Information flow versus divisibility for qubit evolution

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(Received 20 January 2019; published 8 April 2019)

We study the relation between lack of information backflow and completely positive divisibility for noninvertible qubit dynamical maps. Recently, these two concepts were shown to be fully equivalent for the so-called image nonincreasing dynamical maps. Here we show that this equivalence is universal for any qubit dynamical map. A key ingredient in our proof is the observation that there does not exist a completely positive and trace-preserving projector onto a three-dimensional subspace spanned by qubit density operators. Our analysis is illustrated by several examples of qubit evolution, including dynamical maps which are not image nonincreasing.

DOI: 10.1103/PhysRevA.99.042105

I. INTRODUCTION

With the discovery of new experimental techniques, quantum information is no longer a subject of solely theoretical interest. Numerous quantum protocols, principles, and results discovered theoretically are now being tested, verified, and reinforced by experiments. Along with these new developments comes a serious challenge of controlling and understanding real- life quantum systems which are inherently open to the environment. This has resulted in a lot of interest regarding open quantum systems recently [1–6]. The evolution of such systems is represented by a dynamical map, that is, a family of completely positive (CP) trace-preserving (TP) maps Λ_t : $\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ $(t \geqslant 0)$, where $\mathcal{B}(\mathcal{H})$ denotes a linear space of bounded operators acting on the system Hilbert space $\mathcal{H}$. (Actually, in this paper we consider only finite-dimensional cases and hence $\mathcal{B}(\mathcal{H})$ coincides with all linear operators on $\mathcal{H}$.) Moreover, one assumes a natural initial condition $\Lambda_{t=0} =$ id (identity map).

Usually, the origin of a dynamical map is a composed system living in $\mathcal{H} \otimes \mathcal{H}_E$, with $\mathcal{H}_E$ denoting a Hilbert space of environment. Now, if $\mathbf{H} = H_S + H_E + H_{\rm int}$ is the total Hamiltonian of the composed system and $\rho \otimes \rho_E$ is an initial product state, then the standard reduction procedure defined via partial trace operation,

$$\Lambda_t(\rho) = {\rm Tr}_E(e^{-i\mathbf{H}t}\, \rho \otimes \rho_E\, e^{i\mathbf{H}t}), \tag{1}$$

gives rise to a legitimate dynamical map (in the paper we keep $\hbar = 1$).

Recently, the notion of non-Markovian quantum evolution has received considerable attention (see review papers [7–10]). This property, although well defined in the classical regime, has a number of nonequivalent prescriptions in quantum theory. On the level of dynamical maps, two main approaches which turned out to be very influential are based on the concept of CP divisibility [11] and information flow [12]. One calls a dynamical map Λ_t divisible if

$$\Lambda_t = V_{t,s} \circ \Lambda_s\,, \quad (t \geqslant s), \tag{2}$$

where $V_{t,s} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a linear map defined on the entire $\mathcal{B}(\mathcal{H})$. Being divisible the map Λ_t is:

(i) P-divisible if the map $V_{t,s}$ is positive and trace-preserving (PTP), and

(ii) CP-divisible if the map $V_{t,s}$ is CPTP [13]. (For mathematical details of positive and completely positive maps see [14,15].)

In the latter case one may interpret $V_{t,s}$ as a legitimate quantum channel, mapping states at time s into states at time t. Following [11] one calls the quantum evolution to be Markovian if and only if (iff) the corresponding dynamical map is CP divisible.

A second idea developed in [12] is based upon the notion of information flow: for any pair of density operators ρ_1 and ρ_2 one defines an information flow

$$\sigma(\rho_1, \rho_2; t) = \frac{d}{dt} \|\Lambda_t(\rho_1) - \Lambda_t(\rho_2)\|_1, \tag{3}$$

where $\|A\|_1 = {\rm Tr}\sqrt{A^\dagger A}$ denotes the trace norm of A. Actually, $\|\rho_1 - \rho_2\|_1$ represents the distinguishability of ρ_1 and ρ_2. Moreover, $\frac{1}{2}(1 + \|\rho_1 - \rho_2\|_1)$ gives the maximal guessing probability in the unbiased scenario, that is, when ρ_1 and ρ_2 are prepared with the same probability [16]. Following [12] Markovian evolution is characterized by the condition $\sigma(\rho_1, \rho_2; t) \leqslant 0$ for all $t \geqslant 0$ and any pair of initial states ρ_1 and ρ_2. Whenever $\sigma(\rho_1, \rho_2; t) > 0$ one calls it information backflow, meaning that the information flows from the environment back to the system. Note that $\sigma(\rho_1, \rho_2; t) > 0$ implies distinguishability of the time-evolved states $\Lambda_t[\rho_1]$, and $\Lambda_t[\rho_2]$ fails to be monotonically decreasing at time t. A detailed description of how departure from monotonicity can be seen as information backflow is given in [8]. In this case the evolution displays nontrivial memory effects and

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is hence called non-Markovian. One calls $\sigma(\rho_1, \rho_2; t) \leqslant 0$ a Breuer-Laine-Piilo (BLP) condition.

In this paper, we address the problem of analyzing how far these two approaches to Markovianity are equivalent. Actually, the connection between P divisibility and information backflow was first analyzed in [17]. Later on in [18], it was shown that for invertible dynamical maps CP divisibility is equivalent to the lack of information backflow on an extended system comprised of the system and a d-dimensional ancilla in an extended scenario when two states ρ_1 and ρ_2 are prepared with probabilities p_1 and p_2. Bylicka *et al.* [19] observed that one may still use only the unbiased case $p_1 = p_2$, but the price one pays is the use of $(d+1)$-dimensional ancilla. These results were then extended to image nonincreasing dynamical maps [20], which is a large class of dynamical maps including all invertible ones. Also recently, the equivalence between divisibility and a monotonic decrease of information in terms of guessing probability was studied by Buscemi and Datta in [21] for time-discrete dynamical maps.

We show here that the equivalence between CP divisibility and lack of information flow, as described in the previous paragraph, can be extended to an arbitrary dynamical map if we use only dynamical maps on qubits. Our results prove the complete equivalence of the two main approaches to Markovianity for qubit dynamical maps.

The paper is structured as follows: In Sec. II, we review the recent results in this direction so as to provide a background for the paper. Next, in Sec. III we present the main result of our paper and in Sec. IV, we discuss some examples before drawing our conclusions in Sec. V.

II. INVERTIBLE VS NONINVERTIBLE MAPS

By an invertible dynamical map we understand Λ_t such that Λ_t^{-1} exists for all $t > 0$. Note that even if it exists, the inverse need not be completely positive (it is always trace preserving). The inverse is also completely positive if and only if the map Λ_t is unitary, that is, $\Lambda_t(\rho) = U_t \rho U_t^\dagger$, where U_t is a time-dependent unitary operator in $\mathcal{H}$. Now, an invertible map is always divisible; indeed one finds $V_{t,s} = \Lambda_t \Lambda_s^{-1}$. Moreover,

Theorem 1 ([18]). An invertible dynamical map Λ_t is P divisible if and only if

$$\frac{d}{dt}\|\Lambda_t(p_1\rho_1 - p_2\rho_2)\|_1 \leqslant 0 \tag{4}$$

for all probability distributions $p_1 + p_2 = 1$ and density operators ρ_1, ρ_2 in $\mathcal{H}$. Moreover, it is CP divisible if and only if

$$\frac{d}{dt}\|[\mathrm{id}_d \otimes \Lambda_t](p_1\varrho_1 - p_2\varrho_2)\|_1 \leqslant 0 \tag{5}$$

for all probability distributions $p_1 + p_2 = 1$ and density operators ϱ_1, ϱ_2 in $\mathbb{C}^d \otimes \mathcal{H}$ (with $d = \dim \mathcal{H}$).

Note that the BLP Markovianity condition coincides with (4) with $p_1 = p_2$. Interestingly, one may use only the completely unbiased case ($p_1 = p_2$) due to the following:

Theorem 2 ([19]). An invertible dynamical map Λ_t is CP divisible if and only if

$$\frac{d}{dt}\|[\mathrm{id}_{d+1} \otimes \Lambda_t](\varrho_1 - \varrho_2)\|_1 \leqslant 0 \tag{6}$$

for all density operators ϱ_1, ϱ_2 in $\mathbb{C}^{d+1} \otimes \mathcal{H}$.

For maps which are not invertible even divisibility is not evident [19,20]. Note that $V_{t,s}$ is well defined on the image of the maps Λ_s [we denote by $\mathrm{Im}(\Lambda_s)$]. Actually, as shown in [20], divisibility of Λ_t is equivalent to the following property:

$$\mathrm{Ker}(\Lambda_s) \subseteq \mathrm{Ker}(\Lambda_t), \tag{7}$$

for any $s < t$, that is, the map is *kernel nondecreasing*. This condition guarantees that $V_{t,s}$ can be consistently extended from $\mathrm{Im}(\Lambda_s)$ to the whole space $\mathcal{B}(\mathcal{H})$. Chruściński *et al.* [20] analyzed the question of when the extension of $V_{t,s}$ is CPTP. The central result of [20] states the following:

Theorem 3 ([20]). If the dynamical map Λ_t satisfies

$$\frac{d}{dt}\|[\mathrm{id}_d \otimes \Lambda_t](p_1\varrho_1 - p_2\varrho_2)\|_1 \leqslant 0, \tag{8}$$

for all probability distributions $p_1 + p_2 = 1$ and density operators ϱ_1, ϱ_2 in $\mathbb{C}^d \otimes \mathcal{H}$, then it is divisible with $V_{t,s}$ completely positive on $\mathcal{B}(\mathcal{H})$ but trace preserving only on the image $\mathrm{Im}(\Lambda_s)$.

Interestingly, there exists a class of dynamical maps for which the extension of $V_{t,s}$ is not only completely positive but also trace preserving, that is, such maps are CP divisible. One calls a dynamical map Λ_t *image nonincreasing* [20] if

$$\mathrm{Im}(\Lambda_t) \subseteq \mathrm{Im}(\Lambda_s) \tag{9}$$

for $t > s$.

Theorem 4 ([20]). If the dynamical map Λ_t is image nonincreasing and it satisfies

$$\frac{d}{dt}\|[\mathrm{id}_d \otimes \Lambda_t](p_1\varrho_1 - p_2\varrho_2)\|_1 \leqslant 0 \tag{10}$$

for all probability distributions $p_1 + p_2 = 1$ and density operators ϱ_1, ϱ_2 in $\mathbb{C}^d \otimes \mathcal{H}$, then it is CP divisible.

Finally, Theorem 2 may be generalized as follows:

Theorem 5 ([20]). If the dynamical map Λ_t is image nonincreasing and it satisfies

$$\frac{d}{dt}\|[\mathrm{id}_{d+1} \otimes \Lambda_t](\varrho_1 - \varrho_2)\|_1 \leqslant 0 \tag{11}$$

for all density operators ϱ_1, ϱ_2 in $\mathbb{C}^{d+1} \otimes \mathcal{H}$, then it is CP divisible.

III. QUBIT DYNAMICAL MAPS

Now, we consider the simplest scenario—dynamical maps for qubits. The main result of this paper is provided by the following:

Theorem 6. A qubit dynamical map Λ_t is CP divisible if and only if

$$\frac{d}{dt}\|[\mathrm{id}_2 \otimes \Lambda_t](p_1\varrho_1 - p_2\varrho_2)\|_1 \leqslant 0 \tag{12}$$

for all probability distributions $p_1 + p_2 = 1$ and two-qubit density operators ϱ_1, ϱ_2 in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

We stress that this results is universal, that is, we do not assume that the map is invertible (as in Theorem 1) nor that it is image nonincreasing (as in Theorem 4). Of course, for invertible qubit maps it is just a special case of Theorem 1.

The proof of this result consists of the following steps: First it is shown (Proposition 1) that the image of any CPTP qubit projector is never three-dimensional. This observation supplemented by the Alberti-Uhlmann theorem allows to show that condition (12) is indeed equivalent to CP divisibility.

It was shown in [20] that if Λ_t is not invertible and $t_1 > 0$ is the first moment of time such that $\Lambda_{t_1}^{-1}$ does not exist, then condition (12) implies that

$$\Pi_{t_1} = \lim_{\epsilon\to 0^+} V_{t_1,t_1-\epsilon} \tag{13}$$

defines a completely positive projector onto $\mathrm{Im}(\Lambda_{t_1})$.

Proposition 1. There is no CPTP projector $\Pi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ onto a three-dimensional subspace of $M_2(\mathbb{C})$ spanned by density operators.

Actually, the above Proposition follows from the following result of [22] (however, we provide an independent proof in Appendix A). Let Φ be a qubit quantum channel and let $\mathrm{PO}(\Phi)$ be the pure output of Φ, that is, a set of pure state in the image of Φ,

$$\mathrm{PO}(\Phi) = \Phi(\mathbf{B}) \cap \mathbf{S},$$

where $\mathbf{B}$ is a Bloch ball and $\mathbf{S}$ a Bloch sphere—a set of qubit pure states. One proves the following:

Proposition 2 ([22]). Let Φ be a qubit quantum channel such that $\mathrm{PO}(\Phi)$ has more than two elements. Then $\mathrm{PO}(\Phi) = \mathbf{S}$, that is, all pure states belong to the pure output of Φ.

Now, suppose that there exists a CPTP projector Φ such that its image is three-dimensional. Being a projector, it does not change the purity of the input states in subspace $\mathrm{Im}(\Phi)$. It is, therefore, clear that the intersection $\mathrm{Im}(\Phi) \cap \mathbf{S}$ defines a circle on the Bloch sphere. But it contradicts Proposition 2, which requires that in this case $\mathrm{PO}(\Phi) = \mathbf{S}$.

The above observation leads us to the following:

Corollary 1. If the qubit dynamical map Λ_t satisfies (12), then the dimension of its image $\dim \mathrm{Im}(\Lambda_t) \in \{1, 2, 4\}$.

A pictorial representation of the above result is given in Figs. 1(b)–1(d). Another ingredient of the proof of Theorem 6 is based on the following:

Proposition 3 (Alberti-Uhlmann [23]). Let $\{\sigma_1, \sigma_2\}$ and $\{\sigma_1', \sigma_2'\}$ be two sets of qubit states. Then there exists a CPTP map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ connecting them, i.e., $\Phi(\sigma_i) = \sigma_i'$ for $i = 1, 2$, if and only if

$$\|\sigma_1 - \delta\sigma_2\|_1 \geqslant \|\sigma_1' - \delta\sigma_2'\|_1 \tag{14}$$

for all $\delta > 0$.

Note that the above formula can be rewritten as follows:

$$\|p_1\sigma_1 - p_2\sigma_2\|_1 \geqslant \|p_1\sigma_1' - p_2\sigma_2'\|_1 \tag{15}$$

for all probability distributions $p_1 + p_2 = 1$.

Now the proof of Theorem 6 easily follows. Condition (12) implies that Λ_t is divisible or equivalently kernel nondecreasing [20]. Suppose now that $\mathrm{Im}(\Lambda_s)$ is two-dimensional and let ρ_1 and ρ_2 be two density operators such that $\rho_1(s) = \Lambda_s(\rho_1)$ and $\rho_2(s) = \Lambda_s(\rho_2)$ span $\mathrm{Im}(\Lambda_s)$. Inequality (12) implies

$$\|p_1\rho_1(s) - p_2\rho_2(s)\|_1 \geqslant \|p_1\rho_1(t) - p_2\rho_2(t)\|_1, \tag{16}$$

where $\rho_i(t) = \Lambda_t(\rho_i) = V_{t,s}(\rho_i(s))$. Alberti-Uhlmann theorem guaranties that there exists a quantum channel $\widetilde{V}_{t,s} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that $\rho_i(t) = \widetilde{V}_{t,s}(\rho_i(s))$. Clearly, $\widetilde{V}_{t,s}$ is a CPTP extension of $V_{t,s} : \mathrm{Im}(\Lambda_s) \to \mathcal{B}(\mathcal{H})$.

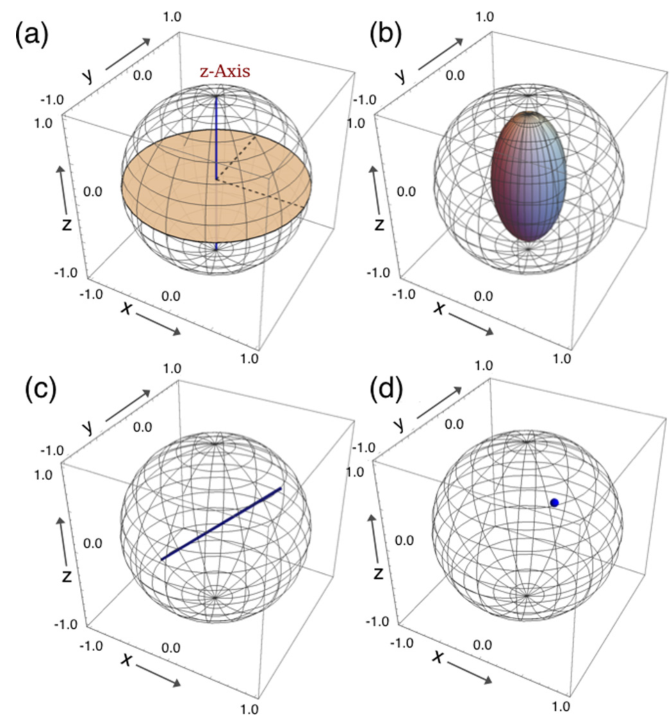

FIG. 1. (a) Bloch ball representation of the action of the PTP map Ψ and the CPTP map $\mathcal{S}_3$. The equatorial brown disk and the thick blue z axis represent the set of density matrices lying in the image of maps Ψ and $\mathcal{S}_3$, respectively. (b, c, d) The allowed structures of density matrices lying in the image of qubit dynamical maps. (b) When the map is invertible the image is an ellipsoid. (c) When the map is noninvertible and its image is two-dimensional it forms a line within the Bloch ball. (d) When the map is noninvertible and its image has dimension 1 it forms a point.

If $\mathrm{Im}(\Lambda_s)$ is one-dimensional, then $\Lambda_s(\rho) = \omega_s \mathrm{Tr}\rho$ for some density operator ω_s. Since the map is divisible it follows that $\mathrm{Im}(\Lambda_t)$ is one-dimensional for all $t > s$, and hence $V_{t,s} = \omega_t \mathrm{Tr}\rho$ is a CPTP projector which proves that the original qubit map Λ_t is CP divisible. □

It should be emphasized that this proof requires that $\mathrm{Im}(\Lambda_s)$ be at most two-dimensional, otherwise one would need more than two density operators to span the image of Λ_s and then the Alberti-Uhlmann theorem is not enough to prove that there exists a universal extension for all states from the image. (See also an interesting discussion of the Alberti-Uhlmann theorem in [24].)

Note, that Proposition 1 does not forbid the existence of qubit quantum channel Φ such that $\dim \mathrm{Im}(\Phi) = 3$. As an example consider

$$\Phi(\rho) = \tfrac{1}{2}\rho + \tfrac{1}{4}(\sigma_1\rho\sigma_1 + \sigma_2\rho\sigma_2). \tag{17}$$

One finds

$$\Phi(\mathbb{1}) = \mathbb{1},\ \Phi(\sigma_1) = \tfrac{1}{2}\sigma_1,\ \Phi(\sigma_2) = \tfrac{1}{2}\sigma_2,\ \Phi(\sigma_3) = 0,$$

which proves that the range of Φ is three-dimensional. It is clear that Φ being a CPTP map is not a CPTP projector (it has two eigenvalues $1/2$) and hence does not preserve the purity of the input states in $\mathrm{Im}(\Phi)$.

Note that since $\dim \mathrm{Im}(\Phi) = 3$, all unit trace trace Hermitian operators in $\mathrm{Im}(\Phi)$ will form a plane which would cut the Bloch sphere along a great circle. As a result, the infinite family of pure states which form the great circle is a subset of $\mathrm{Im}(\Phi)$. This would suggest, erroneously, that the pure output of this channel is this set of pure states. On the contrary, the pure output of any channel is just a set of pure density matrices lying in the image of the Bloch ball under the action of the channel. In this case $\mathrm{PO}(\Phi)$ is the empty set.

It should be stressed that this result is no longer true for positive and trace-preserving projectors. Consider a map

$$\Psi(\rho) = \tfrac{1}{4}(3\rho + \sigma_1\rho\sigma_1 + \sigma_2\rho\sigma_2 - \sigma_3\rho\sigma_3). \tag{18}$$

One finds

$$\Psi(\mathbb{1}) = \mathbb{1},\ \Psi(\sigma_1) = \sigma_1,\ \Psi(\sigma_2) = \sigma_2,\ \Psi(\sigma_3) = 0,$$

and hence Ψ maps a density operator

$$\rho = \tfrac{1}{2}(\mathbb{1} + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3) \tag{19}$$

to a density operator

$$\Psi(\rho) = \tfrac{1}{2}(\mathbb{1} + x_1\sigma_1 + x_2\sigma_2), \tag{20}$$

that is, Ψ projects a Bloch ball into a disk $x_3 = 0$. For more details cf. Appendix B. Interestingly, a map projecting a Bloch ball to the x_3 axis defined by

$$\mathcal{S}_3(\rho) = \tfrac{1}{2}(\rho + \sigma_3\rho\sigma_3) \tag{21}$$

is a CPTP projector satisfying

$$\mathcal{S}_3(\mathbb{1}) = \mathbb{1},\ \mathcal{S}_3(\sigma_1) = 0,\ \mathcal{S}_3(\sigma_2) = 0,\ \mathcal{S}_3(\sigma_3) = \sigma_3,$$

and hence $\dim \mathrm{Im}(\mathcal{S}_3) = 2$. A pictorial representation of the action of Ψ and $\mathcal{S}_3$ is given in Fig. 1(a).

Now, observe that for the map Ψ only points from the equator

$$\tfrac{1}{2}(\mathbb{1} + \cos\phi\sigma_1 + \sin\phi\sigma_2),\quad \phi \in [0, 2\pi),$$

belonging to $\mathrm{PO}(\Psi)$. Hence this map cannot be completely positive. Actually, one proves the following:

Proposition 4. Let $\Pi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be a positive trace-preserving projector onto a three-dimensional subspace. Then $\mathrm{PO}(\Pi)$ is a great circle on the Bloch ball **S**. Equivalently, the subspace $\mathrm{Im}(\Phi) = \Phi(M_2(\mathbb{C}))$ is an operator system (containing $\mathbb{1}$ and is closed under Hermitian conjugation [25]).

For the proof see Appendix B.

IV. EXAMPLES

In this section we illustrate our discussion by three examples:

(1) Commutative and image nonincreasing evolution,
(2) Noncommutative but image nonincreasing, and
(3) Noncommutative and not image nonincreasing.

Recall that if the dynamical map Λ_t is commutative, that is, $\Lambda_t\Lambda_u = \Lambda_u\Lambda_t$, and diagonalizable, meaning that time-independent eigenvectors X_α of Λ_t and Y_α of the dual map $\Lambda_t^\dagger$ (so-called damping basis [26])

$$\Lambda_t[X_\alpha] = \lambda_\alpha(t)X_\alpha\ ,\quad \Lambda_t^\dagger[Y_\alpha] = \lambda_\alpha^*(t)Y_\alpha,$$

span the entire $\mathcal{B}(\mathcal{H})$, the map Λ_t gives rise to the following spectral representation:

$$\Lambda_t(\rho) = \sum_\alpha \lambda_\alpha(t) X_\alpha \mathrm{Tr}(Y_\alpha^\dagger \rho). \tag{22}$$

Moreover, in this case if the map is divisible, that is, kernel nondecreasing, then necessarily it is image nonincreasing.

Example 1. A well-known example of a commutative diagonalizable qubit dynamical map is generated by the following generator (it was already analyzed in [20]):

$$\mathcal{L}_t = \gamma_1(t)\mathcal{L}_1 + \gamma_2(t)\mathcal{L}_2 + \gamma_3(t)\mathcal{L}_3, \tag{23}$$

where $\mathcal{L}_k(\rho) = \frac{1}{2}(\sigma_k\rho\sigma_k - \rho)$. The corresponding dynamical map reads

$$\Lambda_t(\rho) = \sum_{\alpha=0}^{3} p_\alpha(t)\sigma_\alpha\rho\sigma_\alpha, \tag{24}$$

with $\sigma_0 = \mathbb{1}$, and

$$\begin{aligned} p_0(t) &= \tfrac{1}{4}[1 + \lambda_1(t) + \lambda_2(t) + \lambda_3(t)],\\ p_1(t) &= \tfrac{1}{4}[1 + \lambda_1(t) - \lambda_2(t) - \lambda_3(t)],\\ p_2(t) &= \tfrac{1}{4}[1 - \lambda_1(t) + \lambda_2(t) - \lambda_3(t)],\\ p_3(t) &= \tfrac{1}{4}[1 - \lambda_1(t) - \lambda_2(t) + \lambda_3(t)], \end{aligned}$$

and the corresponding eigenvalues $\lambda_\alpha(t)$ read

$$\lambda_i(t) = \exp\left(-\Gamma_j(t) - \Gamma_k(t)\right),$$

where $\{i, j, k\}$ is a permutation of $\{1, 2, 3\}$, and $\Gamma_k(t) = \int_0^t \gamma_k(\tau)d\tau$. The map Λ_t is invertible if all $\Gamma_k(t)$ are finite for finite times. Now, if, for example, one has $\Gamma_1(t') = \infty$, then $\lambda_2(t') = \lambda_3(t')= 0$, which means that the image of $\Lambda_{t'}$ is two-dimensional and, of course, it is orthogonal to the two-dimensional kernel:

$$\mathrm{Im}(\Lambda_{t'}) = \mathrm{span}\{\sigma_0, \sigma_1\}\ ,\quad \mathrm{Ker}(\Lambda_{t'}) = \mathrm{span}\{\sigma_2, \sigma_3\}.$$

Divisibility requires that $\Gamma_1(t) = \infty$ for $t > t'$. Now, if $\Gamma_2(t)$ and $\Gamma_3(t)$ stay finite, then

$$\mathrm{Im}(\Lambda_t) = \mathrm{Im}(\Lambda_{t'})\ ,\quad \mathrm{Ker}(\Lambda_t) = \mathrm{Ker}(\Lambda_{t'})$$

for $t > t'$ and Λ_t is image nonincreasing. If, for example, $\Gamma_2(t'') = \infty$, then the image of $\Lambda_{t''}$ is one-dimensional and it is orthogonal to the three-dimensional kernel:

$$\mathrm{Im}(\Lambda_{t''}) = \mathrm{span}\{\sigma_0\},\quad \mathrm{Ker}(\Lambda_{t''}) = \mathrm{span}\{\sigma_1, \sigma_2, \sigma_3\}.$$

Now, divisibility requires that additionally, $\Gamma_2(t) = \infty$ for $t > t''$, and as a result,

$$\mathrm{Im}(\Lambda_t) = \mathrm{Im}(\Lambda_{t''}),\quad \mathrm{Ker}(\Lambda_t) = \mathrm{Ker}(\Lambda_{t''})$$

for $t > t''$. Thus in this case also Λ_t is image nonincreasing. Finally, the map is CP divisible iff $\gamma_1(t) \geqslant 0$ for $t < t'$ and $\gamma_2(t), \gamma_3(t) \geqslant 0$ for $t < t''$.

The next example goes beyond commutative maps.

Example 2. Consider the following generator [27]:

$$\mathcal{L}_t = \omega(t)\mathcal{L}_0 + \gamma_+(t)\mathcal{L}_+ + \gamma_-(t)\mathcal{L}_- + \gamma_3(t)\mathcal{L}_3, \tag{25}$$

where $\mathcal{L}_0(\rho) = -\frac{i}{2}[\sigma_z, \rho]$ is the Hamiltonian part, and

$$\begin{aligned} \mathcal{L}_+(\rho) &= \tfrac{1}{2}\big(\sigma_+\rho\sigma_- - \tfrac{1}{2}\{\sigma_-\sigma_+, \rho\}\big),\\ \mathcal{L}_-(\rho) &= \tfrac{1}{2}\big(\sigma_-\rho\sigma_+ - \tfrac{1}{2}\{\sigma_+\sigma_-, \rho\}\big),\\ \mathcal{L}_3(\rho) &= \tfrac{1}{2}(\sigma_z\rho\sigma_z - \rho), \end{aligned}$$

with $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$. It defines a noncommutative family in general, that is, $\mathcal{L}_t\mathcal{L}_s \neq \mathcal{L}_s\mathcal{L}_t$. Actually, commutativity holds if and only if $\gamma_-(t) = k\gamma_+(t)$ for a positive constant k. Also negative values of k would violate the CP-divisibility condition of Eq. (25). The corresponding dynamical map $\Lambda_t = \mathcal{T} e^{\int_0^t \mathcal{L}_\tau d\tau}$ is given by

$$\rho = \begin{pmatrix} 1-p & \alpha \\ \alpha^* & p \end{pmatrix} \rightarrow \rho_t = \begin{pmatrix} 1-p(t) & \alpha(t) \\ \alpha(t)^* & p(t) \end{pmatrix}, \tag{26}$$

where

$$p(t) = e^{-\Gamma(t)}[G(t) + p]\ , \quad \alpha(t) = \alpha\, e^{i\Omega(t) - \Gamma(t)/2 - \Gamma_3(t)},$$

with $\Gamma_3(t) = \int_0^t \gamma_3(\tau)d\tau$, $\Omega(t) = \int_0^t 2\,\omega(\tau)d\tau$, and

$$\Gamma(t) = \frac{1}{2}\int_0^t (\gamma_+(\tau) + \gamma_-(\tau))d\tau,$$

$$G(t) = \frac{1}{2}\int_0^t e^{\Gamma(\tau)}\gamma_-(\tau)d\tau.$$

The corresponding time-dependent eigenvalues of Λ_t read: $\lambda_0(t) = 1$, $\lambda_1(t) = e^{i\Omega(t) - \Gamma(t)/2 - \Gamma_3(t)} = \lambda_2^*(t)$, and $\lambda_3(t) = e^{-\Gamma(t)}$ (cf. Appendix C). Finally, one finds for the time-dependent eigenvectors:

$$X_0(t) = \frac{1}{1 - e^{-\Gamma(t)}}\begin{pmatrix} 1 - e^{-\Gamma(t)}[G(t) + 1] & 0 \\ 0 & e^{-\Gamma(t)}G(t) \end{pmatrix},$$

$$X_1 = |0\rangle\langle 1|\ , \ X_2 = |1\rangle\langle 0|\ , \ X_3 = \sigma_3,$$

together with $Y_0 = \mathbb{1}/2$, $Y_1 = X_2$, $Y_2 = X_1$, and

$$Y_3(t) = \frac{1}{1 - e^{-\Gamma(t)}}\begin{pmatrix} e^{-\Gamma(t)}G(t) & 0 \\ 0 & e^{-\Gamma(t)}[G(t) + 1] - 1 \end{pmatrix}.$$

One easily checks that $\{X_\alpha, Y_\beta\}$ defines a damping basis, that is,

$$\mathrm{Tr}(X_\alpha Y_\beta^\dagger) = \delta_{\alpha\beta}.$$

The map is invertible if and only if $\Gamma(t)$ and $\Gamma_3(t)$ are finite for finite t. Now, if $\Gamma_3(t_1) = \infty$, then $\lambda_1(t) = \lambda_2(t) = 0$ and hence $\dim[\mathrm{Im}(\Lambda_{t_1})] = 2$. One finds

$$\mathrm{Im}(\Lambda_{t_1}) = \mathrm{span}\{X_0(t), X_3\}\ , \quad \mathrm{Ker}(\Lambda_{t_1}) = \mathrm{span}\{X_2, X_3\}.$$

Divisibility requires that $\Gamma_3(t) = \infty$ for $t > t_1$, that is,

$$\mathrm{Ker}(\Lambda_t) \supset \mathrm{Ker}(\Lambda_{t_1}) = \mathrm{span}\{X_2, X_3\}.$$

Note that for $t > t_1$ the image $\mathrm{Im}(\Lambda_t)$ is a subset of $\mathrm{Im}(\Lambda_{t_1})$, since $\mathrm{Im}(\Lambda_{t_1})$ spans a set of diagonal matrices. If, moreover, we choose γ_+ and γ_- in such a way that $G(t_2)$ is finite and $\Gamma(t_2) = \infty$, then also $\lambda_3(t_2) = 0$ and hence $\dim[\mathrm{Im}(\Lambda_{t_2})] = 1$, and it is spanned by $X_0(t)$. Again, divisibility requires that $\Gamma(t) = \infty$ for $t > t_2$. Note that in this case for $t > t_2$ one has $X_0(t) = |0\rangle\langle 0|$ and hence

$$\Lambda_t(\rho) = \mathrm{Tr}\,\rho\, |0\rangle\langle 0|$$

for $t \geqslant t_2$, that is, all states are mapped into the $|0\rangle\langle 0|$ state. This proves that also this example being noncommutative gives rise to the image nonincreasing evolution. The evolution is CP divisible if $\gamma_3(t) \geqslant 0$ for $t \leqslant t_1$, and $\gamma_+(t) \geqslant 0$ and $\gamma_-(t) \geqslant 0$ for $t \leqslant t_2$.

The CP-divisibility aspects of the above examples can also be studied using the results in [20]. Therefore, we now consider a map which is neither commutative nor image nonincreasing.

Example 3. Let the dynamical map be a composition of two maps:

$$\Lambda_t = \mathcal{U}_t \circ \Psi_t, \tag{27}$$

where

$$\mathcal{U}_t[\rho] = U_t \rho U_t^\dagger\ ; \ \ U_t = e^{-i\sigma_2 t}, \tag{28}$$

and

$$\Psi_t[\rho] = [1 - p(t)]\rho + p(t)\Phi[\rho], \tag{29}$$

with

$$\Phi[\rho] = |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1| \tag{30}$$

being a totally depolarizing channel. One has $0 \leqslant p(t) \leqslant 1$ with $p(0) = 0$. It is clear that the map Λ_t is CP divisible iff the map Ψ_t is CP divisible. The map Λ_t is invertible only if $p(t) < 1$. Suppose now that $p(t) < 1$ for $t < t_*$ and $p(t) = 1$ for $t \geqslant t_*$. The kernel of the map for $t \geqslant t_*$ is two-dimensional,

$$\mathrm{Ker}(\Lambda_t) = \mathrm{span}\{\sigma_1, \sigma_2\},$$

due to $\Phi(\sigma_1) = \Phi(\sigma_2) = 0$ and hence the map is divisible.

For the image one finds

$$\mathrm{Im}(\Lambda_t) = \mathrm{span}\{\mathbb{1}, X(t)\},$$

with

$$X(t) = \begin{pmatrix} \cos 2t & \sin 2t \\ \sin 2t & -\cos 2t \end{pmatrix}.$$

It is clear that the condition $\mathrm{Im}(\Lambda_t) \subset \mathrm{Im}(\Lambda_{t_*})$ is no longer valid and hence the map is not image nonincreasing.

The corresponding propagator $V_{t,s}$ satisfies

$$V_{t,s} = \mathcal{U}_t \circ W_{t,s} \circ \mathcal{U}_s^{-1},$$

where $W_{t,s}$ is the propagator for the dynamical map Ψ_t, i.e., $\Psi_t = W_{t,s} \circ \Psi_s$. Now, for $t < t_*$ the map Ψ_t is invertible and one can find the corresponding time-local generator [28]

$$\ell_t = \dot{\Psi}_t \Psi_t^{-1}\ , \quad t < t_*.$$

Using

$$\dot{\Psi}_t[\rho] = \dot{p}(t)(\Phi(\rho) - \rho)$$

together with

$$\Psi_t^{-1}[\rho] = \frac{1}{1 - p(t)}(\rho - p(t)\Phi[\rho]),$$

one gets

$$\ell_t[\rho] = \frac{\dot{p}(t)}{1 - p(t)}(\Phi[\rho] - \rho).$$

Now, it is clear that the map Ψ_t is CP divisible iff $\dot{p}(t) \geqslant 0$ for $t < t_*$ and $p(t) = 1$ for $t \geqslant t_*$ [29]. As the corresponding propagator one has

$$W_{t,s} = \Psi_t \Psi_s^{-1}, \quad s < t_*,$$

and

$$W_{t,s} = \Phi, \quad s \geqslant t_*.$$

This implies

$$V_{t,s} = \mathcal{U}_t\big(\Psi_t \Psi_s^{-1}\big)\mathcal{U}_s^{-1}, \quad s < t_*,$$

and

$$V_{t,s} = \mathcal{U}_t \circ \Phi \circ \mathcal{U}_s^{-1}, \quad s \geqslant t_*.$$

Note that $V_{t,t}$ is given by

$$V_{t,t} = \mathrm{id}, \quad t < t_*,$$

and

$$V_{t,t} = \mathcal{U}_t \circ \Phi \circ \mathcal{U}_t^{-1}\ , \quad t \geqslant t_*.$$

It is clear that $V_{t,t}$ always defines a CPTP projector.

V. CONCLUSION

In this paper, we discussed two main approaches to Markovianity based on properties of dynamical maps—CP divisibility and information backflow—and studied conditions under which they are equivalent. This issue has recently been analyzed in several papers [19,20,30], and certain classes of dynamical maps are already known for which the equivalence could be shown, namely, invertible and so-called image nonincreasing maps. Although the image nonincreasing class includes the class of invertible maps and also several dynamical maps known in literature, whether the equivalence could be shown for the general case remains an open question.

Here, we showed the equivalence for general qubit dynamical map (Theorem 6). A key element of the proof is the fact that there are no CPTP projectors onto a qubit subspace of dimension 3, which is spanned by density operators. We also show that there can be positive projectors onto qubit subspaces of dimension 3, but only when the subspace forms an operator system. We expect this result will shed more light on the theoretical understanding of dynamical maps. In a slightly different context, divisibility of qubit channels was recently addressed in [31].

We also discussed the conditions for CP divisibility and the image nonincreasing property for a number of examples of noninvertible dynamical maps. In particular, we presented an example of a qubit dynamical map which is not image nonincreasing to demonstrate the importance of our result.

Finally, we note that our result emphasizes the requirement of further analysis of this topic. Moreover, the question of whether the equivalence could be shown for higher dimensions still remains open. This question, if proved affirmatively, will present a consistent theory of Markovianity based on properties of dynamical maps in the quantum regime.

It should be clear that our result could be immediately applied to a classical case. One can show that for classical dynamical maps of dimension $d \leqslant 3$, P divisibility is equivalent to monotonic decay of distinguishability of any two probability vectors. A detailed discussion can be found in Appendix D. It would be interesting to fully clarify the problem for $d > 2$ in the quantum case and $d > 3$ in the classical case.

ACKNOWLEDGMENTS

D.C. was supported by the National Science Centre through Project No. 2018/30/A/ST2/00837. S.C. is thankful to Suchetana Goswami for useful discussions on outlining and preparing the figure. We thank Ángel Rivas for valuable comments.

APPENDIX A: PROOF OF PROPOSITION 1

Proof 1. We will prove the proposition by contradiction. Let a three-dimensional subspace spanned by qubit density operators be denoted by $\mathcal{M}$. Assume there exists a CPTP projector $\Pi_{\mathcal{M}}$ onto $\mathcal{M}$. Now consider the following arguments.

Since $\mathcal{M}$ has dimension 3, we must have three linearly independent density matrices ρ_1, ρ_2, and ρ_3, which spans $\mathcal{M}$. As the set of all Hermitian operators in $\mathcal{B}(\mathcal{H})$ form a real vector space, we can always find a nonzero Hermitian operator $K \in \mathcal{B}(\mathcal{H})$ which is orthogonal to $\mathcal{M}$, i.e.,

$$\mathrm{Tr}[K\rho_i] = 0\ ;\ i = 1, 2, 3. \tag{A1}$$

Let us now consider the eigenvalue decomposition of K as

$$K = \lambda_0 \left|0\right\rangle \left\langle 0\right| + \lambda_1 \left|1\right\rangle \left\langle 1\right|, \tag{A2}$$

where λ_0, λ_1 are the real eigenvalues and $\{|0\rangle, |1\rangle\}$ are the eigenvectors of K. Now, using Eq. (A1) we find $\lambda_0 \langle 0|\rho_i|0\rangle + \lambda_1 \langle 1|\rho_i|1\rangle = 0$ for $i = 1, 2, 3$. This implies $\lambda_0 \neq 0, \lambda_1 \neq 0$, as any of them being zero will make all the ρ_i's linearly dependent. Therefore, we can choose a Hermitian operator $H \in \mathcal{B}(\mathcal{H})$ which is orthogonal to $\mathcal{M}$ and has the form

$$H = (1/\lambda_0)K = \left|0\right\rangle \left\langle 0\right| + \lambda \left|1\right\rangle \left\langle 1\right|, \tag{A3}$$

where $\lambda \neq 0$ is real. Note that $H \notin \mathcal{M}$ and $\{\rho_1, \rho_2, \rho_3, H\}$ forms a basis for $\mathcal{B}(\mathcal{H})$. Now, using Eqs. (A1) and (A3) we get $\langle 0|\rho_i|0\rangle + \lambda \langle 1|\rho_i|1\rangle = 0$ for $i = 1, 2, 3$. Therefore we can write all the ρ_i's in the basis $\{|0\rangle, |1\rangle\}$ as

$$\rho_1 = \begin{pmatrix} p & x \\ x^* & 1-p \end{pmatrix}; \rho_2 = \begin{pmatrix} p & y \\ y^* & 1-p \end{pmatrix};$$
$$\rho_3 = \begin{pmatrix} p & z \\ z^* & 1-p \end{pmatrix}, \tag{A4}$$

where $x, y, z \in \mathbb{C}$ and

$$p = \frac{\lambda}{\lambda - 1}. \tag{A5}$$

Note that $p \neq 0$, $p \neq 1$ and $x \neq y$, $y \neq z$, $z \neq x$ as ρ_1, ρ_2 and ρ_3 are all independent. Let us define $X = \rho_1 - \rho_2$ and $Y = \rho_1 - \rho_3$. Note that $X, Y \in \mathcal{M}$ are independent and have the form

$$X = \begin{pmatrix} 0 & u_1 + iv_1 \\ u_1 - iv_1 & 0 \end{pmatrix}; \ \ Y = \begin{pmatrix} 0 & u_2 + iv_2 \\ u_2 - iv_2 & 0 \end{pmatrix}, \tag{A6}$$

where $u_1, u_2, v_1, v_2 \in \mathbb{R}$. Note that $u_1 + iv_1 \neq u_2 + iv_2$ as $y \neq z$. This implies either $u_1 \neq u_2$ or $v_1 \neq v_2$. Now since X and Y are independent, we must also have (a) $u_1 \neq 0$ or $u_2 \neq 0$, and (b) $v_1 \neq 0$ or $v_2 \neq 0$. From condition (a) we find that if $u_1 \neq 0$, $(u_2/u_1)X - Y = c\,\sigma_y$, and if $u_2 \neq 0$, $X - (u_1/u_2)Y = c'\,\sigma_y$, where $c, c' \in \mathbb{C}$, and σ_y is the Pauli y matrix. Similarly, from condition (b) we find that if $v_1 \neq 0$ then $(v_2/v_1)X - Y = c'' \sigma_x$, and if $v_2 \neq 0$ then $X - (v_1/v_2)Y = c''' \sigma_x$, where $c'', c''' \in \mathbb{C}$, and σ_x is the Pauli x matrix. This implies $\sigma_x, \sigma_y \in \mathcal{M}$ and hence $|0\rangle\langle 1|, |1\rangle\langle 0| \in \mathcal{M}$. As a result we can define $Z = \rho_1 - x|0\rangle\langle 1| - x^*|1\rangle\langle 0|$, where $Z \in \mathcal{M}$ and has the form

$$Z = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}. \tag{A7}$$

Note that Z and H are independent. Hence we have $|0\rangle\langle 0| = c_1 H + c_2 Z$ and $|1\rangle\langle 1| = c_3 H + c_4 Z$, where

$$c_1 = \frac{(1-p)^2}{1-2p(1-p)},\ c_2 = \frac{p}{1-2p(1-p)},$$
$$c_3 = -\frac{p(1-p)}{1-2p(1-p)},\ c_4 = \frac{1-p}{1-2p(1-p)}. \tag{A8}$$

Let us now consider the action of the CPTP projector $\Pi_{\mathcal{M}}$ on the following operators in $\mathcal{M}$:

$$\Pi_{\mathcal{M}}[|0\rangle\langle 1|] = |0\rangle\langle 1|,\ \Pi_{\mathcal{M}}[|1\rangle\langle 0|] = |1\rangle\langle 0|,$$
$$\Pi_{\mathcal{M}}[Z] = Z. \tag{A9}$$

Let us now consider the Choi matrix $\Gamma(\Pi_{\mathcal{M}})$ given by

$$\begin{aligned}\Gamma(\Pi_{\mathcal{M}}) &= \sum_{i,j=0,1} |i\rangle\langle j| \otimes \Pi_{\mathcal{M}}[|i\rangle\langle j|] \\ &= |0\rangle\langle 0| \otimes \Pi_{\mathcal{M}}[|0\rangle\langle 0|] + |0\rangle\langle 1| \otimes |0\rangle\langle 1| \\ &\quad + |1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes \Pi_{\mathcal{M}}[|1\rangle\langle 1|].\end{aligned} \tag{A10}$$

As $\Pi_{\mathcal{M}}$ is a CPTP map, $\Gamma(\Pi_{\mathcal{M}})$ must be positive. Also note that $\Pi_{\mathcal{M}}[|0\rangle\langle 0|]$ and $\Pi_{\mathcal{M}}[|1\rangle\langle 1|]$ must be density matrices. Let us choose them in the form

$$\Pi_{\mathcal{M}}[|0\rangle\langle 0|] = \begin{pmatrix} q_1 & w_1 \\ w_1^* & 1-q_1 \end{pmatrix};$$
$$\Pi_{\mathcal{M}}[|1\rangle\langle 1|] = \begin{pmatrix} 1-q_2 & w_2 \\ w_2^* & q_2 \end{pmatrix}, \tag{A11}$$

where $0 \leqslant q_1, q_2 \leqslant 1$ and $w_1, w_2 \in \mathbb{C}$. We can now form the Choi matrix:

$$\Gamma(\Pi_{\mathcal{M}}) = \begin{pmatrix} q_1 & w_1 & 0 & 1 \\ w_1^* & 1-q_1 & 0 & 0 \\ 0 & 0 & 1-q_2 & w_2 \\ 1 & 0 & w_2^* & q_2 \end{pmatrix}. \tag{A12}$$

It can be easily seen from the above form that since $\Gamma(\Pi_{\mathcal{M}})$ is positive we must have $q_1 q_2 - 1 \geqslant 0$, which is possible only when $q_1 = q_2 = 1$. As a result, preserving the positivity of $\Pi_{\mathcal{M}}[|0\rangle\langle 0|]$ and $\Pi_{\mathcal{M}}[|1\rangle\langle 1|]$, we get $w_1 = w_2 = 0$. This implies $\Pi_{\mathcal{M}}[|0\rangle\langle 0|] = |0\rangle\langle 0|$ and $\Pi_{\mathcal{M}}[|1\rangle\langle 1|] = |1\rangle\langle 1|$. This in turn implies $\Pi_{\mathcal{M}}[H] = H$. As a result, $\mathrm{Im}(\Pi_{\mathcal{M}}) \not\subset \mathcal{M}$, which is a contradiction.

Proof 2. The proposition also follows from a result in [22]. It was shown in Theorem 4.9 of the paper that if any qubit channel (CPTP map) has more than two pure states in its output it must contain all pure states in its output. From our analysis in the above proof it can be easily seen that the infinite family of pure states

$$\{|\psi_\theta\rangle = \sqrt{p}\,|0\rangle + \sqrt{1-p}e^{i\theta}\,|1\rangle; \theta \in \mathbb{R}\} \subset \mathcal{M} \tag{A13}$$

appears in the output of $\Pi_{\mathcal{M}}$, but the states $|0\rangle$ and $|1\rangle$ do not. Hence $\Pi_{\mathcal{M}}$ cannot be a CPTP map.

APPENDIX B: ARE THERE PTP PROJECTORS ON THREE-DIMENSIONAL SUBSPACES SPANNED BY QUBIT STATES?

We consider a qubit space $\mathcal{H} = \mathbb{C}^2$. Let us consider a PTP projector $\pi_{\mathcal{M}}$ onto $\mathcal{M}$, where $\mathcal{M}$ is as defined in Appendix A. Therefore, $\pi_{\mathcal{M}}$ must have the following properties:

(a) $\mathrm{Im}(\pi_{\mathcal{M}}) \subset \mathcal{M}$.

(b) $\pi_{\mathcal{M}}[X] = X$ if $X \in \mathcal{M}$.

Following the same analysis as in Appendix A, we find

$$\pi_{\mathcal{M}}[|0\rangle\langle 1|] = |0\rangle\langle 1|, \pi_{\mathcal{M}}[|1\rangle\langle 0|] = |1\rangle\langle 0|, \pi_{\mathcal{M}}[Z] = Z. \tag{B1}$$

Now consider the action of $\pi_{\mathcal{M}}$ on H using the form given in Eq. (A3). As $\pi_{\mathcal{M}}$ is TP, without loss of generality we get

$$\pi_{\mathcal{M}}[H] = (1+\lambda)Z + s|0\rangle\langle 1| + s^*|1\rangle\langle 0| = \frac{1-2p}{1-p}Z + s|0\rangle\langle 1| + s^*|1\rangle\langle 0|, \tag{B2}$$

where $s \in \mathbb{C}$. Here we have used the relation between p and λ, as given in Eq. (A5).

Now consider any state $\rho \in \mathcal{P}_+(\mathcal{H})$ having the following form in the $\{|0\rangle, |1\rangle\}$ basis:

$$\rho = \begin{pmatrix} q & r \\ r^* & 1-q \end{pmatrix}, \tag{B3}$$

where $0 \leqslant q \leqslant 1$, $r \in \mathbb{C}$, and $|r|^2 \leqslant q(1-q)$. Note that this is the most general form of a qubit state. Also note that the maximum value that $|r|$ can take is 1/2, which is when $\rho = |\pm\rangle\langle\pm|$, where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. Now, consider the action of $\pi_{\mathcal{M}}$ on ρ:

$$\begin{aligned}\pi_{\mathcal{M}}[\rho] &= q\,\pi_{\mathcal{M}}[|0\rangle\langle 0|] + (1-q)\,\pi_{\mathcal{M}}[|1\rangle\langle 1|] + r|0\rangle\langle 1| + r^*|1\rangle\langle 0| \\ &= q\,\pi_{\mathcal{M}}[c_1 H + c_2 Z] + (1-q)\,\pi_{\mathcal{M}}[c_3 H + c_4 Z] + r|0\rangle\langle 1| + r^*|1\rangle\langle 0| \\ &= [q\,c_1 + (1-q)c_3]\pi_{\mathcal{M}}[H] + [q\,c_2 + (1-q)c_4]Z + r|0\rangle\langle 1| + r^*|1\rangle\langle 0|\end{aligned}$$

$$= [q\,c_1 + (1-q)c_3]\frac{1-2p}{1-p}Z + [q\,c_2 + (1-q)c_4]Z + (r+s)\,|0\rangle\,\langle 1| + (r^*+s^*)\,|1\rangle\,\langle 0|$$
$$= \left[q\left(c_1\frac{1-2p}{1-p}+c_2\right) + (1-q)\left(c_3\frac{1-2p}{1-p}+c_4\right)\right]Z + (r+s)\,|0\rangle\,\langle 1| + (r^*+s^*)\,|1\rangle\,\langle 0|\,. \tag{B4}$$

Now inserting the values of c_1 and c_2, as given in Eq. (A8), we find $c_1\frac{1-2p}{1-p}+c_2 = c_3\frac{1-2p}{1-p}+c_4 = 1$. As a result, we find

$$\pi_{\mathcal{M}}[\rho] = \begin{pmatrix} p & r+s \\ r^*+s^* & 1-p \end{pmatrix}. \tag{B5}$$

In the above form if $\pi_{\mathcal{M}}[\rho]$ is positive, we must have

$$|r+s|^2 \leqslant p(1-p) \leqslant 1/4, \tag{B6}$$

where we have used the fact that $p(1-p) \leqslant 1/4$ and the equality is reached only when $p = 1/2$. Note in Eq. (B2) that s can take any complex value. Let $s = s_1 + is_2$ $(s_1, s_2 \in \mathbb{R})$. If we consider either of the real or imaginary parts of s to be nonzero, we can always choose ρ in Eq. (B3) to have $q = 1/2$ and $r = \frac{s_1}{2|s_1|}$, and as a result, we get $|r+s|^2 = (\frac{1}{2}+|s_1|)^2 + |s_2|^2 > \frac{1}{4}$, which contradicts condition (B6). Therefore, we must have $s = 0$. Now, if we choose ρ in Eq. (B3) to have $q = 1/2$ and $r = 1/2$, the inequality in Eq. (B6) will be satisfied only when $p = 1/2$. Hence we conclude $\pi_{\mathcal{M}}$ is a PTP map only when $p = 1/2$. In that case, $\mathcal{M} = \text{span}\{\frac{\mathbb{1}}{2}, \frac{\mathbb{1}+\sigma_1}{2}, \frac{\mathbb{1}+\sigma_2}{2}\}$ and the PTP projector has the action $\mathbb{1} \to \mathbb{1}, \sigma_1 \to \sigma_1, \sigma_2 \to \sigma_2, \sigma_3 \to 0$. In other words, a PTP projector on a three-dimensional qubit subspace spanned by density matrices will exist only when the subspace is an *operator system* [25].

APPENDIX C: CALCULATIONS FOR EXAMPLE 2

The time-evolved state in Teittinen's paper [27] is given by

$$\rho(t) = \Lambda_t[\rho(0)] = \begin{pmatrix} 1-P_1(t) & \alpha(t) \\ \alpha(t)^* & P_1(t) \end{pmatrix}, \tag{C1}$$

where

$$P_1(t) = e^{-\Gamma(t)}[G(t) + P_1(0)], \tag{C2}$$

$$\alpha(t) = \alpha(0)e^{i\Omega(t)-\Gamma(t)/2-\Gamma_3(t)}. \tag{C3}$$

Note that $G(t)$, $\Omega(t)$, $\Gamma(t)$, and $\Gamma_3(t)$ are as defined in Example 2. Now consider the states

$$|0\rangle\,\langle 0| = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};\ |1\rangle\,\langle 1| = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};\ |+\rangle\,\langle +| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix};\ |+_y\rangle\,\langle +_y| = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}. \tag{C4}$$

Using these states, we can easily find

$$|0\rangle\,\langle 1| = |+\rangle\,\langle +| - i\,|+_y\rangle\,\langle +_y| - \frac{1-i}{2}(|0\rangle\,\langle 0| + |1\rangle\,\langle 1|), \tag{C5}$$

$$|1\rangle\,\langle 0| = |+\rangle\,\langle +| + i\,|+_y\rangle\,\langle +_y| - \frac{1+i}{2}(|0\rangle\,\langle 0| + |1\rangle\,\langle 1|). \tag{C6}$$

Now, using Eq. (C1) we get the time-evolved states,

$$\Lambda_t[|0\rangle\,\langle 0|] = \begin{pmatrix} 1-e^{-\Gamma(t)}G(t) & 0 \\ 0 & e^{-\Gamma(t)}G(t) \end{pmatrix}, \tag{C7}$$

$$\Lambda_t[|1\rangle\,\langle 1|] = \begin{pmatrix} 1-e^{-\Gamma(t)}[G(t)+1] & 0 \\ 0 & e^{-\Gamma(t)}[G(t)+1] \end{pmatrix}, \tag{C8}$$

$$\Lambda_t[|+\rangle\,\langle +|] = \begin{pmatrix} 1-e^{-\Gamma(t)}\big[G(t)+\frac{1}{2}\big] & \frac{1}{2}e^{i\Omega(t)-\Gamma(t)/2-\Gamma_3(t)} \\ \frac{1}{2}e^{-i\Omega(t)-\Gamma(t)/2-\Gamma_3(t)} & e^{-\Gamma(t)}\big[G(t)+\frac{1}{2}\big] \end{pmatrix}, \tag{C9}$$

$$\Lambda_t[|+_y\rangle\,\langle +_y|] = \begin{pmatrix} 1-e^{-\Gamma(t)}\big[G(t)+\frac{1}{2}\big] & \frac{i}{2}e^{i\Omega(t)-\Gamma(t)/2-\Gamma_3(t)} \\ \frac{-i}{2}e^{-i\Omega(t)-\Gamma(t)/2-\Gamma_3(t)} & e^{-\Gamma(t)}\big[G(t)+\frac{1}{2}\big] \end{pmatrix}. \tag{C10}$$

As a result, we can also find the action of Λ_t on operators, which are not density matrices:

$$\Lambda_t[|0\rangle\langle 1|] = \begin{pmatrix} 0 & e^{i\Omega(t)-\Gamma(t)/2-\Gamma_3(t)} \\ 0 & 0 \end{pmatrix}, \quad \text{(C11)}$$

$$\Lambda_t[|1\rangle\langle 0|] = \begin{pmatrix} 0 & 0 \\ e^{-i\Omega(t)-\Gamma(t)/2-\Gamma_3(t)} & 0 \end{pmatrix}. \quad \text{(C12)}$$

Now consider the *operator-vector correspondence* described in the following way: the vector correspondent of an operator $A = \sum_{i,j} |i\rangle\langle j|$ is defined as the vector $\mathrm{vec}(A) = \sum_{i,j} |i\rangle |j\rangle$. Therefore, using this notation we define $\mathcal{N}_t$ in the following way:

$$\mathrm{vec}(\rho(t)) = \mathcal{N}_t[\mathrm{vec}(\rho(0))]. \quad \text{(C13)}$$

Note that $\mathcal{N}_t$ is a 4×4 matrix of the following form,

$$\mathcal{N}_t = \begin{pmatrix} 1-e^{-\Gamma(t)}G(t) & 0 & 0 & 1-e^{-\Gamma(t)}[G(t)+1] \\ 0 & e^{i\Omega(t)-\Gamma(t)/2-\Gamma_3(t)} & 0 & 0 \\ 0 & 0 & e^{-i\Omega(t)-\Gamma(t)/2-\Gamma_3(t)} & 0 \\ e^{-\Gamma(t)}G(t) & 0 & 0 & e^{-\Gamma(t)}[G(t)+1] \end{pmatrix}. \quad \text{(C14)}$$

$\mathcal{N}_t$ has the same eigenvalues as the qubit map Λ_t. We found the eigenvalues and eigenvectors of $\mathcal{N}_t$ to be of the following form given in Table I.

TABLE I. Eigenvalues and eigenvectors of $\mathcal{N}_t$.

Eigenvalues	Eigenvectors
1	$\{\frac{1-e^{-\Gamma(t)}[G(t)+1]}{e^{-\Gamma(t)}G(t)}, 0, 0, 1\}$
$e^{-\Gamma(t)}$	$\{-1, 0, 0, 1\}$
$e^{i\Omega(t)-\Gamma(t)/2-\Gamma_3(t)}$	$\{0, 1, 0, 0\}$
$e^{-i\Omega(t)-\Gamma(t)/2-\Gamma_3(t)}$	$\{0, 0, 1, 0\}$

APPENDIX D: A COROLLARY FOR CLASSICAL DYNAMICAL MAPS

Corollary 2. Consider a classical dynamical map represented by a family of stochastic matrices $T_t : \mathbb{R}^d \to \mathbb{R}^d$ for $t \geqslant 0$ and $T_{t=0} = \mathbb{1}$. The map T_t is called divisible if $T_t = S_{t,s}T_s$, with $S_{t,s} : \mathbb{R}^d \to \mathbb{R}^d$ for $t \geqslant s$. It is called P divisible iff $S_{t,s}$ is a stochastic matrix. Now, if $d \leqslant 3$, then T_t is P divisible iff

$$\frac{d}{dt}\|T_t(x_1\mathbf{p}_1 - x_2\mathbf{p}_2)\|_1 \leqslant 0, \quad \text{(D1)}$$

for arbitrary probability vectors $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{R}^d$, and probability distribution $x_1 + x_2 = 1$. (Recall that the L_1 norm of $v = \{v_1, \ldots, v_d\} \in \mathbb{R}^d$ is given by $\|v\|_1 = \sum_{i=1}^d |v_i|$.)

The proof immediately follows due to the fact that if T_t is noninvertible, then its image has dimension 2 or 1.

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