# Reply to "Comment on 'Coherent-state path integrals in the continuum'"

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In this Reply we briefly clarify the main points of our method to construct coherent-state path integrals in the continuum and we reply to the critique raised by in the preceding Comment by Kochetov [Phys. Rev. A **99**, 026101 (2019)]. By using definite examples, we prove that our approach is capable of resolving the inconsistencies accompanying the standard coherent-state path-integral representation of interacting systems.

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## I. INTRODUCTION

The Feynman path-integral formalism has been proved one of the most powerful methods for the study of quantum mechanics, quantum field theories, and statistical mechanics [1,2]. The extension of path integration into the complex plane $C^1$ through the Glauber coherent states [3], in the complex nonflat manifold $\bar{C}^1$ utilizing the SU(2) spin coherent states [4–6], and to fermionic systems via the fermionic coherent states [1,7] has expanded tremendously its range of applications. However, and despite the progress and the many illuminating contributions [6,8–16], integration over paths constructed in terms of the overcomplete coherent state bases seems to suffer from inconsistencies, even from anomalies [17]. In order to face the pitfalls appearing in coherent-state path integrals, we proposed [18,19] a rather simple way that bypasses the standard construction by making use of the Feynman phase-space integral. In the preceding Comment, Kochetov [20] suggested that our approach to achieve a time-continuous formulation of path integration in complex manifolds still carries the inconsistencies first reported in [17]. In this Reply to the preceding Comment we will prove, in Sec. II, that we did not make the illegal replacements or the unjustified approximations the author suggests we made. In Sec. III we prove that, in contrast to author's allegations, our approach is capable of resolving the coherent-state path-integral (CSPI) inconsistencies in the presence of interactions, inconsistencies that persist when the prescription advocated in [20] is followed.

## II. CONSTRUCTING THE PATH INTEGRAL

A main point of the criticism in [20] is that in constructing the CSPI representation we replace Weyl symbols of powers by powers of Weyl symbols. To prove that this is not the case we will consider as a system of reference the toy Hamiltonian $\hat{H}_0 = \hat{n}^2$, with $\hat{n} = \hat{a}^\dagger\hat{a}$, where $[\hat{a}, \hat{a}^\dagger] = 1$. The partition function of this system is readily found to be

$$Z = \mathrm{Tr}e^{-\beta\hat{H}_0} = \sum_{n=0}^{\infty} e^{-\beta n^2}. \qquad (1)$$

To represent this function as a path integral we follow the standard technique of slicing $\beta/(N+1) = \epsilon \to 0$, but we do not fill the gaps by using the overcomplete coherent-state basis. Instead, in our approach, we rely on the complete bases $\{|p\rangle, |q\rangle\}$ of the Hermitian quadratures $\hat{q} = (\hat{a}^\dagger + \hat{a})/\sqrt{2}$ and $\hat{p} = i(\hat{a}^\dagger - \hat{a})/\sqrt{2}$. In terms of these operators, the Hamiltonian is written as $\hat{H}_0 = (\hat{p}^2 + \hat{q}^2 - 1)^2/4$. Thus, to arrive at the final result we must deal with amplitudes of the form

$$U_{n,n-1} = \langle q_n|e^{-\epsilon(\hat{p}^2+\hat{q}^2-1)^2/4}|q_{n-1}\rangle, \quad \epsilon \to 0. \qquad (2)$$

This is a point where misunderstandings have arisen as the handling of powers of noncommuting operators is not a trivial task. However, in the present case, we have to deal with the second power of a Hermitian operator that possesses a complete set of eigenstates. Thus, we can use the Hubbard-Stratonovich transformation [21–23] to write

$$\langle q_n|e^{-\epsilon(\hat{p}^2+\hat{q}^2-1)^2/4}|q_{n-1}\rangle \sim \int_{-\infty}^{\infty} d\sigma\, e^{-\epsilon\sigma^2/4}\langle q_n|e^{i\epsilon\sigma(\hat{p}^2+\hat{q}^2-1)/2}|q_{n-1}\rangle. \qquad (3)$$

Due to the fact that $\epsilon\sigma = O(\sqrt{\epsilon})$, we can make the approximation

$$\begin{aligned}\langle q_n|e^{i\epsilon\sigma(\hat{p}^2+\hat{q}^2)/2}|q_{n-1}\rangle &\underset{\epsilon\to 0}{\approx} \langle q_n|e^{i\epsilon\sigma\hat{q}^2/2}e^{i\epsilon\sigma\hat{p}^2/2}e^{i\epsilon^2\sigma^2(\hat{q}\hat{p}+\hat{p}\hat{q})/4}|q_{n-1}\rangle \\ &= e^{-\epsilon^2\sigma^2/4}\int_{-\infty}^{\infty}\frac{dp_n}{2\pi}e^{ip_n(q_n-q_{n-1})} \\ &\quad\times e^{i\epsilon\sigma(p_n^2+q_n^2)/2+i\epsilon^2\sigma^2 p_n q_{n-1}/2}. \end{aligned} \qquad (4)$$

Upon integrating over $\sigma$ and omitting $O(\epsilon^2)$ terms, it is a trivial task to arrive at the integral

$$\mathrm{Tr}e^{-\beta\hat{H}_0} = \int Dp \int_{q(0)=q(\beta)} Dq \times \exp\left\{-\int_0^\beta d\tau\left[-ip\dot{q} + H_0^F(p,q)\right]\right\}. \qquad (5)$$

Here the function $H_0^F(p,q) = (p^2+q^2-1)^2/4$ is the classical counterpart of the quantum Hamiltonian $\hat{H}_0 = H_0(\hat{p},\hat{q})$. Although it seems that we have performed the unjustified replacement $[(\hat{p}^2+\hat{q}^2-1)^2]_W = [(\hat{p}^2+\hat{q}^2-1)_W]^2$ as suggested in [20], this is not the case. Instead, for the Hamiltonian $\hat{H}_0 = (\hat{p}^2+\hat{q}^2-1)^2/4$, which is the square of a simple harmonic-oscillator Hamiltonian, we legally used the Hubbard-Stratonovich transformation to transcribe the original system to an effective harmonic-oscillator system

$$e^{-\beta\hat{H}_0^2} \sim \int D\sigma \exp\left(-\frac{1}{4}\int_0^\beta d\tau\, \sigma^2(\tau)\right) \times \exp\left(-i\int_0^\beta d\tau\, \sigma(\tau)\hat{H}_0\right). \tag{6}$$

Then, as explicitly noted in Eq. (4), we neglected the commutator terms having relied on approximations well justified [1] for the simple effective system and for the complete bases we used. We never claimed that the phase-space path integrals are in general indifferent to the quantum Hamiltonian ordering prescription, quite the contrary. We claimed that for the specific system, and consequently for the Bose-Hubbard model [24], neglecting the commutator terms in the discretization process given by Eq. (4) is a legitimate approximation. The quantum content of the system has not been lost as suggested in [20]. Instead, it is present in the nondifferentiability of the paths $q(t)$ that must be integrated out. It is for this reason that the exact calculation of the integral of Eq. (5) yields the correct quantum mechanical result. Up to this point, we have bypassed the difficulties associated with the use of the coherent-state basis. Before proceeding, we must strongly underline the fact that the lattice structure beneath the continuous expression given by Eq. (5) has been fixed to be the symmetric form appearing in Eq. (4). This is quite important, as the invariance of classical mechanics under canonical transformations must be reflected in path integrals [1]. This structure permits the canonical change of variables $q = (\bar{z}+z)/\sqrt{2}$ and $p = i(\bar{z}-z)/\sqrt{2}$ that transcribes the integral of Eq. (5) in terms of the complex coordinates $(\bar{z}, z)$,

$$Z = \int_{\text{periodic}} D^2 z \exp\left\{-\int_0^\beta d\tau \left[\bar{z}\dot{z} + \left(|z|^2 - \frac{1}{2}\right)^2\right]\right\}. \tag{7}$$

This formula is again a source of misunderstandings concerning our approach. It looks like we have replaced $(\hat{n}^2)_W$ by $\hat{n}_W\hat{n}_W$. However, this is not the case, as we have explicitly demonstrated in the preceded analysis. The function that plays the role of the Hamiltonian in the last integral is not taken to be the continuum limit of a discretized expression of the form $\langle z_n|\hat{H}|z_{n-1}\rangle/\langle z_n|z_{n-1}\rangle$, where $\hat{H}$ is a properly (normal, antinormal, Weyl, or otherwise) ordered quantum Hamiltonian. The reason we avoided this limit is that it contains symplectic terms and calls for unwarranted approximations that mix ordering and continuity [25]. Thus, the integral (7) is an inevitable result of the whole construction and not the result of an assumed linearized Weyl reordering of the original quantum Hamiltonian or a retrospective choice for resolving inconsistencies. The construction we followed has bypassed the ambiguities connected with the overcomplete coherent state basis and the classical action has unambiguously been determined. The lattice prescription, associated with the integral of Eq. (7), has been inherited, via a canonical transformation, from the Feynman phase-space integral and has been fixed to be the symmetrical form $|z_n|^4, |z_n|^2 \leftrightarrow |z|^4, |z|^2$. With all these clarifications we will easily confirm, in the next section, that the evaluation of the integral of Eq. (7) produces the correct quantum result.

## III. EVALUATION OF THE PATH INTEGRAL

In [20] it is claimed that our approach is not justified for interacting systems. Although not explicitly stated, the method reviewed in this Comment is advocated as capable of resolving the CSPI inconsistencies even in the presence of interactions. However, the calculations prove that what really happens is quite the contrary. For this and for defending the efficiency of our approach, first, we will briefly sketch the calculation of the coherent-state path integral, pertaining to the toy model in hand, by using the prescription presented in [20] for different orderings of the quantum Hamiltonian, proving that a wrong result is produced in all the cases. Next we will prove that our approach for defining the coherent-state path integral yields the correct answer. The orderings we will examine are the original $\hat{H}_0 = (\hat{a}^\dagger\hat{a})^2 = \hat{n}^2$ and the Weyl one $[\hat{H}]_W = [(\hat{a}^\dagger\hat{a})^2]_W = \hat{n}^2 + \hat{n} + 1/2$. By following the standard discretization procedure we find that the functions that represent these Hamiltonians in the coherent-state path integral are $H_0(\bar{z}, z) = |z|^4 + |z|^2$ and $H_W(\bar{z}, z) = |z|^4 + 2|z|^2 + 1/2$, respectively. The technical difficulty in the calculation of the relevant path integrals is the nonlinearity of these functions, a difficulty that can be faced by using the identity

$$\exp\left(-\int_0^\beta d\tau |z|^4\right) = C\int D\sigma \exp\left(-\int_0^\beta d\tau(\sigma^2/4 + i\sigma|z|^2)\right), \tag{8}$$

where $C^{-1} = \int D\sigma \exp(-\int_0^\beta d\tau\, \sigma^2/4)$. In this way, the coherent-state path integral that must be calculated possesses the form

$$Z_\Omega = \int_{\text{periodic}} D\bar{z}\, Dz \exp\left(-\int_0^\beta d\tau\, \bar{z}[\partial_\tau + \Omega(\tau)]z\right). \tag{9}$$

In this integral the time-dependent frequency is $\Omega = \mu + i\sigma$, where $\mu = 1, 2$ for the original and the Weyl orderings, respectively. The calculation of the integral in Eq. (9) can be easily performed [1], and by adopting the prescription presented in [20] we find that

$$Z_\Omega = \frac{1}{1-\exp\left(-\int_0^\beta d\tau\, \Omega\right)} = \sum_{n=0}^\infty \exp\left(-n\int_0^\beta d\tau\, \Omega\right). \tag{10}$$

Integrating out the auxiliary field $\sigma$, we immediately find for the original Hamiltonian the result $Z_0 = \sum_{n=0}^\infty e^{-\beta n(n+1)}$ and for the Weyl ordered one the result $Z_W = e^{-\beta/2}\sum_{n=0}^\infty e^{-\beta n(n+2)}$. Evidently, both of these outcomes are wrong. Within our approach the function that plays the role of the Hamiltonian in the path integral is, for the original ordering, $H_0^F = (|z|^2 - 1/2)^2$ and for the Weyl one $H_W^F = (|z|^2-1/2)^2 + (|z|^2 - 1/2) + 1/2 = |z|^4 + 1/4$. Introducing the auxiliary field $\sigma$, the coherent-state path integral that must be calculated reads

$$Z_{\Omega^F} = \int_{\text{periodic}} D\bar{z}\, Dz \exp\left[-\int_0^\beta d\tau\, \bar{z}[\partial_\tau + \Omega^F(\tau)]z\right], \tag{11}$$

with $\Omega^F = \mu + i\sigma(\tau)$ and $\mu = -1, 0$ for the original and the Weyl ordering, respectively. As we stressed in Sec. II, our approach not only has fixed the function $H^F_{0,W}$, but also has enforced the underlying lattice prescription to be symmetric. Thus the integration in Eq. (11) yields the result

$$\begin{aligned} Z_{\Omega^F} &= \frac{\exp\left(-\int_0^\beta d\tau\, \Omega^F/2\right)}{1-\exp\left(-\int_0^\beta d\tau\, \Omega^F\right)} \\ &= \sum_{n=0}^{\infty} \exp\left[-\left(n+\frac{1}{2}\right)\int_0^\beta d\tau\, \Omega^F\right] \\ &= \sum_{n=0}^{\infty} \exp\left[-\beta\left(n+\frac{1}{2}\right)\mu - i\left(n+\frac{1}{2}\right)\int_0^\beta d\tau\, \sigma\right]. \end{aligned} \tag{12}$$

Integrating out the auxiliary field, we get the correct result for both orderings, namely,

$$Z_{0;F} = e^{-\beta/4}\sum_{n=0}^{\infty} \exp\left[-\beta\left(n+\frac{1}{2}\right)^2 + \beta\left(n+\frac{1}{2}\right)\right] = \sum_{n=0}^{\infty} e^{-\beta n^2} \tag{13}$$

and

$$Z_{W;F} = e^{-\frac{\beta}{4}}\sum_{n=0}^{\infty} e^{-\beta(n+1/2)^2} = \sum_{n=0}^{\infty} e^{-\beta(n^2+n+1/2)}. \tag{14}$$

Thus, we proved that taking into consideration the underlying time-lattice structure, at least in the way advocated in [20], is not enough to handle inconsistencies in coherent-state path integrals. Instead, for the simple but nontrivial case we presented here as well as for the more interesting cases we have examined [18,19] our approach unambiguously produces the correct results.

## IV. CONCLUSION

Our approach bypasses the problems appearing in the standard construction of coherent-state path integrals and produces, in a complexified phase space, path integrals that, at least for the examined cases, are free of inconsistencies and lead to the correct quantum results.

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