Relativistic finite-nuclear-size corrections to the energy levels in light muonic atoms

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We consider higher-order finite-nuclear-size (FNS) contributions to the energy levels of light ordinary and muonic hydrogenlike atoms. Relativistic corrections to the leading FNS term have been known in terms of an integration in coordinate space. The related results are model dependent. In the meantime, if the data on the nuclear structure are obtained from electron-nucleus scattering, it is a representation of the electric charge form factor in momentum space rather than the charge density. An example is the case of a proton as a nucleus. Often the electric form factor is presented in terms of a dispersion integral or a Padé approximation. We present explicit results for such representations for leading finite-nuclear-size higher-order (in $Z\alpha$ and in $Z\alpha m R_N$) corrections beyond the leading FNS term.

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I. INTRODUCTION

The atoms are electromagnetically bound systems, but the energy levels of even the simplest of them, such as a two-body hydrogenlike one, cannot be studied *ab initio* by means of quantum electrodynamics alone. The atomic energy levels are affected by the finite-nuclear-size (FNS) effects and one has to include a phenomenological consideration of the nuclear-size and nuclear-structure effects in theoretical calculations. The leading FNS contribution to the energy levels in a hydrogen-like atom is a nonrelativistic one. It is of the form

$$\Delta E_{\rm FNS:lead}(nl) = \frac{2}{3}\frac{(Z\alpha)^4 m_r}{n^3}\,(m_r R_N)^2\,\delta_{l0}, \qquad (1)$$

where Z is the nuclear charge (in units of the proton one), R_N is the nuclear radius, and throughout the paper we use the units in which $\hbar = c = 1$. The orbiting particle is either an electron or a muon, depending on whether we study an ordinary atom or a muonic one. We denote the mass of the orbiting particle as m, while m_r stands for the related reduced mass of the two-body system.

There are a number of corrections to the leading term in (1). The expression given there is nonrelativistic and relativistic corrections have to be considered. Consideration of such corrections, i.e., of the corrections of a relative order $(Z\alpha)^2$ is one of the purposes of this paper. In absolute units the leading relativistic FNS correction is of order $(Z\alpha)^6(mR_N)^2m$.

The relativistic corrections were studied with a sufficient accuracy in [1]. The results were adopted for some other light atoms by Borie (see, e.g., [2]). The leading relativistic correction is of the form

$$\Delta E_{\rm FNS:rel}(ns) = (Z\alpha)^2\,\Delta E_{\rm FNS:lead}(ns)\left(\ln\frac{n}{Z\alpha m R_N} + C_{\rm rel}\right), \qquad (2)$$

where $C_{\rm rel}$ is a model-dependent constant which depends on the atomic state and on details of the distribution of the nuclear charge. (When one performs a combined analysis of experimental data on the muonic atom and its ordinary counterpart, such as ordinary and muonic hydrogen, it is important that the relativistic-correction constant $C_{\rm rel}$ is considered within the same nuclear model for both atoms.)

As noted in [3], in the case of muonic hydrogen (see also [4]) the correction is rather small and it is more advantageous to use the result in the logarithmic approximation. In principle, for the Lamb shift of the low levels in light muonic atoms $n = O(1)$, $mR_N = O(1)$, and $Z = O(1)$, and therefore the simplest result in the logarithmic approximation is rather (cf. [3,4])

$$\Delta E^{\rm log}_{\rm FNS:rel}(ns) = (Z\alpha)^2\,\Delta E_{\rm FNS:lead}(ns)\ln\frac{1}{\alpha}. \qquad (3)$$

Still, considering different atoms it is more useful to keep certain nuclear-dependent parameters under the logarithm as

$$\Delta E^{\rm log}_{\rm FNS:rel}(ns) = (Z\alpha)^2\,\Delta E_{\rm FNS:lead}(ns)\ln\frac{n}{Z\alpha m R_N}. \qquad (4)$$

The nonlogarithmic part of the contribution in (2) could be calculated, but it requires a realistic model of the charge

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distribution of the nucleus of interest. We revisit this term in the momentum-space evaluation in which the experimental data on the electric form factor may be applied (see Sec. IV). In particular, we use realistic fits for the proton form factor, i.e., the fits which well describe the existing data of the elastic electron-nucleus scattering and have reasonable behavior at large and small momentum transfers.

The contribution of order $(Z\alpha)^6 m$ for the $l \neq 0$ states

$$\Delta E_{\rm FNS:rel}(np_j) = \frac{n^2-1}{4n^2}\,(Z\alpha)^2\,\Delta E_{\rm FNS:lead}(ns)\delta_{j(1/2)} \quad (5)$$

does not have a logarithmic enhancement and vanishes in the logarithmic approximation.

Considering the relativistic effects, we note that there are two parameters related to the "velocity" of an atomic electron (a muon). One is $Z\alpha$, a characteristic value of the atomic orbiting velocity, and the other is $Z\alpha m R_N$. The expansion in $Z\alpha m R_N$ is an expansion that takes into account the details of the shape of the nuclear charge distribution. In terms of "relativistic" effects that parameter deals with the penetration of the orbiting particle in the nucleus, which corresponds to short distances (comparing to the atomic scale) and therefore to the high-momentum tail of the wave function (cf. [5,6]), which is namely a relativistic tail. In a nonrelativistic (in general) state, the presence of relativistic momenta is suppressed and the above-mentioned parameter $Z\alpha m R_N$ is such a suppression parameter. A study of higher-order (in $Z\alpha m R_N$) corrections is another goal of our paper.

The leading $Z\alpha m R_N$ correction to (1) is the so-called Friar term which can be presented in the coordinate space as [1,7]

$$\Delta E_{\rm FNS:Fr}(ns) = -\frac{(Z\alpha)^5\, m_r^4}{3n^3} \times \int d^3r\, d^3r'\,\hat\rho_E(\mathbf{r})\hat\rho_E(\mathbf{r}')|\mathbf{r}-\mathbf{r}'|^3. \quad (6)$$

The related momentum-space presentation is [4,8]

$$\Delta E_{\rm FNS:Fr}(ns) = -\frac{16(Z\alpha)^5\, m_r^4}{\pi n^3}\int_0^\infty \frac{dq}{q^4} \times \{[G_E(q^2)]^2 - 1 - 2G_E'(0)q^2\}. \quad (7)$$

Here, q^2 is the Euclidean momentum, and $G_E(q^2)$ is the electric form factor, while $\hat\rho_E(\mathbf{r})$ is its Fourier transform which is approximately equal to the charge density. The deviation is the most important at low r and high q due to the nuclear-recoil effects.

While the expressions are generally model independent, their practical application is model dependent because there is no way to extract from experimental data or theoretically calculate either the nuclear density or the form factor in a model-independent way. Note, the numerator in (7) contains a strong cancellation which means that we cannot rely on the experimental data because of their limited accuracy. We cannot even expect that the integral over data is well convergent at low q because any experimentally measured value of the form factor $G(q^2)$ is close to zero only within its experimental accuracy, which would produce large uncertainty of the low-q contribution. Model-dependent consideration allows us to avoid the problem (see, e.g., discussion in [9]).

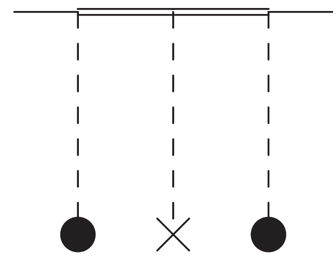

FIG. 1. The third-order perturbation diagram for the logarithmic FNS contribution in (8). The photon lines are for the Coulomb field (with two of the legs related to the extended nucleus and one for the nucleus in the pointlike approximation). The double line is for the Coulomb Green's function of the muon.

The next higher-order term in $Z\alpha m R_N$ is of (absolute) order $(Z\alpha)^6(mR_N)^4 m$. It was also studied in detail in [1] (see also [2]). Similarly to (2), it contains a model-independent logarithm and a model-dependent nonlogarithmic term. Similarly to the case of the pure relativistic correction, it was suggested [3,4] to use only the logarithmic term. Unfortunately, due to a misprint the logarithm factor was lost in [4] and in a number of compilations the expression with a misprint has been utilized. The review [2] is free of this error (see also [10]).

For the light atoms we use an extended logarithmic approximation [cf. (4)]

$$\Delta E_{\rm FNS:rel} = -\frac{2}{3}(Z\alpha)^2\,(m_rR_N)^2\,\Delta E_{\rm FNS:lead}\ln\frac{n}{Z\alpha m R_N}. \quad (8)$$

Due to the misprint mentioned, to be on the safe side we have reproduced this term using the technique developed in [11] (see Fig. 1).

The logarithmic approximation, required to derive (8), may be realized with an essentially nonrelativistic technique [11]. That allows one to restore m_r easily. The appropriate factor $(m_r/m)^k$ is of importance for numerical values of various contributions in light muonic atoms. In particular, the reduced mass factor in (8) for muonic hydrogen is $(m_r/m)^2 \simeq 0.8$.

The result in order $(Z\alpha)^6(mR_N)^4 m$ differs from our earlier result given in [12] because of the misprint with the missing logarithmic factor mentioned above (see also [10]).

Consideration of the nonlogarithmic part of the contribution does not allow us to rescale the result originally obtained with the Dirac equation with appropriate $(m_r/m)^k$ factors. The nonlogarithmic term should include an additional correction due to higher-order effects in m/M. The Friar's calculations [1] of the $(Z\alpha)^6(mR_N)^4 m$ terms, adopted in various compilations, are performed rather for a Dirac-type equation with the reduced mass and therefore cannot deliver a complete account of the m/M terms. One has to use the reduced-mass factors with caution.

We are mostly interested in light muonic atoms, such as muonic hydrogen. In muonic hydrogen, $m_rR_N < 1$ and we focus our consideration on the nonlogarithmic part of the relativistic correction in (2), rather than on the $(Z\alpha)^2(m_rR_N)^2$ correction in (8). In principle, the nonlogarithmic term in (8) could be calculated with a realistic model of the charge distribution of the nucleus similarly to the relativistic correction below in Sec. IV.

II. STANDARD FORM OF PRESENTATIONS FOR THE PROTON ELECTRIC FORM FACTOR (IN MOMENTUM SPACE)

Discussing the nuclei for the light muonic atoms, one may note that for a number of them and, in particular, for the proton, available experimental data from the elastic electron-nucleus scattering deliver us the electric form factor $G_E(q^2)$ rather than the spatial charge distribution $\rho_E(r)$ (see, e.g., Appendix B). That requires a presentation for $G_E(q^2)$ in momentum space and we have to find various higher-order FNS corrections in a way compatible with such a presentation. A straightforward approach to relativistic corrections requires the Dirac-equation wave functions and therefore it is "coordinate-space friendly."

A presentation in momentum space also allows us to control analytic properties and asymptotic behavior of the form factors and their fits. In particular, one can present the electric form factor in terms of the dispersion relations (see, e.g., [13,14])

$$G_E(q^2) = \int_{s_0}^{\infty} ds\, \frac{g_E(s)}{q^2+s}, \tag{9}$$

where we remind one that q^2 is the Euclidean momentum. The dispersion integral is related to the sum over all the intermediate states in the t channel. The density $g_E(s)$ may contain continuous and discrete contributions. The latter are related to vector mesons.

In those terms the normalization condition takes the form

$$G_E(0) = \sum_i \frac{a_i}{s_i} = 1, \tag{10}$$

while the rms charge radius is

$$R_N^2 = 6 \sum_i \frac{a_i}{s_i^2}. \tag{11}$$

The advantage of the presentations based on the dispersion relations is that they have reasonable analytic properties and reasonable asymptotics at high q^2. However, they often have a not very good value of χ^2. Unfortunately, the empiric fits of the world scattering data often have unreasonable asymptotics and incorrect analytic properties.

One can take advantage of the fact that the correct analytic properties of the electric form factor are presented in (9). Using them one can transform various expressions with the form factor into an integral over three-dimensional momentum $\mathbf{q}$. Once such a q integral is produced, we can use any approximation of the form factor in the region of integration, including those with the incorrect analytic properties, like various Padé approximations with complex poles.

A fit, presenting the form factor with a few poles (e.g., due to the dominance of vector-meson contributions to the form factor)

$$G_E(q^2) = \sum_i \frac{a_i}{q^2+s_i}, \tag{12}$$

has a form similar to the one in (9). It has rather reasonable analytic properties if the values of a_i and s_i are real and $s_i > 0$ and therefore the poles are in the timelike domain. Equation (12) decomposes the fit into a sum of simple fractions. However, one can combine all those terms and arrive at one term with a common denominator, which is a Padé form of the fit. In other words, we discuss here not the fits by themselves, but a certain form of their presentation, which allows us to consider various fits existing in literature.

Some fits have reasonable asymptotic behavior at high q^2. (Most of such fits are Padé approximations. Note, that a chain-fraction fit is also a Padé approximation, but specifically presented.) However, many of those Padé approximations do not have reasonable analytic properties and not all the values of a_i and s_i are real. In such case the form of (12) holds; however, for each complex pole s_i there is a complex-conjugate one $s_{i'} = s_i^*$ with $a_{i'} = a_i^*$. (Note, the real negative values of s_i are not available for the fits which cover all the region of $q^2 \leqslant 0$.) In the case of the complex poles, instead of (12) one can also use an equivalent form

$$G_E(q^2) = \mathrm{Re}\left(\sum_i \frac{a_i}{q^2+s_i}\right) = \frac{1}{2}\sum_i \left[\left(\frac{a_i}{q^2+s_i}\right) + \left(\frac{a_i}{q^2+s_i}\right)^*\right]. \tag{13}$$

The form (12) is very useful for an analytic transformation with various FNS contributions. On the other hand, such a form allows one a bunch of possibilities for fitting. Comparing various fits allows us to study the model dependence of the results. We refer to the presentation of the form (12) as the *standard form*, independently of whether the poles and residuals are real or complex. The presentation of some existing "realistic" fits of the proton electric form factor available in literature [15–20], considered in Appendix B, is decomposed to be present in such a form. (The number of fits considered here is somewhat limited because a number of phenomenological fits is designed only for a certain region of the momentum q (e.g., polynomial fits for $q < 1$ GeV/c). When q exceeds the area and in particular in the case of $q \gg 1$ GeV/c, the behavior of those fits may be physically inappropriate. We consider here only fits in literature with a good χ^2 value which are determined for the whole spacelike region of q and have reasonable behavior at low and high q (see Appendix B for details).

For a calculation of various FNS contributions, we may require a correction to the pointlike distribution which is expressed through the difference

$$G_E(q^2) - G_E(0) = -\sum_i \frac{a_i}{s_i}\frac{q^2}{q^2+s_i}. \tag{14}$$

Presentation of the fits in the standard form drastically simplifies the integrations which are required for a calculation of various higher-order FNS effects. In the next two sections we consider the Friar term and the relativistic FNS corrections.

III. THE FRIAR TERM AND THE STANDARD REPRESENTATION

The Friar term is known both in coordinate-space (6) and momentum-space (7) forms. Let us consider the application of a presentation of the form factor in the standard form to that contribution. The Friar term is expressed in terms of the Friar momentum (aka the third Zemach momentum)

$$\langle r^3\rangle_{(2)} = \frac{48}{\pi}\int_0^\infty \frac{dq}{q^4}\{[G_E(q^2)]^2 - 1 - 2G_E'(0)q^2\} = \int d^3r\, d^3r' \hat\rho_E(\mathbf{r})\hat\rho_E(\mathbf{r}')|\mathbf{r}-\mathbf{r}'|^3. \quad (15)$$

Using the standard form, one can easily obtain various relevant integrals in closed analytic form. In particular, we find for the dispersion-relation fits

$$\langle r^3\rangle_{(2)} = 24\int_{s_0}^\infty ds\,\frac{g_E(s)}{s}\int_{s_0}^\infty ds'\frac{g_E(s')}{s'} \times \frac{s^2+s^{3/2}\sqrt{s'}+ss'+s'^{3/2}\sqrt{s}+s'^2}{ss'(s\sqrt{s'}+s'\sqrt{s})}, \quad (16)$$

while for the Padé approximations [in the form of (13)] the results are

$$\langle r^3\rangle_{(2)} = 24\sum_{j,j'}\frac{a_j}{s_j}\frac{a_{j'}}{s_{j'}} \times \frac{s_j^2+s_j^{3/2}\sqrt{s_{j'}}+s_js_{j'}+s_{j'}^{3/2}\sqrt{s_j}+s_{j'}^2}{s_js_{j'}(s_j\sqrt{s_{j'}}+s_{j'}\sqrt{s_j})}. \quad (17)$$

To complete this section, let us mention the so-called dipole approximation

$$G_{\rm dip}(q^2) = \left(\frac{\Lambda^2}{q^2+\Lambda^2}\right)^2. \quad (18)$$

It is not a realistic one; however, being a single-parameter fit it is helpful for rough estimations.

That presentation in (18) is already a kind of standard form. To directly relate it to the expressions obtained for the standard-form presentation we can apply the identity

$$G_{\rm dip}(q^2) - G_{\rm dip}(0) = \left(-1+\Lambda^2\frac{\partial}{\partial\Lambda^2}\right)\frac{q^2}{q^2+\Lambda^2} \quad (19)$$

and reproduce the well-known result

$$\langle r^3\rangle_{(2)}^{\rm dip} = \frac{315}{2\,\Lambda^3}.$$

The closed analytic form of the results obtained above allows one to find the uncertainty of the result once the uncertainty of the fit parameters (a_i, s_i) and their correlations are known.

IV. RELATIVISTIC CONTRIBUTION IN MUONIC HYDROGEN IN MOMENTUM SPACE

There is a number of relativistic evaluations of the FNS contribution in the coordinate space (see, e.g., [1,21]). A calculation of the relativistic effects for the FNS contribution is one of the examples where the standard form of the presentation of the electric form factor may be efficiently utilized. To evaluate the FNS contribution of the first order in the FNS effects, we have to introduce the difference between the extended nucleus and the pointlike one given in (14) as a perturbation. The presentation of the form factor in closed form allows us to develop an effective theory of the relativistic corrections in momentum space.

Using the presentation (12), for the FNS contribution we obtain in the first order

$$\Delta E = \sum_i \frac{a_i}{s_i}\mathcal{I}_i, \quad (20)$$

where

$$\mathcal{I}_i = Z\alpha\int d^3r\,\psi^\dagger(\mathbf{r})\frac{e^{-(s_i)^{1/2}r}}{r}\psi(\mathbf{r}), \quad (21)$$

$(s_i)^{1/2}$ is a square root of s_i which has a positive real part.

That is a result within the external field approximation which is still a function of two parameters, $Z\alpha$ and $Z\alpha m R_N$. The result includes a contribution exact in $Z\alpha$ at the leading order in $Z\alpha m R_N$. The higher-order (in $Z\alpha m R_N$) contributions, such as the Friar term, are accounted for in (20) and (21) only partially. (Those equations present a result of the one-photon exchange. For the complete account of the Friar term one also has to consider the terms of the second order in the FNS effects, which include a two-photon exchange.)

The result for the r integral in (21) is known, e.g., from [22–24]

$$\mathcal{I}_i = \eta^2\frac{\Gamma(2\zeta+n_r+1)(n_r)!}{\frac{Z\alpha}{\eta}-\nu}\sum_{p,k=0}^{n_r}\frac{(-1)^{p+k}}{p!(n_r-p)!k!(n_r-k)!} \times\frac{\Gamma(2\zeta+p+k)}{\Gamma(2\zeta+p+1)\Gamma(2\zeta+k+1)} \times\left\{m\left[\left(\frac{Z\alpha}{\eta}-\nu\right)^2+(n_r-p)(n_r-k)\right] - E_{nlj}\left(\frac{Z\alpha}{\eta}-\nu\right)(2n_r-p-k)\right\}\left(\frac{2}{2+\frac{\sqrt{s_i}}{m}\frac{1}{\eta}}\right)^{p+k+2\zeta}, \quad (22)$$

where

$$\nu = (-1)^{j+l+1/2}(j+1/2),\quad \zeta=\sqrt{\nu^2-(Z\alpha)^2},$$
$$\eta=\sqrt{1-(E_{nlj}/m)^2},\quad n_r = n-|\nu|,$$
$$\widetilde{\kappa}_n = n\eta\kappa_n/(Z\alpha),$$

and E_{nlj} is the exact relativistic energy of the nl_j state for the Dirac-Coulomb problem.

The result for the lowest states $1s$, $2s$, $2p_{1/2}$, $2p_{3/2}$ takes a simple form since summation over p, k has very few terms for $n_r=0$ for $1s$ and $2p$ and for $n_r=1$ for $2s$ (see Appendix A).

In particular, using the standard-form fit we obtain for the relativistic correction in (2)

$$C_{\rm rel}(2s) = \frac{11}{16} - \ln 2 + \frac{3}{R_N^2}\sum_i\frac{a_i}{s_i^2}\ln\left(s_iR_N^2\right). \quad (23)$$

See Appendix A for more results.

In the case of the dipole approximation, the evaluation can be also based on the integral (22), once we present the form factor as in (19). The dipole-form-factor result for the nonlogarithmic part of the relativistic correction is found to

TABLE I. The relativistic constant $C_{\rm rel}(2s)$ for muonic hydrogen applying different fits.

Fit	Ref.	R_E (fm)	$C_{\rm rel}(2s)$
(18)		0.81	0.987
(B6)	[15]	0.90	0.919
(B5)	[16]	0.90	0.925
(B1)	[17]	0.86	0.944
(B2)	[18]	0.88	0.938
(B3)	[19]	0.87	0.954
(B4)	[20]	0.88	0.935

be (see Appendix A for details)

$$C_{\rm rel}(2s) = \tfrac{7}{16} + \tfrac{1}{2}\ln 3 \simeq 0.987. \tag{24}$$

Applying the explicit presentation of various fits for the proton charge form factor in the standard form (see Appendix B) to (23), we obtain numerical results for the constant $C_{\rm rel}(2s)$ of (2) for muonic hydrogen. The results are summarized in Table I. The fits have somewhat different values of the proton charge radius and different asymptotic behavior at high q^2; however, the numerical results are very close.

On the basis of the spread of the results from the realistic form factors, we estimate the constant for the proton electric form factor as

$$C_{\rm rel}(2s) = 0.94(3). \tag{25}$$

The result is obtained by consideration of all the individual results from Table I but the dipole-form-factor one. The dipole form factor, as mentioned above, may be useful for rough estimations but should not be considered as a realistic fit in contrast to all the others in the table.

We have also obtained in Appendix B the results for the $1s$ state by a direct calculation. However, that is not necessary but rather serves for test purposes. The difference $C(1s) - C(2s)$ is model independent [25] (cf. [26]) and can be found in a closed analytic form. The result reads [25] (cf. [1,7])

$$C_{\rm rel}(1s) - C_{\rm rel}(ns) = \sum_1^{n-1}\frac{1}{k} - \frac{(5n+9)(n-1)}{4n^2}. \tag{26}$$

A result for the relativistic constant $C_{\rm rel}$ is required for the Lamb shift in muonic hydrogen. However, parameters $C_{\rm rel}(ns)$ are the same for ordinary and muonic hydrogen. The results on the ns states in hydrogen may be required for a bunch of the states (see, e.g., [27]).

V. CONCLUSIONS

We have obtained above some results for various higher-order FNS contributions in the form friendly to their momentum-space presentations (e.g., for the relativistic correction). That is useful for muonic hydrogen, for which there are a number of parametrizations in momentum space. We have paid special attention to Padé approximations as available phenomenological fits over the electron-proton scattering data. The calculation performed above allows one to express the results for various corrections in terms of the fit parameters. That should help to estimate the uncertainty and the correlations for various FNS contributions.

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APPENDIX A: EXPANSION OF THE RELATIVISTIC SINGLE-PHOTON CONTRIBUTION (20) FOR LOW STATES

The evaluation of (20) for the relativistic FNS contribution to energy for the states $1s$, $2s$, $2p_{1/2}$, $2p_{3/2}$ leads to the result for the two first terms of the $Z\alpha$ expansion

$$\begin{aligned}
\mathcal{I}_i(1s) &= Z\alpha\int dr^3\Psi_{1s}(r)\frac{e^{-\sqrt{s_i}r}}{r}\Psi^*_{1s}(r) \simeq \frac{4(Z\alpha)^4m^3}{s_i}\left\{1-\frac{4mZ\alpha}{\sqrt{s_i}}+(Z\alpha)^2\left[\ln\left(\frac{\sqrt{s_i}}{2Z\alpha m}\right)+\frac{1}{2}+\frac{12m^2}{s_i}\right]\right\},\\
\mathcal{I}_i(2s) &= Z\alpha\int dr^3\Psi_{2s}(r)\frac{e^{-\sqrt{s_i}r}}{r}\Psi^*_{2s}(r) \simeq \frac{(Z\alpha)^4m^3}{2s_i}\left\{1-\frac{4mZ\alpha}{\sqrt{s_i}}+(Z\alpha)^2\left[\ln\left(\frac{\sqrt{s_i}}{Z\alpha m}\right)+\frac{11}{16}+\frac{21m^2}{2s_i}\right]\right\},\\
\mathcal{I}_i(2p_{1/2}) &= Z\alpha\int dr^3\Psi_{2p_{1/2}}(r)\frac{e^{-\sqrt{s_i}r}}{r}\Psi^*_{2p_{1/2}}(r) \simeq \left(\frac{(Z\alpha)^4m^3}{2s_i}\right)\left[(Z\alpha)^2\left(\frac{3}{16}+\frac{m^2}{2s_i}\right)\right],\\
\mathcal{I}_i(2p_{1/2}) &= Z\alpha\int dr^3\Psi_{2p_{3/2}}(r)\frac{e^{-\sqrt{s_i}r}}{r}\Psi^*_{2p_{3/2}}(r) \simeq \left(\frac{(Z\alpha)^4m^3}{2s_i}\right)\left((Z\alpha)^2\frac{m^2}{2s_i}\right).
\end{aligned} \tag{A1}$$

As already mentioned in the text, the expansion in $Z\alpha$ leads to relativistic corrections, while the expansion in $Z\alpha mR_N$, which is presented here with $Z\alpha m/\sqrt{s_i}$ terms, leads to partial higher-order finite-size contributions, such as the Friar term in (7) or a higher-order logarithmic term (8). To avoid double counting we have to stay with the leading term in the finite-size parameter $Z\alpha mR_N$. That reduces the expansion to the result for the FNS contribution in orders $(Z\alpha)^4m(mR_N)^2$ (the leading FNS term) and

$(Z\alpha)^6 m(mR_N)^2$ (the next-to-leading relativistic FNS term) to

$$\mathcal{I}_i(1s) = Z\alpha \int dr^3 \Psi_{1s}(r) \frac{e^{-\sqrt{s_i}r}}{r} \Psi^*_{1s}(r) \doteq \frac{4(Z\alpha)^4 m^3}{s_i} \left\{1 + (Z\alpha)^2 \left[\ln\left(\frac{\sqrt{s_i}}{2Z\alpha m}\right) + \frac{1}{2}\right]\right\},$$
$$\mathcal{I}_i(2s) = Z\alpha \int dr^3 \Psi_{2s}(r) \frac{e^{-\sqrt{s_i}r}}{r} \Psi^*_{2s}(r) \doteq \frac{(Z\alpha)^4 m^3}{2s_i} \left\{1 + (Z\alpha)^2 \left[\ln\left(\frac{\sqrt{s_i}}{Z\alpha m}\right) + \frac{11}{16}\right]\right\},$$
$$\mathcal{I}_i(2p_{1/2}) = Z\alpha \int dr^3 \Psi_{2p_{1/2}}(r) \frac{e^{-\sqrt{s_i}r}}{r} \Psi^*_{2p_{1/2}}(r) \doteq \left(\frac{(Z\alpha)^4 m^3}{2s_i}\right)\left[\frac{3}{16}(Z\alpha)^2\right],$$
$$\mathcal{I}_i(2p_{1/2}) = Z\alpha \int dr^3 \Psi_{2p_{3/2}}(r) \frac{e^{-\sqrt{s_i}r}}{r} \Psi^*_{2p_{3/2}}(r) \doteq 0. \tag{A2}$$

Recalling the definition of the radius (11) and the normalization condition (10), we reproduce the known results (1), (2), and (5) and find the constant $C_{\rm rel}$ in Eq. (2) for the $1s$

$$C_{\rm rel}(1s) = \frac{1}{2} - \ln 2 + \frac{3}{R_N^2} \sum_i \frac{a_i}{s_i^2} \ln\left(s_i R_N^2\right) \tag{A3}$$

and $2s$ states [see (23)], the difference of which perfectly agrees with (26).

The presentation (19) allows us to extend the results obtained in standard form to the dipole approximation. The result of the r integration is

$$\Delta E(nl_j) = \eta^2 \frac{\Gamma(2\zeta + n_r + 1)(n_r)!}{\frac{Z\alpha}{\eta} - \nu} \sum_{p,k=0}^{n_r} \frac{(-1)^{p+k}}{p!(n_r - p)!k!(n_r - k)!} \frac{\Gamma(2\zeta + p + k)}{\Gamma(2\zeta + p + 1)\Gamma(2\zeta + k + p)}$$
$$\times \left\{m\left[\left(\frac{Z\alpha}{\eta} - \nu\right)^2 + (n_r - p)(n_r - k)\right] - E_{nl\,j}\left(\frac{Z\alpha}{\eta} - \nu\right)(2n_r - p - k)\right\}\left(\frac{1}{1 + \frac{\Lambda}{m}\frac{1}{2\eta}}\right)^{1+p+k+2\zeta}$$
$$\times \left[1 + \frac{(2 + p + k + 2\zeta)}{4\eta}\frac{\Lambda}{m}\right], \tag{A4}$$

and, in particular,

$$\Delta E^{\rm dip}(1s) = \frac{2}{3}(Z\alpha)^4 m_r^3 R_N^2 \left[1 + (Z\alpha)^2 \left(\ln \frac{1}{Z\alpha R_N m} + \frac{1}{4} + \frac{1}{2}\ln 3\right)\right],$$
$$\Delta E^{\rm dip}(2s) = \frac{1}{12}(Z\alpha)^4 m_r^3 R_N^2 \left[1 + (Z\alpha)^2 \left(\ln \frac{1}{Z\alpha R_N m} + \frac{7}{16} + \frac{1}{2}\ln 3\right)\right]. \tag{A5}$$

The related $1s$ constant for the relativistic correction in (2) is in the dipole approximation

$$C^{\rm dip}_{\rm rel}(1s) = \tfrac{1}{4} + \tfrac{1}{2}\ln 3 \simeq 0.799. \tag{A6}$$

The results agree with (26).

APPENDIX B: FITS FOR THE ELECTRIC FORM FACTOR OF THE PROTON APPLIED IN THE PAPER

We use for our estimations the fits from the evaluation of the elastic electron-proton scattering data that are available in literature which have good agreement with the data, cover all the spacelike region of q, and have reasonable behavior at low and high q (cf. [9,10,28]). The fits available in literature include various Padé approximations starting with the simplest Kelly's

TABLE II. Parametrization of the empiric fits for the proton form factor in the standard form (12) for the chain fractions.

	(B5), [16]		(B6), [15]	
i	a_i (GeV^2)	s_i (GeV^2)	a_i (GeV^2)	s_i (GeV^2)
1	0.306	0.301	0.299	0.295
2	−0.0746+0.0401i	−0.537−1.59i	−0.0647+0.0421i	−0.744−1.62i
3	−0.0746−0.0401i	−0.537+1.59i	−0.0647−0.0421i	−0.744+1.62i

TABLE III. Parametrization of the empiric fits for the proton form factor in the standard form (12) for the Padé approximations.

	(B1), [17]		(B2), [18]		(B3), [19]		(B4), [20]	
i	a_i (GeV^2)	s_i (GeV^2)	a_i (GeV^2)	s_i (GeV^2)	a_i (GeV^2)	s_i (GeV^2)	a_i (GeV^2)	s_i (GeV^2)
1	0.432	0.364	0.271	0.302	0.413	0.360	0.269	0.299
2	0.0975	49.1	0.0244+0.0195i	−0.853−11.9i	−0.207−0.0953i	1.08+2.13i	0.0168+0.0182i	−0.727−10.9i
3	−0.529	2.83	0.0244−0.0195i	−0.853+11.9i	−0.207+0.0953i	1.08−2.13i	0.0168−0.0182i	−0.727+10.9i
4			−0.160−0.194i	0.705−0.974i			−0.151−0.154i	0.577−1.06i
5			−0.160+0.194i	0.705+0.974i			−0.151+0.154i	0.577+1.06i

fit [17]

$$G_E = \frac{1 - 0.24\tau_p}{1 + 10.98\tau_p + 12.82\tau_p^2 + 0.863\tau_p^3} \tag{B1}$$

through [18]

$$G_E = \frac{1 + 3.439\tau_p - 1.602\tau_p^2 + 0.068\tau_p^3}{1 + 15.055\tau_p + 48.061\tau_p^2 + 99.304\tau_p^3 + 0.012\tau_p^4 + 8.650\tau_p^5} \tag{B2}$$

and [19]

$$G_E(q^2) = \frac{1 - 0.19\tau_p}{1 + 11.12\tau_p + 15.16\tau_p^2 + 21.25\tau_p^3} \tag{B3}$$

to the most recent [20]

$$G_E(q^2) = \frac{1 + 2.909\,66\tau_p - 1.115\,422\,29\tau_p^2 + 3.866\,171 \times 10^{-2}\tau_p^3}{1 + 14.518\,7212\tau_p + 40.883\,33\tau_p^2 + 99.999\,998\tau_p^3 + 4.579 \times 10^{-5}\tau_p^4 + 10.358\,0447\tau_p^5}. \tag{B4}$$

We utilize also two chain-fraction fits [16]

$$G_E(q^2) = \cfrac{1}{1 + \cfrac{3.44Q^2}{1 - \cfrac{0.178Q^2}{1 - \cfrac{1.212Q^2}{1 + \cfrac{1.176Q^2}{1 - 0.284Q^2}}}}} \tag{B5}$$

and [15,16]

$$G_E(q^2) = \cfrac{1}{1 + \cfrac{3.478Q^2}{1 - \cfrac{0.140Q^2}{1 - \cfrac{1.311Q^2}{1 + \cfrac{1.128Q^2}{1 - 0.233Q^2}}}}}. \tag{B6}$$

Here Q is the numerical value for the momentum transfer q in GeV/c and $\tau_p = q^2/4m_p^2$.

The results for the Padé-approximation fits above can be decomposed to the standard form (12). The values of a_i and s_i for the chain fractions are summarized in Table II, while the parameters of the other Padé approximations are given in Table III.

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