta_A^2 + \theta_B^2}/2) f_S(\theta_A, \theta_B)$. The first factor vanishes for points on circles centered on the origin, when $\theta_A^2 + \theta_B^2 = (2\pi n)^2$, $n \in \mathbb{Z}^+$. The zeros of f_S, which lie on the straight lines $\theta_A = \theta = \pm\theta_B$, are integer multiples of $\sqrt{2}\pi$ or satisfy the condition $\tan(\frac{\theta}{2})\tan(\frac{\theta}{2\sqrt{2}}) = -\sqrt{2}$.

No choice of parameters allows one of the meters to measure σ_z while the other measures σ_x. However, the observables $\boldsymbol{a}\cdot\boldsymbol{\sigma}$ and $\boldsymbol{b}\cdot\boldsymbol{\sigma}$ are rotated versions of σ_z and σ_x whenever the nonzero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal. This situation is characterized by

$$f_O = -2\sin^2\left(\frac{\theta_A}{2}\right)\cos\left(\frac{\theta_C}{2}\right) + \theta_A \sin\left(\frac{\theta_A}{2}\right)\text{sinc}\left(\frac{\theta_C}{2}\right)\left[\cos\left(\frac{\theta_A}{2}\right) - \cos\left(\frac{\theta_B}{2}\right)\right]$$
$$+ \text{sinc}\left(\frac{\theta_C}{2}\right)^2\left[\cos\left(\frac{\theta_A}{2}\right)\cos\left(\frac{\theta_B}{2}\right) - \frac{1}{4}\theta_A\theta_B \sin\left(\frac{\theta_A}{2}\right)\sin\left(\frac{\theta_B}{2}\right)\right] + (\theta_A \leftrightarrow \theta_B) = 0,$$

provided that neither θ_A nor θ_B are integer multiples of 2π. As usual, the sine cardinal function is defined by $\text{sinc}(x) = \frac{\sin x}{x}$.

IV. STATE ESTIMATION

When two probes simultaneously interact with a system, generally two observables can be measured. In this section we show that, excluding (infinitely many) exceptional cases, one would have enough information to estimate the probability distributions of σ_x, σ_y, and σ_z; hence, we would be able to estimate the state of the observed system. It is convenient to parametrize the statistical state operator as $\rho = \frac{1}{2}(1 + \boldsymbol{s}\cdot\boldsymbol{\sigma})$ where $\boldsymbol{s}$ is the Bloch vector. Taking into account that for qubits the estimation of a Bloch vector component (e.g., s_z) allows the determination of the probability distribution of the respective operator (e.g., σ_z), we will make little distinction between measurement of observables and estimation of Bloch vector components.

In the preceding section we wrote the long time statistical operator in terms of the Kraus operators E_{kl}, $\rho_\infty = \sum_{k,l=\pm} E_{kl}\rho_S(0)E_{kl}^\dagger \otimes |k_x^A l_x^B\rangle\langle k_x^A l_x^B|$. Kraus operators not only are helpful to express the final state of the system in terms of its initial state

$$\rho_f^S = \sum_{k,l=\pm} E_{kl}\rho_0 E_{kl}^\dagger, \tag{30}$$

but also to calculate the probability that probes A and B end up, respectively, in the states $|k_x^A\rangle$ and $|l_x^B\rangle$:

$$p_{kl} = \text{Tr}(E_{kl}^\dagger E_{kl}\rho_0) = \text{Tr}(M_{kl}\rho_0) \tag{31}$$

$$= \frac{1}{4}s_0 + \sum_{\mu=0}^{3}(a_\mu k + b_\mu l + c_\mu kl)s_\mu. \tag{32}$$

In contrast to the previous approach, where only the separate outcomes of each meter were considered, in this section we take coincidences into account, that is, the joint outcomes of both meters. While the probabilities of each meter's final state, p_k^A and p_k^B, $k = \pm$, do not depend on c_μ, $\mu = 0, 1, 2, 3$, these coefficients are essential for state estimation.

If we define the vectors $\boldsymbol{p} = (p_{++}, p_{+-}, p_{-+}, p_{--})^T$ and $\boldsymbol{S} = (s_0, s_1, s_2, s_3)^T$, Eq. (32) can be written as

$$\boldsymbol{p} = \mathbb{T}\boldsymbol{S}. \tag{33}$$

A simple inversion provides $\boldsymbol{S}$ as $\boldsymbol{S} = \mathbb{T}^{-1}\boldsymbol{p}$. The first component of this vector equation shows that $s_0 = \sum_{ij=\pm} p_{ij} = 1$. The other three components of $\boldsymbol{S}$ can be written as

$$s_q = -\frac{1}{4\Delta}\sum_{k,l=\pm}[C_{q1}(4a_0 - k) + C_{q2}(4b_0 - l)$$
$$+ C_{q3}(4c_0 - kl)]p_{kl}, \tag{34}$$

$q = 1, 2, 3$, where Δ is the determinant of the matrix $\mathbb{D}$:

$$\mathbb{D} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}, \tag{35}$$

and C_{qr} are the co-factors of the same matrix.

Formula (34), which gives the estimation of the state from the probabilities of detection of the two measuring apparatus, constitutes one of the main results of this paper. Equation (34) shows that for generic values of the coupling constants it is possible to estimate the measured state attempting to simultaneously measure σ_x and σ_z (or other noncommuting qubit operators). Of course, when concurrently performed, these measurements are no longer unsharp, because the POVM elements M_{kl} do not mutually commute.

Instead of the purely algebraic approach employed so far in this section, we can follow a more intuitive path. Besides the measurement operators $M_\pm^A$ and $M_\pm^B$ associated with meters A and B, we can define $M_m^C = M_{+,m} + M_{-,-m} = \frac{1}{2}I + 2m\sum_{\mu=0}^{3} c_\mu\sigma_\mu$ with $m = \pm$. M_+^C corresponds to a situation in which the final state of both meters is the same (++ or −−). While the meters A and B measure the observables $\boldsymbol{a}\cdot\boldsymbol{\sigma}$ and $\boldsymbol{b}\cdot\boldsymbol{\sigma}$, the coincidences counts allow the measurement of $\boldsymbol{c}\cdot\boldsymbol{\sigma}$. We write the equations for the expectation values of these operators [see Eq. (29)] as a system of equations

$$\begin{pmatrix} \langle \boldsymbol{a}\cdot\boldsymbol{\sigma}\rangle \\ \langle \boldsymbol{b}\cdot\boldsymbol{\sigma}\rangle \\ \langle \boldsymbol{c}\cdot\boldsymbol{\sigma}\rangle \end{pmatrix} = \begin{pmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{pmatrix}\begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix}$$
$$= \frac{1}{4}\begin{pmatrix} p_{++} + p_{+-} - p_{-+} - p_{--} \\ p_{++} + p_{-+} - p_{+-} - p_{--} \\ p_{++} + p_{--} - p_{-+} - p_{+-} \end{pmatrix} - \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix}. \tag{36}$$

The solution of this equation,

$$\begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix} = (\mathbb{D}^T)^{-1}\begin{pmatrix} \frac{p_{++}+p_{+-}-p_{-+}-p_{--}}{4} - a_0 \\ \frac{p_{++}+p_{-+}-p_{+-}-p_{--}}{4} - b_0 \\ \frac{p_{++}+p_{--}-p_{-+}-p_{+-}}{4} - c_0 \end{pmatrix}, \tag{37}$$

where $\mathbb{D}^T$ denotes the transpose of the matrix defined in (35), is equivalent to the solution given before, Eq. (34).

In general, the measurement described by operators M_{kl}, $k, l = \pm$, is a POVM. However, as shown in Fig. 4, there are points where there is no measurement, points in which the measurement is unsharp (when exactly one component of the Bloch vector of the initial state of the system can be estimated, but not all measurement operators are projectors), and finally points where the measurement is projective. When $\theta_A \approx -5.49388$, $\theta_B \approx 9.0341$ (and its reflections on the lines

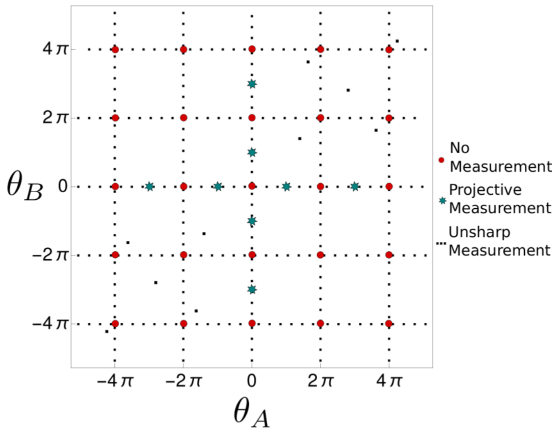

FIG. 4. Type of measurement as a function of the parameters θ_A and θ_B. Generic points in the parameter space correspond to POVMs.

$\theta_A = \pm\theta_B$), vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are approximately orthogonal ($\theta_{\boldsymbol{a},\boldsymbol{b}} = 90.71^\circ, \theta_{\boldsymbol{a},\boldsymbol{c}} = 89.46^\circ, \theta_{\boldsymbol{b},\boldsymbol{c}} = 89.96^\circ$).

We define their Bloch rank of a POVM as the number of independent Bloch components that it allows to estimate. For the model studied in this paper, this number is equal to $d_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}$, the dimension of the space spanned by $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$. If the POVM is informationally complete, that is, if it allows us to completely estimate the state, its Bloch rank attains its maximum value (three, for qubits). If, on the contrary, its Bloch rank is strictly smaller than the maximum, the POVM is called informationally incomplete (IIC).

V. INFORMATIONALLY INCOMPLETE MEASUREMENTS

Under the assumptions of this paper, the final joint state (of the system and both meters) depends on two real parameters, θ_A and θ_B. For generic values of the parameters, it is possible to estimate the state of the system. If at least one of the parameters vanishes (only one meter is coupled to the system), state estimation is impossible. However, given arbitrary (nonzero) values of θ_A and θ_B is full state reconstruction always possible? The answer is negative for parameters in $\mathbb{S}$, a set of dimension $d = 1$. For parameters in $\mathbb{S}$, we find which part of the system's state can be estimated.

Full state reconstruction is not possible when the matrix $\mathbb{D}$, defined in the previous section, becomes singular. The determinant of $\mathbb{D}$ can be geometrically interpreted as the volume of the parallelepiped defined by the vectors $\boldsymbol{a}, \boldsymbol{b}$, and $\boldsymbol{c}$. We can write this determinant as $\Delta = f_0^S f_0^A = f_0^S(f_1^S f_1^A + f_2^S f_2^A)$, where the functions f^S and f^A are, respectively, symmetric and antisymmetric under the permutation of θ_A and θ_B. More explicitly, we have $f_0^S = -\sin^2(\frac{\theta_A}{2})\sin^2(\frac{\theta_B}{2})\,\mathrm{sinc}\,(\frac{\theta_C}{2})/2048$, where $\theta_C = \sqrt{\theta_A^2+\theta_B^2}$. The antisymmetric functions are

$$f_1^A = -\left(\theta_A^2+\theta_A\theta_B-\theta_B^2\right)\sin\theta_A + \left(\theta_A^2-\theta_A\theta_B+\theta_B^2\right)\times\sin(\theta_A-\theta_B) + \left(-\theta_A^2+\theta_A\theta_B+\theta_B^2\right)\sin\theta_B$$

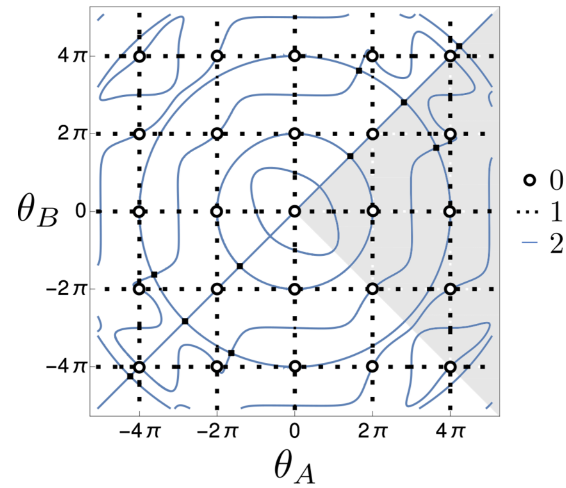

FIG. 5. Number of independent Bloch vector components of the system which can be estimated (Bloch rank) as a function of the parameters θ_A and θ_B.

and $f_2^A = 4\sin(\frac{\theta_A-\theta_B}{2})$. The remaining symmetric functions are more cumbersome, $f_1^S = \mathrm{sinc}\,(\frac{\theta_C}{2})/2$, and

$$f_2^S = \cos\left(\frac{\theta_C}{2}\right)\left[\theta_B\sin\left(\frac{\theta_A}{2}\right)\cos\left(\frac{\theta_B}{2}\right) + \theta_A\cos\left(\frac{\theta_A}{2}\right)\sin\left(\frac{\theta_B}{2}\right)\right].$$

It is easy to prove that the determinant is antisymmetric, respectively symmetric, under reflection with respect to the straight line $\theta_A = \theta_B$, respectively $\theta_A = -\theta_B$. Hence, the determinant on the whole plane θ_A-θ_B can be reconstructed from the quadrant defined by $\theta_A \geqslant 0,\ -\theta_B \leqslant \theta_A \leqslant \theta_B$ (gray area of Fig. 5).

The points of the set $\mathbb{S}$ are those for which the determinant Δ vanishes, that is, when $\sin\frac{\theta_C}{2} = 0$, $\sin\frac{\theta_A}{2} = 0$, or $\sin\frac{\theta_B}{2} = 0$, and when the function f_0^A vanishes. The former case corresponds to the circles $\sqrt{\theta_A^2+\theta_B^2} = 2n_C\pi,\ n_C = 1, 2, \ldots,$ and the straight lines $\theta_A = 2m\pi,\ \theta_B = 2n\pi,\ m, n$ integers, and the latter to $\theta_A = \theta_B$ and to an infinite number of closed curves (like the egg shaped curve of Fig. 5). Thus, state estimation is possible on the whole parameter plane with the exception of the one-dimensional set of points $\mathbb{S}$. The dimension $d_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}$, which ranges from zero (no information is gained about the state) to three (full knowledge of the state), is depicted in Fig. 5. In Table I we list bases for the "singular curves" (curves where the determinant Δ vanishes) which were analytically characterized in this work.

When both θ_A and θ_B are even multiples of π no information is acquired of the system ($\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{c} = \boldsymbol{0}$). When only θ_A (θ_B) is an even multiple of π, both B (A) and C measure the same operator. For points on the parameter plane belonging to circumferences of radii $2\pi n$, $n \in \mathbf{Z}^+$, centered at the origin, though both A and B measure the same observable, C (coincidences) obtains information about a different observable (see Table I). However, $\boldsymbol{c}$ vanishes for points on the circumferences for which the difference between θ_A and θ_B is an even multiple of π; these points correspond to the intersections with the

TABLE I. On each curve on the parameter space (θ_A, θ_B), Bloch vectors which are linear combinations of the basis vectors can be estimated. The subscript gives information about the type of curve: horizontal line (H), vertical line (V), circumference (C), and diagonal line (D). Vectors $\boldsymbol{a}_D, \boldsymbol{b}_D$, and $\boldsymbol{c}_D$ lie on the same plane.

Curve	Nonorthogonal basis vectors
$\theta_B = 2\pi n$ $\theta_C = \sqrt{(2\pi n)^2 + \theta_A^2}$	$\boldsymbol{a}_H = \boldsymbol{c}_H = \frac{1}{8}\sin\frac{\theta_A}{2}\left(2n\pi \operatorname{sinc}\frac{\theta_C}{2}, 0, 2(-1)^n \sin\frac{\theta_A}{2} + \theta_A \operatorname{sinc}\frac{\theta_C}{2}\right)$
$\theta_A = 2\pi n$ $\theta_C = \sqrt{(2\pi n)^2 + \theta_B^2}$	$\boldsymbol{b}_V = \boldsymbol{c}_V = \frac{1}{8}\sin\frac{\theta_B}{2}\left(2(-1)^n \sin\frac{\theta_B}{2} + \theta_B \operatorname{sinc}\frac{\theta_C}{2}, 0, 2n\pi \operatorname{sinc}\frac{\theta_C}{2}\right)$
$\theta_A = 2\pi n\cos\theta$ $\theta_B = 2\pi n\sin\theta$	$\frac{\boldsymbol{a}_C}{\sin\theta_A/2} = \frac{\boldsymbol{b}_C}{(-1)^{n+1}\sin\theta_B/2} = \frac{1}{8}\left[(-1)^n \sin\left(\frac{\theta_B}{2}\right), 0, -\sin\left(\frac{\theta_A}{2}\right)\right]$ $\boldsymbol{c}_C = -\frac{1}{8}\sin\left(\frac{\theta_A-\theta_B}{2}\right)\left(-\cos\frac{\theta_A}{2}\sin\frac{\theta_B}{2}, \sin\frac{\theta_A}{2}\sin\frac{\theta_B}{2}, \sin\frac{\theta_A}{2}\cos\frac{\theta_B}{2}\right)$
$\theta_A = \theta$ $\theta_B = \theta$ $\theta' = \frac{\theta}{\sqrt{2}}$	$\boldsymbol{a}_D = -\frac{1}{8}\sin\frac{\theta}{2}\left(\frac{1}{\sqrt{2}}\cos\frac{\theta}{2}\sin\theta' - \sin\frac{\theta}{2}\cos\theta', \sin\frac{\theta}{2}\sin\theta', \sin\frac{\theta}{2} + \frac{1}{\sqrt{2}}\cos\frac{\theta}{2}\sin\theta'\right)$ $\boldsymbol{b}_D = -\frac{1}{8}\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} + \frac{1}{\sqrt{2}}\cos\frac{\theta}{2}\sin\theta', -\sin\frac{\theta}{2}\sin\theta', \frac{1}{\sqrt{2}}\cos\frac{\theta}{2}\sin\theta' - \sin\frac{\theta}{2}\cos\theta'\right)$ $\boldsymbol{c}_D = -\frac{1}{8\sqrt{2}}\sin\theta\sin(\theta')(1, 0, 1)$

odd-shaped curves. On the diagonal, and on the odd-shaped curves, $\boldsymbol{a}, \boldsymbol{b}$, and $\boldsymbol{c}$ lie on the same plane; hence, only two independent components of the system's Bloch vector can be estimated.

One might think that, due to the symmetry, along the diagonal $\theta_A = \theta_B$ only one component of the Bloch vector could be estimated. In fact, since the interaction Hamiltonians are proportional to σ_z and σ_x, when $\theta_A = \theta_B = \theta$, the Bloch vector components which can be estimated by meters A and B are related: if the former allows the reconstruction of $(n_1, n_2, n_3)\cdot\boldsymbol{\sigma}$, the latter allows the reconstruction of $(n_3, -n_2, n_1)\cdot\boldsymbol{\sigma}$.

Intersection of three or four singular curves occur only when θ_A and θ_B are both even multiples of π (no observable can be estimated). Points where the circles intersect with only one of the other singular curves, correspond to POVMs allowing the reconstruction of a single observable. Finally, the Bloch rank of points at the intersection of only an odd-shaped curve and the diagonal line is two.

On the regular points and on the circles, coincidences add new information. The circles are specially interesting, because both meters allow reconstruction of the same Bloch vector component. In fact, this is a property of any POVM which can be parametrized as

$$\begin{pmatrix} M_{--} \\ M_{-+} \\ M_{+-} \\ M_{++} \end{pmatrix} = \begin{pmatrix} x & y & x \\ az - x & -y & a^2 z - x \\ z - x & -y & az - x \\ x - (a+1)z & y & -za^2 - za + x \end{pmatrix} \boldsymbol{\sigma}, \tag{38}$$

where a, x, y, z are real numbers. Here contributions proportional to the identity were ignored. Either apparatus gives information only on the observable $\sigma_x + a\sigma_z$. If we use the simultaneous information about both apparatuses, we can also obtain information on the observable $(a-1)ax\sigma_x + (a^2+1)y\sigma_y + (1-a)x\sigma_z$.

VI. SIMULTANEOUS STRONG AND WEAK MEASUREMENTS

Strong projective measurements, those for which all of the measurement operators are projectors, occur only for pairs of parameters in which either θ_A or θ_B is an odd multiple of π and the other vanishes, as shown in Fig. 4. In the same figure one can see that if one uses pairs of parameters in which both are odd multiples of π, the resulting measurement is a POVM. However, for these values, the whole initial state cannot be estimated, because they belong to one of the singular curves shown in Fig. 5: either the diagonal line or one of the odd-shaped curves. Thus, "two simultaneous projective measurements" permit the determination of only two components of the Bloch vector.

Weak measurements are associated small values of the system-probe coupling parameters θ_A and θ_B. To estimate the statistics of a single observable using single-probe weak measurements, $\theta_A \ll 1$, it is necessary to perform an expansion to order θ_A^2. The vector of measurement operators $\boldsymbol{M} = (M_{--}, M_{-+}, M_{+-}, M_{++})^T$ is a linear transformation of the matrices $\boldsymbol{\Sigma} = (I, \sigma_x, \sigma_y, \sigma_z)^T$. Hence, we write $\boldsymbol{M} = \mathbb{T}(\theta_A, \theta_B)\boldsymbol{\Sigma}$, where $\mathbb{T}(\theta_A, \theta_B)$ is a 4×4 matrix. Setting $\theta_A = r\cos\phi, \theta_B = r\sin\phi$, and assuming $r \ll 1$ we have, to second order in r, $\mathbb{T}(r,\phi) \approx \mathbb{T}_0 + \frac{r^2}{8}\mathbb{T}_2(\phi)$. Here $\mathbb{T}_0$ is a matrix whose only nonvanishing entry is the component $(1,4)$, $\mathbb{T}_0^{(1,4)} = 1$ (for small r, the probability p_{++} is close to unity, the remaining ones p_{kl} are close to zero). Taking into account that $\mathbb{T}_2(\phi)$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ \cos^2(\phi) & 0 & 0 & \cos^2(\phi) \\ \sin^2(\phi) & \sin^2(\phi) & 0 & 0 \\ -1 & -\sin^2(\phi) & 0 & -\cos^2(\phi) \end{pmatrix}, \tag{39}$$

we see that, unless ϕ is an integer multiple of $\pi/2$, two Bloch vector components can be estimated. Indeed,

$$s_z = \frac{8}{r^2\cos^2(\phi)}P_{-+} - 1, \quad s_x = \frac{8}{r^2\sin^2(\phi)}P_{+-} - 1.$$

The expansion of $\mathbb{T}(r,\phi)$ as a power series of r, shows that, although the third order and the fourth order are both nontrivial, it is necessary to go to fourth order to be able to estimate three independent components of the Bloch vector. In other words, we expect that, in order to estimate the whole system's initial state, many more runs of the experiments will be required than those necessary to estimate a single Bloch component, with a single meter.

VII. GENERALIZATION AND CONCLUSIONS

By studying a particular model we have shown that the attempt to simultaneously measure two noncommuting qubit observables, generally allows state estimation (of the measured system). While each of the meters measures a different observable, coincidences of their outcomes permits the measurement of a third observable. The space of parameters contains lower dimensional subsets where none, one, or two components of the Bloch vector (Bloch rank zero, one, and two) can be estimated. Although the details of these subsets depend on the particular model under study, the qualitative behavior of the measurements on the parameter space is similar to the one discussed in this work.

When the measurement is characterized by four measuring operators, three independent vectors can be defined. They not only determine observables which can be measured, but also allow us to determine the Bloch rank of the measurement (the dimension of the linear space generated by them).

We have explored some models similar to the model described in this paper. Meter-system interaction Hamiltonians were assumed to be separable and proportional to the operator $I + \sigma_z$ of the corresponding meter. Several choices for system operators appearing on the interaction Hamiltonians, initial states of the meters, and decoherence bases were tried. Only the lines $\theta_A = 0$ and $\theta_B = 0$ are singular for any choice of the initial states of the meters and of the decoherence bases; when the same initial state and the same decoherence basis are used for both meters, the line $\theta_A = \theta_B$ is singular. No state estimation is possible if the initial state of the meter is an eigenvector of σ_z or if the decoherence basis is σ_z.

For the model discussed in this work, attempting to perform two projective measurements of different Pauli matrices results in IICMs of Bloch rank two; simultaneous weak measurements allow complete state estimation, but with a slower converge rate than the single-meter weak measurement.

Real experimental setups may require us to explore the interplay between decoherence and system-probe interaction that we have disregarded in this work. An analysis of the convergence rates of the measurement on the space of parameters (how many runs of the experiment are needed to estimate an observable or the statistical operator with a given error) is also lacking. An exploration of different interaction Hamiltonians, as well as the extension to qudits might be interesting.

ACKNOWLEDGMENTS

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APPENDIX: ELEMENTS OF THE MATRIX $\mathbb{D}$

Recall that $\theta_C = \sqrt{\theta_A^2 + \theta_B^2}$. The coefficients not explicitly listed below can be given in terms of other coefficients $b_0(\theta_A,\theta_B) = a_0(\theta_B,\theta_A)$, $b_1(\theta_A,\theta_B) = a_3(\theta_B,\theta_A)$, $b_2(\theta_A,\theta_B) = -a_2(\theta_B,\theta_A)$, $b_3(\theta_A,\theta_B) = a_1(\theta_B,\theta_A)$, $c_3(\theta_A,\theta_B) = c_1(\theta_B,\theta_A)$:

$$a_0 = \frac{1}{8}\cos\left(\frac{\theta_A}{2}\right)\left[\cos\left(\frac{\theta_A}{2}\right) + \frac{\theta_B\sin\left(\frac{\theta_B}{2}\right)\sin\left(\frac{\theta_C}{2}\right)}{\theta_C} + \cos\left(\frac{\theta_B}{2}\right)\cos\left(\frac{\theta_C}{2}\right)\right],$$

$$a_1 = \frac{1}{8}\sin\left(\frac{\theta_A}{2}\right)\left[\sin\left(\frac{\theta_B}{2}\right)\cos\left(\frac{\theta_C}{2}\right) - \frac{\theta_B\cos\left(\frac{\theta_B}{2}\right)\sin\left(\frac{\theta_C}{2}\right)}{\theta_C}\right],$$

$$a_2 = -\frac{\theta_A\sin\left(\frac{\theta_A}{2}\right)\sin\left(\frac{\theta_B}{2}\right)\sin\left(\frac{\theta_C}{2}\right)}{8\theta_C},$$

$$a_3 = -\frac{1}{8}\sin\left(\frac{\theta_A}{2}\right)\left[\frac{\theta_A\cos\left(\frac{\theta_B}{2}\right)\sin\left(\frac{\theta_C}{2}\right)}{\theta_C} + \sin\left(\frac{\theta_A}{2}\right)\right],$$

$$c_0 = \frac{1}{32}\left[4\cos\left(\frac{\theta_C}{2}\right)\cos\left(\frac{\theta_A}{2} + \frac{\theta_B}{2}\right) + \cos(\theta_A - \theta_B) + \cos(\theta_A) + \cos(\theta_B) + 1\right],$$

$$c_1 = \frac{1}{8}\left[\cos\left(\frac{\theta_A}{2}\right)\sin\left(\frac{\theta_B}{2}\right)\sin\left(\frac{\theta_A - \theta_B}{2}\right) - \frac{\theta_B\sin\left(\frac{\theta_C}{2}\right)\sin\left(\frac{\theta_A+\theta_B}{2}\right)}{\theta_C}\right],$$

$$c_2 = -\frac{1}{8}\sin\left(\frac{\theta_A}{2}\right)\sin\left(\frac{\theta_B}{2}\right)\sin\left(\frac{\theta_A - \theta_B}{2}\right).$$

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