Genuine multipartite indistinguishability and its detection via the generalized Hong-Ou-Mandel effect

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The question of whether two indistinguishable particles are bosons or fermions can be answered by observing the Hong-Ou-Mandel effect on a beam splitter. However, already for three particles one can consider symmetries that are neither bosonic nor fermionic. In this work, we propose a notion of a genuine multipartite indistinguishability and describe a simple method of quantifying it experimentally.

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I. INTRODUCTION

The concept of indistinguishability is rooted in the permutation symmetry. In essence, particles are indistinguishable if the underlying state remains unchanged after a permutation of their labels. This means that there are only two types of indistinguishable pairs of particles, bosons and fermions, since there are only two possibilities for such a system not to be affected by a transposition [1]. However, at some points in the history of physics, scientists believed that new types of particles, already discovered or just predicted to exist, could be described by different rules, known as parastatistics.

For instance, quarks seem to be indistinguishable with respect to cyclic permutations rather than transpositions, which turned out to be a signature of a hidden property—color [2]. Their exotic behavior follows from the fact that the requirement of symmetry or antisymmetry applies to the global state of the system, not to the state of a particular degree of freedom. Thus, in principle, exploiting an extra labeling degree of freedom allows one to engineer states that appear to have arbitrary symmetries.

In this work, we describe the indistinguishability present in such states. As already mentioned, this notion is usually defined with respect to the transposition of particles, but we focus on their cyclic permutations instead. Our main result consists in introducing the concept of genuine multipartite indistinguishability. We also provide its clear operational measure by considering unique evolution exhibited by genuine multipartite indistinguishable states on symmetric multiports.

This characteristic behavior stems from a nontrivial multipartite interference effect [3] that connects the system's symmetry with suppressed probabilities of certain outcomes on multiports. In particular, we show that just as the Hong-Ou-Mandel effect [4] can be used to differentiate between two bosons and two fermions, the generalized suppression laws enable the detection and characterization of genuine multipartite indistinguishability. This result provides another perspective on the previous works that proposed the use of the discrete Fourier transform to detect the indistinguishability of a set of particles [5–9].

There are three main motivations behind our research. First, indistinguishability can be considered a potential quantum resource [10–13] related to yet different from entanglement. Since the resource theory of entanglement [14] had a large impact on many fields of quantum science, we believe that a parallel development of the resource theory of indistinguishability can also prove to be a worthy endeavor. To lay its foundations, we discuss indistinguishability outside of a standard bipartite setting. In particular, we provide a rigorous definition of a genuine multipartite indistinguishability and propose its experimentally feasible measure.

Second, indistinguishability is studied as a necessary condition for the multipartite interference [15]. From this perspective, the states we investigate are particularly interesting, as they consist of particles that share some form of global indistinguishability. This leads to peculiar interference effects that might find some applications in quantum information protocols. For example, a form of bunching in which not all the particles tend to group together could be exploited to tweak the evolution of multipartite quantum walks [16,17]. Moreover, multipartite permutation symmetric states can be used in secret-sharing scenarios [18,19].

Finally, we are also interested in the dynamical properties of the parastatistical states in the context of genuine multipartite entanglement. Note that the evolution of correlated noninteracting classical particles can be considered a mixture of independent evolutions, studied particle by particle. On the other hand, due to nonclassical correlations, a collection of noninteracting quantum particles may evolve in a fundamentally multipartite way. An example of such behavior is the bunching of bosons and the antibunching of fermions. Here, we observe nontrivial patterns in the particle-count statistics. If the particles were distinguishable, the corresponding patterns could only result from genuine multipartite entanglement.

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II. INDISTINGUISHABILITY AND PERMUTATIONS

A. Bipartite indistinguishability

Consider a quantum system made of n particles. For simplicity, we assume that it can be described by a pure state

$$|\psi\rangle = \sum_{k_1,\dots,k_n} \alpha_{k_1,\dots,k_n} |k_1 \dots k_n\rangle, \tag{1}$$

where k_i denotes the state of the ith particle.

The goal is to determine if these particles can be considered indistinguishable. As already mentioned in the introduction, this property is connected with permutation invariance. For instance, we say that two particles, i and j, are indistinguishable if the swap of their labels

$$\begin{aligned}\Pi_{ij}|\psi\rangle &= \Pi_{ij} \sum_{k_1,\dots,k_n} \alpha_{k_1,\dots,k_n} |k_1 \dots k_i \dots k_j \dots k_n\rangle \\ &= \sum_{k_1,\dots,k_n} \alpha_{k_1,\dots,k_n} |k_1 \dots k_j \dots k_i \dots k_n\rangle \end{aligned} \tag{2}$$

does not change the state of the system, i.e., $|\langle\psi|\Pi_{jk}|\psi\rangle|^2 = 1$.

The above condition leads to a natural measure of bipartite indistinguishability,

$$\mathcal{I}_{ij} \equiv |\langle\psi|\Pi_{ij}|\psi\rangle|^2. \tag{3}$$

Note that $0 \leqslant \mathcal{I}_{ij} \leqslant 1$. The value 1 is attainable if particles i and j are indistinguishable, i.e., they are in a symmetric (bosonic) or antisymmetric (fermionic) state. On the other hand, the value 0 occurs if the state after the permutation is orthogonal to the initial one, which implies perfect distinguishability. In other words, here we do not ask whether the particles are bosons or fermions but rather whether they are indistinguishable.

As the swap of particle labels is not a physical operation, one may question if there exists an experimentally feasible method to measure $\mathcal{I}_{ij}$. However, it turns out that this value can be easily obtained by investigating the interference phenomenon known as the Hong-Ou-Mandel effect [4]. When two particles are cast into different input ports of a symmetric beam splitter, they leave it through the same output port (bunch) with probability

$$p_B = \frac{1 + \langle\Pi_{ij}\rangle}{2} \tag{4}$$

or remain in different modes (antibunch) with probability

$$p_A = \frac{1 - \langle\Pi_{ij}\rangle}{2}. \tag{5}$$

This means that

$$\mathcal{I}_{ij} = |p_B - p_A|^2. \tag{6}$$

B. Genuine tripartite indistinguishability

We would like to generalize the above operational measure to more than two particles. Before we start, let us note that in multipartite systems indistinguishability is related to all possible symmetries. Since in general permutation operators have only two common eigenvectors, it is natural to consider only the corresponding indistinguishable particles: totally symmetric bosons and totally antisymmetric fermions. However, one may also consider subgroups of permutation operators that define more general types of particle parastatistics. In the following, we will investigate genuinely multipartite indistinguishability stemming from the subgroups generated by cyclic permutations.

The simplest example of such parastatistics can be obtained in a tripartite system. Consider the subgroup made of identity operator $\mathbb{1}$, cyclic permutation $\Pi_{312} = \Pi_{23}\Pi_{12}$, and its inverse $\Pi_{231} = \Pi_{12}\Pi_{23}$. In analogy to Eq. (3), we define a measure of tripartite indistinguishability

$$\mathcal{I}_{123} \equiv |\langle\psi|\Pi_{312}|\psi\rangle|^2 = |\langle\psi|\Pi_{231}|\psi\rangle|^2. \tag{7}$$

The tripartite parastatistics correspond to states that maximize $\mathcal{I}_{123}$. To find them, we consider the eigenstates of cyclic permutation operator, which were also discussed by Peres in [2]

$$|\alpha\rangle = \frac{1}{\sqrt{3}}(|123\rangle + |312\rangle + |231\rangle), \tag{8}$$

$$|\beta\rangle = \frac{1}{\sqrt{3}}\big(|123\rangle + \omega|312\rangle + \omega^2|231\rangle\big), \tag{9}$$

$$|\gamma\rangle = \frac{1}{\sqrt{3}}\big(|123\rangle + \omega^2|312\rangle + \omega|231\rangle\big), \tag{10}$$

$$|\bar{\alpha}\rangle = \frac{1}{\sqrt{3}}(|213\rangle + |132\rangle + |321\rangle), \tag{11}$$

$$|\bar{\beta}\rangle = \frac{1}{\sqrt{3}}\big(|213\rangle + \omega|132\rangle + \omega^2|321\rangle\big), \tag{12}$$

$$|\bar{\gamma}\rangle = \frac{1}{\sqrt{3}}\big(|213\rangle + \omega^2|132\rangle + \omega|321\rangle\big). \tag{13}$$

Here $\omega = e^{i\frac{2\pi}{3}}$ and the notation $|xyz\rangle$ means that the first particle is in mode x, the second in mode y, and the third in mode z. For all these states, $\mathcal{I}_{123} = 1$, but $\mathcal{I}_{12} = \mathcal{I}_{23} = \mathcal{I}_{13} = 0$; therefore, they exhibit a genuinely tripartite indistinguishability without its bipartite counterpart.

C. Symmetric operations

Before we proceed, let us stress that studies on indistinguishability need to be based on symmetric operators. This is because application of asymmetric operators requires the ability to distinguish between the particles. However, we cannot *a priori* assume that the particles are distinguishable. This is what we want to check. Therefore, as long as we do not gain access to a degree of freedom which can be used to effectively label and distinguish between the particles, we are fundamentally limited to symmetric operators.

This restriction makes differentiating between all the states given by Eqs, (8)–(13) impossible. To see this, notice that transpositions preserve the expectation value of any symmetric operator A

$$\mathrm{Tr}\{A\rho\} = \mathrm{Tr}\{(\Pi_{ij}A\Pi_{ij})\rho\} = \mathrm{Tr}\{A(\Pi_{ij}\rho\Pi_{ij})\} \tag{14}$$

and the bipartite permutation operators swap between the states $\{|\alpha\rangle, |\bar{\alpha}\rangle\}$, $\{|\beta\rangle, |\bar{\beta}\rangle\}$, and $\{|\gamma\rangle, |\bar{\gamma}\rangle\}$. This means that the states within each of these pairs cannot be distinguished using symmetric operators. However, one can check that the rank-2 states

$$\rho_\beta = \tfrac{1}{2}(|\beta\rangle\langle\beta| + |\bar{\beta}\rangle\langle\bar{\beta}|), \tag{15}$$

$$\rho_\gamma = \tfrac{1}{2}(|\gamma\rangle\langle\gamma| + |\bar{\gamma}\rangle\langle\bar{\gamma}|), \tag{16}$$

$$\rho_\alpha = \tfrac{1}{2}(|\alpha\rangle\langle\alpha| + |\bar{\alpha}\rangle\langle\bar{\alpha}|) \tag{17}$$

commute with all the permutation operators and thus can be considered as representatives of the tripartite parastatistics. Note that $\mathrm{Tr}\{\Pi_{312}\rho_\alpha\} = \mathrm{Tr}\{\Pi_{312}\rho_\beta\} = \mathrm{Tr}\{\Pi_{312}\rho_\gamma\} = 1$, but $\mathrm{Tr}\{\Pi_{ij}\rho_\alpha\} = \mathrm{Tr}\{\Pi_{ij}\rho_\beta\} = \mathrm{Tr}\{\Pi_{ij}\rho_\gamma\} = 0$ for any $i \neq j$. As a result, the problem of measuring the tripartite indistinguishability reduces to finding a method to perfectly distinguish among ρ_α, ρ_β, and ρ_γ.

III. DETECTION OF TRIPARTITE INDISTINGUISHABILITY

As explained in the previous section, to experimentally measure $\mathcal{I}_{123}$ we should look for a simple natural process capable of perfectly discriminating between the states ρ_α, ρ_β, and ρ_γ. In particular, we ask if it is possible to achieve it with free evolution, i.e., without interaction between the particles. We are going to show that the answer is positive.

Recall that each of the three particles can be in one of the three modes: $|1\rangle$, $|2\rangle$, or $|3\rangle$. We consider a single-partite transformation between these modes given by the quantum Fourier transform (QFT)

$$U^{(3)}_{QFT} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}. \tag{18}$$

This transformation can be implemented with a multiport commonly known as a *tritter* [20]. It can be visualized as a three-port, i.e., a device with three inputs and three outputs. Since all the entries of $U^{(3)}_{QFT}$ have the same modulus, a single particle cast on the tritter is equally likely to end up in each of its output ports. In fact, a tritter can be represented by any unitary 3×3 matrix with this property, as they all generate equivalent dynamics.

Next, we apply $U = U^{(3)}_{QFT} \otimes U^{(3)}_{QFT} \otimes U^{(3)}_{QFT}$ to the states ρ_α, ρ_β, and ρ_γ. In other words, we feed the tritter with the tree particles that are in one of the above three states and observe the output ports. The tritter generates the following transformations on our basis states:

$$|\alpha\rangle \to \frac{|111\rangle + |222\rangle + |333\rangle}{3} + \omega\frac{|213\rangle + |132\rangle + |321\rangle}{3} + \omega^2\frac{|123\rangle + |312\rangle + |231\rangle}{3}, \tag{19}$$

$$|\beta\rangle \to \frac{|211\rangle + \omega|121\rangle + \omega^2|112\rangle}{3} + \frac{|322\rangle + \omega|232\rangle + \omega^2|223\rangle}{3} + \frac{|133\rangle + \omega|313\rangle + \omega^2|331\rangle}{3}, \tag{20}$$

$$|\gamma\rangle \to \frac{|122\rangle + \omega|221\rangle + \omega^2|212\rangle}{3} + \frac{|233\rangle + \omega|332\rangle + \omega^2|323\rangle}{3} + \frac{|311\rangle + \omega|113\rangle + \omega^2|131\rangle}{3}. \tag{21}$$

TABLE I. Probabilities of particle number counts after the tritter transformation on three particles. We consider three states corresponding to different parastatistics, symmetric state, antisymmetric state, state of three distinguishable particles, and an arbitrary state ρ.

	ρ_α	ρ_β	ρ_γ	$\vert+\rangle$	$\vert-\rangle$	$\vert 123\rangle$	ρ
$\{1, 1, 1\}$	2/3	0	0	1/3	1	2/9	p_{111}
$\{3, 0, 0\}$	1/9	0	0	2/9	0	1/27	$(p_\alpha - p_{111})/3$
$\{0, 3, 0\}$	1/9	0	0	2/9	0	1/27	$(p_\alpha - p_{111})/3$
$\{0, 0, 3\}$	1/9	0	0	2/9	0	1/27	$(p_\alpha - p_{111})/3$
$\{2, 1, 0\}$	0	1/3	0	0	0	1/9	$p_\beta/3$
$\{0, 2, 1\}$	0	1/3	0	0	0	1/9	$p_\beta/3$
$\{1, 0, 2\}$	0	1/3	0	0	0	1/9	$p_\beta/3$
$\{1, 2, 0\}$	0	0	1/3	0	0	1/9	$p_\gamma/3$
$\{0, 1, 2\}$	0	0	1/3	0	0	1/9	$p_\gamma/3$
$\{2, 0, 1\}$	0	0	1/3	0	0	1/9	$p_\gamma/3$

Transformations on $|\bar{\alpha}\rangle$, $|\bar{\beta}\rangle$, and $|\bar{\gamma}\rangle$ are the same as on the unbarred states, with the only exception that one needs to swap $\omega \leftrightarrow \omega^2$.

Let us focus on particle number measurements at the output ports of the tritter. We will denote their results as $\{n_1, n_2, n_3\}$, where $n_1 + n_2 + n_3 = 3$ and n_i is the number of particles detected at the ith port. The statistics of such measurements, presented in Table I, offer a clear way of distinguishing between different representatives of the tripartite parastatistics. If we detect only four possible particle number configurations, $\{1, 1, 1\}$, $\{3, 0, 0\}$, $\{0, 3, 0\}$, or $\{0, 0, 3\}$, we know that the corresponding state is a mixture of the totally symmetric and antisymmetric states (like ρ_α) or their superposition. In the case of $\{2, 1, 0\}$, $\{0, 2, 1\}$, or $\{1, 0, 2\}$, we can deduce that the state is ρ_β. Finally, the outcomes $\{1, 2, 0\}$, $\{0, 1, 2\}$, or $\{2, 0, 1\}$ indicate that the state is ρ_γ.

Since we consider a tripartite system in which each particle enters the tritter through a different port, a corresponding input state is a superposition, or a mixture, of the six possible permutations of the state $|123\rangle$. Equivalently, an input state can be also represented in terms of the states given by Eqs. (8)–(13). The formulas (19), (20), and (21) show that their tritter transformations lead to a particle count statistics that is describable by only four parameters (three, if one takes into account normalization). In particular, we observe that some events occur with the same probabilities

$$p_{\{1,1,1\}} \equiv p_{111}, \tag{22}$$

$$p_{\{3,0,0\}} = p_{\{0,3,0\}} = p_{\{0,0,3\}} \equiv \frac{p_\alpha - p_{111}}{3}, \tag{23}$$

$$p_{\{2,1,0\}} = p_{\{0,2,1\}} = p_{\{1,0,2\}} \equiv \frac{p_\beta}{3}, \tag{24}$$

$$p_{\{1,2,0\}} = p_{\{0,1,2\}} = p_{\{2,0,1\}} \equiv \frac{p_\gamma}{3}, \tag{25}$$

where $p_\mu = \mathrm{Tr}\{\rho\rho_\mu\}$ $(\mu = \alpha, \beta, \gamma)$. Therefore, if an arbitrary state of three particles is fed into the tritter, the particle count statistics at the output is determined by p_α, p_β, and p_γ (see the last column in Table I).

The above observations allow us to arrive at the operational formula for the measure of tripartite indistinguishability (7)

$$\mathcal{I}_{123} = |p_\alpha + \omega p_\beta + \omega^2 p_\gamma|^2. \tag{26}$$

Note that $\mathcal{I}_{123} = 1$ for ρ_α, ρ_β, and ρ_γ. In addition, it is also equal to one for the tripartite symmetric and antisymmetric states. On the other hand, for a state of distinguishable particles, like $|123\rangle$ (see the fourth column in Table I), we get $p_\alpha = p_\beta = p_\gamma = 1/3$ and as a consequence $\mathcal{I}_{123} = 0$.

IV. N-PARTITE INDISTINGUISHABILITY

So far, we have considered the simplest case of cyclic indistinguishability of three particles. Now we generalize our results to n-partite systems. Just as before, we define our measure of indistinguishability as the expectation value of the cyclic permutation operator

$$\mathcal{I}_{12\ldots n} \equiv |\langle\psi|\Pi_{n1\ldots(n-1)}|\psi\rangle|^2. \tag{27}$$

Since $(\Pi_{n1\ldots(n-1)})^n = I$, the eigenvalues of the cyclic permutation operator are the nth roots of unity. They are all $(n-1)!$ degenerate and it is easy to verify that the vectors belonging to the same eigenspace can be converted into each other with a proper permutation of the particle labels. Because of that, the eigenvectors corresponding to the same eigenvalue cannot be distinguished with symmetric operators. This means that there are exactly n indistinguishability classes stemming from $\mathcal{I}_{12\ldots n}$.

Let us represent these classes with the eigenvectors of form

$$\begin{aligned}|\lambda^k\rangle = \frac{1}{\sqrt{n}}(&|1,\ 2,\ \ldots,\ n\rangle + \lambda^k|n,\ 1,\ \ldots,\ n-1\rangle\\ &+\lambda^{2k}|n-1,\ n,\ \ldots,\ n-2\rangle + \cdots\\ &+\lambda^{(n-1)k}|2,\ 3,\ \ldots,\ n,\ 1\rangle),\end{aligned} \tag{28}$$

where $k \in \{1,\ldots,n\}$ and $\lambda = e^{i\frac{2\pi}{n}}$. For all these states, the n-partite indistinguishability measure $\mathcal{I}_{1,2,\ldots,n} = \langle\psi|\Pi_{n,1,\ldots,(n-1)}|\psi\rangle = 1$ while $\mathcal{I}_{1,2,\ldots,m} = \langle\psi|\Pi_{m,1,\ldots,(m-1)}|\psi\rangle = 0$ for all $m < n$. This means they are genuinely n-partite indistinguishable.

In order to find an operational formula for $\mathcal{I}_{12\ldots n}$, we study the evolution of the states $|\lambda^k\rangle$ on a Fourier multiport given by

$$U_{QFT}^{(n)}|j\rangle = \frac{1}{\sqrt{n}}\sum_{k=1}^{n} e^{i\frac{2\pi}{n}(j-1)(k-1)}|k\rangle. \tag{29}$$

Just as in the tripartite case, we look for the outputs that are characteristic for each indistinguishability class. This time, however, we will use a suppression law proposed in Ref. [3]. It links the set of allowed output events with the symmetry of the input state and its transformation. Applied to the evolution of the state $|\lambda^k\rangle$ it yields the condition

$$\sum_{i=0}^{n-1} ia_i \equiv k \ (\mathrm{mod}\ n), \tag{30}$$

where $a_0, a_1, \ldots$ denote the number of particles in each consecutive output port. Clearly, if a specific configuration $(a_0, a_1, \ldots, a_{n-1})$ solves the equation for λ^k, it cannot solve it for any other eigenvalue λ^j. This means that the sets

$$A_k := \left\{(a_0, a_1, \ldots, a_{n-1}) : \sum_{i=0}^{n-1} ia_i \equiv k\ (\mathrm{mod}\ n)\right\}, \tag{31}$$

consisting of outputs stemming from the evolution of states of different indistinguishability classes are completely disjoint. Thus, we can define

$$p_k := \sum_{(a_0,a_1,\ldots,a_{n-1})\in A_k} p_{\{a_0,a_1,\ldots,a_{n-1}\}} \tag{32}$$

and use it to obtain the operational formula for $\mathcal{I}_{12\ldots n}$

$$\mathcal{I}_{1,2,\ldots,m} = \left|\sum_{i=0}^{n-1} p_i\lambda^i\right|^2. \tag{33}$$

V. SUMMARY

We have shown that the concept of genuine multipartite indistinguishability naturally emerges if we define indistinguishability with respect to cyclic permutations. Then, we found that this new type of indistinguishability can be tested using symmetric multiports in setups that generalize the Hong-Ou-Mandel one.

The notion of genuine multipartite indistinguishability does not require the usual bosonic or fermionic invariance under all permutations of particle labels. Instead, it focuses on cyclic invariance, which could be engineered in quantum states by exploiting an additional labeling degree of freedom (a relevant idea is explored in Ref. [21]). Given that such states exhibit unique dynamical properties, they may find applications in quantum algorithms based, for example, on multipartite quantum walks. Moreover, our results provide a new way of describing multipartite indistinguishability and, in general, multipartite correlations.

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[1] The reader may be confused that we have omitted anyons [22], which can be observed in some special conditions. This is because here we focus on the properties of the permutation group, which describes the interchange of particles in the three-dimensional space. On the other hand, anyons are in fact complex particles in two-dimensional space whose interchange is described by the braid group.

[2] A. Peres, *Quantum Theory: Concepts and Methods* (Springer, Netherlands, 2002), Chap. 5.

[3] C. Dittel, G. Dufour, M. Walschaers, G. Weihs, A. Buchleitner, and R. Keil, Phys. Rev. Lett. **120**, 240404 (2018).

[4] C. K. Hong, Z. Y. Ou, and L. Mandel, Phys. Rev. Lett. **59**, 2044 (1987).

[5] Y. L. Lim and A. Beige, New J. Phys. **7**, 155 (2005).

[6] M. C. Tichy, M. Tiersch, F. de Melo, F. Mintert, and A. Buchleitner, Phys. Rev. Lett. **104**, 220405 (2010).

[7] M. C. Tichy, M. Tiersch, F. Mintert, and A. Buchleitner, New J. Phys. **14**, 093015 (2012).

[8] I. Jex, E. Andersson, and A. Chefles, J. Mod. Opt. **51**, 505 (2004).

[9] S. Stanisic and P. S. Turner, Phys. Rev. A **98**, 043839 (2018).

[10] R. LoFranco and G. Compagno, Phys. Rev. Lett. **120**, 240403 (2018).

[11] P. Blasiak and M. Markiewicz, arXiv:1807.05546.

[12] M. Karczewski, M. Markiewicz, D. Kaszlikowski, and P. Kurzyński, Phys. Rev. Lett. **120**, 080401 (2018).

[13] M. Karczewski, D. Kaszlikowski, and P. Kurzyński, Phys. Rev. Lett. **121**, 090403 (2018).

[14] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).

[15] D. J. Brod, E. F. Galvão, N. Viggianiello, F. Flamini, N. Spagnolo, and F. Sciarrino, Phys. Rev. Lett. **122**, 063602 (2019).

[16] Y. Omar, N. Paunković, L. Sheridan, and S. Bose, Phys. Rev. A **74**, 042304 (2006).

[17] L. Sansoni, F. Sciarrino, G. Vallone, P. Mataloni, A. Crespi, R. Ramponi, and R. Osellame, Phys. Rev. Lett. **108**, 010502 (2012).

[18] A. Cabello, Phys. Rev. Lett. **89**, 100402 (2002).

[19] Q. Wang, C. Yu, F. Gao, H. Qi, and Q. Wen, Phys. Rev. A **94**, 022333 (2016).

[20] N. Spagnolo, C. Vitelli, L. Aparo, P. Mataloni, F. Sciarrino, A. Crespi, R. Ramponi, and R. Osellame, Nat. Commun. **4**, 1606 (2013).

[21] A. Fedrizzi, T. Herbst, M. Aspelmeyer, M. Barbieri, T. Jennewein, and A. Zeilinger, New J. Phys. **11**, 103052 (2009).

[22] J. M. Leinaas and J. Myrheim, Il Nuovo Cimento B **37**, 1 (1977).