Zitterbewegung in a Kerr medium

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The evolution of a quantum particle propagating in a nonlinear medium and under the action of electric and magnetic fields is studied. For this purpose, the integrable unification of the Dirac and Manakov equations is derived. The integrability conditions of the system and the corresponding representation of the zero curvature of the system are found. We have applied an approach based on the Riemann-Hilbert problem to derive soliton solutions of the unified model. The study of the solution shows that the action of the electric and magnetic fields leads to the Zitterbewegung in the nonlinear interaction mode. It was also found that the collision of two solitons in the presence of an electric field can lead to the rectification of the particle trajectory and the disappearance of its Zitterbewegung.

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I. INTRODUCTION

Originally predicted by Schrödinger in the study of the Dirac equation, Zitterbewegung (ZB) refers to the trembling motion of a freely moving relativistic quantum particle that arises from the interference between the positive and negative energy parts of the spinor wave function [1–3]. The notion of ZB and resulting formalism, however, are not peculiar to relativistic quantum dynamics, and phenomena analogous to ZB, which underlie the same mathematical model of the Dirac equation, have so far predicted in a wide variety of quantum and even classical physical systems, including among others semiconductors and quantum wells [4,5], trapped ions [6], graphene [7,8], cold atoms [9,10], acoustic [11], and photonic [12,13] systems. Simulations of relativistic quantum effects using experimentally accessible physical setups, in which parameter tunability allows access to different physical regimes, have seen in recent years an increasing interest, culminating to the experimental observation of a quantum analog of ZB using a single trapped ion set to behave as a free relativistic quantum particle [14]. In the optical context, the use of photonic systems to mimic quantum phenomena in the laboratory has seen a continuous and increasing interest (see, for instance, Ref. [15] and references therein); in particular, optical analogs of the relativistic ZB have recently been proposed to occur in photonic crystals [11], metamaterial slabs, and binary waveguide array [13]. A classical analog of ZB can be observed in a much simpler and well-known setup of nonlinear optics, namely in the process of sum frequency generation of light waves in a nonlinear $\chi(2)$ medium in the presence of temporal (or spatial) walk off, which has been widely investigated especially in connection to the compression of ultrashort pulses [16,17].

In the past decade, a large variety of experiments have shown the existence of chiral spin selectivity in organic helical molecules [18–20]. This effect results from the spin-orbit coupling (SOC) between the electronic momentum and the molecular electric field created by the helical arrangement of molecular dipoles. Many theoretical models have been proposed to explain these experimental evidences within different frameworks [21–23]. However, none of them was able to provide a good quantitative agreement with experimental data yet. Most recently, a few studies highlight the influence of the electron-lattice interaction on spin transport in organic helical molecules [24]. Accounting for nonlinear effects associated with deformation of the molecular structure can improve the predictive capabilities of such models [25]. Fundamental types of spin-orbit coupling, well known from works on the physics of semiconductors, may be simulated in the atomic Bose-Einstein condensates (BEC); see [26] and references within. In this regard, there are several theoretical studies that predict the existence of a large variety of propagating solitons depending on the interacting parameters in BECs [27–30].

All these processes can be most thoroughly studied analytically in the framework of completely integrable systems of equations. In earlier studies, a nonlinear generalization of the Dirac equation was associated with equations of the relativistic two-dimensional massive spinor field with current-current interaction, such as Thirring equations [31] and others [32,33]. Kartashov and Konotov have studied the dynamics of BECs with helical SOC and found that in the absence of Zeeman splitting and for constant SOC coefficients the equations are integrable [30]. These authors have used linear transform, which allowed them to reduce the evolution equation to the two-component nonlinear Schrödinger (NLS) or Manakov equations [34]. The NLS equations have been widely recognized as an ubiquitous mathematical model for describing the evolution of a slowly varying wave packet in a general nonlinear wave system; thus it plays an important role in a wide range of physical subjects [35]. The inverse scattering transform method (ISTM) was first developed and applied to the one-component NLS by Zakharov and Shabat [36]. In

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certain physical situations, two or more wave packets of different carrier frequencies appear simultaneously, and their interactions are then governed by the coupled NLS equations. Manakov showed that if the coupling is only through cross-phase modulation and the self-phase modulation coefficient, then this system is integrable and developed the respective ISTM [34]. Multisoliton solutions in the Manakov system have also been extensively investigated by Hirota and others methods in numerous studies; see, for instance, Refs. [34,37–39] and an interesting phenomenon of polarization rotation after collision has been found.

It is of interest to find more general integrable nonlinear Dirac equations and to study such phenomena as ZB in nonlinear media. The Riemann-Hilbert (RH) problem is widely used to solve different integrable equations [35], which have application in various parts of physics, including BEC, nonlinear optics, and many others; see, for instance, Refs. [35,40–42]. In this paper, we present the application of the RH problem to the solution of an integrable version of the reduced Dirac equations unified with the Manakov equations, which makes it possible to investigate the ZB and other effects mediated by soliton dynamics.

II. DIRAC-MANAKOV EQUATIONS

The expansion of the Dirac equation for the spin-1/2 particle of its mass and charge under an electromagnetic field using the Foldy-Wouthuysen method [1,2] leads to the Pauli equations $i\partial_t\psi = \widehat{H}_P\psi$ for the upper two components of the four-component spinor. $\widehat{H}_P$ is the Pauli Hamiltonian, which neglects the mass terms mc^2 and the terms on the order higher than m^{-2}, which gives [43]

$$\widehat{H}_P = \frac{\widehat{\vec{P}}^2}{2m} - \frac{Q}{mc}\vec{S}\cdot\vec{B} + V - \frac{Q}{4m^2c^2}[\widehat{\vec{P}}\cdot\vec{S}\times\vec{E} + \vec{S}\times\vec{E}\cdot\widehat{\vec{P}}]. \tag{1}$$

$\widehat{\vec{P}} \equiv -i\nabla - (Q/c)\vec{A}$ is the canonical momentum operator and the magnetic field $\vec{B} = \nabla\times\vec{A}$ is the rotation of the vector potential. $\vec{S} = \vec{\sigma}/2$ is the spin operator and $\vec{\sigma}$ is the vector Pauli matrix. The last term on the right-hand side of Eq. (1) describes the spin-orbit interaction.

We consider a quasi-one-dimensional long thin tube modeling some nonlinear system with the center line oriented along the z axis and assume that the electron propagates inside this tube along the z axis. Vector potential and electric field are assumed to be transverse, i.e., $\vec{E}(z,t) = \vec{e}_xE_x(z,t) + \vec{e}_yE_y(z,t)$, and magnetic field is $\vec{B}(z,t) = \vec{e}_xB_x(z,t) + \vec{e}_yB_y(z,t) + \vec{e}_zB_\parallel(z,t)$; here $\vec{e}_\alpha$, $\alpha = x, y, z$ are the unit vectors of the Cartesian coordinate system. Then the operator $\widehat{\vec{P}}^2$ has the form $\widehat{\vec{P}}^2 = -\partial_z^2 - (\nabla_\perp - iQ\vec{A}_\perp/c)^2$, where $\vec{A}_\perp$ is a transverse part of the vector potential. We add into the Pauli equations a nonlinear part of the electron-matter interaction $NP = \epsilon_{nl}l^{-2}\psi^\dagger\psi$ with a real coefficient ϵ_{nl}; here l is some characteristic length of the medium and a confining in the transverse directions potential $V_\perp$.

Electric and magnetic fields may have different physical sources. However, we assume that transverse electric $E_\perp = E_y - iE_x$ and magnetic $B_\perp = B_x - iB_y$ fields have the same phase, i.e., the transverse polarization components of the magnetic and electric fields satisfy the condition

$$E_xB_x = E_yB_y. \tag{2}$$

Then we can denote

$$\frac{Ql}{2mc^2}E_\perp \equiv E_0e^{-2i\phi}, \tag{3}$$

$$\frac{Ql^2}{c}B_\perp \equiv B_0 2m\, e^{-2i\phi}, \tag{4}$$

$$\frac{Ql^2}{c}B_\parallel \equiv B_z. \tag{5}$$

In the geometry under consideration the longitudinal and transverse variables may be separated. We use the transform

$$\begin{aligned}\vec{\psi} &= \{\psi_u, \psi_g\}^T = \vec{\chi}_\parallel(z,t)\chi_\perp(x,y)\\ &= \{\chi_u(z,t)\exp[-i\phi(z,t)],\ \chi_g(z,t)\exp[i\phi(z,t)]\}^T\chi_\perp(x,y),\end{aligned} \tag{6}$$

where scalar function $\chi_\perp(x,y)$ obeys the equation $(\nabla_\perp - iQ\vec{A}_\perp/c)^2\chi_\perp = V_\perp\chi_\perp$. Thus the Pauli equation takes the form

$$\begin{aligned}\dot{\vec{\chi}} &- [(\phi'' + 2\phi'\partial_z + i\dot{\phi} - iB_z)\sigma_z + (E_0' + 2E_0\partial_z + iB_0)\sigma_x]\vec{\chi}\\ &= i\big[\partial_z^2 + 2\vec{\chi}^\dagger\vec{\chi}\big]\vec{\chi} + [\nabla_\perp^2 + V_\perp(x,y) - \epsilon_0E_0^2 - (\phi')^2]\vec{\chi},\end{aligned} \tag{7}$$

where $\vec{\chi} = \{\chi_u, \chi_g\}^T$. We have changed variables as $t = \tilde{t}2ml^2$, $z = \tilde{z}l$. Below we will omit tildes over time and space variables, i.e., we put $\tilde{z}\to z$, $\tilde{t}\to t$. σ_x, σ_z are the Pauli matrices. We have also denoted $\dot{f} \equiv \partial_t f$, $f' \equiv \partial_z f$ and added the potential to the right side of Eq. (7) $-\epsilon_0E_0^2$, which describes the change in the dielectric constant of the medium under the influence of an external electric field with amplitude E_0; here ϵ_0 is a real constant.

The system of equations (7) may be considered as unification of the two-component reduction of the Dirac equations [1–3]; see the left-hand side of the system, and the Manakov equations [34], which consist of the first terms in the left-hand side and the right-hand sides of system (7). Under some restrictions to the forms of the electric and magnetic fields the Dirac-Manakov equations (DM) (7) become integrable. We can distinguish two cases that correspond to the slightly different representations of the zero curvature (the Lax representations):

$$E_0'(z) = 0,\quad \forall\phi(z),\quad \forall\epsilon_0, \tag{8a}$$

$$\forall E_0(z),\quad \forall\phi(z),\quad \epsilon_0 = -2m. \tag{8b}$$

An additional condition that binds the phase and amplitudes of the fields is the following:

$$[\dot{\phi}(t) + B_z(t)]E_0 = B_0(t)\phi'. \tag{9}$$

All functions in (8) and (9) are real. Thus the DM equations (7) are integrable if the equalities (2), (9) and one of the equalities (8) hold.

III. ISTM APPLICATION

A. Zero-curvature presentation of the model

Consider the following generalization of the DM equations (7):

$$\dot{\vec{r}} + [(h' + 2h\partial_z + iH)\sigma_z + (g' + 2g\partial_z + iG)\sigma_x]\vec{r} = i\big[\partial_z^2 - 2\vec{r}^\dagger\vec{r}_1 - \epsilon_0 g^2 - h^2\big]\vec{r}, \quad (10)$$

$$\dot{\vec{r}}_1 + [(h' + 2h\partial_z - iH)\sigma_z + (g' + 2g\partial_z - iG)\sigma_x]\vec{r}_1 = i\big[\partial_z^2 - 2\vec{r}_1^\dagger\vec{r} - \epsilon_0 g^2 - h^2\big]\vec{r}_1, \quad (11)$$

where $\vec{r} = \{p, q\}^T$, $\vec{r}_1 = \{p_1, q_1\}^T$, G, H are functions of t and g, h are functions of z. For $h = -\phi'(z)$, $g = -E_0(z)$, $G = -B_0$, $H = -B_z - \dot{\phi}$, and $p_1 = -p^*, q_1 = -q^*$ system (10), (11) reduces to equations (7).

The system of equations (10), (11) with the equality, generalizing condition (9),

$$G(t)h(z) = g(z)H(t) \quad (12)$$

are the compatibility conditions of the following two linear systems:

$$\mathbf{\Theta}_z = \mathcal{L}\mathbf{\Theta} \equiv (i\lambda\Lambda + \boldsymbol{U})\mathbf{\Theta}, \quad (13)$$

$$\mathbf{\Theta}_t = (-2i\lambda^2\Lambda + \boldsymbol{V})\mathbf{\Theta}, \quad (14)$$

where $\mathbf{\Theta}$ is a 3×3 matrix-valued function, λ is a spectral parameter, $\Lambda = \mathrm{diag}(1, 1, -1)$, $\boldsymbol{U} = \boldsymbol{U}_0 + \boldsymbol{U}_1$, $\boldsymbol{V} = \boldsymbol{V}_0 + \boldsymbol{W} - 2\lambda\boldsymbol{U}_0$,

$$\boldsymbol{U}_0 = \begin{pmatrix} 0 & 0 & p \\ 0 & 0 & q \\ p_1 & q_1 & 0 \end{pmatrix}, \quad \boldsymbol{U}_1 = i\begin{pmatrix} -h & -g & 0 \\ -g & h & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (15)$$

$$\boldsymbol{V}_0 = i\begin{pmatrix} -pp_1 & -pq_1 & p' \\ -p_1 q & -qq_1 & q' \\ -p_1' & -q_1' & pp_1 + qq_1 \end{pmatrix}, \quad (16)$$

$$\boldsymbol{W} = -\begin{pmatrix} iH & iG & gq + hp \\ iG & -iH & gp - hq \\ gq_1 + hp_1 & gp_1 - hq_1 & -W_{33} \end{pmatrix}, \quad (17)$$

where $W_{33} = -(1 - \epsilon_0)g^2$ and $W_{33} = 0$ for the integrability conditions (51) and (52), respectively.

Consider, for simplicity, a particular case of conditions (51), assuming that $h(z) = \mathrm{const}$, $g(z) = \mathrm{const}$. Let us denote $w = \sqrt{h^2 + g^2}$ and $w_\pm = \sqrt{h^2 + g^2} \pm h$. Use the gauge transform $\mathbf{\Theta} = \mathcal{H}\mathcal{M}\mathbf{\Psi}$, where

$$\mathcal{M} = \begin{pmatrix} 1 & -\gamma_0 & 0 \\ \gamma_0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (18)$$

$\gamma_0 = w_-/g$, $\mathcal{H} = \mathrm{diag}(e^{-i\theta}, e^{i\theta}, 1)$, and

$$\theta = w\left[z + \int_0^t \frac{G(t')}{g}\,dt'\right]. \quad (19)$$

This transform changes the spectral problem (13) as

$$\mathbf{\Psi}_z = \boldsymbol{L}\mathbf{\Psi} \equiv (\mathcal{H}^{-1}\mathcal{M}^{-1}\mathcal{L}\mathcal{M}\mathcal{H} - \mathcal{H}^{-1}\partial_z\mathcal{H})\mathbf{\Psi} = \begin{pmatrix} i\lambda & 0 & L_{13} \\ 0 & i\lambda & L_{23} \\ e^{-i\theta}\left(\frac{w_+ p_1}{g} + q_1\right) & e^{i\theta}\left(q_1 - \frac{w_- p_1}{g}\right) & -i\lambda \end{pmatrix}\mathbf{\Psi}, \quad (20)$$

where

$$L_{13} = \frac{e^{i\theta}(gq + w_+ p)}{2w}, \quad (21)$$

$$L_{23} = \frac{e^{-i\theta}(qw_+ - gp)}{2w}. \quad (22)$$

B. RH problem

We assume that potentials p and q decay to zero sufficiently fast as $z, t \to \pm\infty$. $\mathbf{\Psi}$ in Eq. (20) is a fundamental matrix of the two linear equations. From (20), we note that when $z \to \pm\infty$, one has $\mathbf{\Psi} \to e^{(i\lambda\Lambda z)}$. We introduce the new function

$$\mathbf{\Psi} = \mathbf{\Phi}\, e^{i\lambda\Lambda z}, \quad (23)$$

where $\mathbf{\Phi} \to \boldsymbol{I}$, $z \to \pm\infty$, and $\boldsymbol{I} = \mathrm{diag}(1, 1, 1)$. Inserting (23) into (20) and (14) and taking into account previous gauge transforms we obtain

$$\mathbf{\Phi}_z = i\lambda[\Lambda, \mathbf{\Phi}] + \mathcal{U}\mathbf{\Phi}, \quad (24)$$

$$\mathbf{\Phi}_t = -2i\lambda^2[\Lambda, \mathbf{\Phi}] + \mathcal{V}\mathbf{\Phi}, \quad (25)$$

where $[\Lambda, \mathbf{\Phi}] = \Lambda\mathbf{\Phi} - \mathbf{\Phi}\Lambda$ and $\mathcal{V} = \mathcal{H}^{-1}\mathcal{M}^{-1}\boldsymbol{V}\,\mathcal{M}\,\mathcal{H}$, $\mathcal{U} = \mathcal{H}^{-1}\mathcal{M}^{-1}\boldsymbol{U}\,\mathcal{M}\,\mathcal{H}$.

From (24) we get $\mathrm{tr}\,\mathcal{U} = 0$. Then, for the Jost solutions $\mathbf{\Phi}_\pm$ of (24) with the asymptotic condition $\mathbf{\Phi}_\pm \to \boldsymbol{I}, z \to \pm\infty$, we obtain $\det(\mathbf{\Phi}_\pm) = 1$. Solutions $\mathbf{\Psi} = \mathbf{\Phi}_+ E$ and $\mathbf{\Phi} = \mathbf{\Phi}_- E$ of Eq. (24) are linear dependent

$$\mathbf{\Phi}_-\boldsymbol{E} = \mathbf{\Phi}_+\boldsymbol{E}\,\boldsymbol{S}(\lambda), \quad \lambda \in \mathbb{R}, \quad (26)$$

where $\boldsymbol{E}(z) = e^{i\lambda\Lambda z}$. The components of the scattering matrix $\boldsymbol{S}$ are $s_{ij}(\lambda)$, $i, j = 1, 2, 3$ and its determinant $\det(\boldsymbol{S}(\lambda)bm) = 1$.

Analyzing of the Volterra equations which are obtained by integrating Eq. (24) it can be found that the third column of $\mathbf{\Phi}_-$ and the first two columns of $\mathbf{\Phi}_+$ can be analytically continued to the upper half plane $\mathbf{C}_+$. Similarly, we find that the first two columns of $\mathbf{\Phi}_-$ and the third column of $\mathbf{\Phi}_+$ can be analytically continued to the lower half plane $\mathbf{C}_-$. Presenting $\mathbf{\Phi}_\pm$ as $\mathbf{\Phi}_\pm = [\phi_\pm^{[1]}, \phi_\pm^{[2]}, \phi_\pm^{[3]}]$, where $\phi_\pm^{[j]}$ are the columns, we find that the Jost matrix function

$$\boldsymbol{J}_+(\lambda) = [\phi_-^{[1]}, \phi_-^{[2]}, \phi_+^{[3]}]e^{i\lambda\Lambda z} = \mathbf{\Phi}_+(\boldsymbol{I} - \boldsymbol{P}) + \mathbf{\Phi}_-\boldsymbol{P}, \quad (27)$$

where $\boldsymbol{P} = \mathrm{diag}(1, 1, 0)$ is projector and is analytic in $\lambda \in \mathbf{C}_+$ and $\boldsymbol{J}_+(\lambda) \to \boldsymbol{I}$, $\lambda \in \mathbf{C}_+ \vee \lambda \to \infty$. The adjoint equation of (24) is

$$\tilde{\mathbf{\Phi}}_z = -i\lambda[\Lambda, \tilde{\mathbf{\Phi}}] - \tilde{\mathbf{\Phi}}\mathcal{U}. \quad (28)$$

The respective Jost functions for Eq. (28) are $\tilde{\mathbf{\Phi}}_\pm = \mathbf{\Phi}_\pm^{-1}$. Denote the rows of solutions of Eq. (28) as $\tilde{\mathbf{\Phi}} =$

$[\tilde{\phi}^{[1]}, \tilde{\phi}^{[2]}, \tilde{\phi}^{[3]}]^{\mathsf{T}}$. The Jost functions

$$\boldsymbol{J}_- = e^{-i\lambda\Lambda z}[\tilde{\phi}_+^{[1]}, \tilde{\phi}_+^{[2]}, \tilde{\phi}_-^{[3]}]^{\mathsf{T}} = (\boldsymbol{I}-\boldsymbol{P})\boldsymbol{\Phi}_+^{-1} + \boldsymbol{P}\boldsymbol{\Phi}_-^{-1} \quad (29)$$

are analytic and $\boldsymbol{J}_-(\lambda) \to \boldsymbol{I}$ as $\lambda \to \infty$ for $\lambda \in \mathbf{C}_-$. Thus matrix functions $\boldsymbol{J}_+$ and $\boldsymbol{J}_-$ are analytic in $\mathbf{C}_+$ and $\mathbf{C}_-$, respectively. The respective Riemann-Hilbert problem is

$$\boldsymbol{J}_+(\lambda)\boldsymbol{J}_-(\lambda) = \boldsymbol{G}(\lambda), \quad \lambda \in \mathbb{R}, \quad (30)$$

where

$$\boldsymbol{G}(\lambda) = \boldsymbol{E}\begin{pmatrix} 1 & 0 & s_{13} \\ 0 & 1 & s_{23} \\ \tilde{s}_{31} & \tilde{s}_{32} & 1 \end{pmatrix}\boldsymbol{E}^{-1}, \quad (31)$$

and $\tilde{s}_{31} = s_{21}s_{32} - s_{31}s_{22}$, $\tilde{s}_{32} = s_{31}s_{12} - s_{11}s_{32}$.

From the definition of $\boldsymbol{J}_\pm$ and scattering matrix we have

$$\det \boldsymbol{J}_+(\lambda) = s_{33}, \quad \det \boldsymbol{J}_-(\lambda) = \tilde{s}_{33}, \quad (32)$$

where $\tilde{s}_{33} = s_{11}s_{22} - s_{21}s_{12}$. Suppose that s_{33} has simple zeros $\{\lambda_k \in \mathbf{C}_+, 1 \leqslant k \leqslant N\}$ and $\tilde{s}_{33}$ has simple zeros at $\{\tilde{\lambda}_k \in \mathbf{C}_-, 1 \leqslant k \leqslant N\}$. Kernels $\ker \boldsymbol{J}_+(\lambda_k)$ and $\ker \boldsymbol{J}_-(\tilde{\lambda}_k)$ contain vectors v_k and $\tilde{v}_k$, respectively,

$$\boldsymbol{J}_+(\lambda_k)v_k = 0, \quad \boldsymbol{J}_-(\tilde{\lambda}_k)\tilde{v}_k = 0, \quad 1 \leqslant k \leqslant N. \quad (33)$$

Solution $\boldsymbol{J}_+$ of the spectral problem (24) can be expanded as

$$\boldsymbol{J}_+(\lambda_k, s) = \boldsymbol{I} + \frac{1}{\lambda}\Gamma(s) + O\left(\frac{1}{\lambda^2}\right), \quad \lambda \to \infty. \quad (34)$$

Substituting this series into Eq. (24) we derive for the first degrees of $1/\lambda$

$$\boldsymbol{L}_0 = i[\Lambda, \Gamma] = \begin{pmatrix} 0 & 0 & -2i\Gamma_{13} \\ 0 & 0 & -2i\Gamma_{23} \\ 2i\Gamma_{31} & 2i\Gamma_{32} & 0 \end{pmatrix}. \quad (35)$$

C. Symmetry properties

Assume that

$$\vec{r}_1 = \epsilon\vec{r}^*, \quad \epsilon \in \mathbb{R}. \quad (36)$$

Then from linear system (20) we obtain

$$L_{13} = L_{31}^*\frac{w_-}{2\epsilon w}, \quad L_{23} = L_{32}^*\frac{w_+}{2\epsilon w}. \quad (37)$$

Here L_{ij} are the elements of matrix $\boldsymbol{L}$ in (20). For the system of equations (7), $\epsilon = -1$, so for definiteness we will use this value of ϵ below. Thus the following symmetry property holds:

$$\boldsymbol{L}^\dagger(\lambda^*) = \epsilon\boldsymbol{C}^{-1}\boldsymbol{L}(\lambda)\boldsymbol{C}, \quad (38)$$

where

$$\boldsymbol{C} = \mathrm{diag}\left[\frac{-w_-}{2\epsilon w}, \frac{-w_+}{2\epsilon w}, 1\right]. \quad (39)$$

The symmetry property (38) of the matrix $\boldsymbol{L}$ yields the relation

$$\boldsymbol{\Phi}_\pm^\dagger = \boldsymbol{C}\boldsymbol{\Phi}_\pm^{-1}\boldsymbol{C}^{-1} \quad (40)$$

and the involution property

$$\boldsymbol{J}_+^\dagger(\lambda^*) = \boldsymbol{C}\boldsymbol{J}_-(\lambda)\boldsymbol{C}^{-1}. \quad (41)$$

Using definition of scattering matrix (26) we derive

$$\boldsymbol{S}^\dagger(\lambda^*) = \boldsymbol{C}\boldsymbol{S}^{-1}(\lambda)\boldsymbol{C}^{-1}, \quad (42)$$

which yields $\tilde{\lambda}_k = \lambda_k^*$ for the zeros of $\tilde{s}_{33}(\lambda)$ and $s_{33}(\lambda)$, respectively.

To obtain the symmetry properties for the eigenvectors v_k and $\tilde{v}_k$, we take the Hermitian of the first equation in (33). Upon the use of the involution properties (40) we get

$$v_k^\dagger \boldsymbol{C}\boldsymbol{J}_-(\tilde{\lambda}_k) = 0. \quad (43)$$

Then, comparing it with the second equation in (33), we find the involution property

$$\tilde{v}_k = v_k^\dagger\boldsymbol{C}. \quad (44)$$

IV. SOLITON SOLUTIONS

To obtain soliton solutions, we set $\boldsymbol{G} = \boldsymbol{I}$ in (30). The solutions to this special Riemann-Hilbert problem have been derived following Refs. [35] as

$$\boldsymbol{J}_+(\lambda) = \boldsymbol{I} + \sum_{j,k=1}^{N}\frac{v_j\omega_{jk}^{-1}\tilde{v}_k}{\lambda - \tilde{\lambda}_k}, \quad (45)$$

where

$$\omega_{jk} = \frac{\tilde{v}_j v_k}{\lambda_j^* - \lambda_k}. \quad (46)$$

The zeros λ_k and $\tilde{\lambda}_k$ are time independent. To find the spatial and temporal evolutions for vectors v_k, we take the z and t derivative to the equation (33). By using (24), (25), respectively, one gets

$$v_k(z,t) = e^{i\lambda_k\Lambda z - 2i\lambda_k^2\Lambda + i\nu t}v_k^{(0)}, \quad (47)$$

$$\tilde{v}_k(z,t) = \tilde{v}_k^{(0)}e^{-i\lambda_k^*\Lambda z + 2i\lambda_k^{*2}\Lambda t - i\nu t}C, \quad (48)$$

where $v_k^{(0)}$, $\tilde{v}_k^{(0)}$ are some constants, which are determined by the initial conditions and $\nu = (1-\epsilon_0)g^2$.

Comparing Eqs. (34) and (45) we obtain

$$\Gamma_{13} = i\left[\sum_{j,k=1}^{N} v_j\omega_{jk}^{-1}\tilde{v}_k\right]_{13}, \quad (49)$$

$$\Gamma_{23} = i\left[\sum_{j,k=1}^{N} v_j\omega_{jk}^{-1}\tilde{v}_k\right]_{23}, \quad (50)$$

Denote $\theta_j = i\lambda_j z - 2i\lambda_j^2 t + ig^2(1-\epsilon_0)t$ and $\lambda_j = \xi_j + i\eta_j$, $\xi_j, \eta_j \in \mathbb{R}$. We choose vectors $v_k^{(0)}$ in the form $v_k^{(0)} = [\alpha_k, \beta_k, 1]^{\mathsf{T}}$, where α_k and β_k are arbitrary complex constants. These constants are determined from the initial data.

Compare asymptotic (35) and (20) we derive the general N-soliton solution of system (1):

$$p = -\frac{2i}{w_+}(w_+\Gamma_{13}e^{-i\theta} - g\Gamma_{23}e^{i\theta}), \quad (51)$$

$$q = -\frac{2i}{w_+}(g\Gamma_{13}e^{-i\theta} + w_+\Gamma_{23}e^{i\theta}), \quad (52)$$

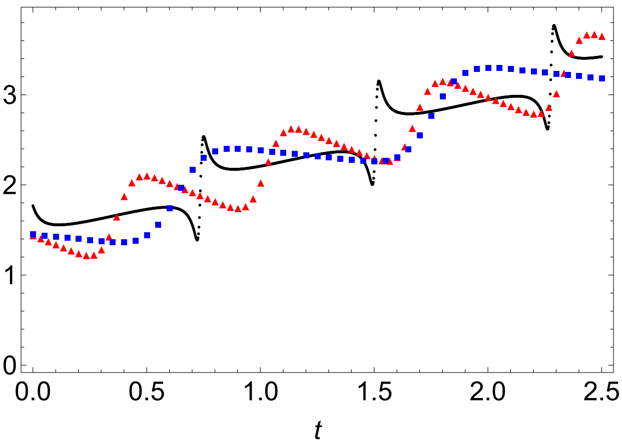

FIG. 1. Image of the average position of a soliton calculated by the formula (57) for $h = H = 0$ in dimensionless units. The black dots, red triangles, and blue squares correspond to the values $g = 0.05, G = -4,$ $g = 0.5, G = -4$, and $g = 0.5, G = -1$, respectively. Soliton parameters are $\eta_1 = 1,\ \xi_1 = 0.4,\ \alpha_1 = 1,$ and $\beta_1 = 1$.

where

$$\begin{bmatrix}\Gamma_{13}\\ \Gamma_{23}\end{bmatrix} = -i\sum_{j,k=1}^{N}\begin{bmatrix}\alpha_j\\ \beta_j\end{bmatrix} e^{\theta_j-\theta_k^*}\omega_{jk}^{-1} \tag{53}$$

and

$$\omega_{jk} = \frac{1}{\lambda_j^* - \lambda_k}[R_{jk}e^{\theta_j^*+\theta_k} + e^{-\theta_j^*-\theta_k}], \tag{54}$$

$$R_{jk} = \frac{w_-\alpha_j^*\alpha_k + w_+\beta_j^*\beta_k}{2w}. \tag{55}$$

Let $N = 1$ and $\epsilon = -1$. Denoting $R_{11} = \exp(-2\rho_{11})$ we found the single pole solution

$$\begin{bmatrix}p\\ q\end{bmatrix} = \frac{2i\,\eta_1 e^{2i(\eta_1^2 t - \xi_1^2 t + w^2 t + \xi_1 z)}}{w_+\cosh(4\eta_1\xi_1 t - 2\eta_1 z + \rho_{11})}\begin{bmatrix}w_+\alpha_1 e^{-\theta} - g\beta_1 e^{\theta}\\ g\alpha_1 e^{-\theta} + w_+\beta_1 e^{\theta}\end{bmatrix}. \tag{56}$$

V. ZB IN THE NONLINEAR SYSTEM

For the classical DM equations, ZB refers to the rapid oscillatory motion of the expectation value of the particle position

$$\langle z\rangle = \frac{\int_{-\infty}^{\infty} z\,\vec{r}^{\dagger}\vec{r}\,dz}{\int_{-\infty}^{\infty}\vec{r}^{\dagger}\vec{r}^{\dagger}\,dz} \tag{57}$$

around its mean trajectory, which arises whenever negative- and positive-energy eigenstates of the Dirac equation are simultaneously excited by the initial condition. Wave packet evolution of an initial solitonic state, showing ZB, is the trembling oscillation of the center of mass; see Fig. 1. It is found that ZB occurs only in the presence of the electric field, i.e., for $g \neq 0$. The solution shows that the amplitude of the ZB increases with increasing amplitudes of the magnetic field. For fixed g the amplitude of the ZB increases with increasing amplitudes of the magnetic field. At a fixed amplitude of the magnetic field and small $|g| \ll 1$, the time dependence of the amplitude ZB is characterized by sharp spikes. As g increases, this dependence becomes smoother and closer to the harmonic form, Fig. 1.

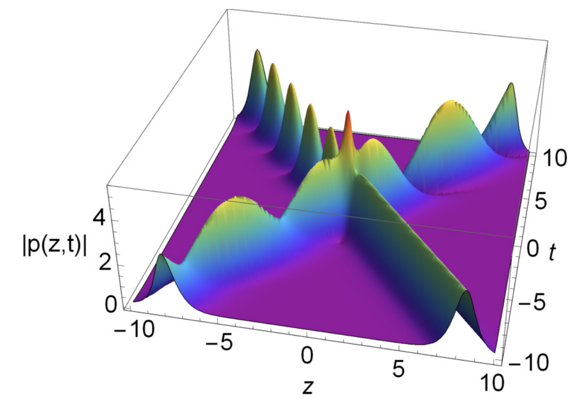

FIG. 2. Rectification of the soliton form due to collision for $h = H = 0,\ g = 0.25,$ and $G = -1$. The dependence of the absolute values of the polarization component $|p|$ on the dimensionless variables x and t is shown. Soliton parameters are $\eta_1 = 1,\ \eta_2 = 1,\ \xi_1 = -0.4,\ \xi_2 = 0.4,\ \alpha_1 = 0,\ \alpha_2 = -2,\ \beta_1 = 1,$ and $\beta_2 = 1$.

Two-pole solutions describing the collision and interaction of two solitons manifest some novel features. In Fig. 2 the collision of two-soliton solutions is depicted. In the process of the solitons' collision, the depth of modulation of the pulse amplitudes can change. The graphical representation of the solution shows that the modulation of the soliton amplitude associated with the action of the electric field can completely disappear; see Fig. 2. This change in the amplitude modulation due to the collision of solitons increases with $|g|$. The ZB effect is directly related to the space-time modulation of the shape of a soliton, which is determined by the amplitudes of the magnetic and electric fields. Analysis of the initial data showed that the most efficient effect is observed for solitons with close amplitudes. Thus solitons' collision may lead to a significant change in the amplitude of the ZB for soliton and straightening the electron trajectory of one of the colliding solitons. Figure 2 shows that the amplitude modulation of one of the solitons disappears after a collision, which corresponds to the rectification of the electron trajectory. For $g = 0$ and for any h, G, H, the collision of solitons has a character close to the well-known one in the theory of the Manakov equations. Since the model does not contain any losses, terms the form of solitons and the modulation of their amplitude after the collision conserve.

VI. CONCLUDING REMARKS

For completely integrable systems of equations, similar to the Manakov equations, the collision of solitons leads to a shift in their phases and a change in the amplitudes of the polarization components. The paper presents the integrable system of equations combining the two-component version of the Dirac equations and the Manakov equations, generalizing that of Kartashov and Konotop [30]. In the case considered above, the gauge transformation leads to a linear combination of potentials with multipliers having phases dependent on the length and time. As a consequence, the interaction of solitons in collisions possessed new features. In particular, the modulation of the parameters of ZB changes when solitons collide. A more general case provides an opportunity to explore new mechanisms for controlling the dynamics of an electron within an exactly solvable model. For instance, it is interesting to find out whether it is possible to control the dynamics of solitons, dynamically influencing their phases. Integrable model (7) allows one to find exact solutions for a more general case describing the space-dependent phase or space-dependent both phase and amplitude of the electric field. This approach can be used to describe the local influence of quantum dots with a large dipole momentum on the evolution of excitons in long curved molecules [44], including in the framework of integrable models [42]. The DM equations can be generalized to the case of some curvilinear nonlinear media and used to analyze the evolution of polarization of an exciton. We believe that results of the present paper may be directly applied to study some one-dimensional models of the BEC with the spin-orbital interactions.

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[1] L. L. Foldy and S. A. Wouthuysen, On the Dirac theory of spin 1/2 particles and its non-relativistic limit, Phys. Rev. **78**, 29 (1950).
[2] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum Electrodynamics* (Butterworth-Heinemann, London, 1982).
[3] W. Greiner, *Relativistic Quantum Mechanics* (Springer, Berlin, 1990).
[4] L. Ferrari and G. Russo, Nonrelativistic zitterbewegung in two-band systems, Phys. Rev. B **42**, 7454 (1990).
[5] J. Schliemann, D. Loss, and R. M. Westervelt, Zitterbewegung of Electronic Wave Packets in III-V Zinc-Blende Semiconductor Quantum Wells, Phys. Rev. Lett. **94**, 206801 (2005).
[6] L. Lamata, J. León, T. Schätz, and E. Solano, Dirac Equation and Quantum Relativistic Effects in a Single Trapped Ion, Phys. Rev. Lett. **98**, 253005 (2007).
[7] J. Cserti and G. David, Unified description of Zitterbewegung for spintronic, graphene, and superconducting systems, Phys. Rev. B **74**, 172305 (2006).
[8] T. M. Rusin and W. Zawadzki, Theory of electron Zitterbewegung in graphene probed by femtosecond laser pulses, Phys. Rev. B **80**, 045416 (2009).
[9] J. Y. Vaishnav and C. W. Clark, Observing Zitterbewegung with Ultracold Atom, Phys. Rev. Lett. **100**, 153002 (2008).
[10] Q. Zhang, J. Gong, and C. H. Oh, Driven Dirac-like equation via mirror oscillation: Controlled cold-atom Zitterbewegung, Phys. Rev. A **81**, 023608 (2010).
[11] X. Zhang and Z. Liu, Extremal Transmission and Beating Effect of Acoustic Waves in Two-Dimensional Sonic Crystals, Phys. Rev. Lett. **101**, 264303 (2008).
[12] X. Zhang, Observing Zitterbewegung for Photons near the Dirac Point of a Two-Dimensional Photonic Crystal, Phys. Rev. Lett. **100**, 113903 (2008).
[13] S. Longhi, Photonic analog of Zitterbewegung in binary waveguide arrays, Opt. Lett. **35**, 235 (2010).
[14] R. Gerritsma, G. Kirchmair, F. Zähringer, E. Solano, R. Blatt, and C. F. Roos, Quantum simulation of the Dirac equation, Nature (London) **463**, 68 (2010).
[15] S. Longhi, Quantum-optical analogies using photonic structure, Laser Photon. Rev. **3**, 243 (2009).
[16] R. W. Boyd, *Nonlinear Optics* (Academic, New York, 2003).
[17] S. Longhi, Zitterbewegung of optical pulses in nonlinear frequency conversion, J. Phys. B **43**, 205402 (2010).
[18] P. C. Mondal, C. Fontanesi, D. H. Waldeck, and R. Naaman, Field and chirality effects on electrochemical charge transfer rates, ACS Nano **9**, 3377 (2015).
[19] V. Kiran, S. P. Mathew, S. R. Cohen, I. Hernández Delgado, J. Lacour, and R. Naaman, Helicenes—a new class of organic spin filter, Adv. Mater. **28**, 1957 (2016).
[20] A. C. Aragonès, E. Medina, M. Ferrer-Huerta, N. Gimeno, M.Teixidó, J. L. Palma, N.Tao, J. M. Ugalde, E. Giralt, I. DíezPérez, and V. Mujica, Measuring the spin-polarization power of a single chiral molecule, Small **13**, 1602519 (2017).
[21] R. Gutierrez, E. Díaz, C. Gaul, T. Brumme, F. Domínguez-Adame, and G. Cuniberti, Modeling spin transport in helical fields: Derivation of an effective low-dimensional hamiltonian, J. Phys. Chem. C **117**, 22276 (2013).
[22] E. Medina, L. A. González-Arraga, D. Finkelstein-Shapiro, B. Berche, and V. Mujica, Continuum model for chiral induced spin selectivity in helical molecules, J. Chem. Phys. **142**, 194308 (2015).
[23] R. A. Caetano, Spin-current and spin-splitting in helicoidal molecules due to spin-orbit coupling, Sci. Rep. **6**, 23452 (2016).
[24] S. Behnia, S. Fathizadeh, and A. Akhshani, DNA spintronics: Charge and spin dynamics in DNA wires, J. Phys. Chem. C **120**, 2973 (2016).
[25] P. Albares, E. Díaz, J. M. Cerveró, F. Domínguez-Adame, E. Diez, and P. G. Estévez, Solitons in a nonlinear model of spin transport in helical molecules, Phys. Rev. E **97**, 022210 (2018).
[26] V. Galitski and I. B. Spielman, Spin-orbit coupling in quantum gases, Nature (London) **494**, 49 (2013).
[27] V. Achilleos, D. J. Frantzeskakis, P. G. Kevrekidis, and D. E. Pelinovsky, Matter-Wave Bright Solitons in Spin-Orbit Coupled Bose-Einstein Condensates, Phys. Rev. Lett. **110**, 264101 (2013).
[28] L. Wen, Q. Sun, Y. Chen, D.-S. Wang, J. Hu, H. Chen, W.-M. Liu, G. Juzeliūnas, B. A. Malomed, and A.-C. Ji, Motion of solitons in one-dimensional spin-orbit-coupled Bose-Einstein condensates, Phys. Rev. A **94**, 061602(R) (2016).
[29] H. Sakaguchi and B. A. Malomed, Flipping-shuttle oscillations of bright one- and two-dimensional solitons in spin-orbit-coupled Bose-Einstein condensates with Rabi mixing, Phys. Rev. A **96**, 043620 (2017).
[30] Y. V. Kartashov and V. V. Konotop, Solitons in Bose-Einstein Condensates with Helicoidal Spin-Orbit Coupling, Phys. Rev. Lett. **118**, 190401 (2017).
[31] A. V. Mikhaĭlov, Integrability of the two-dimensional Thirring model, Pis'ma Zh. Eksp. Teor. Fiz. **23**, 356 (1976) [JETP Lett. **23**, 320 (1976)].
[32] W. H. Steeb, W. Oevel, and W. Strampp, On the integrability of nonlinear Dirac equations, J. Math. Phys. **25**, 2331 (1984).

[33] E. Recami and G. Salesi, Field theory of the spinning electron: II–The new non-linear field equations, in *Electron Theory and Quantum Electrodynamics*, edited by J. P. Dowling, NATO ASI Series (Springer, Boston, MA, 1997), Vol. 358, pp. 253–260.

[34] S. V. Manakov, On the theory of two-dimensional stationary self-focusing of electromagnetic waves, Zh. Eksp. Teor. Fiz. **65**, 505 (1974) [Sov. Phys. JETP **38**, 248 (1974)].

[35] S. P. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, *Theory of Solitons: The Inverse Scattering Method* (Springer-Verlag, Berlin, 1984).

[36] V. E. Zakharov and A. B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, Zh. Eksp. Teor. Fiz. **61**, 118 (1971) [Sov. Phys. JETP **34**, 62 (1972)].

[37] A. P. Sheppard and Y. S. Kivshar, Polarized dark solitons in isotropic Kerr media, Phys. Rev. E **55**, 4773 (1997).

[38] R. Radhakrishnan, M. Lakshmanan, and J. Hietarinta, Exact soliton solutions of coupled nonlinear Schrödinger equations: Shape-changing collisions, logic gates, and partially coherent solitons, Phys. Rev. E **56**, 2213 (1997).

[39] N. Tsuchida, N-soliton collision in the Manakov model, Prog. Theor. Phys. **111**, 151 (2004).

[40] A. V. Mikhailov, The reduction problem and the inverse scattering method, Physica D (Amsterdam, Neth.) **3**, 73 (1981).

[41] V. E. Zakharov and E. I. Schulman, To the integrability of the system of two coupled nonlinear Schrödinger equations, Physica D (Amsterdam, Neth.) **4**, 270 (1982).

[42] A. A. Zabolotskii, Inverse scattering transform for the Yajima-Oikawa equations with nonvanishing boundary conditions, Phys. Rev. A **80**, 063616 (2009).

[43] T. Kosugi, Pauli equation on a curved surface and Rashba splitting on a corrugated surface, J. Phys. Soc. Jpn. **80**, 073602 (2011).

[44] A. A. Zabolotskii, Control of excitons in a bent bunch of molecular aggregates by dipole-dipole interaction with quantum dots, JETP **125**, 572 (2017).

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