Non-Hermitian quantum systems and their geometric phases

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We discuss the basic theoretical framework for non-Hermitian quantum systems with particular emphasis on the diagonalizability of non-Hermitian Hamiltonians and their general linear GL$(1, \mathbb{C})$ gauge freedom, which are relevant to the adiabatic evolution of non-Hermitian quantum systems. We find that the adiabatic evolution is possible only when the eigenenergies are real. The accompanying geometric phase is found to be generally complex and associated with not only the phase of a wave function but also its amplitude. The condition for the real geometric phase is laid out. Our results are illustrated with two examples of non-Hermitian $\mathcal{PT}$-symmetric systems, the two-dimensional non-Hermitian Dirac fermion model and bosonic Bogoliubov quasiparticles.

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I. INTRODUCTION

Nature is fundamentally described by the familiar quantum mechanics, where Hamiltonians and observables are all Hermitian operators. Nevertheless, due to an approximation or interaction with the environment, non-Hermitian quantum systems do arise, such as, for example, bosonic Bogoliubov systems [1,2] and non-Hermitian $\mathcal{PT}$-symmetric systems [3]. There has been tremendous interest recently in these non-Hermitian systems both theoretically [3–17] and experimentally [18–34].

There has also been a growing interest in the topological properties of non-Hermitian Hamiltonians [35–46], where the Chern number associated with the Berry curvature is introduced to characterize the topology and the existence of edge states. However, many basic issues are yet to be clarified. For instance, in Hermitian systems, the Berry phase is defined only when the adiabatic evolution is possible. It is not clear what ensures the adiabatic evolution in non-Hermitian systems.

In this article we investigate the adiabatic evolution and its associated geometric phase in non-Hermitian systems. To set up the theoretical framework for discussion, we first describe the basic features of non-Hermitian systems. They include the diagonalizability of a non-Hermitian Hamiltonian, the general linear GL$(1, \mathbb{C})$ gauge transformation, and the nonorthonormal basis imposed on the Hilbert space by a non-Hermitian Hamiltonian. Due to the last feature, one vector in the Hilbert space has two different forms, covariant and contravariant. We find that the adiabatic evolution is possible only when the eigenenergies are real. In general, non-Hermitian systems have a gauge freedom of GL$(1, \mathbb{C})$, and therefore the geometric phase is generally complex and associated with both the phase and amplitude of the eigenstate. However, the geometric phase can be real when certain conditions are satisfied. Our results are illustrated with two two-mode non-Hermitian systems, the two-dimensional non-Hermitian Dirac fermion model and bosonic Bogoliubov quasiparticles.

II. GENERAL FEATURES OF NON-HERMITIAN SYSTEMS

Non-Hermitian systems share many basic features with the usual Hermitian systems. For example, their states live in Hilbert spaces and all observables except energy are represented by Hermitian operators. At the same time, non-Hermicity brings new features. We discuss the features that are relevant to the adiabatic evolution and geometric phase. The first is the diagonalizability of a non-Hermitian Hamiltonian, which is related to the exceptional points (EPs) in a parameter space. The second is that the non-Hermitian Hamiltonian imposes two sets of nonorthonormal bases, which are biorthonormal to each other, in the Hilbert space. We find it very natural to use a covariant vector and contravariant vector to deal with this issue. The third is the gauge freedom in a non-Hermitian system. The norm is not conserved in a non-Hermitian system, thus its gauge freedom is of GL$(1, \mathbb{C})$ in contrast to $U(1)$ gauge freedom in a Hermitian system. As a result, geometric phases are in general complex. We find that conserved pseudonorms can be defined when eigenvalues of the non-Hermitian Hamiltonian are real. Furthermore, geometric phases may become real when more conditions are satisfied.

A. Diagonalizability of non-Hermitian Hamiltonians

We consider a general n-dimensional matrix $\mathcal{M}$. Its diagonalizability is determined by the algebraic multiplicity and geometric multiplicity of its eigenvalues. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_d$ of matrix $\mathcal{M}$ are the roots of the secular equation,

$$\begin{aligned}\mathrm{Det}(\mathcal{M} - \lambda I) \\ = (\lambda_1 - \lambda)^{\eta(\lambda_1)}(\lambda_2 - \lambda)^{\eta(\lambda_2)} \cdots (\lambda_d - \lambda)^{\eta(\lambda_d)} = 0, \quad (1)\end{aligned}$$

where I is the identity matrix and Det(A) denotes the determinant of A. In the polynomial, the exponent $\eta(\lambda_j)$ is called the algebraic multiplicity of the eigenvalue λ_j [47]. The following relations apparently hold for the algebraic multiplicity,

$$1 \leqslant \eta(\lambda_j) \leqslant n, \quad \sum_{i=1}^{d} \eta(\lambda_i) = n. \tag{2}$$

Corresponding to each eigenvalue λ_j, the maximum number of linearly independent eigenvectors is called the geometric multiplicity $\zeta(\lambda_j)$ [47]. One can prove that for each eigenvalue λ_j its geometric multiplicity cannot exceed its algebraic multiplicity, that is, $\zeta(\lambda_j) \leqslant \eta(\lambda_j)$ [47]. The matrix $\mathcal{M}$ is diagonalizable only when the geometric multiplicity is equal to the algebraic multiplicity for any eigenvalue [47],

$$\zeta(\lambda_j) = \eta(\lambda_j), \quad \text{for } 1 \leqslant j \leqslant d. \tag{3}$$

Consider a family of non-Hermitian Hamiltonians $H(\mathbf{R}) \neq H(\mathbf{R})^\dagger$, which depend on external parameters $\mathbf{R}$. The points in the parameter space $\mathbf{R}$ are called exceptional points (EPs) when $H(\mathbf{R})$ is not diagonalizable at these points. For parameters other than EPs, Eq. (3) holds and there are n linearly independent eigenvectors for an $n \times n$ Hamiltonian matrix H.

B. Covariant and contravariant vectors in Hilbert space

When a non-Hermitian Hamiltonian H is diagonalizable, it has two sets of eigenvectors $|\psi_j\rangle$ and $|\phi^j\rangle$ satisfying [6,7,48]

$$H|\psi_j\rangle = E_j|\psi_j\rangle, \quad H^\dagger|\phi^j\rangle = E_j^*|\phi^j\rangle. \tag{4}$$

They are biorthonormal, $\langle\phi^i|\psi_j\rangle = \delta_{ij}$, and complete

$$\sum_j |\psi_j\rangle\langle\phi^j| = 1. \tag{5}$$

Usually $|\psi_j\rangle$ and $|\phi^j\rangle$ are called the right eigenvector and left eigenvector. We find it more natural to call them contravariant eigenvectors and covariant eigenvectors. Respectively, they form one set of contravariant bases and one set of covariant bases. For a given vector $|\Psi\rangle$ in the Hilbert space, it can be expanded either in the contravariant basis,

$$|\Psi\rangle = \sum_{j=1}^{n} c^j|\psi_j\rangle \equiv \begin{pmatrix} c^1 \\ c^2 \\ \vdots \\ c^n \end{pmatrix}, \tag{6}$$

or in the covariant basis,

$$|\Psi\rangle = \sum_{j=1}^{n} c_j|\phi^j\rangle \equiv \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}. \tag{7}$$

The inner product can be naturally written as

$$\langle\Psi|\Psi\rangle = \sum_{j=1}^{n} c_j^* c^j. \tag{8}$$

Note that in the above we have introduced upper and lower indices to label covariant and contravariant vectors, respectively.

C. Gauge freedom and pseudonorms in non-Hermitian systems

Consider the dynamics of a non-Hermitian system, which is given by the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = H|\Psi\rangle. \tag{9}$$

As H is not Hermitian, the norm $\langle\Psi|\Psi\rangle$ is not conserved during the dynamical evolution. This means that we can carry out a transformation of the wave function $|\Psi'\rangle = f\,|\Psi\rangle$ with $f = |f|e^{i\theta}$ and $|f| \neq 1$. This is a general linear (GL) gauge transformation in a complex domain. Therefore, in general, a non-Hermitian system has $GL(1, \mathbb{C})$ gauge freedom.

However, as we shall show immediately, for a class of non-Hermitian systems, one can define a pseudonorm that is conserved. For the Hilbert space, there always exists a set of complete orthonormal bases $|j\rangle$, $\langle i|j\rangle = \delta_{ij}$. When H is diagonalizable, although the right eigenvectors $|\psi_j\rangle$ are not orthonormal, they are linearly independent and form a set of complete bases. The same is true for the left eigenvectors $|\phi^j\rangle$. Therefore, there exists an invertible matrix A such that

$$|\psi_j\rangle = A|j\rangle, \quad |\phi^j\rangle = (A^{-1})^\dagger|j\rangle. \tag{10}$$

Thus we have $|\phi^j\rangle = (A^{-1})^\dagger A^{-1}|\psi_j\rangle$ and

$$\langle\psi_i|X|\psi_j\rangle = \langle\psi_i|\phi^j\rangle = \delta_{ij}, \tag{11}$$

where $X(\mathbf{R}) = (AA^\dagger)^{-1}$ and is apparently Hermitian. We define the pseudonorm as $\langle\psi|X|\psi\rangle$. One can easily prove that, for arbitrary $|\psi\rangle$,

$$\frac{d}{dt}\langle\psi|X|\psi\rangle = 0, \tag{12}$$

when all the eigenvalues E_j's are real. A special case of such a norm is well known in Bogoliubov systems [2]. We will find later that the Hermitian matrix X plays a crucial role in specifying the condition for the geometric phase to be real.

III. ADIABATIC EVOLUTION

Consider a non-Hermitian Hamiltonian $H(\mathbf{R})$, which depends on external parameters $\mathbf{R}$. We are interested in its adiabatic evolution as $\mathbf{R}$ changes slowly with time and how the geometric phase arises. In conventional quantum mechanics where $H(\mathbf{R})$ is Hermitian, there is an adiabatic theorem which states that the occupation probability at each energy level does not change when there is no degeneracy in the energy levels. Berry later found that a geometric phase can arise when the adiabatic theorem holds. We want to find out under what condition a similar adiabatic theorem holds in non-Hermitian systems.

We first assume that $\mathbf{R}$ is fixed. In this case, as the system is linear, we can always expand a state $|\Psi(t)\rangle$ in terms of the right eigenstates and write the dynamical evolution as

$$|\Psi(t)\rangle = \sum_j c^j \exp\left[-\frac{i}{\hbar}E_j t\right]|\psi_j\rangle. \tag{13}$$

This shows that if E_j's are complex, then the relative probability in each eigenstate $|\psi_j\rangle$ can change with time. The situation may become worse when $\mathbf{R}$ changes. So, for an adiabatic theorem to hold in non-Hermitian systems, the eigenvalues

E_j's must be real and have no degeneracy. This conclusion becomes more evident with the following detailed analysis.

When $\mathbf{R}$ changes with time, all the eigenstates $|\psi_j(t)\rangle$ and eigenvalues $E_j(t)$ become time dependent. In this case, we can write the dynamical evolution as

$$|\psi(t)\rangle = \sum_j c^j(t)\exp\left[-\frac{i}{\hbar}\int_0^t E_j(t')dt'\right]|\psi_j(t)\rangle. \quad (14)$$

We substitute it into the Schrödinger equation (9) and obtain using Eq. (4)

$$i\hbar\sum_j \dot{c}^j(t)\exp\left[-\frac{i}{\hbar}\int_0^t E_n(t')dt'\right]|\psi_j(t)\rangle + i\hbar\sum_j c^j(t)\exp\left[-\frac{i}{\hbar}\int_0^t E_j(t')dt'\right]|\dot{\psi}_j(t)\rangle = 0. \quad (15)$$

Multiplying Eq. (15) with the left eigenstate $\langle\phi^m(t)|$, we have

$$\dot{c}^m = -c^m\langle\phi^m|\dot{\psi}_m\rangle - \sum_{j\neq m} c^j\langle\phi^m|\dot{\psi}_j\rangle \times \exp\left[-\frac{i}{\hbar}\int_0^t [E_j(t') - E_m(t')]dt'\right]. \quad (16)$$

We assume that the system is initially in state $|\psi_m\rangle$. If the adiabatic theorem holds, one would have $|c^j| \ll 1$ ($j \neq m$) during the whole process. When E_j's are all real, the second term on the right-hand side of the above equation can be safely neglected as

$$\left|\frac{\hbar\langle\phi^m|\dot{\psi}_j\rangle}{E_m - E_j}\right| \ll 1, \quad \text{for all } j \neq m. \quad (17)$$

This can be found by integrating Eq. (16). We then have

$$\dot{c}^m = -c^m\langle\phi^m|\dot{\psi}_m\rangle. \quad (18)$$

This is similar to the situation in Hermitian systems. When E_j's are complex, the second term can grow exponentially and cannot be neglected. This means that the adiabatic theorem cannot hold when E_j's are complex. From now on we only consider the real eigenvalue case.

IV. GEOMETRIC PHASE

We are now ready to derive the geometric phase. We assume that the system is in state $|\psi_j\rangle$. When the adiabatic theorem holds, it should evolve with time as

$$|\psi(\mathbf{R})\rangle = |\psi_j(\mathbf{R})\rangle e^{-i\frac{\int E_j(\mathbf{R})dt}{\hbar}} e^{i\beta_j}, \quad (19)$$

where β_j is the geometric phase. According to Eq. (18), we have [49,50]

$$\mathbf{A}_j = \frac{\partial\beta_j}{\partial\mathbf{R}} = i\langle\phi^j(\mathbf{R})|\frac{\partial}{\partial\mathbf{R}}|\psi_j(\mathbf{R})\rangle. \quad (20)$$

The Berry curvature in three-dimensional parameter space therefore takes the following form [49],

$$\mathbf{B}_j = i\langle\nabla\phi^j| \times |\nabla\psi_j\rangle, \quad (21)$$

where $\nabla \equiv \frac{\partial}{\partial\mathbf{R}}$. Because $|\psi_j\rangle$ is usually not equal to $|\phi^j\rangle$ (as $H^\dagger \neq H$), the Berry connection and Berry curvature are generally not real for non-Hermitian systems even when all E_j's are real.

Let us now examine under what condition the Berry connection in (20) is real. Differentiating Eq. (11) with respect to $\mathbf{R}$, we get

$$\langle\psi_j|X\frac{\partial}{\partial\mathbf{R}}|\psi_j\rangle + \left(\langle\psi_j|X\frac{\partial}{\partial\mathbf{R}}|\psi_j\rangle\right)^* + \langle\psi_j|\frac{\partial X}{\partial\mathbf{R}}|\psi_j\rangle = 0, \quad (22)$$

where we have taken advantage of X being Hermitian. Therefore, when X is $\mathbf{R}$ dependent, the following quantity is in general not zero,

$$\langle\psi_j|X\frac{\partial}{\partial\mathbf{R}}|\psi_j\rangle + \left(\langle\psi_j|X\frac{\partial}{\partial\mathbf{R}}|\psi_j\rangle\right)^* \neq 0. \quad (23)$$

This implies that $\langle\psi_j|X\frac{\partial}{\partial\mathbf{R}}|\psi_j\rangle$ may not be purely imaginary and thus $\mathbf{A}_j$ may not be real. The Berry connection is real only if the following identity holds,

$$\langle\psi_j|\frac{\partial X}{\partial\mathbf{R}}|\psi_j\rangle = 0. \quad (24)$$

It is important to note that the above condition is *not* equivalent to $\frac{\partial X}{\partial\mathbf{R}} = 0$. It is possible that the above condition holds when $\frac{\partial X}{\partial\mathbf{R}} \neq 0$. The reason is that the matrix X is independent of choices of $|j\rangle$ and is completely determined by $|\psi_j\rangle$. Another way to understand this is to note that the condition (24) is *not* equivalent to

$$\langle\Psi|\frac{\partial X}{\partial\mathbf{R}}|\Psi\rangle = 0, \quad (25)$$

where $|\Psi\rangle$ is an arbitrary vector in the Hilbert space.

Nevertheless, we find that for many non-Hermitian systems where Eq. (24) holds we can find an $\mathbf{R}$-independent Hermitian matrix Y such that

$$|\phi^j(\mathbf{R})\rangle = \alpha_j Y|\psi_j(\mathbf{R})\rangle = X(\mathbf{R})|\psi_j(\mathbf{R})\rangle, \quad \text{for } j = 1, 2, \ldots, \quad (26)$$

with $\alpha_j = \pm 1$. Differentiating Eq. (26) with respect to $\mathbf{R}$ and left multiplying it with $\langle\phi^j|$, we can still obtain the relation (24) by virtue of Hermiticity of X and Y and $dY/d\mathbf{R} = 0$. With the constant Y, the following relation holds,

$$\langle\psi_j(\mathbf{R})|Y|\psi_j(\mathbf{R})\rangle = \alpha_j, \quad (27)$$

and the Berry connection can be expressed as

$$\mathbf{A}_j = \alpha_j i\langle\psi_j(\mathbf{R})|Y|\frac{\partial}{\partial\mathbf{R}}\psi_j(\mathbf{R})\rangle. \quad (28)$$

According to the gauge freedom in non-Hermitian system considered in Sec. II C, there is a freedom to modify the jth eigenstate by a complex number $f[|f| \neq 1$ and $f \in \mathrm{GL}(1, \mathbb{C})]$,

$$|\psi_j'\rangle = f|\psi_j\rangle, \quad \langle\phi'^j| = \frac{1}{f}\langle\phi^j|. \quad (29)$$

The second equation in (29) is to guarantee the biorthonormal condition. Upon the gauge transformation (29), the Berry connection is modified to

$$\mathbf{A}_j' = \mathbf{A}_j + i\frac{1}{f}\frac{\partial f}{\partial\mathbf{R}}.$$

Writing $f = |f|e^{i\theta}$, we have

$$\mathbf{A}_j' = \mathbf{A}_j + i\frac{1}{|f|}\frac{\partial |f|}{\partial \mathbf{R}} - \frac{\partial \theta}{\partial \mathbf{R}}. \tag{30}$$

When $|f| = 1$, we recover the result for Hermitian systems. Furthermore, it can be checked that

$$|\psi_j'\rangle e^{i\int_{R_1}^{R_2}\mathbf{A}_j' d\mathbf{R}} = |\psi_j\rangle e^{i\int_{R_1}^{R_2}\mathbf{A}_j d\mathbf{R}}, \tag{31}$$

indicating that the current framework for the geometric phase is self-contained.

V. MONOPOLES

We have defined exceptional points (EPs) as points in the parameter space $\mathbf{R}$ where the non-Hermitian matrix $H(\mathbf{R})$ is not diagonalizable. In this section we shall show that they are monopoles in the sense that the divergence of the Berry curvature $\nabla \cdot \mathbf{B}_j$ does not vanish.

According to Eq. (21), the Berry curvature can be written as

$$\mathbf{B}_j = \nabla \times \mathbf{A}_j = i\sum_{j'}\langle \nabla\phi^j|\psi_{j'}\rangle \times \langle \phi_{j'}|\nabla\psi_j\rangle, \tag{32}$$

where the completeness condition in (5) is employed. To calculate the divergence of the Berry curvature, i.e., $\nabla \cdot \mathbf{B}_j$, we introduce an auxiliary operator

$$\mathbf{F} = -i\sum_n |\nabla\psi_n\rangle\langle\phi_n| = i\sum_n |\psi_n\rangle\langle\nabla\phi_n|, \tag{33}$$

where the second equality is ensured by the completeness relation (5). It can be checked that

$$\nabla \times \mathbf{F} = -i\mathbf{F} \times \mathbf{F}. \tag{34}$$

The Berry curvature can be expressed in terms of $\mathbf{F}$ as

$$\mathbf{B}_j = i\sum_{j'}\langle \phi^j|\mathbf{F}|\psi_{j'}\rangle \times \langle \phi_{j'}|\mathbf{F}|\psi_j\rangle = i\langle\phi^j|\mathbf{F}\times\mathbf{F}|\psi_j\rangle. \tag{35}$$

Finally, by virtue of Eq. (34), we find

$$\begin{aligned}\nabla \cdot \mathbf{B}_j &= i[\langle\nabla\phi^j|\cdot(\mathbf{F}\times\mathbf{F})|\psi_j\rangle + \langle\phi^j|(\mathbf{F}\times\mathbf{F})\cdot|\nabla\psi_j\rangle \\ &\quad + \langle\phi^j|\nabla\cdot(\mathbf{F}\times\mathbf{F})|\psi_j\rangle] \\ &= i[-i\langle\phi^j|\mathbf{F}\cdot(\mathbf{F}\times\mathbf{F})|\psi_j\rangle + i\langle\phi^j|(\mathbf{F}\times\mathbf{F})\cdot\mathbf{F}|\psi_j\rangle \\ &\quad + \langle\phi^j|(\nabla\times\mathbf{F})\cdot\mathbf{F}|\psi_j\rangle - \langle\psi_j|\mathbf{F}\cdot(\nabla\times\mathbf{F})|\psi_j\rangle] \\ &= 0. \end{aligned} \tag{36}$$

In the above derivation we have used the completeness relation (5), which is equivalent to $H(\mathbf{R})$ being diagonalizable. Therefore, for all the points in the parameter space $\mathbf{R}$ other than EPs, the divergence of the Berry curvature is zero. In other words, monopoles can only be EPs.

VI. EXAMPLES

In the above we have presented a general framework for geometric phases in non-Hermitian systems. In this section we use two simple examples to illustrate these results. Specifically, these two examples are a Dirac model with non-Hermitian terms and a two-mode Bogoliubov–de Gennes model describing the bosonic Bogoliubov quasiparticles.

A. Non-Hermitian Dirac model

As the first illustrative example, we investigate the Dirac model with a non-Hermitian term. The Hamiltonian is

$$H = p_x\sigma_x + p_y\sigma_y + (p_z + is)\sigma_z, \tag{37}$$

where p_x, p_y, and p_z are the Bloch momentum and s is a real constant, denoting the gain and loss of particles. σ_x, σ_y, and σ_z are Pauli matrices. This non-Hermitian Dirac model has been recently studied to reveal the topology of energy bands and the properties of the edge state [43–46]. The energy bands of H are [with $\mathbf{p} \equiv (p_x, p_y, p_z)$]

$$E_{1(2)} = \pm\sqrt{\mathbf{p}^2 - s^2 + 2i(p_z s)}. \tag{38}$$

They are real when $p_z = 0$ and $p_x^2 + p_y^2 \geqslant s^2$. In particular, $E_1 = E_2 = 0$ on the ring $p_x^2 + p_y^2 = s^2$ at $p_z = 0$. We shall show that this ring is a collection of EPs, where H becomes nondiagonalizable. It is worth noting that we found a disk-shaped monopole in a nonlinear quantum system [51].

For a point on the ring $p_x^2 + p_y^2 = s^2$ at $p_z = 0$, we can obtain the algebraic multiplicity $\eta(0)$ according to Eq. (1), and the geometric multiplicity $\zeta(0)$ by examining the number of linearly independent eigenstates with a zero eigenvalue. The result is $\eta(0) = 2$ and $\zeta(0) = 1$, violating the diagonalizable condition (3). This means that all points on the ring are EPs. For any point off the ring, we have $\eta(E_1) = \zeta(E_1) = \eta(E_2) = \zeta(E_2) = 1$, which satisfies the diagonalizable condition. In other words, all EPs are on the ring $p_x^2 + p_y^2 = s^2$ at $p_z = 0$.

When H in Eq. (37) is diagonalizable, its biorthonormal eigenstates corresponding to the eigenenergy E_1 and E_2 are

$$|\psi_1\rangle = \begin{pmatrix}\sqrt{\mathbf{p}^2 - s^2 + 2ip_z s} + is + p_z \\ p_x + ip_y\end{pmatrix},$$

$$|\phi^1\rangle = \begin{pmatrix}\frac{1}{2\sqrt{\mathbf{p}^2 - s^2 - 2ip_z s}} \\ \frac{\sqrt{\mathbf{p}^2 - s^2 - 2ip_z s} + is - p_z}{2(p_x - ip_y)\sqrt{\mathbf{p}^2 - s^2 - 2ip_z s}}\end{pmatrix}, \tag{39}$$

$$|\psi_2\rangle = \begin{pmatrix}-p_x + ip_y \\ \sqrt{\mathbf{p}^2 - s^2 + 2ip_z s} + is + p_z\end{pmatrix},$$

$$|\phi^2\rangle = \begin{pmatrix}-\frac{\sqrt{\mathbf{p}^2 - s^2 - 2ip_z s} + is - p_z}{2(p_x + ip_y)\sqrt{\mathbf{p}^2 - s^2 - 2ip_z s}} \\ \frac{1}{2\sqrt{\mathbf{p}^2 - s^2 - 2ip_z s}}\end{pmatrix}. \tag{40}$$

The above eigenstates are unique only up to a gauge freedom of $GL(1, \mathbb{C})$ [see Eq. (29)]. Any state in two-dimensional (2D) Hilbert space $|\Psi\rangle = c^1|\psi_1\rangle + c^2|\psi_2\rangle = c_1|\phi^1\rangle + c_2|\phi^2\rangle$ can be expanded on either the contravariant eigenvectors $|\psi_{1(2)}\rangle$ or covariant ones $|\phi^{1(2)}\rangle$, with the norm being $\langle\Psi|\Psi\rangle = c_1^* c^1 + c_2^* c^2$.

According to the Schrödinger equation (9), a state evolves with t as

$$|\Psi(t)\rangle = c^1 \exp\left(-\frac{i}{\hbar}E_1 t\right)|\psi_1\rangle + c^2 \exp\left(-\frac{i}{\hbar}E_2 t\right)|\psi_2\rangle. \tag{41}$$

The norm $\langle\Psi(t)|\Psi(t)\rangle$ is not conserved as $\langle\psi_1|\psi_2\rangle \neq 0$. However, the pseudonorm $\langle\Psi|X|\Psi\rangle$ is conserved when

$p_x^2+p_y^2>s^2$ and $p_z=0$. With Eqs. (39) and (40), we find that

$$X(p_x,p_y)=\frac{1}{2}\begin{pmatrix}1 & -s\frac{p_y+ip_x}{p_x^2+p_y^2}\\ -s\frac{p_y-ip_x}{p_x^2+p_y^2} & 1\end{pmatrix}. \tag{42}$$

We turn to the adiabatic evolution and geometric phase. According to our general theory, the adiabatic evolution is possible only when E_1 and E_2 are real. This means that for this particular model the adiabatic evolution can happen when $p_x^2+p_y^2>s^2$ and $p_z=0$. When p_x and p_y change slowly on the plane $p_z=0$ with $p_x^2+p_y^2>s$, an initial eigenstate $|\psi_{1(2)}[p_x(0),p_y(0),p_z=0]\rangle$ will always be on the instantaneous eigenstate $|\psi_{1(2)}[p_x(t),p_y(t),p_z=0]\rangle$. We study the geometric phase on this plane with constant s.

For this example, Eq. (24) does not hold, indicating that the Berry phase is generally complex. According to Eq. (21), we obtain the purely imaginary Berry curvature

$$B_{1(2)}=\mp\frac{i}{2}\frac{s}{\left(p_x^2+p_y^2-s^2\right)^{\frac{3}{2}}}, \tag{43}$$

with $-/+$ for the state $|\psi_1\rangle/|\psi_2\rangle$. The Berry curvature is in the p_z direction, and is divergent on the EP ring $p_x^2+p_y^2=s^2$, $p_z=0$. One can check that the Berry curvature does not change upon the gauge transformation imposed by f as shown in Eq. (29), whereas the corresponding Berry connection is modified as shown in Eq. (30). Our general theory dictates that, as (p_x,p_y) change adiabatically around a loop $\mathcal{C}$ in the plane $p_x^2+p_y^2>s$, $p_z=0$ without enclosing the disk $p_x^2+p_y^2<s$, $p_z=0$, the state returns to the initial state but with a purely imaginary geometric phase,

$$\begin{aligned}|\psi\rangle&=|\psi_1[p_x(0),p_y(0)]\rangle e^{i\int_S \mathbf{B}_1 dp_x dp_y}\\&=|\psi_1[p_x(0),p_y(0)]\rangle e^{\int_S \frac{s}{2(p_x^2+p_y^2-s^2)^{\frac{3}{2}}}dp_x dp_y},\end{aligned} \tag{44}$$

with S denoting the area enclosed by the loop $\mathcal{C}$. As an example, we consider a circle on the plane $p_z=0$ centered on the origin $p_x=p_y=p_z=0$ with its radius $|\mathbf{p}|>s$ (enclosing the disk $p_x^2+p_y^2<s$, $p_z=0$). For this circle, the Berry phase β can be calculated by integrating the Berry connection derived from the eigenstates (39) and (40) (counterclockwise seen from the positive p_z axis),

$$\beta=\mp\frac{\pi(\sqrt{|\mathbf{p}|^2-s^2}-is)}{\sqrt{|\mathbf{p}|^2-s^2}}, \tag{45}$$

with $-/+$ for the state $|\psi_1\rangle/|\psi_2\rangle$. The complex geometric phase can be viewed as a geometric gain or loss of particles in dissipative systems described by the non-Hermitian Hamiltonian [49].

B. Bogoliubov–de Gennes equation

The second example is the simplest Bogoliubov–de Gennes system, which has only two modes. Its Hamiltonian reads

$$H=\begin{pmatrix}z & y+ix\\ -y+ix & -z\end{pmatrix}=ix\sigma_x+iy\sigma_y+z\sigma_z, \tag{46}$$

where x, y, and z are real parameters. This Bogoliubov–de Gennes Hamiltonian governs the dynamics of bosonic Bogoliubov quasiparticles. Its eigenenergies are

$$E_{1(2)}=\pm\sqrt{z^2-x^2-y^2}, \tag{47}$$

which are real when $z^2\geqslant x^2+y^2$. In the parameter space spanned by (x,y,z), all points on the surface of the cone $z^2=x^2+y^2$ are EPs as one can show that the algebraic multiplicity $\eta(0)=2$ but the geometric multiplicity $\zeta(0)=1$ on the degenerate cone. Off the cone, we have $\eta(E_1)=\zeta(E_1)=\eta(E_2)=\zeta(E_2)=1$.

In a certain gauge, the biorthonormal contravariant and covariant eigenvectors can be worked out as

$$\begin{aligned}|\psi_1\rangle&=\begin{pmatrix}a\\ b\end{pmatrix},\quad |\phi^1\rangle=\begin{pmatrix}a\\ -b\end{pmatrix}\\ |\psi_2\rangle&=\begin{pmatrix}b^*\\ a^*\end{pmatrix},\quad |\phi^2\rangle=\begin{pmatrix}b^*\\ -a^*\end{pmatrix},\end{aligned} \tag{48}$$

where

$$\begin{aligned}a(x,y,x)&=-\frac{1}{\sqrt{2}}\frac{z+\sqrt{z^2-x^2-y^2}}{\sqrt{z^2-x^2-y^2+z\sqrt{z^2-x^2-y^2}}},\\ b(x,y,z)&=\frac{1}{\sqrt{2}}\frac{y-xi}{\sqrt{z^2-x^2-y^2+z\sqrt{z^2-x^2-y^2}}}.\end{aligned} \tag{49}$$

The biorthonormal states can be modified freely by a gauge transformation as shown in Eq. (29).

Under the Bogoliubov–de Gennes equation, the norm $\langle\Psi|\Psi\rangle$ of a general state

$$|\Psi\rangle=c^1\exp\left(-\frac{i}{\hbar}E_1t\right)|\psi_1\rangle+c^2\exp\left(-\frac{i}{\hbar}E_2t\right)|\psi_2\rangle \tag{50}$$

is not conserved. Instead, what is conserved during the temporal evolution is the pseudonorm $\langle\Psi|X|\Psi\rangle$, where

$$X(x,y,z)=\begin{pmatrix}|a|^2+|b|^2 & -2ab^*\\ -2a^*b & |a|^2+|b|^2\end{pmatrix}. \tag{51}$$

The adiabatic evolution can occur when (x,y,z) change slowly inside the cone $z^2>x^2+y^2$ where the eigenenergies are real. In this example, we can find that Eq. (24) holds, i.e., $\langle\psi_{1(2)}|\frac{\partial X}{\partial\mathbf{R}}|\psi_{1(2)}\rangle=0$, indicating that the Berry phase becomes real. According to Eq. (21) we find that the Berry curvature is [1]

$$\mathbf{B}=\mp\frac{(1+\tan^2\theta)^{\frac{3}{2}}}{2(1-\tan^2\theta)^{\frac{3}{2}}|\mathbf{R}|^2}\hat{\mathbf{R}}, \tag{52}$$

with $-/+$ associated with the state $|\psi_1\rangle/|\psi_2\rangle$, $\theta=\arctan(\frac{\sqrt{x^2+y^2}}{z})$, and $\mathbf{R}\equiv(x,y,x)$ ($\hat{\mathbf{R}}$ is the unit vector along $\mathbf{R}$). Upon the gauge transformation (29), the Berry connection is modified according to Eq. (30) but the Berry curvature is fixed. The Berry curvature becomes divergent as $\theta\to\pm\pi/4$, i.e., on the degenerate cone determined by $z=\pm\sqrt{x^2+y^2}$, indicating that these EPs on the cone are monopoles.

In this example, the constant Y matrix as shown in Eq. (27) exists and it is just one of the Pauli matrices,

$$Y = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (53)$$

with $\alpha_1 = 1$ and $\alpha_2 = -1$ defined in Eq. (26). We then have the simple relations $|\phi^1\rangle = Y|\psi_1\rangle$, $|\phi^2\rangle = -Y|\psi_2\rangle$, and the Berry connections according to Eq. (28) [1,52],

$$\mathbf{A}_{1(2)} = \pm i\langle\psi_{1(2)}|\sigma_z \frac{\partial}{\partial \mathbf{R}}|\psi_{1(2)}\rangle, \quad (54)$$

with $+/-$ associated with the state $|\psi_1\rangle/|\psi_2\rangle$. The geometric phase being real indicates that there is no geometric gain or loss of the particles during the adiabatic evolution.

As an example, we consider a circle $z = \sqrt{3}$, $x^2 + y^2 = 1$. As (x, y, z) scans along the circle (counterclockwise seen from above), the Berry phase can be worked out either by the curve integration of the Berry connection or by the area integration of the Berry curvature. The resulted phase $\beta = \mp\pi(\sqrt{3/2} - 1)$ (with $-/+$ associated with the state $|\psi_1\rangle/|\psi_2\rangle$).

The Chern number, which reflects the total magnetic charge contained by the monopole, can be calculated from Eq. (52) as

$$\mathcal{C}_n \to \mp\infty. \quad (55)$$

This is drastically different from the Chern numbers in Hermitian systems, which are always $2n\pi$ with n being integer.

VII. SUMMARY

To summarize, we have studied the adiabatic geometric phase of non-Hermitian quantum mechanics. We show that the structure of the geometric phase of non-Hermitian quantum mechanics is quite different from the unitary quantum mechanics. Since such non-Hermitian dynamics can be generically found or constructed in various physical systems, the current results provide insights into these non-Hermitian systems. The present work also provides a different perspective toward the fundamental understanding of quantum evolution.

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