Effects of gain saturation on the quantum properties of light in a non-Hermitian gain-loss coupler

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Gain saturation is ubiquitous but has not been fully explored in the newly developed field of non-Hermitian optics. On the other hand, the nonclassical properties of light are highly relevant to the gain saturation that influences the quantum noise accompanying light amplification. In this paper, we systematically examine the gain saturation effects in a non-Hermitian system of coupled gain-loss waveguides. We explain the impact of gain saturation on a quantum light field dynamically evolving in the coupled system. In contrast to the ideal situation without gain saturation, one can achieve a quasisteady state under the gain saturation. Moreover, gain saturation reduces the influence of amplification noise and thereby better preserves such quantum features as entanglement. We illustrate the effects of gain saturation by examining the time evolution of the Wigner function, entanglement of the light fields, and the cross-correlation function between the two output modes. Significant differences exist between unsaturated and saturated situations, especially for low photon numbers.

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I. INTRODUCTION

Non-Hermitian optical systems, such as coupled gain-loss waveguides [1–4], photonic lattices [5,6], and whispering-gallery modes [7–9], have attracted considerable attention in recent years. These systems have peculiar properties and many applications in communication, computing, biochemistry, and environmental sensing [10,11]. The most important parameters of non-Hermitian optical systems are the gain and loss rates. Interesting features that are either difficult or impossible to be implemented by Hermitian optical systems exist in the non-Hermitian ones given various combinations of their gains and losses. To mention a few, one has single-mode parity-time lasers [12], unidirectional light propagation [8,13–16], asymmetric mode conversion [17–19], and enhanced sensitivity of sensors near exceptional points [20,21]. For the ideal models, the gain and loss coefficients do not depend on the intensity of the light propagating in the systems. However, in almost all amplifying media, optical gain is a function of field intensity, so that the intensity of the propagating beam does not increase forever. When the intensity of light reaches a steady state, the gain is reduced to its "saturated" value. Although the saturation effect has been examined in some classical non-Hermitian optical systems [8,22–28], the dynamical nature of gain saturation in the quantum regimes has remained unexplored. Especially, since the variation of gain coefficient affects the quantum noise caused by amplification, one expects that the saturation of gain can significantly influence the quantum-noise-sensitive properties of light such as entanglement.

So far, the majority of non-Hermitian optical systems have been studied under the assumption that the light is a classical electromagnetic field, and the gain and loss coefficients are intensity independent (nonsaturable), and the quantum noises due to amplification and dissipation are negligible. In this paper, we dispense with these assumptions by adopting a full quantum mechanical picture, examining the gain saturation effect in a coupled gain-loss waveguide system, and considering the inevitable quantum noise effect (see Fig. 1). We suppose that the gain medium is saturable, but the loss medium is with a constant damping rate. As examples, we show that the gain saturation effect alters the time evolution pattern of the Wigner function and the entanglement of the output fields. In contrast to a constant gain coefficient which often leads to a non-steady-state situation, gain saturation generally prompts a "quasisteady" state of the Wigner function and the entanglement of the light fields. Notably, in some situations, an ideal constant gain rate leads to "entanglement sudden death," but a gain saturation introduces a steady entangled state of the light field by reducing the detrimental quantum noise effect. Moreover, we study the impact of gain saturation on the cross-correlation function between the two output modes for the inputs to the waveguides to be in the Fock state and coherent state. This case also shows that the gain saturation preserves the non-classical features of the light better.

We organize the rest of the paper as follows. In Sec. II, we present the theoretical model of the non-Hermitian optical waveguides. In particular, we compare the effective non-Hermitian Hamiltonian approach to a stochastic Hamiltonian approach involving quantum noises. In Sec. III, we compare the time evolution of the Wigner function in the unsaturated and saturated cases. Section IV is about the impact of gain saturation on the time evolution of the entanglement of the output signals, as well as how the degree of entanglement varies with the saturation intensity. Then, we study the effect of gain saturation of the cross-correlation function in Sec. VI. Finally, we give a brief conclusion in Sec. VII.

II. THEORETICAL MODEL

The non-Hermitian system of coupled, single-mode waveguides is depicted in Fig. 1. Waveguide A carries a

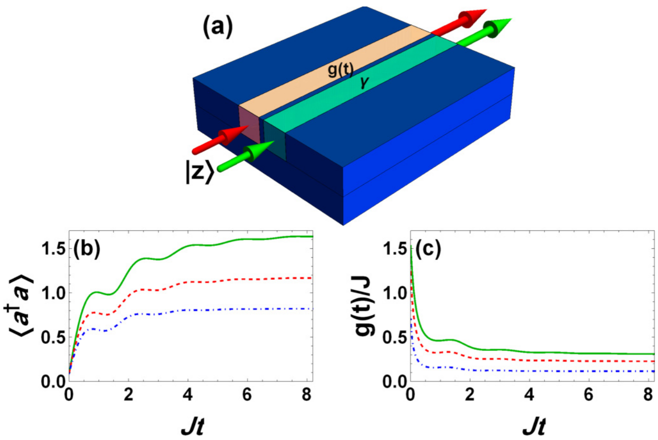

FIG. 1. (a) Non-Hermitian coupled gain-loss waveguides. The input signals are assumed to be squeezed vacuum states $|z\rangle$. Furthermore, the gain medium is saturable, denoted by $g(t)$, whereas the loss coefficient γ is constant. The coupling coefficient of waveguides is J. (b) Three examples of the dynamical behavior of the photon number in the amplifying waveguide for different saturation intensities: $I_{\text{sat}} = 0.25$ (green), 0.15 (dotted-dashed blue), and 0.05 (dashed red) [see Eq. (4)]. The other parameters are $\gamma/J = 0.5$, $z = 0.3$, and $g_0/J = 2$. (c) Time evolution of the normalized gain $g(t)/J$ vs the normalized time Jt (the same parameters are used).

saturable gain medium with a gain coefficient $g(t)$ and waveguide B is a nonsaturable loss medium with a loss coefficient of γ. We denote the light field operator propagating in waveguide A (B) by $\hat{a}$ ($\hat{b}$), which share the same frequency ω_0. The waveguides are coupled via evanescent waves so that the coupling strength J can be adjusted by the gap between them. We assume that the group velocity in both waveguides is equal and the possible superluminal propagation of the evanescent wave as described in [29,30] is negligible.

There are two approaches to dealing with the dynamics of the system. The first one is called the "semiclassical" or mean-field approach, which encapsulates all amplifying and dissipative processes in a non-Hermitian "effective Hamiltonian" ($\hbar = 1$) [10]:

$$H_{\text{eff}} = ig(t)\hat{a}^\dagger\hat{a} - i\gamma\hat{b}^\dagger\hat{b} + J(\hat{a}\hat{b}^\dagger + \hat{a}^\dagger\hat{b}). \tag{1}$$

The first two terms describe the amplification and dissipation of light in waveguides A and B, and the third term characterizes the coupling between the waveguides. The Heisenberg equation of motion determines the time evolution of operators $\hat{a}$ and $\hat{b}$.

If the gain coefficient is constant, one will obtain the eigenvalues of the Hamiltonian in Eq. (1) as $\frac{i}{2}(g-\gamma) \pm \sqrt{4J^2 - (g+\gamma)^2}$. If $g + \gamma = 2J$, not only the eigenvalues but also the corresponding eigenvectors will coalesce. This is a special feature of non-Hermitian systems, and the corresponding transition point is called the exceptional point. In particular, if $g = \gamma$, i.e., if the system respects the parity and time symmetries, it is possible to obtain real eigenvalues below the exceptional point where $g + \gamma < 2J$. However, in the current paper we are considering the gain saturation, and thus the gain coefficient is time dependent. Thus, intuitively one expects the system does not have a stable exceptional point.

While the semiclassical approach is adequate in describing classical light, it does not provide a satisfactory depiction of the nonclassical properties of light. In particular, since Eq. (1) does not explicitly include the quantum noise effect, the quantum correlation functions obtained by the mean-field approach can deviate from the real correlation functions.

The second approach explicitly considers quantum noise and treats it as a stochastic driving force. The equation describing the dynamics of the system in some contexts is called the "stochastic Schrödinger equation" [31]. As we are interested in the dynamical behavior of the quantum features of light, we will follow this approach. The total Hamiltonian includes the system, the reservoir, and the system-reservoir interaction. We suppose that the reservoir is an ensemble of an infinite number of harmonic oscillators with a continuous frequency spectrum (it includes positive as well as negative energies; see [31]). The corresponding Hamiltonian is $H_{\text{R}} = -\int_{-\infty}^{\infty} d\omega\, \omega \hat{f}_a^\dagger(\omega)\hat{f}_a(\omega) + \int_{-\infty}^{\infty} d\omega\, \omega \hat{f}_b^\dagger(\omega)\hat{f}_b(\omega)$, where $\hat{f}_c(\omega)$ is the noise operator satisfying $[\hat{f}_c(\omega), \hat{f}_c^\dagger(\omega')] = \delta(\omega - \omega')$ for $c = a, b$. Using the rotating-wave approximation with smooth system-reservoir coupling and after applying the Markovian approximation, the total Hamiltonian in the interaction picture is [32–34]

$$\begin{aligned} H_{\text{T}} = J(\hat{a}^\dagger\hat{b} + \hat{a}\hat{b}^\dagger) + i\sqrt{2g(t)}\,[\hat{f}_a^\dagger(t)\hat{a}^\dagger - \hat{f}_a(t)\hat{a}] \\ + i\sqrt{2\gamma}\,[\hat{f}_b(t)\hat{b}^\dagger - \hat{f}_b^\dagger(t)\hat{b}]. \end{aligned} \tag{2}$$

Note that the relevant terms to the amplification and dissipation noises are defined such that $[\hat{f}_c(t), \hat{f}_c^{\dagger}(t')] = \delta(t - t')$. The corresponding equations of motion are

$$\begin{aligned} \frac{d}{dt}\hat{a} &= g(t)\hat{a} - iJ\hat{b} + \sqrt{2g}\,\hat{f}_a^\dagger(t), \\ \frac{d}{dt}\hat{b} &= -iJ\hat{a} - \gamma\hat{b} + \sqrt{2\gamma}\,\hat{f}_b(t). \end{aligned} \tag{3}$$

In these equations, $g(t)$ depends on the intensity of the light field propagating in waveguide A [see Eq. (4)]. An example of the gain media is the erbium-doped amplifier [35], as an ensemble of two-level atoms. If a pump laser excites the erbium ions into the higher level so that the population difference between the upper and lower levels is positive, an optical signal propagating in this medium will be amplified exponentially. The amplification is due to the stimulated emission of photons from dopant ions. The excited ions can also decay via spontaneous emission or nonradiative processes that reduce the efficiency of light amplification.

If the length of the medium is long enough or the doping level is high enough, the light eventually reaches an intensity at certain specific lengths such that the energy stored in the upper level is not sufficient to satisfy the exponential growth condition. In other words, when the signal intensity increases to a certain value I_{sat}, the population difference between the upper and lower levels and hence the gain coefficient decreases. This phenomenon is called gain saturation, with I_{sat} being the saturation intensity at the center frequency of the optical beam. The gain coefficient as a function of time (or as a function of saturation intensity) is

$$g(t) = g_0/[1 + I_a(t)/I_{\rm sat}], \tag{4}$$

where g_0 and $I_a(t)$ are the small-signal gain and the intensity of light in waveguide A at time t, respectively. In Eq. (4) the saturation intensity is defined such that the stimulated rate downward equals the normal radiative decay of the upper level. For simplicity, we use the dimensionless saturation intensity in Ref. [8]. Physically, the energy difference between the upper and lower levels, the stimulated cross section, and the lifetime of the upper level determine the saturation intensity [36].

In the following, we study the effect of such gain saturation on the quantum properties of light fields. The approach we use combines two steps: (1) we find the evolved photon number $I_a(t) = \langle \hat{a}^\dagger \hat{a}(t)\rangle$ through the numerical solution of the nonlinear equations obtained by plugging Eq. (4) into Eq. (3); and (2) in determining the contribution to the evolved $\hat{a}(t)$ and $\hat{b}(t)$ from the noise drive terms in Eq. (3), we treat Eq. (3) as a set of closed linear differential equations by assuming a time-dependent coefficient $g(t)$ given by Eq. (4). With the evolved $\hat{a}(t)$ and $\hat{b}(t)$ found this way, one will obtain various correlation functions of these evolved operators, which are relevant to the quantum properties of the evolved light fields.

III. INFLUENCE OF GAIN SATURATION ON WIGNER FUNCTION

The Wigner function of light fields [37–40] provides a visualization of their quantum states. Its general form,

$$W(x, p) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi \, \exp(-ip\xi)\left\langle x + \frac{1}{2}\xi \,|\hat{\rho}|\, x - \frac{1}{2}\xi \right\rangle \tag{5}$$

is defined in terms of density operator $\hat{\rho}$, where $|x - \frac{1}{2}\xi\rangle$ is the eigenket of the position operator. In open systems like that depicted in Fig. 1, the time evolution is affected by the quantum noise in the amplification (dissipation) process. In particular, one may ask how gain saturation affects the dynamical behavior of such Wigner functions. We address this question by comparing the time evolution of the Wigner function in the unsaturated and saturated situations.

The Wigner functions of fields in the coupled gain-loss waveguides have some special features. If the gain rate is fixed, the quantum states of the fields inside the system remain Gaussian, which takes the form [41]

$$W(X) = \frac{\exp\left[-\frac{1}{2}XVX^T\right]}{\pi\sqrt{\mathrm{Det}[V]}}, \tag{6}$$

because the inputs to waveguides A and B are squeezed vacuum states. In this equation, $X \equiv (Q_A, P_A; Q_B, P_B)$, and the parameters inside the parentheses are quadratures of the fields defined as

$$\hat{Q}_m = \frac{1}{\sqrt{2}}(\hat{c} + \hat{c}^\dagger), \;\; \hat{P}_m = -\frac{i}{\sqrt{2}}(\hat{c} - \hat{c}^\dagger), \tag{7}$$

where m stands for A and B, and $c = a, b$. The 4×4 matrix V in Eq. (6) is called the covariance matrix (CM) [42] defined as

$$V = \begin{pmatrix} E & F \\ F^T & G \end{pmatrix}, \tag{8}$$

where E, F, and G are 2×2 matrices. The entries of CM are therefore defined as

$$V_{ij} = \tfrac{1}{2}\langle \hat{X}_i\hat{X}_j + \hat{X}_j\hat{X}_i\rangle - \langle \hat{X}_i\rangle\langle \hat{X}_j\rangle. \tag{9}$$

Here we assume that the input to the waveguides is a single-mode squeezed vacuum state, $|z\rangle = S(z)|0\rangle$, where $S(z) = \exp[\frac{1}{2}z(\hat{c}^\dagger)^2 - \frac{1}{2}z^*\hat{c}^2]$ for $\hat{c} = \hat{a}, \hat{b}$. The squeezing parameter is defined as $z = r\exp(i\theta)$ where r and θ are its magnitude and phase, respectively [43].

One should note that the quadratures involved in Eq. (6) include two parts: contribution of the input that is going to be amplified or dissipated [the homogeneous part of Eq. (3)] and the contribution of the quantum noises [44] [the nonhomogeneous part of Eq. (3)]. Therefore, the gain saturation can directly affect both parts. Particularly, it alters the time evolution caused by the noise terms, which is proportional to the square root of the gain coefficient.

The first and second rows of Fig. 2 show two examples of time evolutions of the Wigner function in the unsaturated and saturated cases, respectively. We assume that, in the top row, the gain coefficient is larger than the loss coefficient. In the unsaturated case, the gain coefficient is constant but, in the saturated case, an effectively time-dependent factor $g(t)$ gives a different evolution pattern. Under the gain saturation, the gain coefficient becomes lower than the loss coefficient, regardless of whether it is initially higher than the loss coefficient. As Figs. 2(a1)–2(a4) show, the nonzero domain of the Wigner function expands with time, but its peak value substantially decreases. On the other hand, neither the domain nor the peak value of the Wigner function changes remarkably with time in the saturated case [Figs. 2(b1)–2(b4)]. In other words, the Wigner function evolves into a quasisteady state. Since the gain coefficient decreases considerably upon saturation, the quantum noise effect also diminishes on the way to evolving into the quasisteady state. One should note that the magnitudes and shapes of the Wigner functions are different because of the absence or presence of the gain saturation. For example, the Wigner function profile in Fig. 2(a4) is circular, but in Fig. 2(b4) it becomes elliptical. Moreover, in the unsaturated case, the Wigner function profile keeps switching between circular and elliptical profiles (Gaussian and non-Gaussian functions) for long times, whereas in the saturated case the profile becomes almost circular after a short time and remains circular.

IV. EFFECT OF GAIN SATURATION ON ENTANGLEMENT

A. Time evolution of entanglement

One expects the coupled gain-loss waveguide system to be a good platform for generating continuous-variable entangled output fields [45]. However, amplifiers add quantum noise to the optical fields, and the noise is especially significant in the amplification of fields with low photon numbers. Even if the input is shot-noise-limited, the output remains noisy [43]. Meanwhile, one should understand that the quantum noise associated with amplification is more detrimental to entanglement because in the amplification process more photons are added to the optical field, while in the dissipation process, a portion of the photons disappear via absorption. Mathematically, one can confirm the dominant role of the

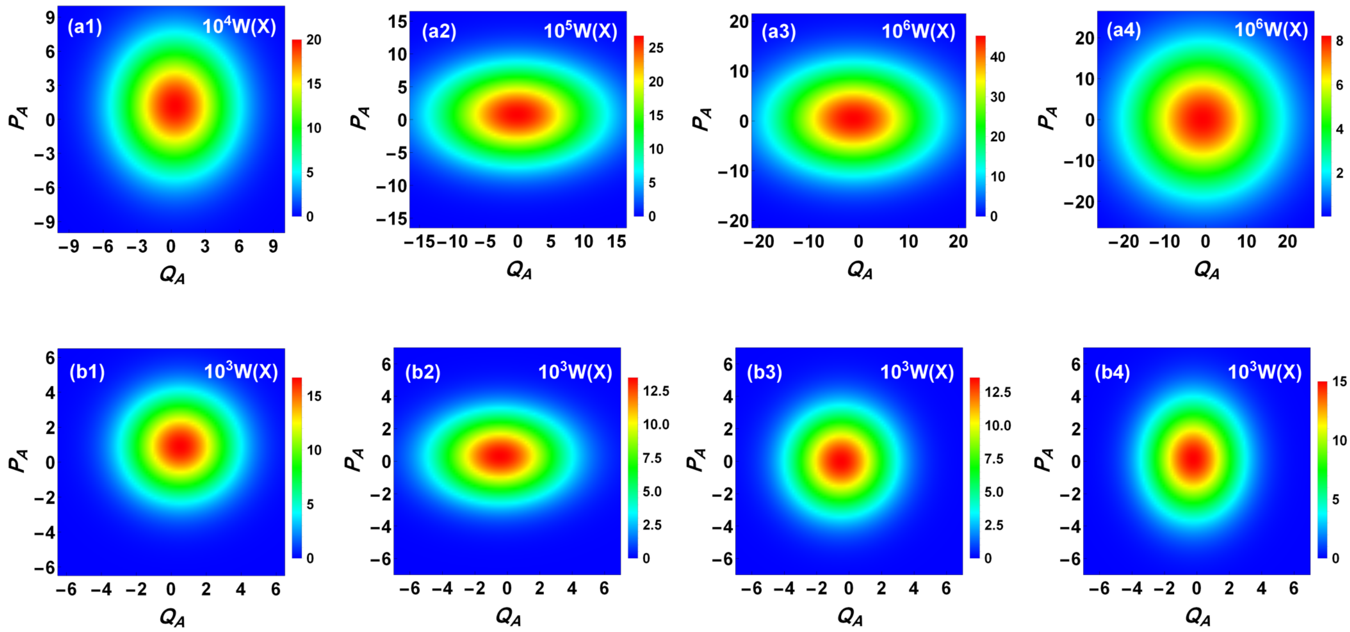

FIG. 2. Projection of the scaled Wigner function of the coupled gain-loss waveguides on the (Q_B, P_B) plane. The top row [(a1)–(a4)] shows the time evolution of the Wigner function in the unsaturated case at times $Jt = 2, 4, 6,$ and 8, respectively. We have assumed the squeezing parameters of both input signals to be equal: $r_1 = r_2 = 0.8$ and $\theta_1 = \theta_2 = 0$. Also, $\tilde{g}/J = 0.5$ and $\gamma/J = 0.3$. The bottom row [(b1)–(b4)] demonstrates the time evolution in the saturated case at the same time intervals. All parameters, except the gain, are equal to those in the unsaturated case. Here, in the normalized version of Eq. (4), $g_0/J = 4.5$, $I_{\text{sat}} = 0.1$, and $I_a(Jt) = \langle \hat{a}^\dagger(Jt)\hat{a}(Jt)\rangle$.

amplification noise by the fact that in the calculation of $\langle \hat{a}^\dagger \hat{a}\rangle$ the noise terms associated with the dissipation vanish because of the commutation relations.

Moreover, the eigenvalues of such a non-Hermitian system are in general complex, and the eigenvectors are nonorthogonal [46]. The main consequence of nonorthogonality of the eigenvectors is that, when random forces due to coupling to the reservoir drive the system, then the noise introduced to the system can be stronger than that in systems with orthogonal eigenvectors [27]. If the quantum noise is intense, it degenerates a nonclassical light field into a classical one. Therefore, one may expect that noise-sensitive quantities such as entanglement deteriorate significantly in non-Hermitian systems. Knowing that the saturation of gain reduces the quantum noise strength, one may ask how the gain saturation impacts the nonclassical features of light fields such as entanglement. In particular, one could ask if the saturation effect can reduce the quantum noise to a level that the sudden death of entanglement [47] is avoided.

Regarding the above points, we show below that the saturation effect can thoroughly alter the time evolution of entanglement. Under some special circumstances, the saturation of gain significantly reduces the quantum noise so that the entanglement of output fields can be well preserved.

To quantify the degree of entanglement of a field in the Gaussian state, we use the logarithmic negativity [42]

$$E_N = \max[0, -\ln 2\eta], \tag{10}$$

where, using the definitions in Eq. (8),

$$\eta = \frac{1}{\sqrt{2}}\sqrt{\sigma - \sqrt{\sigma^2 - 4\det V}}, \tag{11}$$

and

$$\sigma = \det E + \det F - 2\det G. \tag{12}$$

In Fig. 3 we present the numerically calculated time evolutions for some examples of field entanglement, comparing the evolved E_N in the unsaturated case (denoted by a thin, blue curve) to that of the saturated case (shown by a thick, red curve). The gain coefficient in the unsaturated case, $\tilde{g}$, keeps being equal to $g(0)$. Since the input fields are in squeezed vacuum states, in all cases we have chosen small saturation intensities to better demonstrate the difference between the unsaturated and saturated cases. If we assume other inputs like squeezed coherent states, larger saturation intensities yield the same time evolution patterns. In Fig. 3(a) we impose the conditions $\tilde{g} \approx \gamma$ and $\tilde{g} + \gamma < 2J$. One should note that the eigenvalues of Eq. (3) in the unsaturated case are $1/2[\tilde{g} + \gamma \pm \sqrt{(\tilde{g} + \gamma)^2 - 4J^2}]$, so that the exponential factors involved in the solution are pure imaginary. In the saturated case, the exponentials mentioned above are complex because of $g(t) < \gamma$. Moreover, in both cases, we expect to observe oscillations in E_N due to the imaginary parts of the exponentials. Figure 3(a) shows that the output field remains entangled within a finite time range in the unsaturated case. It reaches a maximum value, which depends on the coupling of the waveguides and the squeezing parameters of the input fields. In the beginning, the two fields are independent, but as they propagate, the optical field of waveguide A (B) penetrates the other waveguide, and thus they become entangled. However, due to the domination of quantum noise, E_N vanishes then. This phenomenon is called "entanglement sudden death," and it shows how the quantum noise associated with the amplification is destructive to the entanglement. In the

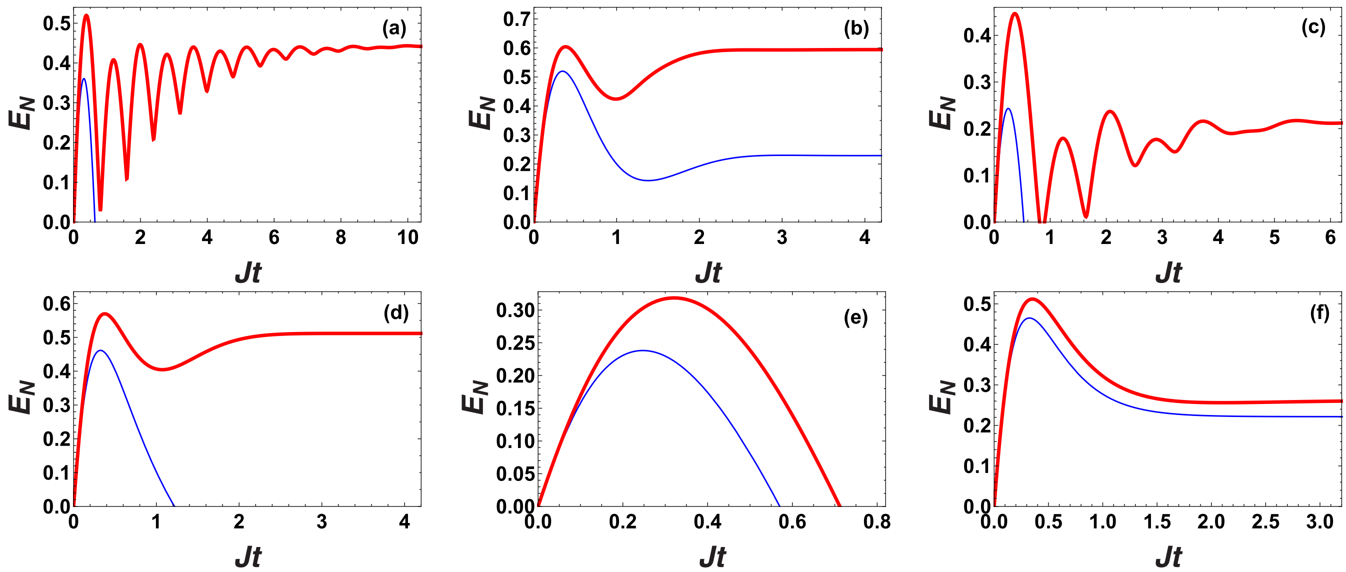

FIG. 3. Time evolutions of the logarithmic negativity E_N in the unsaturated (thin, blue curves) and saturated (thick, red curves) cases. In all figures we assume that $r_1 = r_2 = 0.3$ and $\theta_1 = \theta_2 = \pi/4$. Moreover, we assume that the constant gain coefficient $\tilde{g}$ in each case is equal to the initial value of $g(t) = g_0/(1 + I_a(t)/I_{\text{sat}}$. The other parameters are (a) $g(0)/J = 0.3$, $g_0/J = 0.8$, $\gamma/J = 0.3$; (b) $g(0)/J = 0.3$, $g_0/J = 0.8$, $\gamma/J = 1.3$; (c) $g(0)/J = 0.7$, $g_0/J = 2$, $\gamma/J = 0.5$; (d) $g(0)/J = 0.7$, $g_0/J = 2$, $\gamma/J = 1.5$; (e) $g(0)/J = 1.25$, $g_0/J = 1.5$, $\gamma/J = 0.8$; (f) $g(0)/J = 1.25$, $g_0/J = 1.5$, $\gamma/J = 1.8$. The saturation intensity I_{sat} in (a)–(d) is 0.05. In (e)–(f), $I_{\text{sat}} = 0.5$.

saturated case, however, E_N reaches a quasisteady state after a short time. Also, the oscillatory behavior of E_N is consistent with the expectation. Nevertheless, the oscillations are tiny at long times because the intensity of the field propagating in waveguide A has approached its steady state, and it reaches a constant value since the gain coefficient depends on the intensity. Consequently, the strength of quantum noise substantially decreases so that E_N comes to a quasisteady state.

Without changing $\tilde{g}$, we increase γ in Fig. 3(b) such that $\tilde{g} < \gamma$, but still keep the relation $\tilde{g} + \gamma < 2J$. In contrast to Fig. 3(a) where $g(t)/\gamma < 1$ after a short time, this ratio is always less than one in Fig. 3(b). Since the gain is substantially smaller than loss (in this case, roughly four times), the quantum noise associated with the amplification is not strong enough to erase the entanglement. Therefore, after a long time, a steady state of entanglement is achieved. Here the saturation effect does not change the evolution pattern substantially. The only difference between the unsaturated and saturated cases is that, in the saturated case, E_N in the steady state is higher than that of the unsaturated case because the gain saturation reduces the quantum noise effect.

In Fig. 3(c), there is the relation $\tilde{g} > \gamma$ but one still keeps $\tilde{g} + \gamma < 2J$. Since the gain is higher than the loss in the unsaturated case, we expect that the entanglement vanishes after a finite time. On the other hand, the saturation effect decreases the gain to a value much smaller than the loss, and hence the associated quantum noise is not dominant. Therefore, the entanglement measure E_N again approaches a steady state. Without changing the gain, we increase the loss coefficient in Fig. 3(d) so that there are the relations $\tilde{g} < \gamma$ and $\tilde{g} + \gamma > 2J$. Although the gain is lower than the loss, it is not low enough to avoid the entanglement sudden death. However, the saturation effect reduces the quantum noise and results in a steady state. One should note that $g(t) + \gamma < 2J$ due to the saturation, although one has $g(0) + \gamma > 2J$.

In Fig. 3(e), the system parameters keep the relations $\tilde{g} > \gamma$ and $\tilde{g} + \gamma \approx 2J$. Moreover, we use a higher saturation intensity for Figs. 3(e) and 3(f) (10 times higher than that of the previous figures). In this case, there is no considerable difference between the unsaturated and saturated cases because in both cases the gain factor is large enough, so the associated quantum noise cancels the entanglement. The saturation effect only modifies the maximum value of E_N. In Fig. 3(f), one has the relations $\tilde{g} < \gamma$ and $\tilde{g} + \gamma > 2J$. Interestingly, in both unsaturated and saturated cases, E_N approaches a steady state. Moreover, the difference between these cases is small. In contrast to the previous situations, the relation $g(t) + \gamma > 2J$ is held forever for the saturated cases, the saturation effect only leads to a small modification.

To sum up, one notices that a nonzero E_N is possible when $g(t) \ll \gamma$, provided the initial gain is not too high so that the associated quantum noise quickly erases the entanglement. The numerical calculations are based on parameters achievable in the laboratory. For example, they are close to those in a recent experiment [8] of two high-Q silica-microtoroid resonators with balanced, effective gain, in which $\gamma = 227{-}2210$ MHz, $g = 2835$ MHz, and $J = 8{-}544$ MHz. Therefore, by choosing large coupling coefficients, it is possible to obtain $g/J < 10$ or $\gamma < 10$. One can engineer the gain and saturation intensity by changing the dopant density of the gain medium. We assume low saturation intensity to better demonstrate the difference between the situation of unsaturated gain and that with saturated gain. If the saturation intensity is very high, then $g(t) \approx g_0$ and hence no difference exists in the two situations. Due to the difference between the coupling between microresonators and waveguides, more

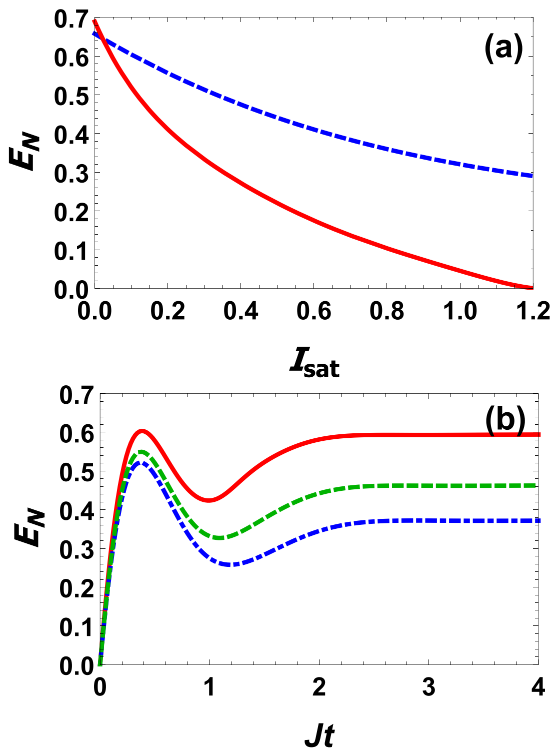

FIG. 4. (a) Variation of the degree of entanglement with saturation intensity. We assume that $r_1 = r_2 = 0.3$, $\theta_1 = \theta_2 = 0$, $Jt = 4$, and $\gamma/J = 1.3$. The red curve corresponds to $g_0/J = 0.8$, and the dashed, blue curve matches $g_0/J = 0.4$. The figure shows that the degree of entanglement decays in a quasiexponential manner to zero as the saturation intensity increases. (b) Variation of the time evolution of E_N with changing saturation intensity. In this case, $g_0/J = 0.8$, and the saturation intensities are 0.05 (red curve), 0.15 (dotted-dashed, blue curve), and 0.25 (dashed, green curve).

exact simulation of possible experiments with waveguides should be based on the actual couplings between waveguides.

B. Entanglement variation with the saturation intensity

After one sees the time-evolution change of entanglement due to gain saturation, it is interesting to know how the degree of entanglement varies with the saturation intensity. For this purpose, we consider a particular time when E_N approaches a steady state. As an example, we adopt the parameters of Fig. 3(b) and select $Jt = 4$ in Fig. 4(a), but the saturation intensity remains a variable. The solid (red) and dashed (blue) curves in the figure correspond to $g_0/J = 0.8$ and $g_0/J = 0.4$, respectively. Figure 4(a) shows that E_N quasiexponentially decays to zero as the saturation intensity increases. Since higher saturation intensity leads to a higher gain coefficient that intensifies the quantum noise, such quasiexponential decay occurs. Also, by comparing the results, one notes that a lower gain demands a higher saturation intensity so that the quantity E_N vanishes. Figure 4(b) shows that the time evolution changes as the saturation intensity increases. For higher saturation intensities, the steady-state value of $g(t)$ and hence the quantum noise increase and the steady-state value of E_N decreases.

V. INFLUENCE OF GAIN SATURATION ON CROSS-CORRELATION FUNCTION

Finally, we examine the Hanbury-Brown–Twiss (HBT) cross correlation [43] between the two modes $\hat{a}$ and $\hat{b}$:

$$A_{a,b} = \frac{\langle \hat{a}^\dagger(t)\hat{b}^\dagger(t)\hat{a}(t)\hat{b}(t)\rangle}{\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle\langle \hat{b}^\dagger(t)\hat{b}(t)\rangle}, \tag{13}$$

where $\langle \hat{a}^\dagger(t)\hat{b}^\dagger(t)\hat{a}(t)\hat{b}(t)\rangle$ is proportional to the probability of simultaneously detecting one photon each in the output of both waveguides. $\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle$ and $\langle \hat{b}^\dagger(t)\hat{b}(t)\rangle$ are the photon numbers in waveguides A and B, respectively (individual detections in the outputs). One can use the quantum regression theorem [43] for Gaussian states to simplify Eq. (13):

$$\begin{aligned}\langle \hat{a}^\dagger(t)\hat{b}^\dagger(t)\hat{a}(t)\hat{b}(t)\rangle &= \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle\langle \hat{b}^\dagger(t)\hat{b}(t)\rangle \\ &\quad + \langle \hat{a}^\dagger(t)\hat{b}(t)\rangle\langle \hat{b}^\dagger(t)\hat{a}(t)\rangle. \end{aligned}\tag{14}$$

We classify the outputs according to their cross-correlation functions (for a single mode, the second-order correlation function). If $A_{a,b} > 1$, the photon statistics is sub-Poissonian. In this case the photons are not equally spaced, but rather appear in bunches. If $A_{a,b} = 1$, the photon statistics is Poissonian, which is the characteristic of a coherent light field. Then the photons are randomly spaced. If $A_{a,b} < 1$, the photon statistics is super-Poissonian, and the photons are antibunched (equally spaced) [48]. The latter is the feature of nonclassical light. One may ask how the gain saturation affects the cross-correlation function. To address this question, we consider two different cases in Fig. 5. In Fig. 5(a), we assume that the input to the waveguides is in the Fock state $|n, m\rangle$ (tensor product of two independent Fock states). Moreover, we assume that the gain coefficient is larger than the loss coefficient. A dashed (solid) curve denotes the unsaturated (saturated) case. Initially, the state is purely quantum mechanical and accordingly $A_{a,b} = 0$, but because of the coupling between the waveguides, the photons can tunnel from one waveguide to the other one. Therefore, the cross-correlation function oscillates in time. In the saturated case, however, the oscillations disappear after a short time, and the cross-correlation function approaches a constant value. Also, the average value of $A_{a,b}$ in the saturated case is less than that of the unsaturated case. This shows that the nonclassical behavior of the fields is preserved better in the saturated case because the quantum noise is damped due to the gain saturation.

In Fig. 5(b) we consider the input to the waveguides to be in a coherent state $|\alpha, \beta\rangle$, which is a quantum state showing classical features. Because of this reason, one expects that the differences between the saturated and unsaturated cases are not significant. As Fig. 5(b) shows, the difference between the two cases is indeed less critical. An exciting feature of the coherent input fields is that the cross-correlation function does not remain equal to 1 throughout the non-Hermitian process. This behavior is in contrast to the case of an unprocessed coherent state whose cross-correlation function remains equal to 1 forever.

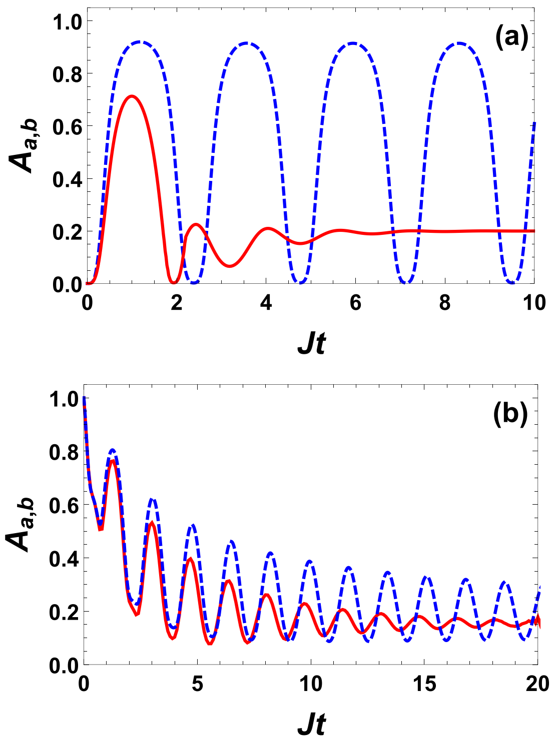

FIG. 5. (a) Cross-correlation function for the input fields to the waveguides in the Fock state, $|n, m\rangle$. The dashed and solid curves demonstrate the unsaturated and saturated cases, respectively. Here we assume $n = m = 10$. The other parameters are $\gamma/J = 0.5$, $g_0/J = 2$, $g(0) = 1$, and $I_{sat} = 10$. 1 (b) Input to the waveguides is in the coherent state $|\alpha, \beta\rangle$. We assume $\alpha = 1$ and $\beta = 2$. The other parameters are $\gamma/J = 0.4$, $g_0/J = 0.5$, $g(0) = 0.42$, and $I_{sat} = 5$.

VI. CONCLUSION

We have demonstrated how gain saturation affects quantum noise in a non-Hermitian system when the quantum properties of light are concerned. As quantum noise alters the nonclassical properties of light, one sees that gain saturation substantially changes the nonclassical features of light. As examples, we first considered the Wigner function, which is an alternative to the state vector and density matrix for the system. We find that the Wigner function evolves into a quasisteady state due to gain saturation, whereas in the unsaturated case, the distribution of the Wigner function in phase space expands, and its peak value drastically decreases. Furthermore, the profile of the evolved Wigner function is different for the saturated and unsaturated cases. The gain saturation reduces the gain coefficient with time, and therefore the quantum noise effect is reduced. Second, we investigated the time evolution of entanglement, which is a pure quantum feature with no classical counterpart. We considered different cases in which the gain, loss, and coupling coefficients are comparable or very different from each other. Generally, as long as the quantum noise level is high, a steady state of entanglement is not achievable. However, gain saturation reduces the quantum noise level, and hence in most cases, one can attain a final steady state. A nonzero entanglement of the output fields is possible only for the cases with $g(t) \ll \gamma$. Then, we showed that gain saturation is especially meaningful when the saturation intensity is low. On the other hand, if the saturation intensity is sufficiently high, the degree of entanglement vanishes, or the difference between the saturated and unsaturated cases is negligible. Finally, we examined the cross-correlation function between the output modes for the inputs in quantum states. All these results indicate that gain saturation does exert considerable impact on the quantum properties of light fields processed by our non-Hermitian setup.

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