Does large quantum Fisher information imply Bell correlations?

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The quantum Fisher information (QFI) of certain multipartite entangled quantum states is larger than what is reachable by separable states, providing a metrological advantage. Are these nonclassical correlations strong enough to potentially violate a Bell inequality? Here, we present evidence from two examples. First, we discuss a Bell inequality designed for spin-squeezed states which is violated only by quantum states with a large QFI. Second, we relax a well-known lower bound on the QFI to find the Mermin Bell inequality as a special case. However, a fully general link between QFI and Bell correlations is still open.

I. INTRODUCTION

The quantum Fisher information (QFI) is an important quantity in the geometry of Hilbert spaces [1,2] and has implications for the foundations of quantum mechanics [3–5] as well as for quantum metrology [6–8] and quantum computation [9,10]. A well-studied case is $\rho_t = \exp(-iAt)\rho_0 \exp(iAt)$, where ρ_0 is an N-partite qubit state and A is a local operator $A = \sum_{i=1}^{N} A^{(i)}$ (with fixed operator norm $\|A^{(i)}\|_\infty = 1/2$ for convenience). Then, the QFI $\mathcal{F}(\rho, A)$ is a function of $\rho_0 \equiv \rho$ and A. This is a typical situation in quantum metrology, where A is the generator of a small perturbation (like a weak external magnetic field) whose strength we would like to measure as precisely as possible. The celebrated quantum Cramér-Rao bound implies that a large QFI is necessary for a high sensitivity [6,11]. It is well known that certain entangled states allow one to go beyond the so-called standard quantum limit for separable states [12–14]. More concretely, it was shown [15,16] that $\mathcal{F}(\rho, A) > N$ implies entanglement between the qubits. The larger the QFI, the larger the entanglement depth of the state [17–19].

Among the nonclassical properties of quantum systems, Bell correlations are of particular importance. On the fundamental side, quantum states exhibiting Bell correlations potentially violate Bell inequalities, thereby proving that nature cannot be modeled with local (hidden) variables [20]. This insight can be used to design device-independent protocols for quantum applications such as secure communication [21] or random number generation [22,23]. Every quantum state with Bell correlations is entangled, but not every entangled quantum state necessarily has Bell correlations [24–26]. Hence, the latter represents a strictly stronger form of quantum correlations.

In the present work we ask whether there exists a connection between large QFI and Bell correlations. Intuitively such a connection can be motivated by the following observation. Both large QFI and Bell correlations are properties of a quantum state associated to specific measurements. That is, they cannot be a property of a quantum state (or measurement) alone, but require the judicious combination of states and measurements.

More specifically, we investigate here whether quantum states with a high enough QFI generically exhibit Bell correlations. If this is the case, then are the same measurements that reveal Bell correlations (potentially in a device-dependent manner) useful to show the presence of a large QFI? While we do not provide a fully general answer to these questions, we discuss two examples that hint at affirmative answers. To this end, we linearize a well-known lower bound on the QFI. First, we take its elements to start with an ansatz for a Bell inequality, which turns out to be of the form of multipartite Bell inequalities based on two-body correlators recently introduced by Tura *et al.* [27]. Considering a multisetting extension of this Bell inequality [28], we show that (i) only quantum states with $\mathcal{F}(\rho, A) > N$ (i.e., beating the standard limit of separable states) can potentially violate the inequality, and (ii) any quantum state with $\mathcal{F}(\rho, A) > 3N$ will violate the Bell inequality. Notably, the same measurements that witness the presence of Bell correlations also demonstrate a large QFI.

The second type of linearization is a relaxation of the QFI bound. For a special case which is optimal for the Greenberger-Horne-Zeilinger (GHZ) state, we show that one side of this linear bound becomes the Bell operator for the Mermin inequality [29], another multipartite Bell inequality specially suited to detect Bell correlations of GHZ states. Again, a very large QFI is necessary for the violation of the Bell inequality and the same measurements that show large QFI are sufficient choices for a potential Bell inequality violation.

II. BELL OPERATORS FROM A QFI BOUND

The QFI $\mathcal{F}(\rho, A)$ is a nonlinear quantity that is defined by measuring the infinitesimal change of ρ evolving under $U = \exp(-iAt)$ with the Bures distance s_B in state space,

$$ds_B = \frac{1}{2}\sqrt{\mathcal{F}(\rho, A)}dt. \tag{1}$$

While the exact value of $\mathcal{F}(\rho, A)$ is generally only accessible with complete knowledge about ρ and A [30], there are powerful lower bounds based on relatively simple measurements. For example, a tighter version of the Heisenberg uncertainty

relation holds for all Hermitian operators B [16,31,32]

$$\mathcal{F}(\rho, A) \geqslant \frac{\langle i[A, B]\rangle_\rho^2}{\langle (B - \langle B \rangle_\rho)^2 \rangle_\rho}, \qquad (2)$$

where, in the following, we restrict ourselves to $\langle B \rangle_\rho = 0$ without loss of generality. There is always an operator B that makes inequality (2) tight for given ρ, A. Hence, a well-chosen B allows one to optimally bound the QFI.

Bell inequalities are bounds on local variable models. Violations of these inequalities are possible in quantum mechanics and imply the presence of Bell correlations. For our purpose, it is sufficient to consider symmetric Bell inequalities of N parties. Following Ref. [27], we define the symmetrized k-body correlators

$$\mathcal{C}_{j_1, \ldots, j_k} = \sum_{\substack{i_1, \ldots, i_k = 1 \\ \text{all } i \text{ different}}}^{N} \langle M_{j_1}^{(i_1)} \cdots M_{j_k}^{(i_k)} \rangle, \qquad (3)$$

where $M_j^{(i)}$ is the measurement operator for setting j at site i. Suppose we have d measurement settings per party. Then, general linear, symmetric Bell inequalities are of the form

$$\sum_{k=1}^{N} \sum_{j_1, \ldots, j_k=0}^{d-1} a_{j_1, \ldots, j_k} \mathcal{C}_{j_1, \ldots, j_k} + a_0 \geqslant 0, \qquad (4)$$

where $a_{j_1, \ldots, j_k}, a_0 \in \mathbb{R}$. They are fulfilled by any local hidden variable model. Here, we are interested in nontrivial Bell inequalities, that is, in those that are violated by some quantum states.

Assuming a connection between large QFI and Bell correlations, one could directly try to turn the right-hand side of Eq. (2) into a Bell inequality up to an additional local bound a_0. Every symmetric, multipartite operator can be expressed in a basis of products of Pauli operators, and its expectation value can be written as a function of correlators (3). However, the nonlinear terms in Eq. (2) render this approach difficult. Therefore, we propose the linear ansatz

$$\alpha + \beta \langle B^2 \rangle_\rho - \gamma \langle C \rangle_\rho \geqslant 0, \qquad (5)$$

with $C = i[A, B]$ and $\alpha, \beta, \gamma > 0$. The idea is that if $\langle B^2 \rangle_\rho$ is sufficiently small and $\langle C \rangle$ sufficiently large, then inequality (5) can be violated which implies Bell correlations and a large QFI via Eq. (2).

This approach turns out to be successful for spin-squeezed states [33]. For concreteness, we choose $A = S_z = \frac{1}{2} \sum_i \sigma_z^{(i)}$ and $B = S_y = \frac{1}{2} \sum_i \sigma_y^{(i)}$, that is, collective spin operators. From the well-known SU(2) commutation relations, one has $C = S_x = \frac{1}{2} \sum_i \sigma_x^{(i)}$. An N-partite qubit state is called spin squeezed if

$$\xi^2 = \frac{N \langle S_y^2 \rangle}{\langle S_x \rangle^2} < 1, \qquad (6)$$

potentially after a suitable change of collective coordinates. Hence, with our choices for A and B, Eq. (5) seems to be a promising candidate for a Bell inequality that can be violated with spin-squeezed states.

However, a direct translation of S_x and S_y^2 into measurement settings cannot lead to nontrivial Bell inequalities because then a local hidden variable model can minimize $\langle S_y^2 \rangle$
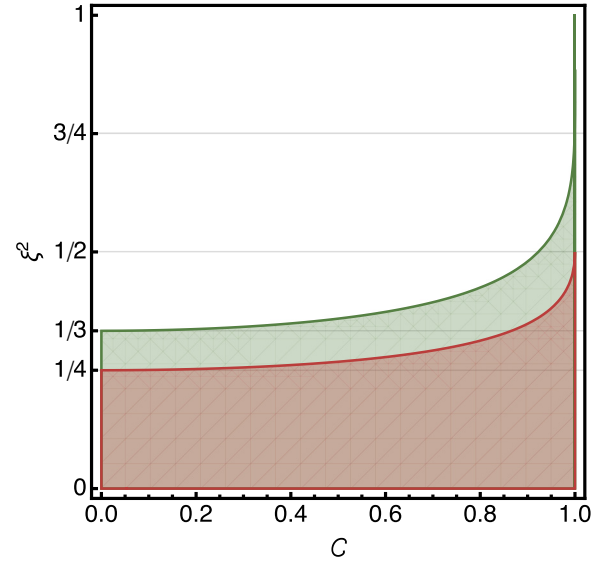

FIG. 1. Regions with Bell correlations. The figure shows regions in the $\mathcal{C}$-ξ^2 plane for which Bell correlations are detected by the witnesses given in Eq. (9) (red) and Eq. (11) (green). Red: For $\xi^2 \leqslant 1/4$ there are Bell correlations. For $1/4 < \xi^2 < 1/2$ the Bell correlation witness Eq. (9) is violated only if $\mathcal{C}$ is sufficiently large, while for $\xi^2 \geqslant 1/2$ there is no violation. Green: For $\xi^2 \leqslant 1/3$ there are Bell correlations. For $1/3 < \xi^2 < 1$ the Bell correlation witness Eq. (11) is violated only if $\mathcal{C}$ is sufficiently large.

and maximize $\langle S_x \rangle$ independently of each other. To couple the two we introduce new measurement bases for every party i,

$$M_0^{(i)} = \cos \phi \, \sigma_y^{(i)} + \sin \phi \, \sigma_x^{(i)},$$
$$M_1^{(i)} = \cos \phi \, \sigma_y^{(i)} - \sin \phi \, \sigma_x^{(i)}. \qquad (7)$$

We note that $4 \sin \phi \langle S_x \rangle = \mathcal{C}_0 - \mathcal{C}_1$ and $4 \cos^2 \phi \langle S_y^2 \rangle = N \cos^2 \phi + \frac{1}{4}(\mathcal{C}_{00} + 2\mathcal{C}_{01} + \mathcal{C}_{11})$. Inserting these relations in Eq. (5) we obtain an inequality of the class recently studied by Tura *et al.* [27], who show that Eq. (5) constitutes a Bell inequality if $\alpha = 2N \sin^2 \phi$, $\beta = 8 \cos^2 \phi$, and $\gamma = 4 \sin \phi$. It reads

$$C_0 - C_1 + \tfrac{1}{2}C_{00} + C_{01} + \tfrac{1}{2}C_{11} + 2N \geqslant 0. \qquad (8)$$

Under the restriction that the measurement settings M_j^i are identical for all parties, the choice of Eq. (7) turns out to be the most general parametrization. The Bell inequality can then be written as a Bell correlation witness which requires collective spin measurements only [28,34]. With the definition of the scaled second moment $\zeta^2 = \langle S_y^2 \rangle/(N/4)$, and of the scaled contrast $\mathcal{C} = \langle S_x \rangle/(N/2)$, the inequality becomes

$$\zeta^2 \geqslant \frac{1}{2}(1 - \sqrt{1 - \mathcal{C}^2}). \qquad (9)$$

From the fact that $\xi^2 = \zeta^2/\mathcal{C}^2$ we can express Eq. (9) as a function of ξ^2 and $\mathcal{C}$, and observe that (see Fig. 1) (i) for $\xi^2 \leqslant 1/4$ the inequality is always violated, independently on $\mathcal{C}$, (ii) for $1/4 < \xi^2 < 1/2$ a minimal $\mathcal{C}$ is needed to violate the inequality, and (iii) for $\xi^2 \geqslant 1/2$ the inequality is never violated. This implies that only states with $\mathcal{F}(\rho, S_z) > 2N$ are able to violate inequality (8). Moreover, all states with

$\mathcal{F}(\rho, S_z) > 4N$ will give violation, that is, for sufficiently squeezed states, such as the one-axis and the two-axes twisted spin-squeezed state [33].

It turns out that these results can be improved by considering a multisetting generalization of the Bell inequality (8) presented in Ref. [28]. Again, this inequality can be written as a Bell correlation witness which requires collective spin measurements only. Specifically, consider the family of m-settings inequalities

$$\sum_{k=0}^{m-1} \alpha_k C_k + \frac{1}{2} \sum_{k,l} C_{k,l} + \beta_c \geqslant 0, \tag{10}$$

with $\alpha_k = m - 2k - 1$ and $\beta_c = \lfloor m^2 N/2 \rfloor$. This inequality can again be written as a witness which in the limit $m \to \infty$ takes the form [28]

$$\zeta^2 \geqslant 1 - \frac{\mathcal{C}}{\text{arctanh}(\mathcal{C})}, \tag{11}$$

which holds for all states featuring local correlations. Performing the same analysis as above, we observe that (see Fig. 1) (i) for $\xi^2 \leqslant 1/3$ the inequality is always violated, independently on $\mathcal{C}$, and (ii) for $1/3 < \xi^2 < 1$ a minimal $\mathcal{C}$ is needed to violate the inequality.

To conclude, from Eqs. (2), (6), and (11), we see that $\mathcal{F}(\rho, S_z) > N$ is a necessary condition for violating the Bell inequality (10). That is, only quantum states that beat the standard limit of separable states can potentially violate the Bell inequality. Moreover, the condition $\mathcal{F}(\rho, S_z) > 3N$ is sufficient for violating the Bell inequality, i.e., all states satisfying it will give violation. Again, here the measurement settings are assumed to be identical for all parties.

III. A BELL INEQUALITY FROM A LINEAR QFI BOUND

In the previous section, we took a rather free inspiration from the Heisenberg uncertainty relation to construct a Bell inequality where only states with a large enough QFI could potentially violate it. Here, we tackle the problem more directly by linearizing the right-hand side of Eq. (2). We simply use $\langle B^2 \rangle_\rho \leqslant \|B\|_\infty^2$, to arrive at a linear lower bound

$$\sqrt{\mathcal{F}(\rho, A)} \geqslant \langle W \rangle_\rho \equiv \frac{1}{\|B\|_\infty} \langle i[A, B] \rangle_\rho. \tag{12}$$

Here, and in the following, we choose the sign of B such that $\langle W \rangle_\rho$ is positive.

Interestingly, since $\mathcal{F}(\rho, A) \leqslant N$ for all separable states, Eq. (12) can be turned into an entanglement witness with operator $\mathcal{W} = \sqrt{N} - W$ for all B. In other words, the correctness of the Heisenberg uncertainty relation gives us a constructive tool to derive new entanglement witnesses.

This linearization seems to come at the price that the bound is now much looser, but it turns out that, at least for pure states, there always exists a B to achieve tightness.

Observation 1. For $\rho = |\psi\rangle\langle\psi| \equiv \psi$, the choice

$$B = -i[A, \psi] \tag{13}$$

implies tightness of Eq. (12).

Proof. This can be proved by direct calculation. For this, we define the orthogonal state $|\psi^\perp\rangle = 1/(\Delta_\psi A)(A - \langle A \rangle_\psi)|\psi\rangle$ with $\Delta_\psi A = \sqrt{\langle (A - \langle A \rangle_\psi)^2 \rangle_\psi}$ and find that $B = i\Delta_\psi A(|\psi^\perp\rangle\langle\psi| - |\psi\rangle\langle\psi^\perp|)$ and $\|B\|_\infty = \Delta_\psi A$. This leads to $i[A, B] = A^2\psi - 2A\psi A + \psi A^2$. Hence, one has $\langle W \rangle_\psi = 2V(\psi, A)$. Since for pure states $\mathcal{F}(\psi, A) = 4V(\psi, A)$ [1], this implies equality in Eq. (12). ∎

We study a specific example for the choice of Eq. (13). We consider the GHZ state,

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}). \tag{14}$$

This state has the maximal QFI for $A = S_z$ with $\mathcal{F}(\text{GHZ}, S_z) = N^2$. Again, direct calculation shows that $W = N(|\text{GHZ}\rangle\langle\text{GHZ}| - |\text{GHZ}^\perp\rangle\langle\text{GHZ}^\perp|)$, where $|\text{GHZ}^\perp\rangle = 1/\sqrt{2}(|0\rangle^{\otimes N} - |1\rangle^{\otimes N})$.

Interestingly, the very same W as in Eq. (12) appears when quantum mechanics is applied to the Bell inequality of Mermin [29], up to a constant and an irrelevant phase. With our choice of the normalization, Mermin's inequality shows Bell correlations of $\rho = |\text{GHZ}\rangle\langle\text{GHZ}|$ whenever

$$\langle W \rangle_\rho \leqslant \begin{cases} N2^{-N/2+1}, & N \text{ is even,} \\ N2^{-N/2+1/2}, & N \text{ is odd} \end{cases} \tag{15}$$

is violated. We compare this to the witness of large QFI

$$\langle W \rangle_\rho \leqslant \sqrt{\mathcal{F}(\rho, S_z)}. \tag{16}$$

We observe a connection between the Bell inequality and the lower bound on the QFI. The GHZ state maximally violates the Bell inequality and makes Eq. (16) tight. We see that a certain minimal QFI is necessary to potentially violate the Bell inequality. However, the fact that W is tailored to the GHZ state makes both inequalities not very useful for other states. Furthermore, a QFI beyond the shot-noise limit N is not necessary in this case. To illustrate this, we consider the quantum state

$$\rho = \frac{1+p}{2}|\text{GHZ}\rangle\langle\text{GHZ}| + \frac{1-p}{2}|\text{GHZ}^\perp\rangle\langle\text{GHZ}^\perp| \tag{17}$$

with $p \in [0, 1]$. Using the positive partial transpose (PPT) criterion, one easily convinces oneself that the state has bipartite entanglement for any $p > 0$. It violates the Mermin inequality if $p > 2^{-N/2+1}N$. Last, the state has a QFI of $\mathcal{F}(\rho, S_z) = p^2 N^2$, implying that $p > 1/\sqrt{N}$ is necessary to have a QFI that is larger than for any separable state. For large N, the latter bound is exponentially more restrictive than the local bound of the Mermin inequality.

IV. DISCUSSION

We investigated whether Bell correlations and large QFI are connected. For two examples of Bell inequalities, one instance from a class studied in [27] and the Mermin inequality [29], we showed that a sufficiently large QFI is necessary for a violation. How generic is this connection? Is it possible to find a Bell inequality that is violated for any quantum state with large QFI? Both approaches presented in this Rapid Communication give hope to find further Bell inequalities designed for such states like the Dicke states. Currently, however, we are not aware of a constructive method to conjecture and prove these inequalities. This is mainly due to the step of finding good measurement basis for a quantum operator like in Eq. (5) for a nontrivial Bell inequality.

Finally, note that a large QFI is generally not necessary for a quantum state to violate a Bell inequality. Indeed, every pure entangled state can violate a Bell inequality [35–37], but not every pure entangled state has a QFI beyond the standard quantum limit if only collective measurements are performed [38].

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