Dynamical symmetries hidden in the form of the potential

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A general approach to conserved physical quantities originating from dynamical symmetries is presented for quantum-mechanical systems. It is illustrated that a general ansatz for the Hamiltonian leads to a differential equation for the central potential that can be solved analytically. This indicates that additional integrals of motion are closely connected to the functional form of the potential. In nonrelativistic three-dimensional quantum mechanics, we show that, besides the trivial case of a constant potential, the Coulomb and harmonic potentials are the only two examples that give rise to additional integrals of motion known as the Runge-Lenz-Laplace vector and the Demkov-Fradkin second-rank tensor, which is in agreement with earlier results. Tensors of rank higher than 2 are only conserved quantities for a constant potential and basically denote a generalization of momentum conservation for higher ranks. We also consider the relativistic case by studying the Hamiltonian form of the Dirac equation. Here we show that only a constant and the Coulomb potential lead to conserved quantities. In the case of a Coulomb potential, this is the pseudoscalar Johnson-Lippmann-Biedenharn operator, which reduces to a spin projection of the Runge-Lenz-Laplace vector in the nonrelativistic limit.

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I. INTRODUCTION

It is well known in both classical and quantum mechanics that continuous global symmetries are connected to conserved integrals of motion according to Noether's theorem [1]. In quantum mechanics, these conserved observables (i.e., the Noether charges) are the generators of a Lie group that span the corresponding (Lie) algebra. However, there are specific cases in which a higher symmetry than anticipated is present that is not directly related to the geometrical symmetry of the system. These continuous and dynamical symmetries give rise to additional constants of motion and are typically related to the specific form of a given potential in the Hamiltonian. A very instructive example in this sense is the nonrelativistic two-body problem. In classical mechanics, Bertrand's theorem [2] states that only the gravitational (or Coulomb) $[V(r) \propto -1/r]$ and the harmonic potential $[V(r) \propto r^2]$ give rise to closed orbits with no perihelion shift. In the case of the gravitational or Coulomb potential, the corresponding integral of motion for this constraint is the well-known Runge-Lenz-Laplace vector directing from the focal point to the perihelion [3–5]. For the harmonic oscillator, this additional conserved quantity is the symmetric second-rank Demkov-Fradkin tensor [6–9]. Struckmeier and Riedel illustratively showed for classical mechanics that the underlying dynamical symmetry transformation does not leave the Lagrangian itself invariant, but rather the action $\delta S = L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)dt$ and thus derived the connected dynamical Noether currents from the invariance of time-dependent Hamiltonian systems [10]. A similarly general approach to time-dependent systems in quantum mechanics was demonstrated earlier by Castaños *et al.* by the correspondence principle [11].

The previously described dynamical quantities retain their meaning in quantum mechanics and can be basically translated by means of the correspondence principle. In quantum mechanics, the Runge-Lenz-Laplace vector has gained special attention since it has been noticed that it may be used to construct a simple algebraic solution of the Schrödinger equation of the hydrogen atom by Pauli [12], Fock [13], and Bargmann [14]. Moreover, it turned out to be related to the exceptionally high degeneracy of the hydrogenic energy eigenstates.

In complete analogy, the extraordinarily high symmetry of the three-dimensional isotropic harmonic oscillator gives rise to the Demkov-Fradkin tensor operator [6–8]. Once special relativity is included, both the Runge-Lenz-Laplace vector and the Demkov-Fradkin tensor are no longer conserved quantities and thus lead to a precession of the elliptical orbits in the Kepler problem. Fradkin also derived the corresponding dynamical constants of motion for relativistic mechanics [9]. Moving to quantum theory, we will discuss the Hamiltonian form of the Dirac equation, which is often used to evaluate relativistic effects in an expansion in v/c, the ratio of a typical velocity v of the particles and the speed of light c. For the Dirac equation with a Coulomb potential, Johnson and Lippmann reported a dynamical constant of motion [15] that was later derived by Biedenharn in a more explicit manner [16,17]. The corresponding pseudoscalar Johnson-Lippmann-Biedenharn operator is also responsible for the extraordinary degeneracy of the relativistic hydrogen atom where levels with equal total angular momentum number j but different $\ell = j \pm \frac{1}{2}$ are degenerate in the absence of radiative corrections. However, due to the implicit presence of spin-orbit coupling in the Dirac equation, the degeneracy is not as high as for the nonrelativistic hydrogen atom. It turns out that the Johnson-Lippmann-Biedenharn operator reduces to a spin projection

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of the Runge-Lenz-Laplace vector in the nonrelativistic limit [16–19].

The Johnson-Lippmann-Biedenharn operator has gained special attraction due to its role as one of the supercharges connecting the doubly degenerate excited states in the relativistic hydrogen atom for a given j in an $S(2)$ supersymmetry [18,20–22]. As a similar dynamical quantity for a relativistic harmonic oscillator, which is basically related to the linear energy-momentum relation in the Dirac equation, had not been described, Moshinsky and Szczepaniak therefore suggested a Dirac oscillator model, which is also linear in position and shows an exceptionally high degeneracy [23,24]. Nonetheless, no conserved dynamical quantity had been discussed in relation to that feature, although it was shown to exhibit supersymmetric features [25–27]. Also, a connection to pseudospin symmetry was suggested [28,29].

Since the previously described dynamical symmetries are related to specific potentials, this raises the question whether there is a fundamental connection between the specific functional form of a potential and the corresponding dynamical symmetry generator. We will restrict ourselves to the case of central potentials $V(\boldsymbol{x}) = V(|\boldsymbol{x}| = r)$. For a general nonrelativistic Hamiltonian, Truax could show by means of group algebra that only the cases of a constant, Coulomb, and harmonic potential give rise to additional dynamical symmetries [30]. Our approach presented here starts from a physically more intuitive point of view and has the advantage that it can also be easily extended to relativistic Hamiltonians. By a very general ansatz for the additionally conserved observable, a constraining differential equation for the potential can be derived, which is typically analytically solvable. Thus, the generators of the dynamical quantities are shown to be clearly connected to the explicit functional form of a central potential $V(r)$.

This paper is organized as follows. After a brief review of the appearance and properties of the dynamical constants of motion in quantum mechanics, their general derivation and connection to the central potential is presented for both the nonrelativistic and relativistic cases. In the case of relativistic quantum mechanics, we will restrict ourselves to the Dirac equation as the equation of motion for spin-$\frac{1}{2}$ particles. Our presented approach is however easily extendable to any other equation of motion as well.

II. DYNAMICAL SYMMETRIES IN QUANTUM MECHANICS

A. Nonrelativistic quantum mechanics

We start with a nonrelativistic Hamiltonian $\hat{H}_{\text{nr}}$ for a single particle moving in a potential

$$\hat{H}_{\text{nr}} = \frac{\hat{\boldsymbol{p}}^2}{2m} + V(r), \tag{1}$$

where $\hat{\boldsymbol{p}}$ is the momentum operator of a single particle, m is its respective mass, and $V(r)$ is a general central potential. Due to rotational invariance of the Hamiltonian (1), the total orbital angular momentum $\hat{\boldsymbol{L}}$ is conserved and its components span an so(3) Lie algebra of the SO(3) rotational group. If $V(r)$ is a Coulomb-type potential [$V(r) = -k/r$, $k \in \mathbb{R}$], there exists an additional integral of motion, the Runge-Lenz-Laplace vector $\hat{\boldsymbol{Q}}$. For the given Hamiltonian (1) with $V(r) = -k/r$, it reads

$$\hat{\boldsymbol{Q}} = \frac{1}{2m}(\hat{\boldsymbol{p}} \times \hat{\boldsymbol{L}} - \hat{\boldsymbol{L}} \times \hat{\boldsymbol{p}}) - \frac{k}{r}\hat{\boldsymbol{x}}, \tag{2}$$

where $\hat{\boldsymbol{x}}$ is the position operator.

It can be shown that its square is related to the Hamiltonian (1) and may be written as (see the Supplemental Material [31])

$$\hat{\boldsymbol{Q}}^2 = k^2\left[\frac{2\hat{H}_{\text{nr}}}{mk^2}(\hat{\boldsymbol{L}}^2 + \hbar^2) + 1\right]. \tag{3}$$

Since $\hat{\boldsymbol{Q}}$ commutes with $\hat{H}_{\text{nr}}$, the Hamiltonian in Eq. (3) may also be replaced by the energy eigenvalues. For bound states (like in the hydrogen atom), the Runge-Lenz-Laplace vector may therefore be normalized to

$$\hat{\boldsymbol{Q}}_0 = \frac{\hat{\boldsymbol{Q}}}{\sqrt{-2E}}, \quad E < 0. \tag{4}$$

Together with the three components $\hat{L}_i$ of the orbital angular momentum, the components of $\hat{Q}_{0i}$ span an so(4) Lie algebra for the bound states of the hydrogen atom given by the algebraic relations (see the Supplemental Material [31]) [32]

$$[\hat{L}_j, \hat{L}_k] = i\hbar\varepsilon_{jkl}\hat{L}_l, \tag{5a}$$

$$[\hat{L}_j, \hat{Q}_{0k}] = i\hbar\varepsilon_{jkl}\hat{Q}_{0l}, \tag{5b}$$

$$[\hat{Q}_{0j}, \hat{Q}_{0k}] = i\hbar\varepsilon_{jkl}\hat{L}_l, \tag{5c}$$

which are the generators of a dynamical SO(4) symmetry of the bound states of the hydrogen atom. The operators $\hat{\boldsymbol{L}}^2$ and $\hat{\boldsymbol{Q}}_0^2$ are the two Casimir operators of this SO(4). Moreover, the Runge-Lenz-Laplace vector is orthogonal to $\hat{\boldsymbol{L}}$,

$$\hat{\boldsymbol{Q}}_0 \cdot \hat{\boldsymbol{L}} = \hat{\boldsymbol{L}} \cdot \hat{\boldsymbol{Q}}_0 = 0, \tag{6}$$

and thus lies in the plane spanned $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{p}}$, i.e., $\hat{\boldsymbol{Q}}_0$ is also a vector with odd parity. Equations (3)–(6) lay the foundations for an algebraic solution of the Schrödinger equation of the hydrogen atom by Pauli [12], Fock [13] and Bargmann [14]. Thus, the Hamiltonian $\hat{H}_{\text{nr}}$ with $V(r) = -k/r$ is invariant under global SO(4) transformations, which include SO(3) as a subgroup. This higher symmetry is also the reason for the exceptionally high degeneracy of the hydrogenic energy levels of n^2 instead of the anticipated $2\ell + 1$ eigenstates only. For the specific case of an electrostatic Coulomb potential, it is $k = Z\alpha\hbar c$ and the energy eigenvalues of the hydrogenlike system then read

$$E_n = -\frac{mc^2}{2}\frac{(Z\alpha)^2}{n^2}, \quad n \in \mathbb{N},\ 0 \leqslant \ell \leqslant n-1, \tag{7}$$

where $\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c}$ is the electromagnetic fine-structure constant. The higher degeneracy is related to the fact that, according to Eq. (3), the Hamiltonian of the hydrogen atom may be constructed by both the pseudovector $\hat{\boldsymbol{L}}$ and vector $\hat{\boldsymbol{Q}}_0$ such that states with both even and odd parity (represented by even or odd values of ℓ) are degenerate for a given quantum number $n \in \mathbb{N}$.

Initially, Jauch and Hill [33,34] and later on especially Demkov [6,7] and Fradkin [8] independently showed that also a dynamical symmetry exists in the case of the three-dimensional isotropic harmonic oscillator. Alliluev [35] and McIntosh [36] indicated the connection to the high degree of degeneracy found in this quantum-mechanical system. For the three-dimensional isotropic harmonic oscillator, it is $V(r) = \frac{m\omega^2}{2} r^2$, with ω the eigenfrequency of the oscillator. The corresponding dynamical constant of motion turns out to be the symmetric second-rank Demkov-Fradkin tensor $\hat{Q}_{ij}$ [6–8], which may be defined by $(i, j = \{1, 2, 3\})$

$$\hat{Q}_{ij} = \frac{1}{2m}\hat{p}_i\hat{p}_j + \frac{m\omega^2}{2}\hat{x}_i\hat{x}_j, \tag{8}$$

with its trace being the Hamiltonian $\hat{H}_{\mathrm{nr}}$. Together with the three components $\hat{L}_i$ of the orbital angular momentum, its five other independent components span a closed su(3) algebra (see the Supplemental Material [31]) [32]

$$[\hat{L}_j, \hat{L}_k] = i\hbar\varepsilon_{jkl}\hat{L}_l, \tag{9a}$$

$$[\hat{L}_j, \hat{Q}_{kl}] = 2i\hbar\varepsilon_{jkr}\hat{Q}_{rl}, \tag{9b}$$

$$[\hat{Q}_{jk}, \hat{Q}_{lr}] = i\hbar\frac{\omega^2}{4}(\delta_{jl}\varepsilon_{krs} + \delta_{jr}\varepsilon_{kls} + \delta_{kl}\varepsilon_{jrs} + \delta_{kr}\varepsilon_{jls})\hat{L}_s, \tag{9c}$$

where $j, k, l, r, s = \{1, 2, 3\}$. The dynamical SU(3) symmetry is not directly evident from the algebraic relations (9a)–(9c). They can, however, be rewritten in the standard form; the Cartan subalgebra of the SU(3) group is obtained by the linear combinations [8]

$$\hat{L}_\pm = \hat{L}_1 \pm i\hat{L}_2, \tag{10a}$$

$$\hat{Q}_\pm = \mp\frac{1}{\omega}(\hat{Q}_{13} \pm i\hat{Q}_{23}), \tag{10b}$$

$$\hat{Q}_{\pm 2} = \frac{1}{\omega}(\hat{Q}_{11} - \hat{Q}_{22} \pm 2i\hat{Q}_{12}), \tag{10c}$$

$$\hat{Q}_3 = \frac{1}{\omega}(2\hat{Q}_{33} - \hat{Q}_{11} - \hat{Q}_{22}), \tag{10d}$$

while the Cartan generators are

$$\hat{H}_1 = \tfrac{1}{6}(\sqrt{3}\hat{L}_3 \cos\phi + \hat{Q}_3 \sin\phi), \tag{11a}$$

$$\hat{H}_2 = \tfrac{1}{6}(\sqrt{3}\hat{L}_3 \sin\phi - \hat{Q}_3 \cos\phi) \tag{11b}$$

and the roots are given by

$$\hat{E}^\lambda_\pm = \frac{1}{4\sqrt{3}}(\hat{L}_\pm \pm \lambda\hat{Q}_\pm), \tag{12a}$$

$$\hat{E}_{\pm 2} = \frac{1}{2\sqrt{6}}\hat{Q}_{\pm 2}, \tag{12b}$$

with $\lambda = \pm 1$ and ϕ an arbitrary angle (see [8,9]).

The properties of the Demkov-Fradkin tensor $\hat{Q}_{ij}$ [see Eq. (8)] are similar to those of the Runge-Lenz-Laplace vector. The most obvious one is that it is orthogonal to $\hat{\boldsymbol{L}}$ in the sense that its contraction with the components $\hat{L}_i$ $(i = \{1, 2, 3\})$ yields zero,

$$\hat{Q}_{ji}\hat{L}_i = \hat{L}_i\hat{Q}_{ij} = 0\,. \tag{13}$$

Correspondingly, two eigenvectors of the Demkov-Fradkin tensor $\hat{Q}_{ij}$ lie in a plane with $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{p}}$ and define the major and minor axes of the elliptical orbit of the particle in the classical two-body problem. The total contraction of the Demkov-Fradkin tensor is also related to the Hamiltonian since

$$\hat{Q}_{ij}\hat{Q}_{ji} = \hat{H}^2_{\mathrm{nr}} - \frac{\omega^2}{2}(\hat{\boldsymbol{L}}^2 + 3\hbar^2). \tag{14}$$

This contraction and $\hat{\boldsymbol{L}}^2$ again represent two possible Casimir operators of the su(3) algebra.

Due to their construction according to Eq. (8), the components $\hat{Q}_{ij}$ of the Demkov-Fradkin tensor have even parity like the orbital angular momentum $\hat{\boldsymbol{L}}$. This is also reflected in the ℓ degeneracy of the eigenstates $|n\rangle$ of the isotropic harmonic oscillator. For a given $n = \sum_i n_i$ $(i = \{1, 2, 3\})$, the possible ℓ values read

$$\ell = \begin{cases} n, n-2, \dots, 0, & n = 2k \\ n, n-2, \dots, 1, & n = 2k+1, \end{cases} \tag{15}$$

with $k \in \mathbb{N}_0$. Thus, the energy eigenvalues of the harmonic oscillator,

$$E_n = \left(n + \tfrac{3}{2}\right)\hbar\omega, \quad n \in \mathbb{N}_0, \tag{16}$$

have a degeneracy of $\frac{1}{2}(n+1)(n+2)$ for a given n. Like for the hydrogen atom, the eigenstates of the isotropic harmonic oscillator show a higher degeneracy than the $2\ell + 1$ typically anticipated for the SO(3) group. However, eigenstates of different parities are not degenerate in the isotropic oscillator [see Eq. (15)]. This is related to the fact that the three orbital angular momentum components $\hat{L}_i$ and the five components of the Demkov-Fradkin tensor $\hat{Q}_{ij}$, i.e., all eight generators of the su(3) algebra, have even parity. Truax analyzed the symmetry structure of the Schrödinger equation from a group-theoretic point of view and could show that the two previously described potentials and the case of a constant potential are the only central potentials that can actually give rise to a dynamical symmetry in nonrelativistic quantum mechanics at all [30]. In Sec. III A we will confirm his results from a systematic study of generalized tensor operators defined in phase space.

B. Relativistic quantum mechanics

Relativistic effects are often taken into account by studying relativistic wave equations such as the Klein-Gordon or the Dirac equation. However, it is well known that all these equations cannot be interpreted as a Schrödinger equation for a single particle, since particle number is not conserved in a relativistic theory. To this end, the relativistic wave equations must be interpreted as an equation of motion of a relativistic quantum field theory.

Nevertheless, the solutions of the relativistic wave equations may still serve as the starting point of an expansion in v/c, where v is the typical velocity of the particle and c is the speed of light. Starting from the Dirac equation and including a Coulomb potential, one can expand in $Z\alpha$ (which is v/c in atomic units) to obtain the full expression for the fine structure of the hydrogen atom. From this point of view we consider it useful to study the Dirac equation in the Schrödinger form, where the Hamiltonian reads

$$\hat{H}_{\mathrm{D}} = c(\hat{\boldsymbol{\alpha}} \cdot \hat{\boldsymbol{p}} + \beta mc) + V(r), \tag{17}$$

with $\hat{\alpha}_i = \gamma_0\gamma_i$ and $\beta = \gamma_0$ as the usual Dirac matrices. For any arbitrary central potential $V(r)$, the Dirac operator $\hat{\mathcal{K}}$ representing spin-orbit coupling,

$$\hat{\mathcal{K}} = \beta\left(\frac{\hat{\boldsymbol{\Sigma}} \cdot \hat{\boldsymbol{L}}}{\hbar} + 1\right) = \beta\left(\frac{\hat{\boldsymbol{J}}^2 - \hat{\boldsymbol{L}}^2}{\hbar^2} + \frac{1}{4}\right), \tag{18}$$

commutes with the Dirac Hamiltonian (17) and has the eigenvalues $\kappa = \pm(j + \frac{1}{2})$, with $j = |\ell \pm \frac{1}{2}|$. In Eq. (18), $\hat{\boldsymbol{\Sigma}} = \gamma_5\hat{\boldsymbol{\alpha}}$ is the non-normalized Pauli spin operator

$$\hat{\boldsymbol{\Sigma}} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \quad \hat{\boldsymbol{S}} = \frac{\hbar}{2}\hat{\boldsymbol{\Sigma}}, \tag{19}$$

where $\boldsymbol{\sigma}$ represents the vector containing the three Pauli matrices as its components. Since the Runge-Lenz vector (2) is not a constant of motion anymore, the degeneracy of the relativistic hydrogenic energy eigenstates is lowered. However, the energies still have a higher degeneracy with $\mathrm{sgn}(\kappa)$ than actually anticipated from the SU(2) symmetry [37],

$$E_{nj} = mc^2\left[1 + \left(\frac{Z\alpha}{n - |\kappa| + \sqrt{\kappa^2 - (Z\alpha)^2}}\right)^2\right]^{-1/2}. \tag{20}$$

This degeneracy is also related to a dynamical Noether charge. The corresponding constant of motion for a relativistic Coulomb problem $[V(r) = -k/r,\ k \in \mathbb{R}]$ is given by the so-called Johnson-Lippmann-Biedenharn operator $\hat{\mathcal{R}}$ [15,16],

$$\begin{aligned} \hat{\mathcal{R}} &= \frac{i\hbar}{mc}\hat{\mathcal{K}}\gamma_5(\hat{H}_{\mathrm{D}} - \beta mc^2) - \frac{k}{r}(\hat{\boldsymbol{\Sigma}} \cdot \hat{\boldsymbol{x}}) \\ &= \hat{\boldsymbol{\Sigma}} \cdot \left[\frac{\beta}{2m}(\hat{\boldsymbol{p}} \times \hat{\boldsymbol{L}} - \hat{\boldsymbol{L}} \times \hat{\boldsymbol{p}}) - \frac{k}{r}\hat{\boldsymbol{x}}\right] - \frac{i\hbar}{mc}\frac{k}{r}\hat{\mathcal{K}}\gamma_5. \end{aligned} \tag{21}$$

The equality of both representations of $\hat{\mathcal{R}}$ is shown in the Supplemental Material [31]. In the nonrelativistic limit, it is $c \to \infty$. Moreover, the antiparticle solutions are suppressed and thereby $\beta \to \mathbb{1}_{2\times 2}$, which finally leads to the reduction $\hat{\mathcal{R}} \to \hat{\boldsymbol{\Sigma}} \cdot \hat{\boldsymbol{Q}}$, with $\hat{\boldsymbol{Q}}$ as given in Eq. (2). It should be noted that also $\hat{\mathcal{R}}^2$ is linked to $\hat{H}_{\mathrm{D}}$ in a manner similar to how the Runge-Lenz-Laplace vector is linked to the nonrelativistic Hamiltonian (1),

$$\hat{\mathcal{R}}^2 = k^2\left\{1 - \left(\frac{\hbar c}{k}\right)^2\hat{\mathcal{K}}^2\left[1 - \left(\frac{\hat{H}_{\mathrm{D}}}{mc^2}\right)^2\right]\right\}, \tag{22}$$

which may be in essence rearranged to the Sommerfeld formula (20) since all operators in Eq. (22) commute or anticommute with each other [18]. The eigenvalues ϱ of $\hat{\mathcal{R}}$ are thus related to the principal quantum number n in Eq. (20).

The Johnson-Lippmann-Biedenharn operator connects the doubly degenerate states for a given $|\kappa|$ and thus acts as one of the two supercharges of an $S(2)$ supersymmetry in the relativistic hydrogen atom [18,19,38]

$$\hat{Q}_\alpha = \hat{\mathcal{R}}, \quad \hat{Q}_\beta = i\hat{\mathcal{R}}\frac{\hat{\mathcal{K}}}{|\kappa|}. \tag{23}$$

With $\hat{Q}_\pm = \hat{Q}_\alpha \pm i\hat{Q}_\beta$, these supercharges fulfill the common Witten superalgebra [39]

$$\{\hat{Q}_+\hat{Q}_-\} = \hat{H}_{\mathrm{SUSY}} = \hat{\mathcal{R}}^2, \tag{24a}$$

$$\hat{Q}_\pm^2 = 0, \tag{24b}$$

$$[\hat{Q}_\pm, \hat{H}_{\mathrm{SUSY}}] = 0, \tag{24c}$$

with $\hat{H}_{\mathrm{SUSY}}$ the supersymmetric Hamiltonian, which is formally equivalent to $\hat{H}_{\mathrm{D}}^2$ by the relation (22). The generated supersymmetry contains a $\mathbb{Z}_2$ grading related to the parity operator

$$\hat{\mathcal{P}}_\kappa = \frac{\hat{\mathcal{K}}}{|\kappa|}, \tag{25}$$

with eigenvalues $\mathrm{sgn}(\kappa) = \pm 1$. Correspondingly, the parity operator anticommutes with the Johnson-Lippmann-Biedenharn operator $\hat{\mathcal{R}}$ (see the Supplemental Material [31]) [18].

Together with the three components $\hat{J}_i$ of the total angular momentum, the three rescaled operators [18,38]

$$\hat{M}_1 = \frac{\hbar}{2}\frac{\hat{\mathcal{R}}}{(\hat{\mathcal{R}}^2)^{1/2}} = \frac{\hbar}{2}\frac{\hat{\mathcal{R}}}{|\varrho|}, \tag{26a}$$

$$\hat{M}_2 = \frac{\hbar}{2}\frac{\hat{\mathcal{K}}}{(\hat{\mathcal{K}}^2)^{1/2}} = \frac{\hbar}{2}\frac{\hat{\mathcal{K}}}{|\kappa|}, \tag{26b}$$

$$\hat{M}_3 = -\frac{2i}{\hbar}\hat{M}_1\hat{M}_2 \tag{26c}$$

span an $\mathrm{su}(2) \times \mathrm{su}(2)$ algebra that is locally isomorphic to an so(4) algebra [38],

$$[\hat{M}_j, \hat{M}_k] = i\hbar\varepsilon_{jkl}\hat{M}_l, \tag{27a}$$

$$[\hat{J}_j, \hat{J}_k] = i\hbar\varepsilon_{jkl}\hat{J}_l, \tag{27b}$$

$$[\hat{J}_j, \hat{M}_k] = 0. \tag{27c}$$

Unlike for its nonrelativistic counterpart, no corresponding dynamical constant of motion has been described so far for a relativistic isotropic harmonic oscillator. For the sake of completeness, we mention that Itô *et al.* [23] as well as Moshinsky and Szczepaniak [24] discussed an alternative and recently even experimentally realized model [40] of a Dirac oscillator, which is linear in both momentum and position and obtained by the nongauge transformation

$$\hat{\boldsymbol{p}} \to \hat{\boldsymbol{p}} + i\sqrt{\frac{m}{2}}\omega\beta\hat{\boldsymbol{r}} \tag{28}$$

of the momentum operator in the Dirac Hamiltonian [see Eq. (18)]. In fact, this model system shows an extraordinarily high symmetry similar to the nonrelativistic isotropic harmonic oscillator [25–27], but no specific dynamical constant of motion has been discussed for this case so far. It has however been argued that the interpretation of this type of potential may be regarded as an anomalous magnetic moment generated by an electrostatic harmonic potential [29,41]. In contrast to nonrelativistic quantum mechanics, no restriction on the type of central potentials for the presence of dynamical Noether charges has been reported so far. Thus, it also seemed interesting to us to compare our methodology to the case of a relativistic spin-$\frac{1}{2}$ particle subject to a general central potential $V(r)$ in order to elucidate the types of potentials giving rise to dynamical symmetries in a similarly systematic manner as known for nonrelativistic quantum mechanics [30].

III. GENERAL DERIVATION OF DYNAMICAL SYMMETRIES FOR CENTRAL POTENTIALS

A. Nonrelativistic case

1. Prerequisites

Our purpose is to derive dynamical integrals of motion for a general nonrelativistic Hamiltonian (1) without specification of the central potential $V(r)$. Since the operator should describe a physical observable, the operator is chosen to be Hermitian. We assume that it may be constructed as a tensor of rank l from the position operator $\hat{\boldsymbol{x}}$ and the momentum operator $\hat{\boldsymbol{p}}$ in phase space. This tensor should be symmetric in its variables since any antisymmetric combination is proportional to the angular momentum operators that act as the generators of a common so(3) algebra.

2. Rank $l = 0$

This introductory example nicely illustrates the general procedure in the derivation of dynamical Noether charges, since it will reveal the conservation of orbital angular momentum (not being, however, a dynamical Noether charge in this specific example). We start with an ansatz for a scalar Hermitian operator acting on a Hilbert space $\mathcal{H}$,

$$\hat{Q} = a_1^{(0)}\hat{\boldsymbol{x}}^2 + a_2^{(0)}\hat{\boldsymbol{p}}^2 + \tfrac{1}{2}a_3^{(0)}(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{p}} + \hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{x}}) + a_4^{(0)}, \tag{29}$$

where the $a_i^{(0)}$ with $i = \{1, 2, 3, 4\}$ are real scalar, eventually operator-valued functions. The coefficient functions $a_i^{(0)}$ are now chosen such that $\hat{Q}$ is a constant of motion, that is,

$$[\hat{Q}, \hat{H}_{\text{nr}}] = 0. \tag{30}$$

With the canonical commutators

$$[\hat{p}_j, \hat{H}_{\text{nr}}] = -i\hbar\frac{\partial\hat{H}_{\text{nr}}}{\partial\hat{x}_j} = -\frac{i\hbar}{r}\frac{\partial V}{\partial r}\hat{x}_j, \tag{31a}$$

$$[\hat{x}_j, \hat{H}_{\text{nr}}] = i\hbar\frac{\partial\hat{H}_{\text{nr}}}{\partial\hat{p}_j} = \frac{i\hbar}{m}\hat{p}_j \tag{31b}$$

and the common product rule for commutators affording

$$\begin{aligned}[\hat{\boldsymbol{p}}^2, \hat{H}_{\text{nr}}] &= -i\hbar\left(\frac{1}{r}\frac{\partial V}{\partial r}\right)(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{p}} + \hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{x}}) \\ &\quad - \hbar^2\left(\frac{\partial^2 V}{\partial r^2} - \frac{1}{r}\frac{\partial V}{\partial r}\right) \\ &= -i\hbar(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{p}} + \hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{x}})\left(\frac{1}{r}\frac{\partial V}{\partial r}\right) \\ &\quad + \hbar^2\left(\frac{\partial^2 V}{\partial r^2} - \frac{1}{r}\frac{\partial V}{\partial r}\right),\end{aligned} \tag{32a}$$

$$[\hat{\boldsymbol{x}}^2, \hat{H}_{\text{nr}}] = \frac{i\hbar}{m}(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{p}} + \hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{x}}), \tag{32b}$$

$$\begin{aligned}[\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{p}}, \hat{H}_{\text{nr}}] = [\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{x}}, \hat{H}_{\text{nr}}] &= i\hbar\left(\frac{\hat{\boldsymbol{p}}^2}{m} - r\frac{\partial V}{\partial r}\right) \\ &= 2i\hbar\left\{\hat{H}_{\text{nr}} - \left(V(r) + \frac{r}{2}\frac{\partial V}{\partial r}\right)\right\},\end{aligned} \tag{32c}$$

the following restrictions on the other coefficient functions may be derived from Eq. (30):

$$\left[a_1^{(0)}, \hat{H}_{\text{nr}}\right] = \hbar^2 a_2^{(0)}\frac{1}{r^2}\left(\frac{\partial^2 V}{\partial r^2} - \frac{1}{r}\frac{\partial V}{\partial r}\right) + i\hbar a_3^{(0)}\left(\frac{1}{r}\frac{\partial V}{\partial r}\right), \tag{33a}$$

$$\left[a_2^{(0)}, \hat{H}_{\text{nr}}\right] = -\frac{i\hbar}{m}a_3^{(0)}, \tag{33b}$$

$$\left[a_3^{(0)}, \hat{H}_{\text{nr}}\right] = -\frac{2i\hbar}{m}a_1^{(0)} + 2i\hbar a_2^{(0)}\left(\frac{1}{r}\frac{\partial V}{\partial r}\right), \tag{33c}$$

$$\left[a_4^{(0)}, \hat{H}_{\text{nr}}\right] = 0. \tag{33d}$$

If these equations can be solved nontrivially (i.e., $a_i^{(0)} \neq 0$ for at least one i), another dynamical constant of motion is found. It is clear from inspection of Eq. (33d) that $a_4^{(0)} \in \mathbb{R}$ for a Hermitian operator. A special case is given for $V(r) = V_0 \in \mathbb{R}$. Then Eqs. (33a)–(33d) are consistently solved for $a_1^{(0)} = a_3^{(0)} = 0$ and

$$\hat{Q} = a_2^{(0)}\hat{\boldsymbol{p}}^2 + a_4^{(0)}, \quad a_2^{(0)}, a_4^{(0)} \in \mathbb{R}, \tag{34}$$

where $a_2^{(0)}$ and $a_4^{(0)}$ commute with the Hamiltonian (1). In particular, one may set $a_2^{(0)} = 1$ and $a_4^{(0)} = 0$ and Eq. (34) then states conservation of momentum in the case of a constant potential as is expected.

Another possible choice is $a_1^{(0)} = \hat{\boldsymbol{p}}^2$, which implies $a_2^{(0)} = \hat{\boldsymbol{x}}^2 = r^2$ and $a_3^{(0)} = -(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{p}} + \hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{x}})$ and is consistent with Eqs. (33a)–(33c) irrespective of the functional form of the potential $V(r)$. Thus, the corresponding Noether charge reads

$$\hat{Q} = \hat{\boldsymbol{p}}^2 r^2 + r^2\hat{\boldsymbol{p}}^2 - \tfrac{1}{2}(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{p}} + \hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{x}})^2 + a_4^{(0)}. \tag{35}$$

Upon usage of $\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{p}} = \hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{x}} + 3i\hbar$ and the fact that $\hat{\boldsymbol{L}}^2$ is Hermitian, this may be rearranged to

$$\begin{aligned}\hat{Q} &= [r^2\hat{\boldsymbol{p}}^2 - (\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{p}})(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{x}})] + [\hat{\boldsymbol{p}}^2 r^2 - (\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{x}})(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{p}})] + \tfrac{9}{2}\hbar^2 \\ &\quad + a_4^{(0)} \\ &= 2\left(\hat{\boldsymbol{L}}^2 - \tfrac{3}{4}\hbar^2\right) + a_4^{(0)}, \quad a_4^{(0)} \in \mathbb{R},\end{aligned} \tag{36}$$

with the relation between the scalar operators and $\hat{\boldsymbol{L}}^2$ shown in the Supplemental Material [31]. Since $a_4^{(0)}$ may be arbitrarily chosen as long as it commutes with $\hat{H}_{\text{nr}}$, we may set $a_4^{(0)} = \frac{3}{2}\hbar^2$. Thus, our ansatz (29) for a scalar Hermitian Noether charge reveals the conservation of orbital angular momentum for any given central potential $V(r)$.

3. Rank $l = 1$

The next possible choice is a polar vector operator. Since the orbital angular momentum $\hat{\boldsymbol{L}}$ transforms as a pseudovector with even parity, it is excluded from the basis set of Hermitian vector operators used for the construction of the dynamical constant of motion. The simplest and most general choice is then

$$\hat{Q}_i = a_1^{(1)}\hat{x}_i + a_2^{(1)}\hat{p}_i, \tag{37}$$

where $a_1^{(1)}$ and $a_2^{(1)}$ are again scalar, eventually operator-valued functions that do not necessarily commute with the Hamiltonian (1). With the commutation relations (31a) and (31b), it is simple to derive the conditions for the coefficient functions

$$\left[a_1^{(1)}, \hat{H}_{\rm nr}\right] = i\hbar a_2^{(1)}\left(\frac{1}{r}\frac{\partial V}{\partial r}\right), \tag{38a}$$

$$\left[a_2^{(1)}, \hat{H}_{\rm nr}\right] = -\frac{i\hbar}{m}a_1^{(1)}. \tag{38b}$$

Note that we recover the orbital angular momentum if the a_i^1 are allowed to be vector components of the momentum and position operator, respectively [for that set $a_1^1 = \hat{p}_j$ and $a_2^1 = -\hat{x}_j$ and identify Eqs. (38a) and (38b) with relations (31a) and (31b)]. The only consistent solution with scalar operator-valued coefficients is obtained for the choice

$$a_2^{(1)} = -\frac{\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{x}}}{m}, \tag{39}$$

which then leads to the other coefficient

$$a_1^{(1)} = \frac{\hat{\boldsymbol{p}}^2}{m} - r\frac{\partial V}{\partial r} = 2\left(\hat{H}_{\rm nr} - V(r) - \frac{r}{2}\frac{\partial V}{\partial r}\right). \tag{40}$$

Upon insertion into Eq. (38a), the relation can then be transformed into

$$\left[a_1^{(1)}, \hat{H}_{\rm nr}\right] = i\hbar\left(-\frac{\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{x}}}{m}\right)\left(\frac{1}{r}\frac{\partial V}{\partial r}\right) - \frac{i\hbar}{2m}\left[\left(\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r}\right)\times(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{p}}) + \text{H.c.}\right] + \frac{\hbar^2}{2m}\left(\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r}\right), \tag{41}$$

where H.c. denotes the Hermitian conjugate operator. Thus, an additional Noether charge is found if and only if $V(r)$ is the solution of the differential equation

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} = 0. \tag{42}$$

A trivial solution is $V(r) = V_0,\ V_0 \in \mathbb{R}$. In that case, $a_1^{(1)}$ must be a scalar that commutes with the Hamiltonian. For the given choice (39), this implies $a_1^{(1)} = \frac{\hat{\boldsymbol{p}}^2}{m}$ [see Eq. (32c)] and thus leads to

$$\hat{\boldsymbol{Q}} = \frac{1}{m}[\hat{\boldsymbol{p}}^2\hat{\boldsymbol{x}} - (\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{x}})\hat{\boldsymbol{p}}] = \frac{1}{2m}(\hat{\boldsymbol{p}}\times\hat{\boldsymbol{L}} - \hat{\boldsymbol{L}}\times\hat{\boldsymbol{p}}), \tag{43}$$

where in the last step the Grassmann identity has been used. This is already a special form of the Runge-Lenz-Laplace vector (2) in the case of a constant potential.

A nontrivial solution is easily accessible, however, upon identification of Eq. (42) as the well-known Laplace equation in spherical coordinates (with no boundary conditions). A special solution is the Coulomb potential $V(r) = -k/r$, with $k \in \mathbb{R}$ being the Green's function of the Laplace operator. The corresponding constant of motion is then

$$\hat{\boldsymbol{Q}} = \left(\frac{\hat{\boldsymbol{p}}^2}{m} - \frac{k}{r}\right)\hat{\boldsymbol{x}} - \frac{\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{x}}}{m}\hat{\boldsymbol{p}} = \frac{1}{2m}(\hat{\boldsymbol{p}}\times\hat{\boldsymbol{L}} - \hat{\boldsymbol{L}}\times\hat{\boldsymbol{p}}) - \frac{k}{r}\hat{\boldsymbol{x}}, \tag{44}$$

which is the known form of the Runge-Lenz-Laplace vector $\hat{\boldsymbol{Q}}$ given in Eq. (2). Thus, its existence is clearly connected to the functional form of the Coulomb potential.

4. Rank $l = 2$

A symmetric tensor operator of rank 2 is most generally defined as

$$\hat{Q}_{ij} = a_1^{(2)}\hat{x}_i\hat{x}_j + a_2^{(2)}\hat{p}_i\hat{p}_j, \tag{45}$$

with $a_1^{(1)}$ and $a_2^{(1)}$ scalar operator-valued functions. Again, we exclude any antisymmetric operators in the ansatz since they recover the generalization of orbital angular momentum being the so(3) generators. Upon usage of Eq. (30), the following conditions on the coefficients follow:

$$\left[a_1^{(2)}, \hat{H}_{\rm nr}\right] = \hbar^2 a_2^{(2)}\frac{1}{r^2}\left(\frac{\partial^2 V}{\partial r^2} - \frac{1}{r}\frac{\partial V}{\partial r}\right), \tag{46a}$$

$$\left[a_2^{(2)}, \hat{H}_{\rm nr}\right] = 0, \tag{46b}$$

$$a_1^{(2)} = m a_2^{(2)}\left(\frac{1}{r}\frac{\partial V}{\partial r}\right). \tag{46c}$$

Insertion of Eqs. (46b) and (46c) then yields

$$\left[a_1^{(2)}, \hat{H}_{\rm nr}\right] = \hbar^2 a_2^{(2)}\frac{1}{r^2}\left(\frac{\partial^2 V}{\partial r^2} - \frac{1}{r}\frac{\partial V}{\partial r}\right) + \frac{i\hbar}{2}a_2^{(2)}\frac{1}{r^2}\left[\left(\frac{\partial^2 V}{\partial r^2} - \frac{1}{r}\frac{\partial V}{\partial r}\right)(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{p}}) + \text{H.c.}\right] - 2\hbar^2 a_2^{(2)}\frac{1}{r^2}\left(\frac{\partial^2 V}{\partial r^2} - \frac{1}{r}\frac{\partial V}{\partial r}\right), \tag{47}$$

with H.c. again denoting the Hermitian conjugate part. Equation (47) is consistent with the condition (46a) if and only if the potential $V(r)$ solves the differential equation

$$\frac{1}{r^2}\left(\frac{\partial^2 V}{\partial r^2} - \frac{1}{r}\frac{\partial V}{\partial r}\right) = 0. \tag{48}$$

Again, a trivial solution is $V(r) = V_0,\ V_0 \in \mathbb{R}$. Then $a_1^{(1)}$ is zero and $a_2^{(2)}$ may be any arbitrary scalar that commutes with $\hat{H}_{\rm nr}$, i.e., the solution for this case may be simply written as

$$\hat{Q}_{ij} = a_2^{(2)}\hat{p}_i\hat{p}_j, \tag{49}$$

which may be easily rescaled to the physical form $\hat{Q}_{ij} = \frac{1}{2m}\hat{p}_i\hat{p}_j$. A special solution of the differential equation (48) is the harmonic potential $V(r) = kr^2,\ k \in \mathbb{R}$. Correspondingly, the dynamical Noether charge then reads

$$\hat{Q}_{ij} = 2m a_2^{(2)}\left(\frac{1}{2m}\hat{p}_i\hat{p}_j + k\hat{x}_i\hat{x}_j\right), \tag{50}$$

with any $a_2^{(2)}$ that commutes with the Hamiltonian [see Eq. (46b)]. The dynamical charge can then be rescaled to the Demkov-Fradkin tensor given in Eq. (8). Its existence is therefore also related to the specific functional form of the harmonic potential.

5. Rank $l = 3$

The next-higher-order generalization of a possible dynamical tensor operator of rank 3 would be

$$\hat{Q}_{ijk} = a_1^{(3)}\hat{x}_i\hat{x}_j\hat{x}_k + a_2^{(3)}\hat{p}_i\hat{p}_j\hat{p}_k, \tag{51}$$

with the coefficients $a_1^{(3)}$ and $a_2^{(3)}$ again scalar operator-valued functions. Insertion of Eq. (51) into Eq. (30) then leads to the conditions

$$\left[a_1^{(3)}, \hat{H}_{\mathrm{nr}}\right] = -i\hbar a_2^{(3)} \frac{1}{r^3}\left(\frac{\partial^3 V}{\partial r^3} - \frac{3}{r}\frac{\partial^2 V}{\partial r^2} + \frac{3}{r^2}\frac{\partial V}{\partial r}\right), \tag{52a}$$

$$\left[a_2^{(3)}, \hat{H}_{\mathrm{nr}}\right] = 0, \tag{52b}$$

$$a_1^{(3)} = i\hbar m a_2^{(3)}\left[\frac{1}{r^2}\left(\frac{\partial^2 V}{\partial r^2} - \frac{1}{r}\frac{\partial V}{\partial r}\right)\right], \tag{52c}$$

$$a_2^{(3)}\left(\frac{1}{r}\frac{\partial V}{\partial r}\right) = 0. \tag{52d}$$

The condition (52d) only allows the case

$$\frac{1}{r}\frac{\partial V}{\partial r} = 0 \tag{53}$$

for a nontrivial solution of the given system of conditions. This is however only fulfilled for the case of a constant potential $V(r) = V_0$, $V_0 \in \mathbb{R}$. In this case, $a_1^{(3)}$ is zero again and $a_2^{(3)}$ can be chosen as an arbitrary scalar that commutes with $\hat{H}_{\mathrm{nr}}$. The corresponding Noether charge is

$$\hat{Q}_{ijk} = a_2^{(3)}\hat{p}_i\hat{p}_j\hat{p}_k. \tag{54}$$

6. Arbitrary rank $l > 3$

The previously described procedure can be generalized to any symmetric tensor of arbitrary rank l in phase space given by

$$\hat{Q}_{ijk\cdots r} = a_1^{(l)}\hat{x}_i\hat{x}_j\hat{x}_k\cdots\hat{x}_r + a_2^{(l)}\hat{p}_i\hat{p}_j\hat{p}_k\cdots\hat{p}_r. \tag{55}$$

For any rank $l > 0$, an ansatz for a dynamical constant of motion with two operator-valued coefficients $a_1^{(l)}$ and $a_2^{(l)}$ and usage of Eq. (30) will lead to $l + 1$ conditions. If $l \geqslant 3$, two specific conditions are always

$$\left[a_2^{(l)}, \hat{H}_{\mathrm{nr}}\right] = 0, \tag{56a}$$

$$\frac{1}{r}\frac{\partial V}{\partial r} = 0, \tag{56b}$$

which only allow a constant potential $V(r) = V_0$, $V_0 \in \mathbb{R}$, and due to that, the generalized constant of motion (despite its unphysical meaning) is

$$\hat{Q}_{ij\cdots r} = a_2^{(l)}\hat{p}_i\hat{p}_j\cdots\hat{p}_r, \quad l \geqslant 3, \tag{57}$$

as anticipated from the form of the nonrelativistic Hamiltonian (1). The results thus confirm Truax's group algebraic findings that a dynamical Noether charge for a general nonrelativistic Hamiltonian (1) is only found for the cases of constant, Coulomb, and harmonic potentials [30].

B. Relativistic case

1. Prerequisites

In the following, we will restrict ourselves to spin-$\frac{1}{2}$ particles, for which the Dirac Hamiltonian (17) properly describes the dynamics. Like in the nonrelativistic case, we aim at a derivation of dynamical Noether charges constructed as tensors of rank l from the position operator $\hat{\boldsymbol{x}}$ and the momentum operator $\hat{\boldsymbol{p}}$ as well as their dependence on the specific functional form of the central potential $V(r)$.

2. Rank $l = 0$

A generic and very insightful derivation of a rank-0 dynamical constant of motion was reported by Khachidze and Khelashvili [19]. However, the presence of a Coulomb potential was implicitly assumed and several operators such as the unit position vector and $\hat{\mathcal{K}}(\hat{\boldsymbol{\Sigma}}\cdot\hat{\boldsymbol{p}})$ were taken intentionally for the ansatz of a dynamical constant of motion in order to make it similar to the spin projection of the nonrelativistic Runge-Lenz-Laplace vector. In contrast, our purpose is to derive both the form of the Johnson-Lippmann-Biedenharn operator from a most general ansatz and its uniqueness for a Coulomb potential.

Due to the SU(2) invariance of the Dirac Hamiltonian (17), the total angular momentum $\hat{\boldsymbol{J}}$ is a conserved quantity. Under Lorentz transformations, it behaves like a pseudovector. In addition, the Dirac operator $\hat{\mathcal{K}}$ [see Eq. (18)] commutes with the Dirac Hamiltonian (17) irrespective of the explicit form of a central potential $V(r)$. However, the SU(2) double cover also allows for operators such as $\hat{\boldsymbol{\Sigma}}\cdot\hat{\boldsymbol{x}}$ or $\hat{\boldsymbol{\Sigma}}\cdot\hat{\boldsymbol{p}}$ as possible choices for a dynamical constant of motion of rank 0. Since any operator containing $\hat{\boldsymbol{\Sigma}}$ transforms like a pseudoscalar, the most general ansatz for an integral of motion may also include the γ_5 matrix and thus

$$\hat{\mathcal{R}} = b_1^{(0)}(\hat{\boldsymbol{\Sigma}}\cdot\hat{\boldsymbol{x}}) + b_2^{(0)}(\hat{\boldsymbol{\Sigma}}\cdot\hat{\boldsymbol{p}}) + b_3^{(0)}\gamma_5, \tag{58}$$

where the coefficients b_i^0 ($i = \{1, 2, 3\}$) are again scalar, eventually operator-valued functions that are chosen such that the commutator between $\hat{\mathcal{R}}$ and $\hat{H}_{\mathrm{D}}$ vanishes,

$$[\hat{\mathcal{R}}, \hat{H}_{\mathrm{D}}] = 0. \tag{59}$$

With the commutation relations

$$[\hat{\boldsymbol{\Sigma}}\cdot\hat{\boldsymbol{p}}, \hat{H}_{\mathrm{D}}] = -\frac{i\hbar}{r}\frac{\partial V}{\partial r}(\hat{\boldsymbol{\Sigma}}\cdot\hat{\boldsymbol{x}}), \tag{60a}$$

$$[\hat{\boldsymbol{\Sigma}}\cdot\hat{\boldsymbol{x}}, \hat{H}_{\mathrm{D}}] = i\hbar c\gamma_5(2\beta\hat{\mathcal{K}} + 1), \tag{60b}$$

$$[\gamma_5, \hat{H}_{\mathrm{D}}] = 2mc^2\gamma_5\beta \tag{60c}$$

and insertion into Eq. (59), it is possible to derive the necessary restrictions

$$\left[b_2^{(0)}, \hat{H}_{\mathrm{D}}\right] = 0, \tag{61a}$$

$$b_3^{(0)} = \frac{i\hbar}{mc} b_1^{(0)}\hat{\mathcal{K}}, \tag{61b}$$

which yields

$$[\hat{\mathcal{R}},\hat{H}_{\mathrm{D}}]=\big[b_1^{(0)},\hat{H}_{\mathrm{D}}\big](\hat{\boldsymbol{\Sigma}}\cdot\hat{\boldsymbol{x}})+\frac{i\hbar}{mc}\big[b_1^{(0)},\hat{H}_{\mathrm{D}}\big]\hat{\mathcal{K}}\gamma_5 + i\hbar c b_1^{(0)}\gamma_5 - i\hbar b_2^{(0)}\left(\frac{1}{r}\frac{\partial V}{\partial r}\right)(\hat{\boldsymbol{\Sigma}}\cdot\hat{\boldsymbol{x}})=0. \quad (62)$$

This equation is only solved nontrivially if $b_1^{(0)}$ is a function of r, $b_1^{(0)}=b(r)$. With

$$[b(r),\hat{H}_{\mathrm{D}}]=i\hbar c\gamma^5\left(\frac{1}{r}\frac{\partial b}{\partial r}\right)(\hat{\boldsymbol{\Sigma}}\cdot\hat{\boldsymbol{x}}) \quad (63)$$

and $(\hat{\boldsymbol{\Sigma}}\cdot\hat{\boldsymbol{x}})(\hat{\boldsymbol{\Sigma}}\cdot\hat{\boldsymbol{x}})=r^2$, it follows that

$$[\hat{\mathcal{R}},\hat{H}_{\mathrm{D}}]=i\hbar c\gamma_5\left(b(r)+r\frac{\partial b}{\partial r}\right)+i\hbar\left[\frac{i\hbar}{m}\hat{\mathcal{K}}\left(\frac{1}{r}\frac{\partial b}{\partial r}\right) - b_2^{(0)}\left(\frac{1}{r}\frac{\partial V}{\partial r}\right)\right](\hat{\boldsymbol{\Sigma}}\cdot\hat{\boldsymbol{x}})\stackrel{!}{=}0, \quad (64)$$

where the fact that $[\hat{\mathcal{K}},\gamma_5(\hat{\boldsymbol{\Sigma}}\cdot\hat{\boldsymbol{x}})]=0$ has been used. A trivial solution to this equation is $b(r)=0$ and $V(r)=V_0$, $V_0\in\mathbb{R}$. Then the corresponding dynamical Noether charge is

$$\hat{\mathcal{R}}=b_2^{(0)}(\hat{\boldsymbol{\Sigma}}\cdot\hat{\boldsymbol{p}}), \quad (65)$$

with any $b_2^{(0)}$ that commutes with $\hat{H}_{\mathrm{D}}$ [see Eq. (61a)]. This includes two special cases. On the one hand, for any spin-$\frac{1}{2}$ particle subject to a constant potential, not only the total momentum $\hat{\boldsymbol{p}}$, but also its helicity $\hat{\boldsymbol{\Sigma}}\cdot\hat{\boldsymbol{p}}$ is a conserved quantity in the framework of relativistic quantum mechanics (given the choice $b_2^{(0)}=\frac{1}{|\boldsymbol{p}|}$). It should be noted that chirality is only conserved in the case of a massless particle and coincides with its helicity in that case. On the other hand, however, the specific choice $b_2^{(0)}=\frac{i\hbar}{m}\hat{\mathcal{K}}$ leads to the conservation of the quantity

$$\hat{\mathcal{R}}=\frac{i\hbar}{m}\hat{\mathcal{K}}(\hat{\boldsymbol{\Sigma}}\cdot\hat{\boldsymbol{p}})=\hat{\boldsymbol{\Sigma}}\cdot\frac{\beta}{2m}(\hat{\boldsymbol{p}}\times\hat{\boldsymbol{L}}-\hat{\boldsymbol{L}}\times\hat{\boldsymbol{p}}), \quad (66)$$

which goes to the spin projection $\hat{\mathcal{R}}\to\hat{\boldsymbol{\Sigma}}\cdot\hat{\boldsymbol{Q}}$ of the Runge-Lenz-Laplace vector (43) for the case of a constant potential in the nonrelativistic limit $\beta\to\mathbb{1}_{2\times2}$.

A nontrivial solution is obtained for $b(r)=V(r)$ and if the potential fulfills the differential equation

$$V(r)+r\frac{\partial V}{\partial r}=0, \quad (67)$$

with the Coulomb potential $V(r)=-k/r$, $k\in\mathbb{R}$, as its solution. For that case, $b_2^{(0)}=\frac{i\hbar}{m}\hat{\mathcal{K}}$, and with Eq. (61b) the corresponding dynamical Noether charge reads

$$\hat{\mathcal{R}}=-\frac{k}{r}(\hat{\boldsymbol{\Sigma}}\cdot\hat{\boldsymbol{x}})+\frac{i\hbar}{m}\hat{\mathcal{K}}(\hat{\boldsymbol{\Sigma}}\cdot\hat{\boldsymbol{p}})-\frac{i\hbar}{mc}\frac{k}{r}\hat{\mathcal{K}}\gamma_5, \quad (68)$$

in agreement with Eq. (21). Thus, the existence of the Johnson-Lippmann-Biedenharn operator is connected to the presence of a Coulomb potential only.

3. Rank $l = 1$

The next-higher-order dynamical constant of motion is a tensor of rank 1. The total angular momentum $\hat{\boldsymbol{J}}=\hat{\boldsymbol{L}}+\hat{\boldsymbol{S}}$ is a conserved quantity, but behaves like a pseudovector under Lorentz transformations. Thus, it is feasible to use the general ansatz for a dynamical Noether charge

$$\hat{\mathcal{R}}_i=b_1^{(1)}\hat{x}_i+b_2^{(1)}\hat{p}_i+b_3^{(1)}\gamma_i, \quad (69)$$

where the γ_i matrices ($i=\{1,2,3\}$) are used since they also transform as polar vectors under Lorentz transformations. Note that the Dirac matrices $\hat{\alpha}_i$ and β are closely related to them as $\hat{\alpha}_i=\beta\gamma_i$ and $\beta=\gamma_0$. With the relations

$$[\hat{x}_j,\hat{H}_{\mathrm{D}}]=i\hbar\frac{\partial\hat{H}_{\mathrm{D}}}{\partial\hat{p}_j}=i\hbar c\hat{\alpha}_j, \quad (70a)$$

$$[\hat{p}_j,\hat{H}_{\mathrm{D}}]=-i\hbar\frac{\partial\hat{H}_{\mathrm{D}}}{\partial\hat{x}_j}=-\frac{i\hbar}{r}\frac{\partial V}{\partial r}\hat{x}_j, \quad (70b)$$

$$[\gamma_j,\hat{H}_{\mathrm{D}}]=2c\beta(\hat{p}_j-mc\gamma_j), \quad (70c)$$

the following restrictions on the operator-valued scalar coefficient functions follow from Eq. (59):

$$\big[b_1^{(1)},\hat{H}_{\mathrm{D}}\big]=i\hbar b_2^{(1)}\left(\frac{1}{r}\frac{\partial V}{\partial r}\right), \quad (71a)$$

$$\big[b_2^{(1)},\hat{H}_{\mathrm{D}}\big]=-2cb_3^{(1)}\beta, \quad (71b)$$

$$\big[b_3^{(1)},\hat{H}_{\mathrm{D}}\big]=c\big(2mcb_3^{(1)}-i\hbar b_1^{(1)}\big)\beta. \quad (71c)$$

This system of equations is consistently solved for the choice $b_2^{(1)}=\beta+b\hat{\mathcal{K}}$, $b\in\mathbb{R}$, which subsequently implies $b_3^{(1)}=\hat{\boldsymbol{\alpha}}\cdot\hat{\boldsymbol{p}}$ upon insertion of $b_2^{(1)}$ into Eq. (71b). Substituting these relations into Eq. (71c) then yield

$$b_1^{(1)}=-\frac{1}{c}\beta(\hat{\boldsymbol{\alpha}}\cdot\hat{\boldsymbol{x}})\left(\frac{1}{r}\frac{\partial V}{\partial r}\right). \quad (72)$$

Final usage of this relation in Eq. (71a) and several rearrangements (see the Supplemental Material [31]) then lead to

$$\big[b_1^{(1)},\hat{H}_{\mathrm{D}}\big]=i\hbar\beta\left(\frac{1}{r}\frac{\partial V}{\partial r}\right)+\beta\left[\left(\frac{1}{r}\frac{\partial V}{\partial r}\right)(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{p}})+\text{H.c.}\right] - 2mc\left(\frac{1}{r}\frac{\partial V}{\partial r}\right)(\hat{\boldsymbol{\alpha}}\cdot\hat{\boldsymbol{x}})-i\hbar\beta\left(\frac{1}{r}\frac{\partial V}{\partial r}\right). \quad (73)$$

This is consistent with Eq. (71a) as well as the choices of $b_2^{(1)}$ and $b_3^{(1)}$ if and only if

$$\frac{1}{r}\frac{\partial V}{\partial r}=0, \quad (74)$$

i.e., if the potential is constant, $V(r)=V_0\in\mathbb{R}$. Then $b_1^{(1)}=0$ and the corresponding Noether charge of rank 1 reads

$$\hat{\mathcal{R}}_i=\beta\hat{p}_i+(\hat{\boldsymbol{\alpha}}\cdot\hat{\boldsymbol{p}})\gamma_i=\beta[\hat{p}_i-(\hat{\boldsymbol{\alpha}}\cdot\hat{\boldsymbol{p}})\hat{\alpha}_i]. \quad (75)$$

Besides these three components of a polar vector and the three components of the pseudovectorial total angular momentum $\hat{J}_i$, no other dynamical constants of motion of rank 1 are found.

4. Rank $l = 2$

Like in the aforementioned cases, our ansatz includes a symmetric tensor of second rank ($l=2$),

$$\hat{\mathcal{R}}_{ij}=b_1^{(2)}\hat{x}_i\hat{x}_j+b_2^{(2)}\hat{p}_i\hat{p}_j+\tfrac{1}{2}b_3^{(2)}(\hat{x}_i\gamma_j+\gamma_i\hat{x}_j) + \tfrac{1}{2}b_4^{(2)}(\hat{p}_i\gamma_j+\gamma_i\hat{p}_j), \quad (76)$$

since any antisymmetric tensor (as represented by $\sigma_{ij} = \frac{i}{2}[\gamma_i, \gamma_j]$) will lead to a generalization of total angular momentum conservation. Usage of Eq. (59) leads to the restrictions

$$\big[b_1^{(2)}, \hat{H}_{\mathrm{D}}\big] = \hbar^2 b_2^{(2)} \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial r^2} - \frac{1}{r}\frac{\partial V}{\partial r}\right), \tag{77a}$$

$$\big[b_2^{(2)}, \hat{H}_{\mathrm{D}}\big] = -2cb_4^{(2)}\beta, \tag{77b}$$

$$\big[b_3^{(2)}, \hat{H}_{\mathrm{D}}\big] = 2i\hbar c b_1^{(2)}\beta + 2mc^2 b_3^{(2)}\beta + i\hbar b_4^{(2)}\left(\frac{1}{r}\frac{\partial V}{\partial r}\right), \tag{77c}$$

$$\big[b_4^{(2)}, \hat{H}_{\mathrm{D}}\big] = 2mc^2 b_4^{(2)}\beta, \tag{77d}$$

$$2cb_3^{(2)}\beta = -i\hbar b_2^{(2)}\left(\frac{1}{r}\frac{\partial V}{\partial r}\right), \tag{77e}$$

$$b_3^{(2)}\beta\delta_{ij} = 0, \tag{77f}$$

which are however consistently solvable if and only if Eq. (74) is fulfilled. Thus, a second-rank Noether charge for a spin-$\frac{1}{2}$ particle is only found for a constant potential $V(r) = V_0$, $V_0 \in \mathbb{R}$ and then given by

$$\hat{\mathcal{R}}_{ij} = \beta\hat{p}_i\hat{p}_j + \tfrac{1}{2}(\hat{\boldsymbol{\alpha}}\cdot\hat{\boldsymbol{p}})(\hat{p}_i\gamma_j + \gamma_i\hat{p}_j) = \beta[\hat{p}_i\hat{p}_j - \tfrac{1}{2}(\hat{\boldsymbol{\alpha}}\cdot\hat{\boldsymbol{p}})(\hat{p}_i\hat{\alpha}_j + \hat{\alpha}_i\hat{p}_j)]. \tag{78}$$

5. Arbitrary rank $l > 2$

The previously described ansatz can be extended to any symmetric tensor of arbitrary rank $l > 2$,

$$\hat{\mathcal{R}}_{ijk\cdots r} = b_1^{(l)}\hat{x}_i\hat{x}_j\hat{x}_k\cdots\hat{x}_r + b_2^{(l)}\hat{p}_i\hat{p}_j\hat{p}_k\cdots\hat{p}_r + \frac{1}{l}b_3^{(l)} \times \sum_\pi \gamma_i\hat{x}_j\hat{x}_k\cdots\hat{x}_r + \frac{1}{l}b_4^{(l)}\sum_\pi \gamma_i\hat{p}_j\hat{p}_k\cdots\hat{p}_r, \tag{79}$$

where every index runs from 1 to 3 and the sums go over all possible permutations π respecting the order of indices. This means, for a combination of, e.g. three different operator components,

$$\sum_\pi \hat{a}_i\hat{b}_j\hat{c}_k = \hat{a}_i\hat{b}_j\hat{c}_k + \hat{a}_i\hat{c}_j\hat{b}_k + \hat{b}_i\hat{a}_j\hat{c}_k + \hat{b}_i\hat{c}_j\hat{a}_k + \hat{c}_i\hat{a}_j\hat{b}_k + \hat{c}_i\hat{b}_j\hat{a}_k. \tag{80}$$

Insertion of this tensor operator into Eq. (59) reveals the necessary condition (74) again among others (see the Supplemental Material [31]). This sets $b_1^{(l)} = b_3^{(l)} = 0$ and leads to conservation of the quantity

$$\hat{\mathcal{R}}_{ijk\cdots r} = \beta\hat{p}_i\hat{p}_j\hat{p}_k\cdots\hat{p}_r + \frac{1}{l}(\hat{\boldsymbol{\alpha}}\cdot\hat{\boldsymbol{p}})\sum_\pi\gamma_i\hat{p}_j\hat{p}_k\cdots\hat{p}_r = \beta\left[\hat{p}_i\hat{p}_j\hat{p}_k\cdots\hat{p}_r - \frac{1}{l}(\hat{\boldsymbol{\alpha}}\cdot\hat{\boldsymbol{p}})\sum_\pi\hat{\alpha}_i\hat{p}_j\hat{p}_k\cdots\hat{p}_r\right]. \tag{81}$$

Thus, the general symmetrized (dynamical) constant of motion of rank $l \geqslant 2$ incorporates both momentum and generalized helicity conservation in the case of a constant potential $V(r) = V_0$, $V_0 \in \mathbb{R}$. With this procedure, it turns out that the mathematical structure of the Dirac Hamiltonian (17) only allows dynamical Noether charges in the presence of a Coulomb or constant potential. For the case of a Coulomb potential, this is the Johnson-Lippmann-Biedenharn operator $\hat{\mathcal{R}}$ [see Eq. (68)], whereas the constant potential reveals generalized conservation of momentum and helicity.

IV. CONCLUSION

In this paper, the appearance of dynamical Noether charges was related to the specific functional form of central potentials in both nonrelativistic and relativistic quantum mechanics. Moreover, a general derivation scheme was depicted that starts from a very general ansatz in phase space within the Hamiltonian formalism. The condition of a vanishing commutator between this constructed constant of motion and the Hamiltonian leads to a differential equation, which has to be fulfilled by the potential in order to generate a dynamical symmetry. In nonrelativistic quantum mechanics, these differential equations only allow for a Coulomb-like [$V(r) \propto 1/r$], harmonic [$V(r) \propto r^2$], or constant [$V(r) = V_0 \in \mathbb{R}$] potential in order to find dynamical Noether charges of rank $l = 1, 2$ or $l > 2$, respectively. The nontrivial cases are known as the Runge-Lenz-Laplace vector ($l = 1$) and the symmetric Demkov-Fradkin tensor ($l = 2$). Together with the orbital angular momentum components, both mentioned constants of motion are generators spanning a higher-dimensional dynamical Lie algebra that contains so(3) as a subalgebra. For constant potentials, the correspondingly conserved higher-rank tensors merely represent a generalization of momentum conservation.

Generalizing to relativistic quantum mechanics based on the Dirac equation, only a Coulomb-like and a constant potential give rise to additional dynamical Noether charges of rank $l = 0$ or $l > 0$, respectively. For the Coulomb potential, the corresponding constant of motion can be shown to be the well-known Johnson-Lippmann-Biedenharn pseudoscalar that reduces to a spin projection of the Runge-Lenz-Laplace vector in the nonrelativistic limit. Like in the nonrelativistic limit, the higher-rank tensor operators represent a generalization of momentum conservation, but also allow for helicity conservation. These Noether charges are however only given for constant potentials.

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