Time-dependent physical Stokes parameters and degree of polarization of light

Jari Turunen*
Department of Physics and Mathematics, University of Eastern Finland, P.O. Box 111, FI-80101 Joensuu, Finland

Frank Wyrowski
Institute of Applied Physics, Friedrich-Schiller University, Albert-Einstein-Straße 15, D-07745 Jena, Germany

(Received 7 December 2018; published 13 February 2019)

We extend the concept of the Eberly-Wódkiewicz time-dependent physical spectrum of light to electromagnetic fields by considering the observable time dependence of four appropriately defined Stokes parameters. We also define the concept of time-dependent physical degree of polarization of light by means of these parameters, and discuss the measurement of these quantities using a tunable spectral filter and a detector with a finite response time. The concepts are illustrated by examples with spectrally phase-modulated fully and partially coherent model pulse trains.

DOI: 10.1103/PhysRevA.99.023824

I. INTRODUCTION

The statistical properties of nonstationary light fields, such as trains of ultrashort pulses, can be described (up to second order) by means of two-time and two-frequency correlation functions in temporal and spectral domains, respectively [1]. In their classic paper, Eberly and Wódkiewitz [2] introduced the concept of time-dependent physical spectrum of light to describe the outcome of an experiment in which a pulse train with any state of temporal (or spectral) coherence passes through a spectrally selective (tunable) filter and the time dependence of the output pulses is measured by a square-law detector. Because of the time-frequency uncertainly principle, high spectral resolution implies low temporal resolution and vice versa [2–5].

Correlations between the temporal intensity and spectrum of light are also manifested in spectrograms used in modern ultrashort-pulse characterization techniques [1,6]. If the pulse train is fully coherent (chronocyclic), the spectral and temporal amplitudes and phases of the input pulses can be determined from spectrograms by numerical retrieval algorithms, though the characterization of extremely complex pulses is a challenging task [7]. If the pulse train is partially coherent, an ensemble of pulses retrieved from single-shot measurements is typically needed to construct the second-order temporal and spectral correlations functions [1]. Such partially coherent pulse trains are generated by many important light sources, including supercontinuum pulses generated in bulk media and optical fibers [8], chirped-pulse amplification systems [9] and free-electron lasers [10]. These pulses can be extremely complex, individual pulses can be too weak to allow single-shot characterization, or the (mean) wavelength of the pulses may be in such a domain (such as the x-ray regime) that experimental generation of a spectrogram is difficult. Instead of direct construction of correlation functions by single-shot measurements and retrieval, alternative techniques can be applied at least in the visible region [11,12]. If one is primarily interested in the spectral and temporal intensity profiles of the pulse train (instead of coherence properties), straightforward measurements of time-dependent physical spectra using band-pass filters and square-law detectors remain important, and can sometimes be the only viable option.

The theory of time-dependent physical spectra was given by Eberly and Wódkiewitz in a scalar form, i.e., the polarization of light was not considered. It is, however, conceivable that the polarization properties of pulsed fields depend on frequency. For example, polarization-sensitive spectral modifications of light can take place in transmission though scattering media, or be introduced on purpose by passing the pulses through polarization-modulating optical elements with spectrally variable properties [13–16]. In this case not only the observable (physical) spectrum but also the state of polarization of pulses becomes time-dependent, which implies a time-dependent degree of polarization in the case of partially coherent pulse trains.

In this paper we present an electromagnetic extension of the Eberly-Wódkiewitz theory by introducing a full set of four "physical" Stokes parameters. These parameters enable us to characterize the time-dependent state of polarization of light, including the physically observable time-dependent degree of polarization of random pulse trains.

In general, the state of polarization of an ensemble of plane-wave electromagnetic pulses is described by a 2×2 polarization matrix with three independent components. This matrix can be defined for pulsed fields either in the temporal or the spectral domain [17]. Equivalently, in either domain, one can characterize the state of polarization by four Stokes parameters S_0, S_1, S_2, and S_3.

In order to fully characterize nonstationary fields of any (second-order) spectral and temporal coherence, one needs to employ matrices that depend on two frequency coordinates in the spectral domain, and on two time coordinates in the temporal domain. Such two-coordinate correlation matrices

*Jari.Turunen@uef.fi

are, in fact, required to determine the spectral state of polarization from the temporal properties of the pulse ensemble, or to determine the time-domain state of polarization from the spectral properties of the ensemble. In analogy with the single-coordinate case, the state of coherence and polarization of nonstationary light can alternatively be characterized by two-frequency and two-time Stokes parameters, which will defined in Secs. II and III, respectively, in analogy with their spatial two-point counterparts [18,19].

The rest of the present paper is organized as follows. In Sec. IV we extend the concept of Eberly-Wódkiewicz time-dependent physical spectrum into the electromagnetic case by introducing four time-dependent physical Stokes parameters $\mathcal{S}_0$, $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$. Of these, $\mathcal{S}_0$ is the electromagnetic extension of the Eberly-Wódkiewitz physical spectrum, while $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ specify the time-dependent physical state of polarization of the field. In Sec. V we show that these parameters can be measured by passing the pulsed field through suitable polarization-modulating elements, in analogy with the measurement of the state of polarization of stationary light (see, e.g., Sec. 6.2 of Ref. [20]). The type of effects that may be expected will be presented in Sec. VI by means of a mathematically convenient Gaussian Schell model for pulse trains, which is an electromagnetic extension of the model used to illustrate time-dependent physical spectra in Ref. [5].

II. SPECTRAL STOKES PARAMETERS

Considering pulsed optical fields in the space-frequency domain, we denote the three-dimensional electric field associated with a single realization (an individual pulse) at point $\mathbf{r}$ and frequency ω by a column vector $\mathbf{E}(\mathbf{r},\omega)$. Both coherence and polarization properties of the entire ensemble of realizations in the spectral domain are fully described by 3×3 cross-spectral density matrix (CSDM) defined as

$$\mathbf{W}(\mathbf{r}_1,\mathbf{r}_2;\omega_1,\omega_2) = \langle \mathbf{E}^*(\mathbf{r}_1;\omega_1)\mathbf{E}^{\mathrm{T}}(\mathbf{r}_2;\omega_2)\rangle, \qquad (1)$$

where the asterisk and superscript T mean the complex conjugate and the transpose, respectively, and the brackets denote ensemble averaging over all realizations. Because of Maxwell's divergence equation and the fact that the CSDM is Hermitian, only three of the nine components of $\mathbf{W}(\mathbf{r}_1,\mathbf{r}_2;\omega_1,\omega_2)$ are independent. It is therefore possible to characterize the coherence and polarization properties of the field using four space-frequency domain Stokes parameters $\mathcal{S}_j(\mathbf{r}_1,\mathbf{r}_2;\omega_1,\omega_2)$, $j=0,1,2,3$, that depend on two spatial positions and two frequencies [17].

In the present context we are interested in measurements of fields at a single spatial point $\mathbf{r}$, which we leave implicit from now on to shorten the notation (with the understanding that all quantities to be discussed may depend on $\mathbf{r}$). We assume that the three independent components of the CSDM are its cartesian xx, yy, and xy components, and therefore consider the two-frequency CSDM

$$\mathbf{W}(\omega_1,\omega_2) = \langle \mathbf{E}^*(\omega_1)\mathbf{E}^{\mathrm{T}}(\omega_2)\rangle = \begin{bmatrix} W_{xx}(\omega_1,\omega_2) & W_{xy}(\omega_1,\omega_2) \\ W_{yx}(\omega_1,\omega_2) & W_{yy}(\omega_1,\omega_2)\end{bmatrix}, \qquad (2)$$

where $\mathbf{W}^*(\omega_2,\omega_1) = \mathbf{W}(\omega_1,\omega_2)$ due to Hermiticity. From now on we will also assume that the field is beamlike, so that the remaining components of the full 3×3 CSDM can be neglected; collimated beams are typically used in practical measurements of the spectrum and the state of polarization. However, the results that follow can be extended to nonparaxial fields without formal difficulty.

The spectral polarization matrix of a pulsed beam is defined by setting $\omega_1=\omega_2$ in (2):

$$\mathbf{\Phi}(\omega) = \mathbf{W}(\omega,\omega) = \begin{bmatrix} \Phi_{xx}(\omega) & \Phi_{xy}(\omega) \\ \Phi_{yx}(\omega) & \Phi_{yy}(\omega)\end{bmatrix}, \qquad (3)$$

where $\Phi^*_{yx}(\omega) = \Phi_{xy}(\omega)$. The spectral density (or spectrum) of the pulsed beam is defined as

$$S_0(\omega) = \mathrm{tr}\,\mathbf{W}(\omega,\omega) = \mathrm{tr}\,\mathbf{\Phi}(\omega), \qquad (4)$$

where tr denotes the trace of a matrix, and the spectral degree of polarization of a beamlike field is

$$P(\omega) = \left[1 - \frac{4\det\mathbf{\Phi}(\omega)}{\mathrm{tr}^2\mathbf{\Phi}(\omega)}\right]^{1/2}. \qquad (5)$$

Finally, one can introduce the two-frequency Stokes parameters

$$\mathcal{S}_0(\omega_1,\omega_2) = W_{xx}(\omega_1,\omega_2) + W_{yy}(\omega_1,\omega_2), \qquad (6a)$$

$$\mathcal{S}_1(\omega_1,\omega_2) = W_{xx}(\omega_1,\omega_2) - W_{yy}(\omega_1,\omega_2), \qquad (6b)$$

$$\mathcal{S}_2(\omega_1,\omega_2) = W_{xy}(\omega_1,\omega_2) + W_{yx}(\omega_1,\omega_2), \qquad (6c)$$

$$\mathcal{S}_3(\omega_1,\omega_2) = i[W_{yx}(\omega_1,\omega_2) - W_{xy}(\omega_1,\omega_2)] \qquad (6d)$$

as extensions of the more usual single-frequency Stokes parameters

$$S_0(\omega) = \Phi_{xx}(\omega) + \Phi_{yy}(\omega), \qquad (7a)$$

$$S_1(\omega) = \Phi_{xx}(\omega) - \Phi_{yy}(\omega), \qquad (7b)$$

$$S_2(\omega) = \Phi_{xy}(\omega) + \Phi_{yx}(\omega), \qquad (7c)$$

$$S_3(\omega) = i[\Phi_{yx}(\omega) - \Phi_{xy}(\omega)] \qquad (7d)$$

for pulsed fields. This extension is fully analogous to the generalization introduced in Ref. [18] in the spatial domain. Obviously, the first single-point Stokes parameter $S_0(\omega)$ is precisely the spectrum defined in (4), and the spectral degree of polarization in (5) takes the form

$$P(\omega) = \frac{\left[S_1^2(\omega) + S_2^2(\omega) + S_3^2(\omega)\right]^{1/2}}{S_0(\omega)} \qquad (8)$$

when expressed in terms of the single-frequency Stokes parameters.

III. TEMPORAL STOKES PARAMETERS

Second-order correlations of pulsed electromagnetic fields in the time domain are described by the mutual coherence matrix (MCM)

$$\mathbf{\Gamma}(t_1,t_2) = \langle \mathbf{E}^*(t_1)\mathbf{E}^{\mathrm{T}}(t_2)\rangle = \begin{bmatrix} \Gamma_{xx}(t_1,t_2) & \Gamma_{xy}(t_1,t_2) \\ \Gamma_{yx}(t_1,t_2) & \Gamma_{yy}(t_1,t_2)\end{bmatrix}, \qquad (9)$$

where

$$\mathbf{E}(t) = \int_0^\infty \mathbf{E}(\omega)\exp(-i\omega t)\,d\omega \tag{10}$$

is the time-domain electric field vector (we have again dropped the position dependence for brevity). By setting $t_1 = t_2$ in (9) we obtain the temporal polarization matrix

$$\mathbf{J}(t) = \boldsymbol{\Gamma}(t,t) = \begin{bmatrix} J_{xx}(t) & J_{xy}(t) \\ J_{yx}(t) & J_{yy}(t) \end{bmatrix}. \tag{11}$$

The temporal intensity distribution is defined as

$$I(t) = \mathrm{tr}\,\boldsymbol{\Gamma}(t,t) = \mathrm{tr}\,\mathbf{J}(t) \tag{12}$$

and the time-domain degree of polarization as

$$P(t) = \left[1 - \frac{4\det\mathbf{J}(t)}{\mathrm{tr}^2\mathbf{J}(t)}\right]^{1/2}. \tag{13}$$

In analogy with definitions (6a)–(6d) in the spectral domain, we can introduce the two-time Stokes parameters

$$\mathcal{S}_0(t_1,t_2) = \Gamma_{xx}(t_1,t_2) + \Gamma_{yy}(t_1,t_2), \tag{14a}$$

$$\mathcal{S}_1(t_1,t_2) = \Gamma_{xx}(t_1,t_2) - \Gamma_{yy}(t_1,t_2), \tag{14b}$$

$$\mathcal{S}_2(t_1,t_2) = \Gamma_{xy}(t_1,t_2) + \Gamma_{yx}(t_1,t_2), \tag{14c}$$

$$\mathcal{S}_3(t_1,t_2) = i[\Gamma_{yx}(t_1,t_2) - \Gamma_{xy}(t_1,t_2)] \tag{14d}$$

as extensions of the equal-time Stokes parameters

$$S_0(t) = J_{xx}(t) + J_{yy}(t), \tag{15a}$$

$$S_1(t) = J_{xx}(t) - J_{yy}(t), \tag{15b}$$

$$S_2(t) = J_{xy}(t) + J_{yx}(t), \tag{15c}$$

$$S_3(t) = i[J_{yx}(t) - J_{xy}(t)], \tag{15d}$$

with which the time-domain degree of polarization in (13) can be expressed in the alternative form

$$P(t) = \frac{\left[S_1^2(t) + S_2^2(t) + S_3^2(t)\right]^{1/2}}{S_0(t)}. \tag{16}$$

Obviously, the first equal-time Stokes parameter $S_0(t)$ is precisely the temporal intensity distribution defined in (12).

Using the definitions in Eqs. (2) and (9), Eq. (10), and its Fourier inverse, we obtain the relations

$$\boldsymbol{\Gamma}(t_1,t_2) = \iint_0^\infty \mathbf{W}(\omega_1,\omega_2)\exp[i(\omega_1 t_1 - \omega_2 t_2)]\,d\omega_1\,d\omega_2 \tag{17}$$

and

$$\mathbf{W}(\omega_1,\omega_2) = \frac{1}{(2\pi)^2}\iint_{-\infty}^\infty \boldsymbol{\Gamma}(t_1,t_2) \times \exp[i(\omega_2 t_2 - \omega_1 t_1)]\,dt_1\,dt_2 \tag{18}$$

between the two-time MCM and the two-frequency CSDM. Since these relations apply to the matrices elementwise, and the Stokes parameters are linear combinations of the matrix elements, we also have the relations

$$\mathcal{S}_j(t_1,t_2) = \iint_0^\infty \mathcal{S}_j(\omega_1,\omega_2)\exp[i(\omega_1 t_1 - \omega_2 t_2)]\,d\omega_1\,d\omega_2 \tag{19}$$

and

$$\mathcal{S}_j(\omega_1,\omega_2) = \frac{1}{(2\pi)^2}\iint_{-\infty}^\infty \mathcal{S}_j(t_1,t_2) \times \exp[i(\omega_2 t_2 - \omega_1 t_1)]\,dt_1\,dt_2 \tag{20}$$

between the two-time and two-frequency Stokes parameters $\mathcal{S}_j$, $j = 0, 1, 2, 3$.

We can write $t_1 = t_2$ in (17) to obtain an expression for the temporal polarization matrix in terms of the CSDM:

$$\mathbf{J}(t) = \iint_0^\infty \mathbf{W}(\omega_1,\omega_2)\exp[i(\omega_1 - \omega_2)t]\,d\omega_1\,d\omega_2. \tag{21}$$

We can also write $\omega_1 = \omega_2$ in (18) to express the spectral polarization matrix as

$$\boldsymbol{\Phi}(\omega) = \frac{1}{(2\pi)^2}\iint_{-\infty}^\infty \boldsymbol{\Gamma}(t_1,t_2)\exp[i\omega(t_2 - t_1)]\,dt_1\,dt_2. \tag{22}$$

Analogously, we can set $t_1 = t_2$ is (19) to get

$$S_j(t) = \iint_0^\infty \mathcal{S}_j(\omega_1,\omega_2)\exp[i(\omega_1 - \omega_2)t]\,d\omega_1\,d\omega_2, \tag{23}$$

or $\omega_1 = \omega_2$ in (20) to arrive at

$$S_j(\omega) = \frac{1}{(2\pi)^2}\iint_{-\infty}^\infty \mathcal{S}_j(t_1,t_2)\exp[i\omega(t_2 - t_1)]\,dt_1\,dt_2. \tag{24}$$

Thus we can express the temporal polarization matrix with the aid of the CSDM, and the spectral polarization matrix with the aid of the MCM [17]. Similarly, the equal-time Stokes parameters can be expressed in terms of the two-frequency Stokes parameters, and the single-frequency Stokes parameters in terms of the two-time Stokes parameters. However, the knowledge of the spectral polarization properties of the field is not sufficient for the determination of the temporal polarization properties, or vice versa.

IV. SPECTRALLY FILTERED STOKES PARAMETERS

Let us assume that the field $\mathbf{E}(\omega)$ passes through a deterministic frequency-selective filter characterized by a spectral 2×2 transmission matrix $\mathbf{T}(\omega,\omega_\mathrm{F})$, where ω_F denotes the "setting frequency" of the filter in the same sense as in Ref. [2]. In the determination of the physical spectrum $\mathbf{T}(\omega,\omega_\mathrm{F})$ represents a narrow-band filter, for instance a tunable Fabry-Perot filter, a prism, or a grating [2–4], with the passband centered around $\omega = \omega_\mathrm{F}$. In the ideal case $\mathbf{T}(\omega,\omega_\mathrm{F}) = \mathbf{I}T(\omega,\omega_\mathrm{F})$, where $\mathbf{I}$ is a unit matrix; such a polarization-insensitive response is well approximated by a Fabry-Perot filter. However, the spectral response of a grating spectrometer is typically polarization-dependent and one should write $\mathbf{T}(\omega,\omega_\mathrm{F}) = \mathbf{G}(\omega)T(\omega,\omega_\mathrm{F})$, where $\mathbf{G}(\omega)$ is the grating response and $T(\omega,\omega_\mathrm{F})$ is determined by the spectral resolution of the device. If conical mounting of the grating(s) is used, $\mathbf{G}(\omega)$ is a full 2×2 matrix. In a single-grating spectrometer with nonconical mounting it is a diagonal matrix, where the diagonal elements are (generally complex-valued) functions with magnitudes $\sqrt{\eta_\mathrm{TE}(\omega)}$ and $\sqrt{\eta_\mathrm{TM}(\omega)}$, where $\eta_\mathrm{TE}(\omega)$ and $\eta_\mathrm{TM}(\omega)$ are the spectral diffraction efficiency curves of the grating under TE and TM polarized illumination, respectively.

A. Basic definitions

The spectral field after transmission though the filter is given by

$$\mathbf{E}(\omega, \omega_{\mathrm{F}}) = \mathbf{T}(\omega, \omega_{\mathrm{F}})\mathbf{E}(\omega), \tag{25}$$

where we have indicated explicitly the dependence of the output field on the setting frequency. Using the definition in (2) we then find that the CSDM of the transmitted field is

$$\mathbf{W}(\omega_1, \omega_2, \omega_{\mathrm{F}}) = \langle \mathbf{E}^*(\omega_1, \omega_{\mathrm{F}})\mathbf{E}^{\mathrm{T}}(\omega_2, \omega_{\mathrm{F}})\rangle = \mathbf{T}^*(\omega_1, \omega_{\mathrm{F}})\mathbf{W}(\omega_1, \omega_2)\mathbf{T}^{\mathrm{T}}(\omega_2, \omega_{\mathrm{F}}). \tag{26}$$

The two-frequency spectral Stokes parameters of this field are defined as

$$\mathcal{S}_0(\omega_1, \omega_2, \omega_{\mathrm{F}}) = W_{xx}(\omega_1, \omega_2, \omega_{\mathrm{F}}) + W_{yy}(\omega_1, \omega_2, \omega_{\mathrm{F}}), \tag{27a}$$

$$\mathcal{S}_1(\omega_1, \omega_2, \omega_{\mathrm{F}}) = W_{xx}(\omega_1, \omega_2, \omega_{\mathrm{F}}) - W_{yy}(\omega_1, \omega_2, \omega_{\mathrm{F}}), \tag{27b}$$

$$\mathcal{S}_2(\omega_1, \omega_2, \omega_{\mathrm{F}}) = W_{xy}(\omega_1, \omega_2, \omega_{\mathrm{F}}) + W_{yx}(\omega_1, \omega_2, \omega_{\mathrm{F}}), \tag{27c}$$

$$\mathcal{S}_3(\omega_1, \omega_2, \omega_{\mathrm{F}}) = i[W_{yx}(\omega_1, \omega_2, \omega_{\mathrm{F}}) - W_{xy}(\omega_1, \omega_2, \omega_{\mathrm{F}})] \tag{27d}$$

in analogy with Eqs. (6a)–(6d). The single-frequency Stokes parameters of the transmitted field and its spectral degree of polarization can be defined in a strictly corresponding manner.

The mutual coherence matrix $\mathbf{\Gamma}(t_1, t_2, \omega_{\mathrm{F}})$ of the field transmitted by the filter is obtained by applying (the obvious extension of) (17) to (26). The associated two-time temporal Stokes parameters $\mathcal{S}_j(t_1, t_2, \omega_{\mathrm{F}})$ can then be defined in analogy with Eqs. (14a)–(14d), or alternatively by applying relations of the form of (19) to Eqs. (27a)–(27d). In the present context we are primarily interested in the equal-time Stokes parameters observed after the spectral filter, defined as

$$S_0(t, \omega_{\mathrm{F}}) = J_{xx}(t, \omega_{\mathrm{F}}) + J_{yy}(t, \omega_{\mathrm{F}}), \tag{28a}$$

$$S_1(t, \omega_{\mathrm{F}}) = J_{xx}(t, \omega_{\mathrm{F}}) - J_{yy}(t, \omega_{\mathrm{F}}), \tag{28b}$$

$$S_2(t, \omega_{\mathrm{F}}) = J_{xy}(t, \omega_{\mathrm{F}}) + J_{yx}(t, \omega_{\mathrm{F}}), \tag{28c}$$

$$S_3(t, \omega_{\mathrm{F}}) = i[J_{yx}(t, \omega_{\mathrm{F}}) - J_{xy}(t, \omega_{\mathrm{F}})], \tag{28d}$$

where $J_{ij}(t, \omega_{\mathrm{F}})$ are elements of the physically observed polarization matrix

$$\mathbf{J}(t, \omega_{\mathrm{F}}) = \iint_0^\infty \mathbf{T}^*(\omega_1, \omega_{\mathrm{F}})\mathbf{W}(\omega_1, \omega_2)\mathbf{T}^{\mathrm{T}}(\omega_2, \omega_{\mathrm{F}}) \times \exp[i(\omega_1 - \omega_2)t]\, d\omega_1\, d\omega_2. \tag{29}$$

The parameter $\mathcal{S}_0(t, \omega_{\mathrm{F}}) = \mathrm{tr}\, \mathbf{J}(t, \omega_{\mathrm{F}})$ is the electromagnetic extension of the Eberly-Wódkiewitz time-dependent physical spectrum of light. The parameters $\mathcal{S}_j(t, \omega_{\mathrm{F}})$, $j = 1, 2, 3$, characterize the physically observable time-dependent polarization state of the field, reflecting the time-frequency uncertainty relationship. With the aid of these physical Stokes parameters, we can define the time-dependent physical degree of polarization of light as

$$P(t, \omega_{\mathrm{F}}) = \frac{\left[S_1^2(t, \omega_{\mathrm{F}}) + S_2^2(t, \omega_{\mathrm{F}}) + S_3^2(t, \omega_{\mathrm{F}})\right]^{1/2}}{S_0(t, \omega_{\mathrm{F}})}, \tag{30}$$

in analogy with (16). When the passband of the filter is narrow, this quantity is a measure of the spectral degree of polarization, which generally depends on the setting frequency ω_{F}. Of course, most of the information on any possible time dependence of the degree of polarization is then lost.

In the definitions given above the physical Stokes parameters were expressed in terms of the two-frequency CSDM (or two-frequency Stokes parameters) of the incident field. Alternatively, they can be expressed in terms of the two-time MCM (or two-time Stokes parameters) of the incident field since the CSDM and MCM are connected by Eqs. (17) and (18). Let us define the temporal filter response matrix by a Fourier integral

$$\mathbf{t}(t, \omega_{\mathrm{F}}) = \int_0^\infty \mathbf{T}(\omega, \omega_{\mathrm{F}}) \exp(-i\omega t)\, d\omega \tag{31}$$

with inverse

$$\mathbf{T}(\omega, \omega_{\mathrm{F}}) = \frac{1}{2\pi} \int_{-\infty}^\infty \mathbf{t}(t, \omega_{\mathrm{F}}) \exp(i\omega t)\, dt, \tag{32}$$

where (and in what follows) the lower temporal integration limit extends to $-\infty$ only formally. The temporal electric field after the filter is, in view of (25) and the convolution theorem, given by

$$\mathbf{E}(t, \omega_{\mathrm{F}}) = \frac{1}{2\pi} \int_{-\infty}^\infty \mathbf{t}(t - t', \omega_{\mathrm{F}})\mathbf{E}(t')\, dt' \tag{33}$$

and the observed temporal polarization matrix

$$\mathbf{J}(t, \omega_{\mathrm{F}}) = \langle \mathbf{E}^*(t, \omega_{\mathrm{F}})\mathbf{E}^{\mathrm{T}}(t, \omega_{\mathrm{F}})\rangle \tag{34}$$

takes the form

$$\mathbf{J}(t, \omega_{\mathrm{F}}) = \frac{1}{(2\pi)^2} \iint_{-\infty}^\infty \mathbf{t}^*(t - t_1, \omega_{\mathrm{F}}) \times \mathbf{\Gamma}(t_1, t_2)\mathbf{t}^{\mathrm{T}}(t - t_2, \omega_{\mathrm{F}})\, dt_1\, dt_2, \tag{35}$$

which is the desired alternative representation of (29). Whether the form given in (29) or the form in (35) is more convenient depends on the circumstances and the models under consideration.

B. Genuine representations

It should be stressed that not any form of $\mathbf{W}(\omega_1, \omega_2)$ defined in (2) is physically meaningful since the CSDM must be non-negative definite. In fact, the CSDM must be of "genuine" form as discussed in the spatial domain in Refs. [21,22]. On transforming the spatial-domain definition of a genuine CSDM into the spectral domain we have

$$\mathbf{W}(\omega_1, \omega_2) = \int_D \mathbf{H}^*(\omega_1, v)\mathbf{p}(v)\mathbf{H}^{\mathrm{T}}(\omega_2, v)\, dv, \tag{36}$$

where the kernel $\mathbf{H}(\omega, v)$ is arbitrary, the matrix $\mathbf{p}(v)$ has non-negative diagonal elements and trace, and D is the domain where $\mathbf{p}(v)$ is nonzero.

The two-frequency Stokes parameters also have genuine representations, which follow immediately by substituting appropriate elements of the CSDM from (36) into Eqs. (6a)–(6d). On the other hand, on inserting from (36) into (17) we find that the MCM has a genuine representation

$$\mathbf{\Gamma}(t_1, t_2) = \int_D \mathbf{h}^*(t_1, v)\mathbf{p}(v)\mathbf{h}^{\mathrm{T}}(t_2, v)\, dv, \tag{37}$$

where

$$\mathbf{h}(t, v) = \int_0^\infty \mathbf{H}(\omega, v) \exp(-i\omega t)\, d\omega. \tag{38}$$

Also the two-time Stokes parameters have genuine forms, which follow by substituting from (37) into Eqs. (14a)–(14d).

An expression for the physically observable polarization matrix in terms of the genuine representation of the CSDM of the incident field can be obtained straightforwardly on inserting from (36) into (29). Rearranging the order of integrations and defining

$$\mathbf{h}(t, \omega_F, v) = \int_0^\infty \mathbf{T}(\omega, \omega_F)\mathbf{H}(\omega, v) \exp(-i\omega t)\, d\omega \tag{39}$$

we arrive at

$$\mathbf{J}(t, \omega_F) = \int_D \mathbf{h}^*(t, \omega_F, v)\mathbf{p}(v)\mathbf{h}^T(t, \omega_F, v)\, dv. \tag{40}$$

Alternatively, we may insert from (37) into (35). If we then define

$$\mathbf{h}(t, \omega_F, v) = \frac{1}{2\pi}\int_{-\infty}^\infty \mathbf{t}(t - t', \omega_F)\mathbf{h}(t', v)\, dt', \tag{41}$$

we arrive at (40) again. The equivalence of the definitions (39) and (41) can be easily seen with the aid of (31) or (32).

The genuine representations (36) and (37) have some important and physically transparent special cases, which follow from specific forms of the matrix $\mathbf{H}$. These special cases, sometimes called elementary-field representations, have been considered in some detail (in the scalar domain) in Ref. [23]. Even though these representations are not general, they apply to a large class of fields with practical significance in spectral, temporal, and spatial domains, including several mathematically convenient model fields.

Let us first assume that the matrix $\mathbf{h}(t, v)$ is of the shift-invariant form $\mathbf{h}(t, v) = \mathbf{e}(t - v)$. Then, in view of (37), the MCM has an expression

$$\mathbf{\Gamma}(t_1, t_2) = \int_D \mathbf{e}^*(t_1 - v)\mathbf{p}(v)\mathbf{e}^T(t_2 - v)\, dv. \tag{42}$$

If $\mathbf{p}(v) = \mathbf{I}\delta(v)$, where $\mathbf{I}$ is a unit matrix, we obtain $\mathbf{\Gamma}(t_1, t_2) = \mathbf{e}^*(t_1)\mathbf{e}^T(t_2)$, which describes a completely coherent and polarized field. Since $\mathbf{J}(t) = \mathbf{e}^*(t)\mathbf{e}^T(t)$, the polarization properties (all four equal-time Stokes parameters) of this field may depend on time. Hence, in general, (42) describes a partially coherent and partially polarized pulsed beam as a continuous incoherent superposition of fully coherent and polarized "elementary pulses" that are all identical but centered temporally at instants of time $t = v$. The elementary pulses are weighted by the matrix $\mathbf{p}(v)$, where v is interpreted physically as a time variable. In this representation the physical polarization matrix is of the form of (40) with

$$\mathbf{h}(t, \omega_F, v) = \frac{1}{2\pi}\int_{-\infty}^\infty \mathbf{t}(t - t', \omega_F)\mathbf{e}(t' - v)\, dt'. \tag{43}$$

This expression can be interpreted as the physically observable elementary pulse.

We may alternatively assume that the matrix $\mathbf{H}(\omega, v)$ is of the shift-invariant form $\mathbf{H}(\omega, v) = \mathbf{g}(\omega - v)$. Then, according to (36),

$$\mathbf{W}(\omega_1, \omega_2) = \int_D \mathbf{g}^*(\omega_1 - v)\mathbf{p}(v)\mathbf{g}^T(\omega_2 - v)\, dv. \tag{44}$$

Hence the CSDM is an incoherent superposition of a continuum of spectrally shifted but functionally identical elementary contributions $\mathbf{g}(\omega)$, which are centered at frequencies $\omega = v$ and weighted by the matrix $\mathbf{p}(v)$, where v is now interpreted physically as a frequency variable. Defining

$$\mathbf{h}(t, \omega_F, v) = \int_0^\infty \mathbf{T}(\omega, \omega_F)\mathbf{e}(\omega - v) \exp(-i\omega t)\, d\omega \tag{45}$$

we again have (40).

C. Coherent-mode representations

The time-dependent physical polarization matrix and the Stokes parameters can be expressed in yet other fully general forms, which are sometimes convenient. Every (genuine) two-frequency CSDM has a unique series representation of the form [24]

$$\mathbf{W}(\omega_1, \omega_2) = \sum_n \alpha_n \boldsymbol{\psi}_n^*(\omega_1)\boldsymbol{\psi}_n^T(\omega_2), \tag{46}$$

where the (column) vectors $\boldsymbol{\psi}_n(\omega)$ represent fully coherent electric fields and α_n are non-negative coefficients. The coefficients α_n and the so-called coherent modes $\boldsymbol{\psi}_n(\omega)$ are the eigenvalues and eigenfunctions, respectively, of a Fredholm integral equation

$$\int_0^\infty \boldsymbol{\psi}_n^T(\omega_1)\mathbf{W}(\omega_1, \omega_2)\, d\omega_1 = \alpha_n \boldsymbol{\psi}_n^T(\omega_2). \tag{47}$$

On inserting from (46) into (21) we arrive immediately at a representation

$$\mathbf{J}(t, \omega_F) = \sum_n \alpha_n \boldsymbol{\psi}_n^*(t_1, \omega_F)\boldsymbol{\psi}_n^T(t_2, \omega_F), \tag{48}$$

where the vectors

$$\boldsymbol{\psi}_n(t, \omega_F) = \int_0^\infty \mathbf{T}(\omega, \omega_F)\boldsymbol{\psi}_n(\omega) \exp(-i\omega t)\, d\omega \tag{49}$$

represent the observable coherent modes transmitted by the filter.

It follows directly from Eqs. (17) and (46) that the MCM has a coherent-mode representation [24]

$$\mathbf{\Gamma}(t_1, t_2) = \sum_n \alpha_n \boldsymbol{\psi}_n^*(t_1)\boldsymbol{\psi}_n^T(t_2), \tag{50}$$

where

$$\boldsymbol{\psi}_n(t) = \int_0^\infty \boldsymbol{\psi}_n(\omega) \exp(-i\omega t)\, d\omega. \tag{51}$$

Inserting this representation into (35) and defining

$$\boldsymbol{\psi}_n(t, \omega_F) = \frac{1}{2\pi}\int_{-\infty}^\infty \mathbf{T}(t - t', \omega_F)\boldsymbol{\psi}_n(t')\, dt' \tag{52}$$

leads to (48). Hence we again have two alternative representations for the time-dependent physical polarization matrix.

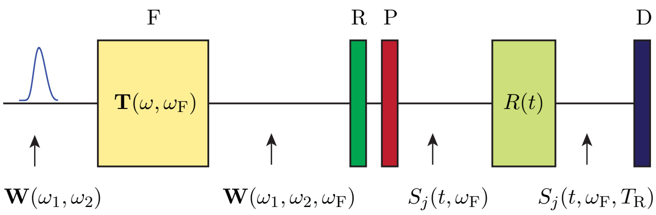

FIG. 1. Schematic diagram for the measurement of time-dependent physical Stokes parameters of a pulse train using a spectral filter F with spectral response $\mathbf{T}(\omega, \omega_F)$, a retarder R with retardation δ, a polarizer P oriented at an angle θ, and a square-law detector D with temporal response $R(t)$.

V. OBSERVABLE STOKES PARAMETERS

The measurement of the time-dependent physical Stokes parameters can be performed in full analogy with the measurement of the usual equal-time Stokes parameters of stationary fields (see Fig. 1), using linear polarization-modulating elements to convert polarization modulation into intensity variations that can be observed using a square-law detector (see Sec. 6.2 of Ref. [20]). Typically one employs a cascade of a compensator with retardation δ and a linear polarizer with its transmission axis rotated at an angle θ with respect to the x axis. The action of such a cascade is described by a Jones matrix

$$\mathbf{K} = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta\exp(i\delta) \\ \sin\theta\cos\theta & \sin^2\theta\exp(i\delta) \end{bmatrix}, \tag{53}$$

which generally depends on frequency since δ varies with ω at least weakly. However, we leave this dependence implicit (assuming effectively that the retarder is achromatic over the spectral range of concern).

One of the typical schemes to measure the Stokes parameters is to carry out six separate measurements M_q with different forms $\mathbf{K}_q$ of the transformation matrix: M_1 with parameter choices $(\theta, \delta) = (0, 0)$, M_2 with $(\theta, \delta) = (\pi/2, 0)$, M_3 with $(\theta, \delta) = (\pi/4, 0)$, M_4 with $(\theta, \delta) = (-\pi/4, 0)$, M_5 with $(\theta, \delta) = (\pi/4, \pi/2)$, and M_6 with $(\theta, \delta) = (-\pi/4, \pi/2)$. The observed time-dependent intensity distributions then become

$$I_q(t, \omega_F) = \mathrm{tr}\big[\mathbf{K}_q^*\mathbf{J}(t, \omega_F)\mathbf{K}_q^{\mathrm{T}}\big]. \tag{54}$$

Explicitly, we have

$$I_1(t, \omega_F) = J_{xx}(t, \omega_F), \tag{55a}$$

$$I_2(t, \omega_F) = J_{yy}(t, \omega_F), \tag{55b}$$

$$I_3(t, \omega_F) = \frac{1}{2}\mathrm{tr}\,\mathbf{J}(t, \omega_F) + \frac{1}{2}[J_{xy}(t, \omega_F) + J_{yx}(t, \omega_F)], \tag{55c}$$

$$I_4(t, \omega_F) = \frac{1}{2}\mathrm{tr}\,\mathbf{J}(t, \omega_F) - \frac{1}{2}[J_{xy}(t, \omega_F) + J_{yx}(t, \omega_F)], \tag{55d}$$

$$I_5(t, \omega_F) = \frac{1}{2}\mathrm{tr}\,\mathbf{J}(t, \omega_F) - \frac{i}{2}[J_{yx}(t, \omega_F) - J_{xy}(t, \omega_F)], \tag{55e}$$

$$I_6(t, \omega_F) = \frac{1}{2}\mathrm{tr}\,\mathbf{J}(t, \omega_F) + \frac{i}{2}[J_{yx}(t, \omega_F) - J_{xy}(t, \omega_F)], \tag{55f}$$

from which the time-dependent physical Stokes parameters can be retrieved:

$$S_0(t, \omega_F) = I_1(t, \omega_F) + I_2(t, \omega_F), \tag{56a}$$

$$S_1(t, \omega_F) = I_1(t, \omega_F) - I_2(t, \omega_F), \tag{56b}$$

$$S_2(t, \omega_F) = I_3(t, \omega_F) - I_4(t, \omega_F), \tag{56c}$$

$$S_3(t, \omega_F) = I_6(t, \omega_F) - I_5(t, \omega_F). \tag{56d}$$

The use of measurement schemes described above requires, of course, that the detector is sufficiently fast to follow the temporal variations of the spectrally filtered field. This is true, to a good approximation, if the filter passband is sufficiently narrow. If not, the detector will perform some time averaging. Let us assume that the detector has an impulse response function $R(t)$ characterized by response time T_R. Then the detected intensity distributions are

$$I_q(t, \omega_F, R) = \int_{-\infty}^{\infty} R(t - t')I_q(t', \omega_F)\, dt', \tag{57}$$

where $I_q(t, \omega_F)$, $q = 1, \ldots, 6$, represent the quantities defined in Eqs. (55a)–(55f). The polarization matrix observed with a detector with finite response time is

$$\mathbf{J}(t, \omega_F, R) = \int_{-\infty}^{\infty} R(t - t')\mathbf{J}(t', \omega_F)\, dt'. \tag{58}$$

More explicit forms of $\mathbf{J}(t, \omega_F, T_R)$ can be obtained using any of the expressions for $\mathbf{J}(t, \omega_F)$ derived in Sec. IV. For example, using (29), defining the frequency response $Q(\omega)$ of the time-averaging detector via

$$R(t) = \int_0^{\infty} Q(\omega)\exp(-i\omega t)\, d\omega \tag{59}$$

and assuming an ideal spectral device with $\mathbf{T} = \mathbf{I}T(\omega, \omega_F)$, we can express the observed polarization matrix in the form

$$\mathbf{J}(t, \omega_F, R) = 2\pi \iint_0^{\infty} \mathbf{W}(\omega_1, \omega_2)T^*(\omega_1, \omega_F)T(\omega_2, \omega_F) \times Q(\omega_2 - \omega_1)\exp[-i(\omega_2-\omega_1)t]\, d\omega_1\, d\omega_2. \tag{60}$$

The Stokes parameters are then

$$S_j(t, \omega_F, R) = 2\pi \iint_0^{\infty} \mathrm{tr}[\mathbf{L}_j(\omega_1, \omega_2)]T^*(\omega_1, \omega_F)T(\omega_2, \omega_F) \times Q(\omega_2 - \omega_1)\exp[-i(\omega_2 - \omega_1)t]\, d\omega_1\, d\omega_2, \tag{61}$$

where $j = 0, \ldots, 3$ and

$$\mathbf{L}_0(\omega_1, \omega_2) = \mathbf{K}_1^*\mathbf{W}(\omega_1, \omega_2)\mathbf{K}_1^{\mathrm{T}} + \mathbf{K}_2^*\mathbf{W}(\omega_1, \omega_2)\mathbf{K}_2^{\mathrm{T}}, \tag{62a}$$

$$\mathbf{L}_1(\omega_1, \omega_2) = \mathbf{K}_1^*\mathbf{W}(\omega_1, \omega_2)\mathbf{K}_1^{\mathrm{T}} - \mathbf{K}_2^*\mathbf{W}(\omega_1, \omega_2)\mathbf{K}_2^{\mathrm{T}}, \tag{62b}$$

$$\mathbf{L}_2(\omega_1, \omega_2) = \mathbf{K}_3^*\mathbf{W}(\omega_1, \omega_2)\mathbf{K}_3^{\mathrm{T}} - \mathbf{K}_4^*\mathbf{W}(\omega_1, \omega_2)\mathbf{K}_4^{\mathrm{T}}, \tag{62c}$$

$$\mathbf{L}_3(\omega_1, \omega_2) = \mathbf{K}_6^*\mathbf{W}(\omega_1, \omega_2)\mathbf{K}_6^{\mathrm{T}} - \mathbf{K}_5^*\mathbf{W}(\omega_1, \omega_2)\mathbf{K}_5^{\mathrm{T}}. \tag{62d}$$

The time-dependent physical degree of polarization in the presence of a detector with finite response time is defined, in analogy with (30), as

$$P(t,\omega_F,R) = \frac{\left[S_1^2(t,\omega_F,R)+S_2^2(t,\omega_F,R)+S_3^2(t,\omega_F,R)\right]^{1/2}}{S_0(t,\omega_F,R)}. \quad (63)$$

In summary, the "work flow" for the determination of the observable Stokes parameters proceeds as follows:

(1) Construction of a model for the pulse train to be measured either is spectral or in temporal domain, i.e., the matrix $\mathbf{W}(\omega_1,\omega_2)$ or $\mathbf{\Gamma}(t_1,t_2)$.

(2) Model for the spectral filter: matrix $\mathbf{T}(\omega,\omega_F)$. Determination of the spectrally filtered polarization matrix $\mathbf{J}(t,\omega_F)$.

(3) Model for the impulse response of the detector: function $R(t)$. Determination of the observable polarization matrix $\mathbf{J}(t,\omega_F,R)$, the associated Stokes parameters, and the degree of polarization.

VI. APPLICATION TO VECTORIAL SCHELL-MODEL PULSE TRAINS

We proceed to apply the formalism introduced above to a class of vectorial model fields described in the spectral domain by

$$\mathbf{W}(\omega_1,\omega_2)=\mathbf{M}^*(\omega_1)\mathbf{\Phi}^{(0)}\mathbf{M}^{\mathrm{T}}(\omega_2)W^{(0)}(\omega_1,\omega_2), \quad (64)$$

where $\mathbf{M}(\omega)$ is a deterministic matrix, $\mathbf{\Phi}^{(0)}$ is a constant polarization matrix, and $W^{(0)}(\omega_1,\omega_2)$ is a scalar function. This representation can be interpreted by considering the transmission of an incident pulse train with $\mathbf{W}^{(0)}(\omega_1,\omega_2)=\mathbf{\Phi}^{(0)}W^{(0)}(\omega_1,\omega_2)$ through a spectrally selective deterministic polarization device that could be implemented, e.g., using the methods described in Refs. [13–16].

We assume that the scalar part of the CSDM describes a Gaussian Schell-model plane-wave pulse train [25]:

$$W^{(0)}(\omega_1,\omega_2)=W_0\exp\left[-\frac{(\omega_1-\omega_0)^2+(\omega_2-\omega_0)^2}{\Omega^2}\right]\times\exp\left[-\frac{(\omega_1-\omega_2)^2}{2\Omega_\mu^2}\right]. \quad (65)$$

Here ω_0 is the center frequency of the pulses, Ω is the expectation value of the pulse length, and Ω_μ represents the spectral coherence width of the pulse train. It will be convenient to use average and difference spectral coordinates $\bar{\omega}=\frac{1}{2}(\omega_1+\omega_2)$ and $\Delta\omega=\omega_2-\omega_1$ because (65) is separable in these coordinates: the transformation gives

$$W^{(0)}(\bar{\omega},\Delta\omega)=W_0\exp\left[-\frac{2(\bar{\omega}-\omega_0)^2}{\Omega^2}\right]\exp\left(-\frac{\Delta\omega^2}{2\Omega^2\beta^2}\right) \quad (66)$$

with the parameter

$$\beta=[1+(\Omega/\Omega_\mu)^2]^{-1/2} \quad (67)$$

introduced for brevity. This parameter represents the spectral coherence of the incident field in a sense that it is limited to the range $0\leqslant\beta\leqslant 1$, with the lower and upper bounds representing spectral incoherence and full spectral coherence of the pulse train, respectively.

If $\mathbf{M}(\omega)=\mathbf{I}$ is a frequency-independent unit matrix, the MCM given by (17) is of the form $\mathbf{\Gamma}(t_1,t_2)=\mathbf{\Phi}^{(0)}\Gamma^{(0)}(t_1,t_2)$. In average and difference temporal coordinates $\bar{t}=\frac{1}{2}(t_1+t_2)$ and $\Delta t=t_2-t_1$ we have

$$\Gamma^{(0)}(\bar{t},\Delta t)=\iint_{-\infty}^{\infty}W^{(0)}(\bar{\omega},\Delta\omega)\times\exp[-i(\bar{\omega}\Delta t+\Delta\omega\bar{t})]\,d\bar{\omega}\,d\Delta\omega. \quad (68)$$

In writing this expression we assume that the lower bound of integration with respect to $\bar{\omega}$ can be replaced with $-\infty$. On inserting from (66) into (68) we then find that

$$\Gamma^{(0)}(\bar{t},\Delta t)=\Gamma_0\exp\left(-\frac{2\bar{t}^2}{T^2}\right)\exp\left(-\frac{\Delta t^2}{2T^2\beta^2}\right)\times\exp(-i\omega_0\Delta t), \quad (69)$$

where $\Gamma_0=2\pi W_0\Omega/T$ and

$$T=2/\Omega\beta. \quad (70)$$

In absolute temporal coordinates the scalar part of the MCM in (69) takes the form

$$\Gamma^{(0)}(t_1,t_2)=\Gamma_0\exp\left(-\frac{t_1^2+t_2^2}{T^2}\right)\exp\left[-\frac{(t_1-t_2)^2}{2T_\gamma^2}\right]\times\exp(-i\omega_0\Delta t) \quad (71)$$

with

$$T_\gamma=T\Omega_\mu/\Omega. \quad (72)$$

The quantities T and T_γ thus represent the mean pulse duration and coherence time of the incident Gaussian Schell-model pulse train, respectively [25]. With the help of (72), we can write the parameter β in an alternative form

$$\beta=[1+(T/T_\gamma)^2]^{-1/2}. \quad (73)$$

In view of (70), the time-bandwidth product of the pulse train is $\Omega T=2/\beta$, and in the spectrally incoherent (stationary) limit $\Omega_\mu\to 0$ we have $T\to\infty$ and $T_\gamma\to 2/\Omega$. Equation (70) establishes a temporal equivalence law for scalar Gaussian Schell-model pulse trains: a family of pulse trains with the same mean temporal pulse duration T (but different coherence times T_γ) is obtained if the spectral bandwidth is chosen according to $\Omega=2/T\beta$.

From now on we assume that the incident pulse train is spectrally unpolarized,

$$\mathbf{\Phi}^{(0)}=\frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad (74)$$

and that $\mathbf{M}(\omega)$ is, instead of a unit matrix, of the form

$$\mathbf{M}(\omega)=\begin{bmatrix}1 & 0\\ 0 & A\exp[i\phi(\omega)]\end{bmatrix} \quad (75)$$

with

$$\phi(\omega)=\phi_0+(\omega-\omega_0)\tau+\frac{\kappa}{\Omega^2}(\omega-\omega_0)^2, \quad (76)$$

where τ is a constant with dimensions of time and κ is (a dimensionless) chirp coefficient. Hence the polarization device modulates the amplitude of the y component of the electric field by a constant real factor $A \leqslant 1$ and its phase by a quadratic function $\phi(\omega)$, while leaving the x component unchanged.

In view of (64), the field that we wish to analyze is now described by

$$\mathbf{W}(\bar{\omega}, \Delta\omega) = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & A^2 \exp[i\varphi(\bar{\omega}, \Delta\omega)] \end{bmatrix} W^{(0)}(\bar{\omega}, \Delta\omega), \tag{77}$$

where

$$\varphi(\bar{\omega}, \Delta\omega) = \Delta\omega\tau + \frac{2\kappa}{\Omega^2}(\bar{\omega} - \omega_0)\Delta\omega. \tag{78}$$

The single-frequency spectral Stokes parameters now read as

$$S_0(\omega) = \frac{1}{2}(1 + A^2)W_0 \exp\left[-\frac{2}{\Omega^2}(\omega - \omega_0)^2\right], \tag{79a}$$

$$S_1(\omega) = \frac{1}{2}(1 - A^2)W_0 \exp\left[-\frac{2}{\Omega^2}(\omega - \omega_0)^2\right], \tag{79b}$$

$$S_2(\omega) = S_3(\omega) = 0. \tag{79c}$$

The spectral degree of polarization, defined in (5), is

$$P(\omega) = \frac{1 - A^2}{1 + A^2}, \tag{80}$$

thus being independent on frequency and the phase modulation. Obviously, $P(\omega) = 0$ if $A = 1$, meaning that the field is spectrally unpolarized.

The mutual coherence matrix of our model field can be determined using (17). It is also a diagonal matrix, with

$$\Gamma_{xx}(\bar{t}, \Delta t) = \tfrac{1}{2}\Gamma^{(0)}(\bar{t}, \Delta t) \tag{81}$$

and

$$\Gamma_{yy}(\bar{t}, \Delta t) = \frac{1}{2}\Gamma_0 A^2 B \exp\left[-\frac{2B^2}{T^2}(\bar{t} - \tau)^2\right] \times \exp\left(-\frac{B^2}{2T^2\beta^2}\Delta t^2\right) \times \exp\left[-\frac{i2\kappa B^2}{T^2}(\bar{t} - \tau)\Delta t\right]\exp(-i\omega_0\Delta t), \tag{82}$$

where

$$B = [1 + (2\kappa/\Omega T)^2]^{-1/2} = [1 + (\kappa\beta)^2]^{-1/2}. \tag{83}$$

The equal-time Stokes parameters are therefore given by

$$S_0(t) = \frac{\Gamma_0}{2}\left\{\exp\left(-\frac{2t^2}{T^2}\right) + A^2 B \exp\left[-\frac{2B^2}{T^2}(t - \tau)^2\right]\right\}, \tag{84a}$$

$$S_1(t) = \frac{\Gamma_0}{2}\left\{\exp\left(-\frac{2t^2}{T^2}\right) - A^2 B \exp\left[-\frac{2B^2}{T^2}(t - \tau)^2\right]\right\}, \tag{84b}$$

$$S_2(t) = S_3(t) = 0. \tag{84c}$$

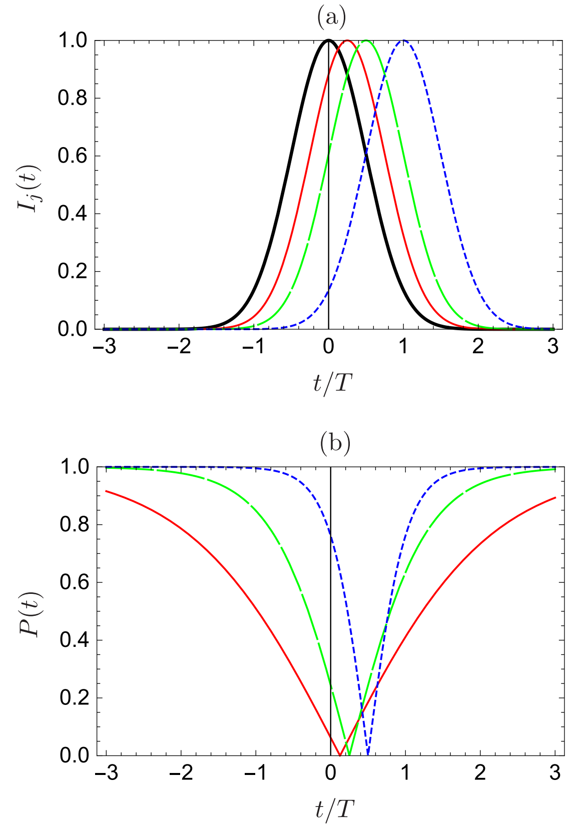

FIG. 2. (a) Distributions of the x (thick solid) and y (colored) components of the temporal intensity profile, normalized by $I_0 = \Gamma_0/2$, and (b) the time-dependent degree of polarization $P(t)$ for the model pulse train. Here $\beta = 1$, $A = 1$, $\kappa = 0$, and $\tau = 0.25T$ (solid red), $\tau = 0.5T$ (dashed green), $\tau = T$ (dotted blue).

Clearly, in the absence of chirp, $C = 1$ and then the temporal intensity distribution $I_y(t) = J_{yy}(t) = \Gamma_{yy}(t, 0)$ associated with the yy component of the MCM is simply an attenuated and temporally shifted replica of the intensity $I_x(t) = J_{xx}(t) = \Gamma_{xx}(t, 0)$ of the xx component. The time-domain degree of polarization of our model field, defined in (16), has the form

$$P(t) = \frac{|\exp(-2t^2/T^2) - A^2 B \exp[-2B^2(t - \tau)^2/T^2]|}{\exp(-2t^2/T^2) + A^2 B \exp[-2B^2(t - \tau)^2/T^2]}. \tag{85}$$

Thus, even with $A = 1$, the time-domain degree of polarization can be varied within the temporal duration of the pulse by spectral phase modulation. This occurs because the equal-time temporal Stokes parameters depend on the two-frequency spectral Stokes parameters.

Step 1 of the work flow is now complete. Figure 2 illustrates the distributions of $I_x(t)$, $I_y(t)$, and $P(t)$ when there is no chirp ($\kappa = 0$) but the y component of the electric field is delayed in time by various amounts τ. The degree of polarization $P(t)$ has a zero at the instant of time when $I_x(t) = I_y(t)$, but has a finite value for all other values of t. Hence the field is temporally unpolarized at only one instant of time. The temporal distribution $P(t)$ is independent on spectral (or temporal coherence of the pulse train (i.e., parameter β) when plotted in normalized time units t/T. When chirp is present, as

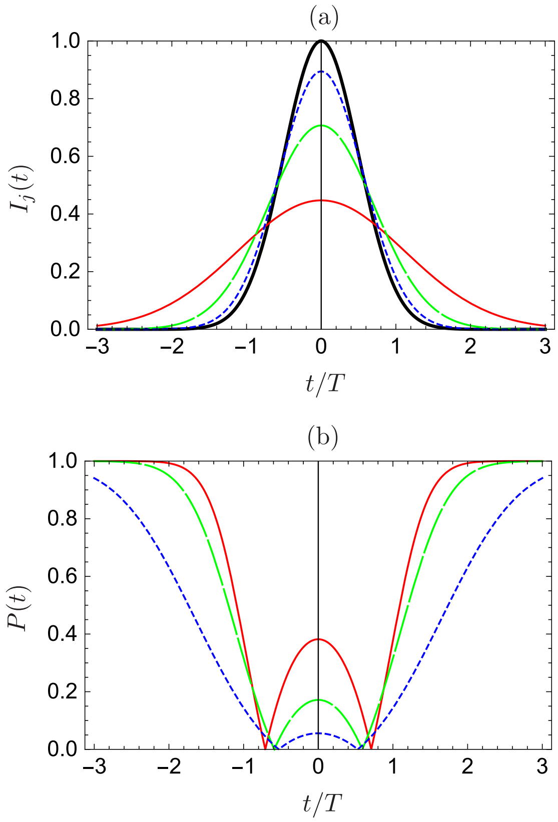

FIG. 3. Same as Fig. 2 but $\tau = 0$ and $\kappa = 2$. Solid red: $\beta = 1$. Dashed green: $\beta = 0.5$. Dotted blue: $\beta = 0.25$.

shown in Fig. 3, $P(t)$ depends also on the degree of coherence since the width of $I_y(t)$ varies with β. Again, zeros are seen for each chosen value of β at the (two) instants of time when $I_x(t) = I_y(t)$.

Proceeding to consider the physically observable Stokes parameters, we assume that the spectral filter has the ideal form $\mathbf{T} = \mathbf{I}T(\omega, \omega_F)$ with a Gaussian spectral passband

$$T(\omega, \omega_F) = \exp\left[-\frac{(\omega-\omega_F)^2}{\Omega_F^2}\right], \tag{86}$$

characterized by bandwidth Ω_F (Step 2 of the work flow). Furthermore, we assume (Step 3) that the temporal response of the detector has a Gaussian form

$$R(t) = \sqrt{\frac{2}{\pi}}\frac{1}{T_R}\exp\left(-\frac{2t^2}{T_R^2}\right) \tag{87}$$

with characteristic width T_R. The spectral response of the detector is then, in view of the inverse of (59),

$$Q(\omega) = \frac{1}{2\pi}\exp\left(-\frac{1}{8}T_R^2\omega^2\right). \tag{88}$$

This kind of Gaussian filter functions have been used in, e.g., Ref. [1], for their mathematical convenience. We stress, however, that they do not fully satisfy the causality requirements.

We can now apply Eqs. (60) and (61) to evaluate the observed polarization matrix and physical Stokes parameters (Step 4). When expressed in average and difference coordinates, (60) reads as

$$\begin{aligned}\mathbf{J}(t, \omega_F, R) = 2\pi \iint_{-\infty}^{\infty} &\mathbf{W}(\bar\omega, \Delta\omega)T^*(\bar\omega - \Delta\omega/2, \omega_F)\\ &\times T(\bar\omega + \Delta\omega/2, \omega_F)Q(\Delta\omega)\\ &\times \exp(-i\Delta\omega t)\, d\bar\omega\, d\Delta\omega,\end{aligned} \tag{89}$$

where we extend frequency integration limits to $\pm\infty$ to keep the mathematics simple. Thus also the negative frequency components in Eqs. (86) and (88) are formally retained, which is a good approximation provided that the filter passbands are much narrower than the center frequencies.

Obviously, since $\mathbf{W}(\bar\omega, \Delta\omega)$ is diagonal in our example, so is $\mathbf{J}(t, \omega_F, R)$. On inserting from Eqs. (77), (86), and (88) in (89) we first obtain an expression for its xx component in the form

$$\begin{aligned}J_{xx}(t, \omega_F, R) = \pi W_0 \frac{\Omega\Omega_F}{T_X\Omega_X}\exp\left(-\frac{2t^2}{T_X^2}\right)\\ \times \exp\left[-\frac{2}{\Omega_X^2}(\omega_F - \omega_0)^2\right],\end{aligned} \tag{90}$$

where we have defined

$$\Omega_X = \sqrt{\Omega^2 + \Omega_F^2} \tag{91}$$

and

$$T_X^2 = \frac{4}{\Omega^2\beta^2} + \frac{4}{\Omega_F^2} + T_R^2 = T^2 + \frac{4}{\Omega_F^2} + T_R^2. \tag{92}$$

Thus the time-dependent (scalar) physical spectrum associated with this field component is a simple product of Gaussian spectral and temporal intensity profiles with characteristic widths Ω_X and T_X that depend of the spectral width and the degree of coherence of the incident pulse train, as well as on the spectral and temporal resolutions Ω_F and T_R of the measurement device.

Correspondingly, we may evaluate the yy component of $\mathbf{J}(t, \omega_F, R)$. The result may be expressed in the form

$$\begin{aligned}J_{yy}(t, \omega_F, R) = \pi W_0 A^2 \frac{\Omega D}{T_X}\exp\left[-\frac{2}{T_Y^2}(t-\tau)^2\right]\\ \times \exp\left[-\frac{2}{\Omega_Y^2}(\omega_F - \omega_0)^2\right]\\ \times \exp[C(t-\tau)(\omega_F - \omega_0)],\end{aligned} \tag{93}$$

where we have defined

$$\Omega_Y = \frac{\Omega_F}{\sqrt{1 - D^2(\Omega/\Omega_F)^2}}, \tag{94}$$

$$T_Y = \frac{T_X}{\sqrt{1 - D^2(2\kappa/T_X\Omega)^2}}, \tag{95}$$

$$C = \frac{8\kappa D^2}{T_X^2\Omega_F^2}, \tag{96}$$

$$\frac{1}{D^2} = 1 + \left(\frac{\Omega}{\Omega_F}\right)^2 + \left(\frac{2\kappa}{T_X\Omega}\right)^2. \tag{97}$$

Clearly, Ω_Y is the observed width of the spectral profile at $t = \tau$ and T_Y is the temporal width of the detected pulses at

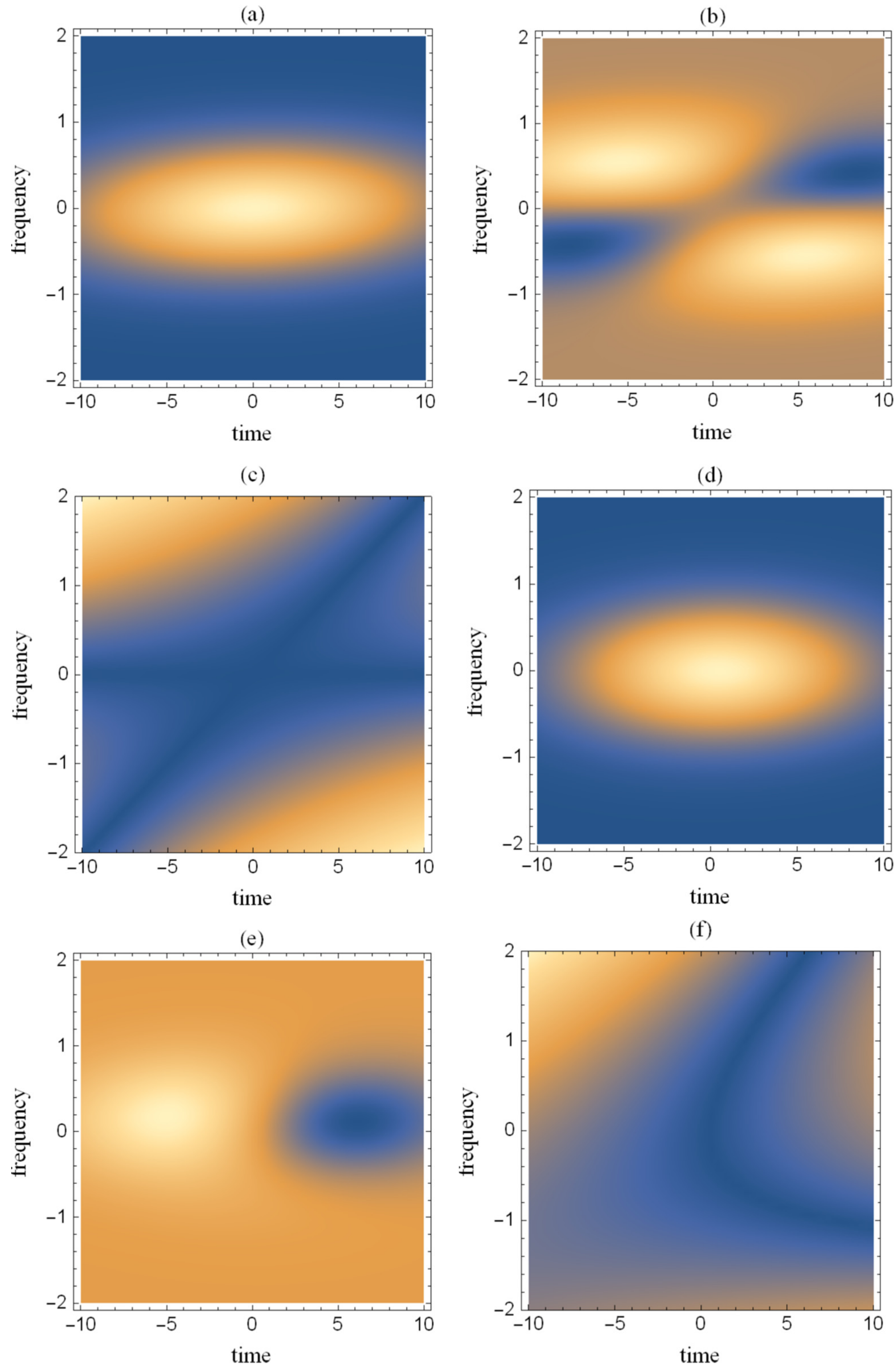

FIG. 4. Distributions of (normalized) time-dependent physical Stokes parameters (a, d) $S_0(t, \omega_F, R)$ and (b, e) $S_1(t, \omega_F, R)$, and (c, f) the degree of polarization $P(t, \omega_F, R)$ plotted as a function of t/T (time axis) and $(\omega_F - \omega_0)/\Omega$ (frequency axis). Left column: parameters $A = 1$, $\beta = 1$, $\tau = 0$, $\kappa = 5$, $\Omega_F = 0.1\Omega$, and $T_R = 10T$. Right column: parameters $A = 1$, $\beta = 0.5$, $\tau = T$, $\kappa = 5$, $\Omega_F = 0.1\Omega$, and $T_R = 10T$. The plot range is [0,1] in (a, c, d, f), [−0.025, 0.04] in (b), and [−0.055, 0.055] in (e).

$\omega_F = \omega_0$. The new feature that arises due to chirp is the time-frequency cross term.

It is readily seen from the formulas given above that, if an instantaneous detector were available ($T_R \to 0$) and a wide-band spectral filter $\Omega_F \gg \Omega$ were used, the temporal variation of the degree of polarization $P(t)$ could be measured. Indications of the temporal variation of $P(t)$ remain also in physical measurements, are we will shortly demonstrate, if we filter the spectrum sufficiently to make the temporal evolution of the ensuing pulses observable with available detectors.

Figure 4 illustrates the nonvanishing time-dependent Stokes parameters $S_0(t,\omega_F,R) = J_{xx}(t,\omega_F,R) + J_{yy}(t,\omega_F,R)$ and $S_1(t,\omega_F,R) = J_{xx}(t,\omega_F,R) - J_{yy}(t,\omega_F,R)$, as well as the associated degree of polarization

$$P(t,\omega_F,R) = \frac{|S_1(t,\omega_F,R)|}{S_0(t,\omega_F,R)}, \tag{98}$$

as functions of normalized time and frequency coordinates t/T and $(\omega_F - \omega_0)/\Omega$. In the left column of Fig. 4 we consider a fully coherent chirped ($\kappa = 5$) pulse train ($\beta = 1$) with zero delay ($\tau = 0$). We assume that $\Omega_F = 0.1\Omega$ and $T_R = 10T$; such a response time is realistic for picosecond-range pulses. The chirp (linear temporal change of carrier frequency) is manifested in the (small) tilt of the physical spectrum $S_0(t,\omega_F,R)$, which is seen more clearly in the plot of parameter $S_1(t,\omega_F,R)$ and in the inclined cross-shape of the minima of the physical degree of polarization. The horizontal arm of this cross moves up or down in frequency scale when $\tau \neq 0$ (depending on its sign). When the degree of spectral coherence is reduced, the cross becomes blurred and bent as illustrated in right column of Fig. 4, where we have chosen $\beta = 0.5$ and $\tau = T$.

VII. DISCUSSION AND CONCLUSIONS

We have introduced the concepts of time-dependent physical Stokes parameters and the associated degree of polarization of light as extensions of the concept of time-dependent physical spectrum in scalar theory on pulsed light. We have also demonstrated that it is possible to find indications of time variation of the degree of polarization using simple experimental techniques involving spectral filtering and standard polarization components. The analytical model presented here could be extended to partially polarized fields by assuming a general form for the polarization matrix $\mathbf{\Phi}^{(0)}$. Spectral filters and detector responses of any realistic form could be treated by numerical simulations. Experimental verification of the results given here could be provided in laboratory conditions by modulating a pulsed beam from a mode-locked laser by systems described in Refs. [13–15]. The degree of spectral coherence (the parameter β) of the pulsed beam could be controlled using the techniques proposed in Ref. [26]. We intend to carry out such simulations and experiments elsewhere.

[1] I. A. Walmsley and C. Dorrer, Characterization of ultrashort electromagnetic pulses, Adv. Opt. Phot. **1**, 308 (2009).

[2] J. H. Eberly and K. Wódkiewicz, The time-dependent physical spectrum of light, J. Opt. Soc. Am. **67**, 1252 (1977).

[3] J. Durnin and J. H. Eberly, Limits to resolution in time-dependent spectral analysis of pulses, J. Opt. Soc. Am. A **3**, 2019 (1986).

[4] R. Gase, Time-dependent spectrum of linear optical systems, J. Opt. Soc. Am. A **8**, 850 (1991).

[5] K. Saastamoinen, J. Turunen, and P. Vahimaa, Time-dependent physical spectra of Gaussian Schell-model sources, Opt. Commun. **271**, 309 (2007).

[6] R. Trebino, *Frequency-Resolved Optical Gating: The Measurement of Ultrashort Laser Pulses* (Kluwer, Boston, 2012).

[7] X. Gu, L. Xu, M. Kimmel, E. Zeek, P. O'Shea, A. P. Shreenath, R. Trebino, and R. S. Windeler, Frequency-resolved optical gating and single-shot spectral measurements reveal fine structure in microstructure-fiber continuum, Opt. Lett. **13**, 1174 (2002).

[8] J. M. Dudley and J. R. Taylor, eds. *Supercontinuum Generation in Optical Fibers* (Cambridge University Press, Cambridge, 2010).

[9] C. Dorrer, B. deBeauvoir, C. Le Blanc, J.-P. Rousseau, S. Ranc, P. Rousseau, J.-P. Chambaret, and F. Salin, Characterization of chirped-pulse amplification systems with spectral phase interferometry for direct electric-field reconstruction, Appl. Phys. B **70**, S77 (2000).

[10] S. Roling, B. Siemer, M. Wöstmann, H. Zacharias, R. Mitzner, A. Singer, K. Tiedtke, and I. A. Vartanyants, Temporal and spatial coherence properties of free-electron laser pulses in the extreme ultraviolet region, Phys. Rev. Special Topics Accel. Beams **14**, 080701 (2011).

[11] G. Genty, M. Surakka, J. Turunen, and A. T. Friberg, Complete characterization of supercontinuum coherence, J. Opt. Soc. Am. B **28**, 2301 (2011).

[12] M. Närhi, J. Turunen, A. T. Friberg, and G. Genty, Experimental Measurement of the Second-Order Coherence of Supercontinuum, Phys. Rev. Lett. **116**, 243901 (2016).

[13] M. Plewicki, F. Weise, S. M. Weber, and A. Lindinger, Phase, amplitude, and polarization shaping with a pulse shaper in a Mach-Zehnder interferometer, Appl. Opt. **45**, 8354 (2006).

[14] R. Selle, P. Nuernberger, F. Langhofer, F. Dimler, S. Fechner, G. Gerber, and T. Brixner, Generation of polarization-shaped ultraviolet femtosecond pulses, Opt. Lett. **33**, 803 (2008).

[15] M. T. Seidel, Z. Zhang, S. Yan, K. L. Wells, and H.-S. Tan, Characterization of polarization shaped ultraviolet femtosecond laser pulses, J. Opt. Soc. Am. B **28**, 2718 (2011).

[16] K. Misawa, Applications of polarization-shaped femtosecond pulses, Adv. Phys. X **1**, 544 (2016).

[17] T. Voipio, T. Setälä, and A. T. Friberg, Partial polarization theory of pulsed optical beams, J. Opt. Soc. Am. A **30**, 71 (2013).

[18] O. Korotkova and E. Wolf, Generalized Stokes parameters of random electromagnetic beams, Opt. Lett. **30**, 198 (2005).

[19] T. Setälä, J. Tervo, and A. T. Friberg, Contrasts of Stokes parameters in Young's interference experiment and electromagnetic degree of coherence, Opt. Lett. **31**, 2669 (2006).

[20] L. Mandel and E. Wolf, *Coherence and Quantum Optics* (Cambridge University Press, Cambridge, 1995).

[21] F. Gori, V. Ramírez-Sánchez, M. Santarsiero, and T. Shirai, On genuine cross-spectral density matrices, J. Opt. A **11**, 085706 (2009).

[22] T. Shirai, Interpretation of the recipe for synthesizing genuine cross-spectral density matrices, Opt. Commun. **283**, 4478 (2010).

[23] J. Turunen, Elementary-field representations in partially coherent optics, J. Mod. Opt. **58**, 509 (2011).

[24] T. Voipio, T. Setälä, and A. T. Friberg, Coherent-mode representation of partially polarized pulsed electromagnetic beams, J. Opt. Soc. Am. A **30**, 2433 (2013).

[25] P. Pääkkönen, J. Turunen, P. Vahimaa, A. T. Friberg, and F. Wyrowski, Partially coherent Gaussian pulses, Opt. Commun. **204**, 53 (2002).

[26] C. Ding, M. Koivurova, J. Turunen, T. Setälä, and A. T. Friberg, Coherence control of pulse trains by spectral phase modulation, J. Opt. **19**, 095501 (2017).