Eddy magnetization from the chiral Barnett effect

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We discuss the spin, the angular momentum, and the magnetic moment of rotating chiral fermions using a kinetic theory. We find that, in addition to the chiral vortical contribution along the rotation axis, finite circular spin polarization is induced by the spin-momentum correlation of chiral fermions, which is canceled by a change in the orbital angular momentum. We point out that the eddy magnetic moment is nonvanishing due to the g factors, exhibiting the chiral Barnett effect.

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I. INTRODUCTION

The Barnett effect refers to the magnetization induced by mechanical rotation of a charge neutral object [1]. The Einstein–de Haas effect is an inverse phenomenon [2], that is, a finite rotation attributed to a change in the magnetization. We can regard these two closely related effects as realization of transmutation between the spin $\boldsymbol{S}$ and the orbital angular momentum $\boldsymbol{L}$ via the LS coupling. Because of the conservation law of the total angular momentum $\boldsymbol{J}$, a change in the magnetization or $\boldsymbol{S}$ must be compensated by a change in $\boldsymbol{L}$. For a historical summary of the theory and the experiments, Ref. [3] is one of the most comprehensive reviews, in which the gyromagnetic effects, including not only the above-mentioned two effects but also Maxwell's experiment and the gyromagnetic magnetization by rotating magnetic fields are explained from a general point of view.

Before relativistic generalization of the Barnett effect, which is the central subject studied in the current work, it would be useful to review key equations briefly for the conventional Barnett effect. A finite rotation with the angular velocity vector $\boldsymbol{\omega}$ would shift the one-particle energy by $\boldsymbol{\omega}\cdot\boldsymbol{J}$. This energy shift should be equated to the magnetic energy of $\boldsymbol{\mu}\cdot\boldsymbol{H}_{\rm eff}$, with the magnetic moment $\boldsymbol{\mu}$ and the effective magnetic field $\boldsymbol{H}_{\rm eff}$ corresponding to the magnetization. We note that the vacuum permeability is $\mu_0 = 1$ in our convention of the natural unit. With the magnetic susceptibility χ_B, the magnetization is given as $\boldsymbol{M} = \chi_B \boldsymbol{H}_{\rm eff}$ and the magnetic moment is $\boldsymbol{\mu} = \gamma \boldsymbol{J}$, where γ denotes the gyromagnetic ratio. Combining these relations to eliminate $\boldsymbol{H}_{\rm eff}$ and $\boldsymbol{\mu}$, we finally get the well-known formula, i.e., $\boldsymbol{M} = (\chi_B/\gamma)\boldsymbol{\omega}$.

The Barnett and Einstein–de Haas effects have attracted attention in general physics fields including condensed-matter physics for years. Theoretical studies are found, for example, in the rotational states of nanostructured magnetic systems [4–7]. It has also been pointed out in Refs. [7–9] that both the Barnett and the Einstein–de Haas effects are governed by the same gyromagnetic tensor components which satisfy the Onsager reciprocal relations, i.e., the gyromagnetic relation. In experiments, therefore, confirming one of them could be sufficient instead of measuring both effects for the same physical system. Here, we lay out several examples of experimental realization: The Einstein–de Haas effect has been observed in thin film deposited on a microcantilever [10]. There are several proposals for experiments in an atomic gas with Bose-Einstein condensate [11,12] and in a ferromagnetic insulator with phonons [13]. In contrast to the Einstein–de Haas effect, the Barnett effect has been measured in systems such as magnetic nanostructures [14], nuclear magnetic resonance [15], paramagnetic materials [16], etc. Furthermore, circular spin-current generation has been theoretically predicted as a result of spin-orbit interaction and the mechanical rotation [17–19], which has an analogous feature to what we are going to discuss in the present work. Interestingly, the theoretical predictions have been experimentally confirmed recently, see Refs. [20–22]. For more details, interested readers can consult a recent textbook [23].

Possible extension of the Barnett effect to systems with massless or chiral fermions is an intriguing problem, and theoretical investigations are demanded by recent experimental developments. In high-energy experiments with almost massless quarks involved, the most pertinent effort lies in the measurement of Λ and $\bar{\Lambda}$ global polarization conducted by the STAR Collaboration of the Relativistic Heavy-Ion Collider (RHIC) [24,25]. In noncentral collisions, two nuclei collide with a huge orbital angular momentum [26–31], creating the "most vortical fluid" and inducing a nonzero value of Λ and $\bar{\Lambda}$ global polarization.

Many works have been published to formulate the transfer from the orbital angular momentum to the spin carried by hot and dense hadronic matter. Some examples include the microscopic spin-orbital coupling model [26–28], the statistical hydrodynamical model [32–38], and the quantum kinetic theory with Wigner functions [39–41]. Moreover, it was proposed in Refs. [26,27] that the local polarization of Λ and vector mesons could also be experimentally sensitive to the net orbital angular momentum. For more relevant references and discussions, see the review of Ref. [42]. Although there are theoretical simulations for the observed polarization of hadrons, to deepen our understanding from the fundamental level, it would be instructive to analyze an idealized environment of noninteracting and rotating chiral fermions, as done in this work.

In relativistic systems the chiral anomaly plays an important role for inducing an imbalance with respect to chirality [43,44]. A clear manifestation of the chiral anomaly can be found in a system with electromagnetic fields and/or finite vorticity: one can easily understand the chiral anomaly in such a system in terms of helicity conservation, and the helicity of fermions can be interchanged with the magnetic helicity and/or the fluid helicity. Then helicity transmutation results in the chiral magnetic effect (CME) [45–47], the chiral vortical effect (CVE) [48–50], and related topological effects (see Refs. [51–53] for reviews). These topological effects induce nondissipative currents and survive in the hydrodynamic limit. Thus, the quantum anomaly could be macroscopically manifested. Interestingly, it has been argued within the framework of hydrodynamics [54] that the transmutation between the helicity of fermions and the fluid helicity, which is related to the CVE, can be regarded as an analog of the Barnett/Einstein–de Haas effects.

In this paper we will first discuss the properties of rotating chiral fermions using the kinetic theory and then address a possible connection to the hydrodynamic counterpart. We take this strategy since in this way a physical interpretation of the spin and the orbital parts would be transparent combined with field-theoretical considerations. The Boltzmann equation assumes a quasiparticle approximation, which is a semiclassical treatment of dynamics. Recently it has been established how to implement the spin degrees of freedom in the Boltzmann equation. Such an augmented Boltzmann equation is commonly called the chiral kinetic theory (CKT) in the high-energy physics community. There are a number of literature sources using different ways to derive the CKT—e.g., effective field theories [55–58], path integrals [59–61], and Wigner functions [62–68].

Once we have the CKT, it is straightforward to derive the macroscopic currents and the energy-momentum tensor. By integrating physical observables weighted with the distribution function over the momentum, one will obtain the expectation values of the observables such as the vector and axial vector currents with quantum corrections, which are identified as the CME and the CVE [39,60–63]. Remarkably, the correct transport coefficient of the CVE contains two origins. The first one comes from a shift in the particle energy dispersion modified by the rotation. The nontrivial Lorentz transformations for massless particles, called the side-jump effects [61–64], make the second contribution to the CVE coefficient. Since the description of the CVE in terms of the CKT has been fully established, it is natural for us to employ the CKT for the Barnett effect that is related to the CVE.

The present paper is organized as follows. In Sec. II we will give a brief review on the total angular momentum, the orbital angular momentum, and the spin of chiral fermions. In Sec. III we will write down the expressions of the orbital angular momentum and the spin operators in the kinetic theory language. In Sec. IV we will consider the CKT in a globally rotating chiral system and will compute the orbital angular momentum and the spin. Next, we will relate our results to the Einstein–de Haas and the Barnett effects and will discuss their chiral extensions in Sec. V. We will also make a comment on the anomalous hydrodynamics in Sec. VI. Finally we summarize our results in Sec. VII. Throughout this paper we use the natural unit for the speed of light, $c = 1$, while we retain $\hbar$.

II. ANGULAR MOMENTUM DECOMPOSITION

The angular momentum is a conserved quantity, but its decomposition into the spin and the orbital components is not unique in relativistic theories. In this section, we clarify our convention and explain its physical interpretation. Let us start with a free Dirac field (where the generalization to include interaction is not difficult by $\partial_\mu \to D_\mu$), whose Lagrangian density is

$$\mathcal{L} = \bar{\psi}(i\hbar\gamma^\mu\partial_\mu - m)\psi. \tag{1}$$

This Lagrangian is invariant under an infinitesimal rotation,

$$x^\mu \;\to\; x'^\mu = x^\mu + \epsilon^\mu_{\ \nu}x^\nu, \tag{2}$$

where $\epsilon^\mu_{\ \nu}$ is an antisymmetric tensor whose magnitude is infinitesimally small. The angular momentum tensor is the Noether current associated with rotation symmetry. We note that, under Eq. (2), the spinor transforms as

$$\psi(x) \to \psi'(x') = \psi(x) - \frac{i}{2}\epsilon_{\mu\nu}\Sigma^{\mu\nu}\psi(x), \tag{3}$$

where $\Sigma^{\mu\nu} = (i/4)[\gamma^\mu, \gamma^\nu]$. Correspondingly, the Noether current with current index λ gets two contributions: the coordinate part from Eq. (2) and the spinor part from Eq. (3) as

$$J^{\lambda\mu\nu} = L^{\lambda\mu\nu} + S^{\lambda\mu\nu}. \tag{4}$$

We can express the first term $L^{\lambda\,\mu\nu}$ using the canonical energy-momentum tensor,

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\frac{\partial\psi}{\partial x_\nu} = \bar{\psi}\, i\hbar\gamma^\mu\partial^\nu\psi, \tag{5}$$

as the following form:

$$\begin{aligned} L^{\lambda\mu\nu} &= x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu} \\ &= \bar{\psi}\, i\hbar(\gamma^\lambda x^\mu\partial^\nu - \gamma^\lambda x^\nu\partial^\mu)\psi. \end{aligned} \tag{6}$$

For the second term of Eq. (4), the explicit form reads,

$$S^{\lambda\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\lambda\psi)}\left[-\frac{i}{2}\Sigma^{\mu\nu}\psi(x)\right] = \frac{1}{4}\bar{\psi}\, i\hbar\gamma^\lambda[\gamma^\mu, \gamma^\nu]\psi. \tag{7}$$

One could also obtain another form of the spin tensor from the symmetrized Dirac Lagrangian, $\mathcal{L} = \frac{i\hbar}{2}(\bar{\psi}\gamma^\mu\overrightarrow{\partial_\mu}\psi - \bar{\psi}\gamma^\mu\overleftarrow{\partial_\mu}\psi)$, that is, $S^{\lambda\mu\nu} = \frac{1}{8}\bar{\psi}\, i\hbar\{\gamma^\lambda, [\gamma^\mu, \gamma^\nu]\}\psi$, but we are interested in $S^{0\mu\nu}$ components for later discussions, and the difference from Eq. (7) is vanishing and the choice of the Lagrangian is irrelevant for physical quantities as it should.

Now, the total-angular-momentum tensor is

$$J^{\lambda\mu\nu} = \bar{\psi}\, i\hbar\big(\gamma^\lambda x^\mu\partial^\nu - \gamma^\lambda x^\nu\partial^\mu + \tfrac{1}{4}\gamma^\lambda[\gamma^\mu, \gamma^\nu]\big)\psi, \tag{8}$$

whose $\lambda = 0$ component is the conserved charge density, i.e., the conserved angular momentum. Using the Dirac equation, we can easily check that

$$\partial_\lambda L^{\lambda\mu\nu} = -\partial_\lambda S^{\lambda\mu\nu} = \bar{\psi}\, i\hbar(\gamma^\mu\partial^\nu - \gamma^\nu\partial^\mu)\psi, \tag{9}$$

from which $\partial_\lambda J^{\lambda\mu\nu} = 0$ immediately follows. If the surface term is irrelevant, we can then arrive at the angular momentum conservation law:

$$\frac{d}{dt}\int d^3x\, J^{0\mu\nu} = 0. \tag{10}$$

One might have thought that the above identification of $L^{0\mu\nu}$ and $S^{0\mu\nu}$ as the orbital and the spin components would be the most natural. Indeed, in the nonrelativistic limit, $L^{0\mu\nu}$ and $S^{0\mu\nu}$ amount to the orbital and the spin components, respectively. Nevertheless, in relativistic theories, no unique decomposition is guaranteed.

Actually, the energy-momentum tensor always has ambiguity by an arbitrary antisymmetric tensor $\Sigma^{\mu\nu\lambda}$ as

$$\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\lambda \Sigma^{\mu\nu\lambda}. \tag{11}$$

It is obvious that $\Theta^{\mu\nu}$ also satisfies the conservation law, so it is equally qualified as the energy-momentum tensor. In particular, with an appropriate choice of $\Sigma^{\mu\nu\lambda}$, one can make $\Theta^{\mu\nu}$ symmetric as

$$\Theta^{\mu\nu} = \tfrac{1}{2}\bar{\psi}\, i\hbar(\gamma^\mu\partial^\nu + \gamma^\nu\partial^\mu)\psi. \tag{12}$$

The corresponding "orbital" component of the angular momentum, deduced from Eq. (6) with $T^{\mu\nu}$ replaced by $\Theta^{\mu\nu}$, is

$$\tilde{L}^{\lambda\mu\nu} = \tfrac{1}{2}L^{\lambda\mu\nu} + \tfrac{1}{2}\bar{\psi}\, i\hbar[(x^\mu\gamma^\nu - x^\nu\gamma^\mu)\partial^\lambda]\psi, \tag{13}$$

and the "spin" component is inferred from $\tilde{S}^{\lambda\mu\nu} = J^{\lambda\mu\nu} - \tilde{L}^{\lambda\mu\nu}$. Interestingly, using the Dirac equation again, we can prove

$$\partial_\lambda \tilde{L}^{\lambda\mu\nu} = \partial_\lambda \tilde{S}^{\lambda\mu\nu} = 0. \tag{14}$$

In contrast to Eq. (9), the above relation (14) indicates that, in this construction, the orbital and the spin components of the angular momentum are separately conserved (see Ref. [69] for a related discussion on electron vortices), while the canonical ones, $L^{\lambda\mu\nu}$ and $S^{\lambda\mu\nu}$, are not. However, this fact does not mean any superiority of $\tilde{L}^{\lambda\mu\nu}$ and $\tilde{S}^{\lambda\mu\nu}$, because neither of them is a true symmetry generator alone. The situation is quite similar to the decomposition of the optical spin and the optical orbital angular momentum. For free electromagnetic fields one can generally define individually conserved spin and the orbital-angular-momentum operator, but due to the transversality constraint, only their combination, i.e., the total angular momentum, is the physically meaningful quantity [70,71].

Throughout this work we adopt the canonically defined spin $S^{\lambda\mu\nu}$ and orbital angular momentum $L^{\lambda\mu\nu}$, because these are the definitions with most natural connection to their nonrelativistic counterparts. Another advantage is that S^{0ij}, or $\boldsymbol{S}$, turns out to be nothing but the axial current,

$$S^{0ij} = \epsilon^{ijk}\frac{\hbar}{2}\bar{\psi}\gamma^k\gamma_5\psi = \epsilon^{ijk}\frac{j_5^k}{2}, \tag{15}$$

$$S^k \equiv \tfrac{1}{2}\epsilon^{ijk}S^{0ij}. \tag{16}$$

Thus it has an interpretation evidently related to the chiral anomaly. Similarly, we define the orbital-angular-momentum vector $\boldsymbol{L}$ as

$$L^k \equiv \tfrac{1}{2}\epsilon^{ijk}L^{0ij}. \tag{17}$$

Equation (15) also implies that, if the axial current is a measurable physical observable, so will $\boldsymbol{S}$ and then $\boldsymbol{L}$ be.

III. TRANSCRIPTION TO KINETIC THEORY

Since we will deal with our problem in terms of kinetic theory, we should seek corresponding expressions for $L^{\lambda\mu\nu}$ and $S^{\lambda\mu\nu}$ involving the distribution function, $f(\boldsymbol{p}, \boldsymbol{x}, t)$. We note that the spin and the orbital angular momentum are the properties of matter in equilibrium unrelated to the collisions, once the corresponding operators for $L^{\lambda\mu\nu}$ and $S^{\lambda\mu\nu}$ are identified. Although we discuss the kinetic theory transcription, we are not studying the off-equilibrium dynamics, but we are considering the operators in terms of the kinetic theory in this section and in terms of hydrodynamics in Sec. VI.

To this end, we consider the single-particle angular momentum tensor as done in Refs. [58,72], i.e.,

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu + S^{\mu\nu}, \tag{18}$$

where $p^\mu = (p = |\boldsymbol{p}|,\, \boldsymbol{p})$. Comparing Eq. (18) with Eq. (8), given the correspondence of $i\hbar\partial^\mu \to p^\mu$, we can identify the first two terms as the orbital part $L^{0\mu\nu}$ of our choice. Then, the last term represents the spin part, whose specific form, according to Refs. [58,63,64], is fixed up to a frame vector n_β. We choose the laboratory frame with $n_\beta = (1, \boldsymbol{0})$, which simplifies the concrete expression of $S^{\mu\nu}$, leading to the following operator decomposition:

$$\boldsymbol{L} = \boldsymbol{x}\times\boldsymbol{p}, \quad \boldsymbol{S} = \hbar\lambda\left(\hat{\boldsymbol{p}} - \hbar\lambda\frac{\hat{\boldsymbol{p}}}{p}\times\nabla\right). \tag{19}$$

Here λ is the helicity, i.e., $\lambda = \pm 1/2$, and $\hat{\boldsymbol{p}} = \boldsymbol{p}/|\boldsymbol{p}|$ is the unit momentum vector.

We emphasize the importance of the second term in $\boldsymbol{S}$ to make the computation consistent with the CVE and the relation (15). This additional term originates from a gyromagnetic effect and is nothing but a familiar Rashba spin-orbit coupling. Another way to think of the field-theoretical origin of this term is the current expectation value as a derivative with respect to the vector potential. Then, as discussed in Refs. [61,63,64], the current reads

$$\boldsymbol{j} = \int_p \left(\hat{\boldsymbol{p}} - \hbar\lambda\frac{\hat{\boldsymbol{p}}}{p}\times\nabla\right)f, \tag{20}$$

where the second term in the parentheses appears from a magnetic-dependent term, $-\lambda\hat{\boldsymbol{p}}\cdot\boldsymbol{B}/|\boldsymbol{p}|$, in the energy dispersion relation, which is eventually transcribed into the additional term in $\boldsymbol{S}$ as seen above. An interesting point worth mentioning is that ∇ is the spatial derivative and a finite rotation would indeed induce spatial inhomogeneity.

We note that one can understand Eqs. (19) and (20) easily from the well-known Gordon decomposition on the vector current with Dirac spinors at momentum $\boldsymbol{p}$ [73], i.e.,

$$\hat{\boldsymbol{j}} = \hbar\bar{\psi}\boldsymbol{\gamma}\psi = \frac{\hbar}{2ip}[\psi^\dagger\nabla\psi - (\nabla\psi^\dagger)\psi] + \frac{\hbar}{2p}\nabla\times(\psi^\dagger\boldsymbol{\Sigma}\psi), \tag{21}$$

where $\Sigma^k = \epsilon^{ijk}\Sigma^{ij}$. This is the mathematical background for Eq. (20). Because extra γ_5 is irrelevant for a system with either left- or right-handed particles only, the argument on the vector current can be straightforwardly translated to the axial current in Eq. (19).

It should be noted that $\boldsymbol{L}$ and $\boldsymbol{S}$ have the same physical unit, but $\hbar$ in $\boldsymbol{L}$ is hidden in the momentum $\boldsymbol{p}$, which looks like $O(\hbar^0)$ in a semiclassical treatment. Such $\hbar$ counting is consistent with our intuition that the spin is a quantum effect but the orbital angular momentum is a macroscopic observable, while the full consistent treatment would require the derivative expansion.

IV. ROTATING CHIRAL FERMIONS

In this work we study the effect of bulk rotation of chiral matter at constant angular velocity $\boldsymbol{\omega}$ rather than a fluid with local vorticity. We turn electromagnetic fields off for simplicity, and if necessary, the generalization including electromagnetic fields is straightforward.

For an equilibrium state in the absence of rotation, the distribution function f is homogeneous in coordinate space and isotropic in momentum space, which means that f should be a function of single-particle energy ε, i.e., $f = f(\varepsilon)$.[1] Let us consider what would change if we introduce $\omega \neq 0$ into the system. For this purpose we put ourselves into a comoving frame that rotates together with matter. We can thereby postulate that the local thermal equilibrium is reached after a sufficiently long time so that $f = f(\varepsilon_{\rm rot})$ with $\varepsilon_{\rm rot}$ defined in the comoving frame (which is implicitly assumed in the implementation of finite-temperature field theory in Ref. [48]). We can solve a free Weyl equation in the rotating frame to find $\varepsilon_{\rm rot}$ as

$$\varepsilon_{\rm rot} = p - \boldsymbol{\omega}\cdot(\boldsymbol{x}\times\boldsymbol{p} + \hbar\lambda\hat{\boldsymbol{p}}) \tag{22}$$

using the laboratory-frame (nonrotating) coordinates $\boldsymbol{x}$ and momenta $\boldsymbol{p}$. We note that the energy shift in Eq. (22) takes a standard cranking form, $-\boldsymbol{\omega}\cdot\boldsymbol{J}$. In terms of laboratory-frame variables $f(\varepsilon_{\rm rot})$ is neither homogeneous in coordinate space nor isotropic in momentum space due to finite rotation; thus the spin and the orbital angular momentum derived from $f(\varepsilon_{\rm rot})$ can be nonzero. We begin with calculating the spin expectation value under an assumption that ω is small. Up to the linear order of ω we get

$$\begin{aligned}\langle\boldsymbol{S}\rangle &= \int_{\boldsymbol{p}} \lambda\hbar\left(\hat{\boldsymbol{p}} - \lambda\hbar\frac{\hat{\boldsymbol{p}}}{p}\times\nabla\right) f(\varepsilon_{\rm rot})\\ &\approx -\hbar\lambda(\boldsymbol{\omega}\times\boldsymbol{x})\int_{\boldsymbol{p}}\frac{p}{3}f'(p) - \hbar^2\lambda^2\boldsymbol{\omega}\int_{\boldsymbol{p}} f'(p),\end{aligned} \tag{23}$$

where $f'(p) = \partial f(p)/\partial p$. It should be mentioned that our "expectation value" involves only the momentum integration, $\int_{\boldsymbol{p}} = \int d^3p/(2\pi\hbar)^3$, but not the coordinate integration, which is denoted later by $\int_V = \int d^3x$.

Here, we briefly mention the difference between setups in Refs. [58,61] and ours. If the rotation effects are introduced by a local vorticity vector as in Refs. [58,61], physical quantities can be homogeneous. However, to characterize the Einstein–de Haas effect, we implicitly assume a finite-size system for which the center of rotation is well defined. Then, the velocity of rotating particles depends on the distance from the center of rotation, and physical quantities, including the spin expectation value, can be dependent on x, as seen in the first term in Eq. (23).

We shall make a remark about our power counting of $\hbar$ order. In the last section we found the operators for the spin and the orbital angular momentum in a heuristic way. In principle, one could utilize the Wigner function to take account of quantum corrections systematically in the $\hbar$ expansion. Then, $\boldsymbol{S}$ and $\varepsilon_{\rm rot}$ may have $O(\hbar^3)$ and $O(\hbar^2)$ corrections, respectively, and they contribute to an $O(\hbar^3)$ correction to Eq. (23).

We next turn to the orbital angular momentum. In the same way we expand the distribution function with respect to $\boldsymbol{\omega}$ and obtain

$$\begin{aligned}\langle\boldsymbol{L}\rangle &\approx \int_{\boldsymbol{p}}(\boldsymbol{x}\times\boldsymbol{p})f'(p)(-\boldsymbol{\omega})\cdot(\boldsymbol{x}\times\boldsymbol{p}+\hbar\lambda\hat{\boldsymbol{p}})\\ &= -\boldsymbol{x}\times(\boldsymbol{\omega}\times\boldsymbol{x})\int_{\boldsymbol{p}}\frac{p^2}{3}f'(p) + \hbar\lambda(\boldsymbol{\omega}\times\boldsymbol{x})\int_{\boldsymbol{p}}\frac{p}{3}f'(p).\end{aligned} \tag{24}$$

Equations (23) and (24) are our central results in this paper. In the following sections we shall expound their physical interpretations.

V. APPLICATIONS—CHIRAL EINSTEIN–DE HAAS AND BARNETT EFFECTS

We utilize our results for $\langle\boldsymbol{L}\rangle$ and $\langle\boldsymbol{S}\rangle$ to discuss the relativistic extension of the Einstein–de Haas effect and the Barnett effect.

A. Chiral Einstein–de Haas effect

The physical meaning of Eq. (23) becomes transparent once we add up both left-handed and right-handed contributions. After an integration by parts, the first term in Eq. (23), which is denoted by $\langle\boldsymbol{S}\rangle_\perp$ hereafter, takes the following form as

$$\begin{aligned}\langle\boldsymbol{S}\rangle_\perp &= -\hbar\sum_{R,L}\lambda(\boldsymbol{\omega}\times\boldsymbol{x})\int_{\boldsymbol{p}}\frac{p}{3}f'_\lambda(p)\\ &= \frac{\hbar}{2}(\boldsymbol{\omega}\times\boldsymbol{x})\int_{\boldsymbol{p}}[f_R(p)-f_L(p)] = \frac{\hbar}{2}(\boldsymbol{\omega}\times\boldsymbol{x})n_5,\end{aligned} \tag{25}$$

where f_R and f_L refer to the distribution functions of right-handed and left-handed particles, respectively, and thus $n_5 = n_R - n_L$ means the chirality density. Such a rotation-induced spin alignment is intuitively understood as follows. For massless fermions, the spin and the momentum directions are locked up. In this way, the angular momentum is related to the translational motion. Therefore, if we macroscopically move our chiral matter with the velocity $\boldsymbol{u} = \boldsymbol{\omega}\times\boldsymbol{x}$, the spin will be tilted along $\boldsymbol{u}$. In this sense $\langle\boldsymbol{S}\rangle_\perp$ is a unique result inherently for chiral fermions. Interestingly, this transverse eddy spin alignment requires a finite chiral imbalance. We present a

[1]According to some references [61,63,64] our assumption corresponds to the "global equilibrium" case because our distribution function is independent of n_β up to the $\hbar$ order. In the "local equilibrium" case the distribution function may depend on n_β, which is generally a function of spatial coordinates. For more discussions on polarization effects in the local equilibrium case, see Ref. [72].

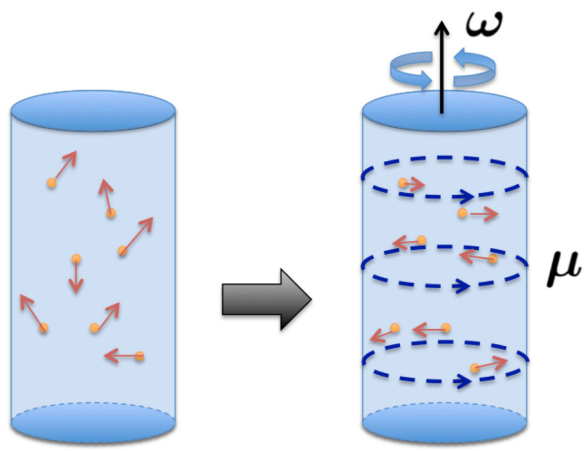

FIG. 1. A schematic illustration for an intuitive picture to understand the circular spin polarization and the associated eddy magnetization $\boldsymbol{\mu}$ in a rotating chiral system with the angular velocity vector $\boldsymbol{\omega}$. For simplicity we consider only the right-handed fermions in the illustration. The red arrows stand for the direction of particle momentum and spin.

schematic illustration in Fig. 1 to explain how $\langle \boldsymbol{S} \rangle_\perp$ appears in a rotating chiral system.

After similar algebra, we rewrite the orbital angular momentum Eq. (24) as

$$\begin{aligned} \langle \boldsymbol{L} \rangle &= \boldsymbol{x} \times (\boldsymbol{\omega} \times \boldsymbol{x}) \frac{4}{3} \int_{\boldsymbol{p}} p[f_R(p) + f_L(p)] - \langle \boldsymbol{S} \rangle_\perp \\ &= \langle \boldsymbol{L} \rangle_{\text{mech}} - \langle \boldsymbol{S} \rangle_\perp . \end{aligned} \tag{26}$$

Here, $\langle \boldsymbol{L} \rangle_{\text{mech}}$ represents the first term involving $\boldsymbol{\omega} \times \boldsymbol{x}$ in the above expression. We shall illuminate the physical interpretation of $\langle \boldsymbol{L} \rangle_{\text{mech}}$ in what follows. For concreteness we will consider a cylindrically symmetric system which rotates rigidly around the z axis, i.e., $\boldsymbol{\omega} = \omega \hat{z}$. Then in such a setup the volume integration of $\langle \boldsymbol{L} \rangle_{\text{mech}}$ yields

$$\int_V \langle \boldsymbol{L} \rangle_{\text{mech}} = \omega \hat{z} \int_V r^2 \frac{4}{3} \int_{\boldsymbol{p}} p[f_R(p) + f_L(p)]. \tag{27}$$

Since p is the energy for chiral fermions, the p integration gives the energy density or the mass distribution, together with which the volume integration leads to the moment of inertia. To see this clearly, let us assume that the distribution function features Fermi degeneracy to a chemical potential μ, and then the energy density $\mathcal{E}$ is calculated as $\mathcal{E} = \frac{3}{4}\mu n$, where n is the number density. Consequently, $\frac{4}{3}\int_{\boldsymbol{p}} p[f_R(p) + f_L(p)]$ reduces to a relativistic counterpart of the mass density, $\mu_R n_R + \mu_L n_L$. From this argument it is clear that $\langle \boldsymbol{L} \rangle_{\text{mech}}$ corresponds to the mechanically induced orbital angular momentum, which is naturally of $O(\hbar^0)$.

Next, we delve into the second term in $\langle \boldsymbol{L} \rangle$ given by $-\langle \boldsymbol{S} \rangle_\perp$. This term has an intriguing interpretation as the "chiral Einstein–de Haas effect." Let us consider the following thinking experiment. We rotate the fermionic system from the initial condition, $\langle \boldsymbol{L} \rangle = \langle \boldsymbol{S} \rangle = 0$. Apparently, the total angular momentum carried by rotating chiral matter should be $\langle \boldsymbol{J} \rangle = \langle \boldsymbol{L} \rangle_{\text{mech}}$. However, as mentioned above, due to the spin and momentum lockup, the transverse motion results in $\langle \boldsymbol{S} \rangle_\perp \neq 0$. This nonzero $\langle \boldsymbol{S} \rangle_\perp$ must be canceled by a change in the orbital part so that the total-angular-momentum conservation can be satisfied. In this way, a shift by $-\langle \boldsymbol{S} \rangle_\perp$ should arise in $\langle \boldsymbol{L} \rangle$. Such a physical mechanism is comparable to the Einstein–de Haas effect. In the nonrelativistic case the spin is controlled by an external magnetic field, but it can be changed by the momentum direction for chiral fermions, which induces an orbital rotation.

We make two comments on the second term in Eq. (23). The first one is that this term corresponding to the CVE can be also exactly canceled in a finite-size system by surface states not to violate the angular momentum conservation [74]. The second comment is that, if we consider the zero n_5 limit, the second term in Eq. (23) would dominate and lead to the local spin polarization proposed in Ref. [39].

B. Chiral Barnett effect

Along similar lines, we can apply our formula to address the Barnett effect for chiral fermions. That is, a finite magnetization is generated by rotation [1], which can be quantified with our results. For this purpose we need the gyromagnetic ratio to convert the angular momentum into the magnetic moment. For nonrelativistic fermions, the gyromagnetic ratio is derived from the Dirac equation as

$$\boldsymbol{\mu} = \boldsymbol{\mu}_L + \boldsymbol{\mu}_S = g_L \frac{q_e}{2m} \boldsymbol{L} + g_S \frac{q_e}{2m} \boldsymbol{S}, \tag{28}$$

where q_e and m are, respectively, the electric charge and the mass of the considered particle. For noninteracting Dirac fermions the g factors are $g_L = 1$ and $g_S = 2$. Since $g_L \neq g_S$, the right-hand side of Eq. (28) is not parallel to $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$. Once one takes an expectation value with the $\boldsymbol{J}^2$ and J_z eigenstates, however, one can show that the right-hand side is projected onto the $\boldsymbol{J}$ direction, which is guaranteed by the Wigner-Eckart theorem, and the effective g factor becomes the Landé g factor.

For chiral fermions Eq. (28) should be modified. In the chiral limit Eq. (28) turns into (see Refs. [73,75])[2]

$$\boldsymbol{\mu} = \boldsymbol{\mu}_L + \boldsymbol{\mu}_S = g_L \frac{q_e}{2p} \boldsymbol{L} + g_S \frac{q_e}{2p} \boldsymbol{S}. \tag{29}$$

The g factors remain the same, and from now on we plug $g_L = 1$ and $g_S = 2$ into $\boldsymbol{\mu}_L$ and $\boldsymbol{\mu}_S$. We note that Eq. (29) is a local relation, and so we compute the expectation value as we did in the previous sections. The results up to $\hbar$ order are

$$\begin{aligned} \langle \boldsymbol{\mu}_L \rangle = &-\frac{q_e}{6} \boldsymbol{x} \times (\boldsymbol{\omega} \times \boldsymbol{x}) \int_{\boldsymbol{p}} p f'(p) \\ &+ \hbar\lambda \frac{q_e}{6} (\boldsymbol{\omega} \times \boldsymbol{x}) \int_{\boldsymbol{p}} f'(p), \end{aligned} \tag{30}$$

$$\langle \boldsymbol{\mu}_S \rangle = -\hbar\lambda \frac{q_e}{3} (\boldsymbol{\omega} \times \boldsymbol{x}) \int_{\boldsymbol{p}} f'(p). \tag{31}$$

[2]Infrared singularity in Eq. (29) is regularized by the Debye screening in many-body systems. In other words, the momentum convoluted with a distribution function has an infrared cutoff by gT, where g is the coupling constant of the theory and T is the temperature.

We can immediately identify the first term of $\langle \boldsymbol{\mu}_L \rangle$ as the mechanical contribution. The integration by parts makes it more visible as

$$\langle \boldsymbol{\mu}_L \rangle_{\text{mech}} = \tfrac{1}{2} \boldsymbol{x} \times (\boldsymbol{\omega} \times \boldsymbol{x})\, n_e, \tag{32}$$

where n_e represents the electric charge density. Given that $\boldsymbol{\omega} \times \boldsymbol{x}$ is the velocity vector associated with the rotating motion, the above expression is exactly the one known as the magnetic dipole moment from the Ampére loop in classical electromagnetism.

The second term of $\langle \boldsymbol{\mu}_L \rangle$ is at the same order as $\langle \boldsymbol{\mu}_S \rangle$, but they do not cancel out. The total magnetization reads

$$\langle \boldsymbol{\mu} \rangle = \langle \boldsymbol{\mu}_L \rangle + \langle \boldsymbol{\mu}_S \rangle = \langle \boldsymbol{\mu}_L \rangle_{\text{mech}} - \hbar\lambda \frac{q_e}{6} (\boldsymbol{\omega} \times \boldsymbol{x}) \int_{\boldsymbol{p}} f'(p). \tag{33}$$

This result exhibits the chiral Barnett effect. What is nontrivial in the relativistic case is the second term. It is proportional to $\boldsymbol{\omega} \times \boldsymbol{x}$, and thus has the circular orientation around the rotation axis, just like previously discussed $\langle \boldsymbol{S} \rangle_\perp$ as shown in Fig. 1. Since the magnetic moment is a source for the magnetic field, we can expect a generation of eddy magnetic field in rotating chiral media. Further exploration on this point and applications to astrophysical objects will be reported elsewhere [76].

VI. COMMENTS ON HYDRODYNAMICAL FORMULATION

In this section we briefly address the problem of calculating the orbital angular momentum in anomalous hydrodynamics [50]. In the framework of anomalous hydrodynamics, the energy-momentum tensor reads (see Refs. [39,77])

$$T^{\mu\nu}_{\text{hydro}} = (E + P) u^\mu u^\nu - P\, g^{\mu\nu} + \hbar\, n_5 (u^\mu \omega^\nu + u^\nu \omega^\mu), \tag{34}$$

where E and P are the energy density and the pressure, respectively, and $u^\mu = \gamma(1, \boldsymbol{u})$ denotes the fluid velocity. For simplicity we assume the small velocity limit, i.e., $|\boldsymbol{u}| \ll 1$ and $u^\mu \approx (1, \boldsymbol{u})$. In such a limit the vorticity is $\omega^\nu = \frac{1}{2}\epsilon^{\nu\alpha\beta\gamma} u_\alpha \partial_\beta u_\gamma \approx (0,\, \boldsymbol{\nabla} \times \boldsymbol{u}) + O(|\boldsymbol{u}|^2)$ by definition. Using this energy-momentum tensor to define the hydro-orbital angular momentum, we find

$$\begin{aligned} L^{ij}_{\text{hydro}} &= x^i T^{0j}_{\text{hydro}} - x^j T^{0i}_{\text{hydro}} \\ &= x^i [(E + P) u^j + \hbar\, n_5 \omega^j] - (i \leftrightarrow j). \end{aligned} \tag{35}$$

For a mechanically rotating fluid, we specify $u^\mu = (1, \boldsymbol{\omega} \times \boldsymbol{x})$ and $\omega^\mu = (0, \boldsymbol{\omega})$, which leads to

$$\boldsymbol{L}_{\text{hydro}} = (E + P)(\boldsymbol{x} \times \boldsymbol{u}) - \hbar\, n_5 (\boldsymbol{\omega} \times \boldsymbol{x}). \tag{36}$$

To see the connection between Eq. (36) and our results in a kinetic picture, we remember that massless noninteracting systems have the equation of state as $P = E/3$, and in a kinetic framework E can be expressed as

$$E = \int_{\boldsymbol{p}} \frac{(u \cdot p)^2}{p^0} (f_R + f_L) \approx \int_{\boldsymbol{p}} p\, (f_R + f_L). \tag{37}$$

Then, eventually, the hydroangular momentum takes the form of

$$\boldsymbol{L}_{\text{hydro}} = \boldsymbol{x} \times (\boldsymbol{\omega} \times \boldsymbol{x})\, \frac{4}{3} \int_{\boldsymbol{p}} p (f_R + f_L) - 2 \langle \boldsymbol{S} \rangle_\perp. \tag{38}$$

Now we can make a direct comparison between Eq. (38) and our results in Eq. (26). We find that the first term corresponding to $\langle \boldsymbol{\mu}_L \rangle_{\text{mech}}$ exactly agrees, but the coefficient for $\langle \boldsymbol{S} \rangle_\perp$ is different.

This discrepancy originates from different definitions of the energy-momentum tensor. The energy-momentum tensor operator $T^{\mu\nu}$ obtained from the Noether theorem in Eq. (5) is not symmetric and does not correspond to the hydrodynamic energy-momentum tensor in Eq. (34). In fact, it is the symmetrized energy-momentum tensor operator $\Theta^{\mu\nu}$ in Eq. (12) that corresponds to Eq. (34). Such a symmetrized definition is prevalently adopted in anomalous hydrodynamics (see Refs. [58,63,64] for examples). Accordingly, for the orbital angular momentum, $\boldsymbol{L}_{\text{hydro}}$ corresponds to $\tilde{L}^{0\mu\nu}$ in Eq. (13) rather than the canonical one $L^{0\mu\nu}$ in Eq. (6) used for our CKT computation.

Here we make a comment on the approximate spin conservation law. Under such circumstances as massless fermions and vanishing electromagnetic fields, the axial current j_5^μ is conserved and so does the spin. In hydrodynamics one can show that this conservation law of $\langle \boldsymbol{S} \rangle$ follows from the expansions with respect to $\hbar$ and ω. In terms of the fluid velocity u^μ and the vorticity ω^μ, the conservation law of j_5^μ is [39,50]

$$\partial_\mu j_5^\mu = \partial_\mu (n_5 u^\mu + \hbar\, \xi_5\, \omega^\mu) = 0, \tag{39}$$

where ξ_5 is the CVE coefficient as a function of the temperature, the chemical potentials, etc. For a slowly rotating fluid with $u^\mu \approx (1, \boldsymbol{\omega} \times \boldsymbol{x})$, Eq. (39) tells us that

$$\frac{d}{dt} n_5 = -\boldsymbol{\nabla} \cdot (n_5 \boldsymbol{\omega} \times \boldsymbol{x}) + \hbar\, \partial_\mu (\xi_5 \omega^\mu). \tag{40}$$

Then, for small ω, it is easy to verify that

$$\frac{d}{dt} \int_V \langle \boldsymbol{S} \rangle = \frac{\hbar}{2} \int_V \frac{d}{dt} (n_5 \boldsymbol{\omega} \times \boldsymbol{x}) + O(\hbar^2) = O(\hbar^2, \omega^2). \tag{41}$$

We thus conclude that, under the approximation to drop terms of $O(\omega^2)$ and/or $O(\hbar^2)$, both our $\langle \boldsymbol{L} \rangle$ in the canonical definition and $\langle \boldsymbol{L} \rangle_{\text{hydro}}$ are equally qualified as the orbital angular momentum. It should be noted that in the above argument we implicitly assumed that x is of the order of unity. This means that the system size must not be as large as $1/\hbar$ or $1/\omega$; otherwise, the surface state is not negligible [74].

VII. CONCLUSION

In this work we systematically discussed the spin, the angular momentum, and the magnetic momentum for rotating chiral fermions using the framework of the CKT. First, we gave a brief review of deriving the angular momentum tensors as Noether's currents. Although the decomposition into the orbital and the spin components is not unique, we adopted the canonical definition in which the spin is directly related to the axial current. Next, we considered a globally rotating chiral system. Combining the contributions from a shift in the particle energy dispersion and an extra spin-orbital coupling term in the CKT, we identified the expectation values of the spin [see $\langle \boldsymbol{S} \rangle$ in Eq. (23)] and the orbital angular momentum [see $\langle \boldsymbol{L} \rangle$ in Eq. (24)].

Based on these two expressions for $\langle \boldsymbol{S} \rangle$ and $\langle \boldsymbol{L} \rangle$, we developed a physical picture of the relativistic extension of the Einstein–de Haas effect. Up to $O(\hbar)$ terms, the circular spin alignment is induced by the mechanical rotation as illustrated in Fig. 1, which can be intuitively understood through the fact that the rotation is accompanied by the axial current. Then, a shift in $\langle \boldsymbol{L} \rangle$ is caused by $\langle \boldsymbol{S} \rangle$ to maintain the conservation law of the total angular momentum, which can be regarded as a relativistic counterpart of the Einstein–de Haas effect realized in a chiral medium.

Furthermore, we applied our results to address the Barnett effect for chiral fermions. We computed the magnetic moments, $\langle \boldsymbol{\mu}_L \rangle$ and $\langle \boldsymbol{\mu}_S \rangle$, proportional to the orbital angular momentum and the spin, respectively. The leading-order term in $\langle \boldsymbol{\mu}_L \rangle$ of $O(\hbar^0)$ is exactly the one from the magnetic dipole moment obtained in classical electromagnetism. The next order terms in $\langle \boldsymbol{\mu}_L \rangle$ and $\langle \boldsymbol{\mu}_S \rangle$ of $O(\hbar)$ will not cancel out, and this nonvanishing magnetic moment exhibits what we call the chiral Barnett effect.

Before closing our discussions in the end, we supplemented some discussions on the anomalous hydrodynamics. We pointed out that the symmetric energy-momentum tensor adopted in the anomalous hydrodynamics does not correspond to the canonical one derived from the Noether theorem. Using the hydrodynamical energy-momentum tensor, we could define another form of the orbital angular momentum $\boldsymbol{L}_{\rm hydro}$, which is approximately a conserved quantity of $O(\hbar)$.

There will be many possible extensions and applications of our work. As discussed in Sec. V B, the rotation may induce the eddy magnetic fields, which could explain the internal structures of the neutron star [78]. In astrophysics, generally, the magnetic field and the rotation are commonly found in macroscopic objects, and the chiral Barnett effect may play an intriguing role [76]. Another interesting direction lies in possible generalization to the nonequilibrium situation. In this study we assumed only a near-equilibrium distribution function to compute $\langle \boldsymbol{S} \rangle$ and $\langle \boldsymbol{L} \rangle$ in a steady state. The full real-time evolution of $\langle \boldsymbol{S} \rangle$ and $\langle \boldsymbol{L} \rangle$ starting with some initial condition would be a challenging future problem.

In the future approximations made in the present work should be relaxed. Our treatment of fermions is limited to the massless case only, and the inclusion of finite mass effects would be a crucial step toward phenomenological applications to relativistic heavy-ion collision experiments. In this work, we neglected surface terms, and we should amend this with finite-size effects taken into account. So far, our analysis is only up to $O(\hbar)$ apart from the CVE term. Thus, we have not included higher-order nontrivial effects, e.g., the local spin-polarization effect [39]. We are currently making progress to incorporate these effects dropped in the present work.

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[1] S. J. Barnett, Phys. Rev. **6**, 239 (1915).
[2] A. Einstein and W. J. de Haas, Verh. Dtsch. Phys. Ges **17**, 152 (1915).
[3] S. J. Barnett, Rev. Mod. Phys. **7**, 129 (1935).
[4] A. A. Kovalev, G. E. W. Bauer, and A. Brataas, Phys. Rev. B **75**, 014430 (2007).
[5] R. Jaafar, E. M. Chudnovsky, and D. A. Garanin, Phys. Rev. B **79**, 104410 (2009).
[6] E. M. Chudnovsky and D. A. Garanin, Phys. Rev. B **81**, 214423 (2010).
[7] G. E. W. Bauer, S. Bretzel, A. Brataas, and Y. Tserkovnyak, Phys. Rev. B **81**, 024427 (2010).
[8] G. E. Bauer, E. Saitoh, and B. J. Van Wees, Nat. Mater. **11**, 391 (2012).
[9] L. Landau, E. Lifshitz, and L. Pitaevskii, *Electrodynamics of Continuous Media, Course of Theoretical Physics* (Butterworth-Heinemann, Oxford, UK, 1995).
[10] T. M. Wallis, J. Moreland, and P. Kabos, Appl. Phys. Lett. **89**, 122502 (2006).
[11] Y. Kawaguchi, H. Saito, and M. Ueda, Phys. Rev. Lett. **96**, 080405 (2006).
[12] U. Ebling and M. Ueda, arXiv:1701.05446.
[13] L. Zhang and Q. Niu, Phys. Rev. Lett. **112**, 085503 (2014).
[14] S. Bretzel, G. E. Bauer, Y. Tserkovnyak, and A. Brataas, Appl. Phys. Lett. **95**, 122504 (2009).
[15] H. Chudo, M. Ono, K. Harii, M. Matsuo, J. Ieda, R. Haruki, S. Okayasu, S. Maekawa, H. Yasuoka, and E. Saitoh, Appl. Phys. Exp. **7**, 063004 (2014).
[16] M. Ono, H. Chudo, K. Harii, S. Okayasu, M. Matsuo, J. Ieda, R. Takahashi, S. Maekawa, and E. Saitoh, Phys. Rev. B **92**, 174424 (2015).
[17] M. Matsuo, J. Ieda, E. Saitoh, and S. Maekawa, Phys. Rev. Lett. **106**, 076601 (2011).
[18] M. Matsuo, J. Ieda, K. Harii, E. Saitoh, and S. Maekawa, Phys. Rev. B **87**, 180402 (2013).
[19] M. Matsuo, Y. Ohnuma, and S. Maekawa, Phys. Rev. B **96**, 020401 (2017).
[20] R. Takahashi, M. Matsuo, M. Ono, K. Harii, H. Chudo, S. Okayasu, J. Ieda, S. Takahashi, S. Maekawa, and E. Saitoh, Nat. Phys. **12**, 52 (2015).
[21] D. Kobayashi, T. Yoshikawa, M. Matsuo, R. Iguchi, S. Maekawa, E. Saitoh, and Y. Nozaki, Phys. Rev. Lett. **119**, 077202 (2017).
[22] A. Hirohata, Y. Baba, B. A. Murphy, B. Ng, Y. Yao, K. Nagao, and J.-y. Kim, Sci. Rep. **8**, 1974 (2018).
[23] S. Maekawa, S. O. Valenzuela, T. Kimura, and E. Saitoh, *Spin Current* (Oxford University Press, Oxford, UK, 2017), Vol. 22.
[24] L. Adamczyk, J. K. Adkins, G. Agakishiev, M. M. Aggarwal, Z. Ahammed, N. N. Ajitanand, I. Alekseev, D. M. Anderson,

R. Aoyama, A. Aparin *et al.* (STAR Collaboration), Nature (London) **548**, 62 (2017).
[25] B. I. Abelev, M. M. Aggarwal, Z. Ahammed, B. D. Anderson, D. Arkhipkin, G. S. Averichev, Y. Bai, J. Balewski, O. Barannikova, L. S. Barnby *et al.* (STAR Collaboration), Phys. Rev. C **76**, 024915 (2007); **95**, 039906(E) (2017).
[26] Z.-T. Liang and X.-N. Wang, Phys. Rev. Lett. **94**, 102301 (2005); **96**, 039901(E) (2006).
[27] Z.-T. Liang and X.-N. Wang, Phys. Lett. B **629**, 20 (2005).
[28] J.-H. Gao, S.-W. Chen, W.-t. Deng, Z.-T. Liang, Q. Wang, and X.-N. Wang, Phys. Rev. C **77**, 044902 (2008).
[29] S. A. Voloshin, arXiv:nucl-th/0410089.
[30] B. Betz, M. Gyulassy, and G. Torrieri, Phys. Rev. C **76**, 044901 (2007).
[31] F. Becattini, F. Piccinini, and J. Rizzo, Phys. Rev. C **77**, 024906 (2008).
[32] C. G. Van Weert, Ann. Phys. **140**, 133 (1982).
[33] D. N. Zubarev, A. V. Prozorkevich, and S. A. Smolyanskii, Theor. Math. Phys. **40**, 821 (1979).
[34] F. Becattini and L. Tinti, Ann. Phys. **325**, 1566 (2010).
[35] F. Becattini, Phys. Rev. Lett. **108**, 244502 (2012).
[36] F. Becattini, V. Chandra, L. Del Zanna, and E. Grossi, Ann. Phys. **338**, 32 (2013).
[37] F. Becattini and E. Grossi, Phys. Rev. D **92**, 045037 (2015).
[38] T. Hayata, Y. Hidaka, T. Noumi, and M. Hongo, Phys. Rev. D **92**, 065008 (2015).
[39] J.-H. Gao, Z.-T. Liang, S. Pu, Q. Wang, and X.-N. Wang, Phys. Rev. Lett. **109**, 232301 (2012).
[40] R.-h. Fang, L.-g. Pang, Q. Wang, and X.-n. Wang, Phys. Rev. C **94**, 024904 (2016).
[41] L.-G. Pang, R.-H. Fang, H. Petersen, Q. Wang, and X.-N. Wang, J. Phys.: Conf. Ser. **779**, 012069 (2017).
[42] Q. Wang, Nucl. Phys. A **967**, 225 (2017).
[43] S. L. Adler, Phys. Rev. **177**, 2426 (1969).
[44] J. S. Bell and R. Jackiw, Nuovo Cim. A **60**, 47 (1969).
[45] A. Vilenkin, Phys. Rev. D **22**, 3080 (1980).
[46] K. Fukushima, D. E. Kharzeev, and H. J. Warringa, Phys. Rev. D **78**, 074033 (2008).
[47] K. Fukushima, D. E. Kharzeev, and H. J. Warringa, Phys. Rev. Lett. **104**, 212001 (2010).
[48] A. Vilenkin, Phys. Rev. D **20**, 1807 (1979).
[49] J. Erdmenger, M. Haack, M. Kaminski, and A. Yarom, J. High Energy Phys. 01 (2009) 055.
[50] D. T. Son and P. Surowka, Phys. Rev. Lett. **103**, 191601 (2009).
[51] D. E. Kharzeev, J. Liao, S. A. Voloshin, and G. Wang, Prog. Part. Nucl. Phys. **88**, 1 (2016).
[52] J. Liao, Pramana **84**, 901 (2015).
[53] A. A. Burkov, J. Phys.: Condens. Matter **27**, 113201 (2015).
[54] V. I. Zakharov, J. Exp. Theor. Phys. **120**, 428 (2015).
[55] D. T. Son and N. Yamamoto, Phys. Rev. Lett. **109**, 181602 (2012).
[56] M. Stone and V. Dwivedi, Phys. Rev. D **88**, 045012 (2013).
[57] D. T. Son and N. Yamamoto, Phys. Rev. D **87**, 085016 (2013).
[58] J.-Y. Chen, D. T. Son, and M. A. Stephanov, Phys. Rev. Lett. **115**, 021601 (2015).
[59] M. A. Stephanov and Y. Yin, Phys. Rev. Lett. **109**, 162001 (2012).
[60] J.-W. Chen, J.-y. Pang, S. Pu, and Q. Wang, Phys. Rev. D **89**, 094003 (2014).
[61] J.-Y. Chen, D. T. Son, M. A. Stephanov, H.-U. Yee, and Y. Yin, Phys. Rev. Lett. **113**, 182302 (2014).
[62] J.-W. Chen, S. Pu, Q. Wang, and X.-N. Wang, Phys. Rev. Lett. **110**, 262301 (2013).
[63] Y. Hidaka, S. Pu, and D.-L. Yang, Phys. Rev. D **95**, 091901 (2017).
[64] Y. Hidaka, S. Pu, and D.-L. Yang, Phys. Rev. D **97**, 016004 (2018).
[65] Y. Hidaka and D.-L. Yang, Phys. Rev. D **98**, 016012 (2018).
[66] J.-h. Gao, S. Pu, and Q. Wang, Phys. Rev. D **96**, 016002 (2017).
[67] J.-H. Gao, Z.-T. Liang, Q. Wang, and X.-N. Wang, Phys. Rev. D **98**, 036019 (2018).
[68] A. Huang, S. Shi, Y. Jiang, J. Liao, and P. Zhuang, Phys. Rev. D **98**, 036010 (2018).
[69] S. M. Barnett, Phys. Rev. Lett. **118**, 114802 (2017).
[70] S. J. van Enk and G. Nienhuis, Europhys. Lett. **25**, 497 (1994).
[71] S. M. Barnett, J. Mod. Opt. **57**, 1339 (2010).
[72] D.-L. Yang, Phys. Rev. D **98**, 076019 (2018).
[73] M. Stone, Int. J. Mod. Phys. B **30**, 1550249 (2015).
[74] H.-L. Chen, K. Fukushima, D. Kharzeev, Y. Hirono, X.-G. Huang, and K. Mameda (unpublished).
[75] D. E. Kharzeev, M. A. Stephanov, and H.-U. Yee, Phys. Rev. D **95**, 051901 (2017).
[76] K. Fukushima, S. Pu, and Z. Qiu (unpublished).
[77] S. Pu, J.-h. Gao, and Q. Wang, Phys. Rev. D **83**, 094017 (2011).
[78] K. Makishima, T. Enoto, J. S. Hiraga, T. Nakano, K. Nakazawa, S. Sakurai, M. Sasano, and H. Murakami, Phys. Rev. Lett. **112**, 171102 (2014).