Peculiarities of Cherenkov radiation from a charge moving through a dielectric cone

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Radiation generated by a charge moving in a vacuum channel through a dielectric cone is analyzed. The size of the conical target is assumed to be much greater than the wavelengths under consideration. We calculate the wave field outside the target using Stratton-Chu formulas ("aperture integrals"). This work focuses mainly on investigation of the electromagnetic field in the far-field area (Fraunhofer area). We calculate representative radiation patterns and show that typically the maximal radiation is generated in the direction of one of the refracted rays. However, in the specific case where the refracted ray is parallel to the symmetry axis, the maximal radiation is observed at some small (but nonzero) angle with respect to this axis. The maximum radiation in this case is much more intensive compared to all other situations. This phenomenon can be referred to as the "Cherenkov spotlight."

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I. INTRODUCTION

Radiation of charged particles moving in the presence of dielectric objects ("targets") of complicated form is of interest for various applications [1–5]. As an example, one can mention a new method of bunch diagnostics which requires calculation of Cherenkov radiation outside dielectric objects [2]. Typically, the size of the target is much larger than the wavelengths under consideration. On the one hand, this fact considerably complicates computer simulations because they require very large amount of resources and time. On the other hand, this fact gives us an obvious small parameter of the problem and allows developing approximate methods of analysis.

Previously, we have offered two methods for the solution of such problems which can be called the "ray-optic method" and "aperture method" [6–12]. Both these methods are valid for objects which are much larger than the wavelengths under consideration.

The first step is the same for both methods: We should solve the "etalon problem" which does not take into account "external" boundaries of the target. For example, if a charge moves in the vacuum channel inside the target, then we consider a channel border but do not take into account the external boundary of the object. In other words, initially we consider the problem for an infinite medium with the boundary nearest to the charge trajectory and obtain the field inside the bulk of the target. This field can be called the "incident" field.

As a second step, we select a part of the external surface of the object which is illuminated by Cherenkov radiation (this part will be called the "aperture"). Then, we use the fact that the object is large in comparison with the wavelengths under consideration. More exactly, we assume that

(1) the size of the aperture is much larger than the wavelength;

(2) the main part of the aperture is far from the path of the charge (in the wavelength scale).

Therefore we can neglect the quasi-static (quasi-Coulomb) field and replace the incident field with its asymptotic (i.e., the wave field). Further we calculate the field at the aperture of the object using Snell's and Fresnel's laws.

The last steps of these methods are different. The ray-optic method uses the ray-optic laws for calculation of the wave field outside the object [6,7]. However, this technique has essential additional limitations. The distance L from the aperture to the observation point should not be very large, i.e., the so-called "wave parameter" D should be small: $D \sim \lambda L/\Sigma \ll 1$, where Σ is the aperture area, and λ is a wavelength under consideration. Also, the observation point cannot be close to the focuses and caustics.

The aperture method is more general [8–12]. It is valid for observation points with arbitrary wave parameter D, including the Fraunhofer area (or far-field area) where $D \gg 1$, as well as in neighborhoods of focuses and caustics. At the final step of this technique we calculate the field outside the target using known Stratton-Chu formulas ("aperture integrals"). These formulas allow determining the field in the surrounding space if tangential components of electric and magnetic fields on the aperture are known.

Note that using the method described above, we neglect interior re-reflections which affect the field on the aperture. In principle, it is possible to take into account several re-reflections, but this is a cumbersome enough procedure. Meanwhile, as a rule, their role is inessential because each re-reflected wave is attenuated by corresponding reflection and transition coefficients. This fact is confirmed, in particular, by the results of Ref. [11] where we have verified the aperture method for the cone target using simulations in COMSOL MULTIPHYSICS. Comparing results of both techniques, we have concluded that the aperture method can be applied even for relatively small objects having the size of several wavelengths only. Another important result of [11] is that the accuracy of analytical calculations increases with an increase in the

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distance from the aperture. Note as well that the Appendix to this paper includes results of additional verification of the aperture method.

Note that in [11] we considered both a conventional conical target and a "truncated" target which does not have a "nose." It seems that the accuracy of the method under consideration should be higher for the truncated target, since its aperture does not have any region where the wave approximation for the incident field is incorrect. However, contrary to expectations, it has been shown that the accuracy of the method is approximately the same for both targets. This means that the "nose" does not significantly affect the radiation (obviously, due to the smallness of the region where the wave approximation for the incident field is incorrect).

Here we consider the dielectric cone with the axisymmetric vacuum channel where a charge moves. In contrast to [11], we focus mainly on the analysis of radiation in the far-field zone. This issue is the most important for both a description of the radiation properties and potential applications.

Note that a dielectric cone itself is widely used in optics where it is usually referred to as an "axicon" [13]. The most common application of this optical element is generation of the Bessel beam from a parallel laser beam illuminating the axicon [14]. Other modern applications include formation of vortex beams [15] and creation of hollow-plasma channels [16]. The paper [17] discussed the possibility of collimating of Cherenkov radiation (generated in a separate dielectric radiator) using a "grating axicon." Contrary to [17], in the present paper we consider the situation where both generation of Cherenkov radiation and transformation of its wavefront is implemented with the help of the same dielectric target of conical form.

II. APERTURE INTEGRALS: GENERAL FORM AND APPROXIMATION IN FRAUNHOFER ZONE

Aperture integrals (or Stratton-Chu formulas) for Fourier transform of electric fields [18] can be written in the following general form (we use the Gaussian system of units):

$$\begin{aligned}
\vec{E}(\vec{R}) &= \vec{E}^{(h)}(\vec{R}) + \vec{E}^{(e)}(\vec{R}),\\
\vec{E}^{(h)}(\vec{R}) &= \frac{ik}{4\pi}\int_\Sigma \Big\{[\vec{n}'\times\vec{H}(\vec{R}')]G(|\vec{R}-\vec{R}'|)\\
&\quad + \frac{1}{k^2}([\vec{n}'\times\vec{H}(\vec{R}')]\cdot\nabla')\nabla' G(|\vec{R}-\vec{R}'|)\Big\}d\Sigma',\\
\vec{E}^{(e)}(\vec{R}) &= \frac{1}{4\pi}\int_\Sigma \{[\vec{n}'\times\vec{E}(\vec{R}')]\times\nabla' G(|\vec{R}-\vec{R}'|)\}d\Sigma', \qquad (1)
\end{aligned}$$

where $\vec{E}(\vec{R}')$, $\vec{H}(\vec{R}')$ is the field on the aperture, $k = \omega/c$ is a wave number of the outer space (vacuum), $\vec{n}'$ is a unit external normal to the aperture in the point $\vec{R}'$, $G(R) = \exp(ikR)/R$ is a Green's function of the Helmholtz equation, and ∇' is a gradient, $\nabla' = \vec{e}_x\partial/\partial x' + \vec{e}_y\partial/\partial y' + \vec{e}_z\partial/\partial z'$. Analogous formulas are known for the magnetic field as well; however, we do not write them here because we are interested in a far-field zone where $|\vec{E}| \approx |\vec{H}|$. Note that in the paper [11] the expressions (1) have been written as well in the specific form for the conical target.

It should be noted that, in Stratton-Chu formulas, the surface Σ should completely cover the region where the sources are located. Such a surface can be formed by the "aperture," i.e., the part of the object's surface illuminated by Cherenkov radiation, and the other part which can be constructed arbitrarily. In our approximation, it is assumed that the contribution of this part is unimportant and integration is performed only over the aperture. This approximation is analogous to the Kirchhoff approximation, which is very widely used in optics, diffraction theory, antenna theory, etc.

Now we are interested in the case when the observation point is often located far from the target, in other words, in the region where so-called "wave parameter" D is large:

$$D \sim \lambda R/\Sigma \sim \lambda R/d^2 \gg 1, \qquad (2)$$

where λ is a wavelength under consideration, R is a distance from the target to the observation point, and $\Sigma \sim d^2$ is a square of an aperture. (We assume that the origin of the coordinate frame is located in the vicinity of the target.) This region is sometimes named a Fraunhofer area, or far-field area. It is interesting to simplify the general formulas (1) in this area.

Note that condition (2) automatically results in the inequality

$$R \gg d(d/\lambda) \gg d, \qquad (3)$$

because $d \gg \lambda$ in the problem under consideration. Using the inequalities (2) and (3) and taking into account that $|\vec{R}'| \sim d$, one can apply the following approximation in the formulas (1):

$$G(|\vec{R}-\vec{R}'|) \approx \frac{\exp(ikR - ik\vec{R}\vec{R}'/R\,)}{R}. \qquad (4)$$

As a result, we obtain the following formulas for Fraunhofer area:

$$\begin{aligned}
\vec{E}^{(h)} &\approx \frac{ike^{ikR}}{4\pi R}\int_\Sigma \{[\vec{n}'\times\vec{H}(\vec{R}')] - \vec{e}_R(\vec{e}_R\cdot[\vec{n}'\times\vec{H}(\vec{R}')])\}\\
&\quad\times e^{-ik\vec{e}_R\vec{R}'}d\Sigma',\\
\vec{E}^{(e)} &\approx \frac{ike^{ikR}}{4\pi R}\int_\Sigma \{\vec{e}_R\times[\vec{n}'\times\vec{E}(\vec{R}')]\}e^{-ik\vec{e}_R\vec{R}'}d\Sigma', \qquad (5)
\end{aligned}$$

where $\vec{e}_R = \vec{R}/R$. The formulas (5) can have essential advantages in comparison with (1) for specific objects because we can hope to evaluate these integrals analytically.

III. APERTURE INTEGRALS FOR THE CASE OF A CONE WITH A CHANNEL

We analyze the radiation of a charge moving along the axis of a cylindrical channel of radius a in the conical object (Fig. 1). The target is made of dielectric with permittivity ε and permeability μ (the conductivity is assumed to be negligible). The charge q moves with constant velocity $\vec{V} = c\beta\vec{e}_z$ along the z axis, and this velocity exceeds the "Cherenkov threshold," i.e., $\beta > 1/n$, where $n = \sqrt{\varepsilon\mu}$ is the refractive index of the prism material. Note that the form of the back border of the target (dotted line in Fig. 1) is not important. But it is important that the lateral border with the size d (bold

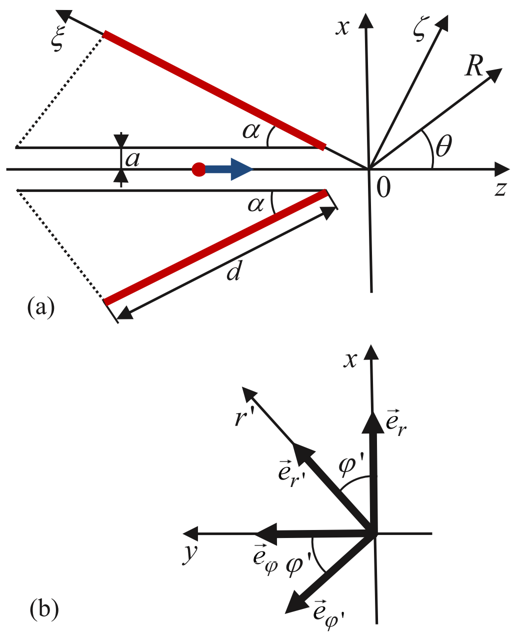

FIG. 1. (a) Cross section of the cone with channel; the aperture is marked by the bold red line. (b) The mutual arrangement of the vectors used in calculating the aperture integrals.

red line in Fig. 1) is illuminated by Cherenkov radiation. This border is the aperture Σ used in (1).

Recall that the object is much greater than the wavelength under consideration λ. This means that $d \gg \lambda$, and, moreover, $\alpha d \gg \lambda$, that is, the cone angle α is not very small. As a consequence, almost the whole aperture is far from the charge path, and the incident field can be approximately replaced with its wave part. This approximation allows using the aperture method in the form described above.

For definiteness, we will deal with a point charge having the charge density $\rho = q\delta(x)\delta(y)\delta(z-Vt)$, where $\delta(z-Vt)$ is a Dirac delta function. However, the results obtained below can be easily generalized for the thin bunch with finite length because we consider Fourier transforms of the field components. Note that the Fourier transform of the charge field is equivalent to the field of the threadlike current with given frequency ω which is localized on the z axis.

According to the aperture method, first we must find the "incident" field, i.e., the field in the infinite medium having constants ε and μ (considering the channel but not taking into account other borders of the cone). For the considered case, this field is known [7,19]. We write here Fourier transform of the magnetic component at the distance r, which is much larger than the wavelength under consideration (the cylindrical coordinates r, φ, z are used):

$$H_\varphi^{(i)} \approx \frac{iq}{c}\eta\sqrt{\frac{s}{2\pi r}}\exp\left\{i\left(sr+\frac{\omega}{V}z-\frac{3\pi}{4}\right)\right\}, \tag{6}$$

$$\eta=-\frac{2i}{\pi a}\left[\kappa\frac{1-n^2\beta^2}{\varepsilon(1-\beta^2)}I_1(\kappa a)H_0^{(1)}(sa)+sI_0(\kappa a)H_1^{(1)}(sa)\right]^{-1}, \tag{7}$$

where $s(\omega)=\frac{\omega}{V}\sqrt{n^2\beta^2-1}$, $\kappa(\omega)=\frac{\omega}{V}\sqrt{1-\beta^2}$, $I_{0,1}$ are modified Bessel functions, and $H_{0,1}^{(1)}$ are Hankel functions. Note that $\mathrm{Im}\, s(\omega) \geqslant 0$ if we take into account a small dissipation. If dissipation tends to zero, then this condition results in the rule $\mathrm{sgn}[s(\omega)] = \mathrm{sgn}(\omega)$ (we exclude the exotic case of so-called "left-handed" medium). The result (2) is valid for $|sr| \gg 1$. Electric field $\vec{E}^{(i)}$ can be easily found, because vectors $\vec{E}^{(i)}$, $\vec{H}^{(i)}$ and the wave vector of Cherenkov radiation $\vec{k}^{(i)} = s\vec{e}_r + \vec{e}_z\omega/V$ form the right-hand orthogonal triad in this area: $\vec{E}^{(i)} = -\sqrt{\mu/\varepsilon}[\vec{k}^{(i)}/k^{(i)} \times \vec{H}^{(i)}]$. The angle between the wave vector $\vec{k}^{(i)}$ and the charge velocity $\vec{V}$ is $\theta_p = \arccos[1/(n\beta)]$.

The wave (6) falls on the cone boundary at the angle $\theta_i = \frac{\pi}{2} - \alpha - \theta_p$ and is refracted at the angle $\theta_t = \arcsin(n\sin\theta_i)$ with respect to the boundary normal $\vec{e}_\zeta$. Because of the axial symmetry of the problem, the refracted field has the same ("vertical") polarization as the incident one, that is, outside the object we have

$$\vec{H} = T_{\rm v}H_\varphi^{(i)}\vec{e}_\varphi, \quad \vec{E} = -[\vec{k}^{(t)}/k^{(t)} \times \vec{H}], \tag{8}$$

where $\vec{k}^{(t)}$ is a wave vector of refracted wave ($k^{(t)} = k = \omega/c$), and $T_{\rm v}$ is a corresponding Fresnel coefficient:

$$T_{\rm v} = 2\sqrt{\mu/\varepsilon}\cos\theta_i\big/\left(\sqrt{\mu/\varepsilon}\cos\theta_i + \cos\theta_t\right). \tag{9}$$

The field on the aperture can be presented in the form

$$H_\varphi(\vec{R}) \approx \frac{T_{\rm v}q\eta\sqrt{s}}{c\sqrt{2\pi\xi\sin\alpha}}\exp\left\{i\frac{\omega}{V}\left(\sqrt{n^2\beta^2-1}\sin\alpha-\cos\alpha\right)\xi - \frac{i\pi}{4}\right\}. \tag{10}$$

Furthermore, we need the tangential electric component E_ξ (see Fig. 1) as well. Since the field is practically transversal plane wave, we can use the following relation for this component:

$$E_\xi(\vec{R}) = E_r(\vec{R})\sin\alpha - E_z(\vec{R})\cos\alpha = H_\varphi\cos\theta_t. \tag{11}$$

We are interested in the far-field zone where the general formula (1) has the form (5). Besides cylindrical coordinates r, φ, z we will use spherical coordinates $R, \theta = \arccos(z/R), \varphi$. Because of the axial symmetry of the problem one can assume that the observation is in the xz plane, i.e., $r = x$, $\varphi = 0$, and $\vec{e}_r = \vec{e}_x$, $\vec{e}_\varphi = \vec{e}_y$. Taking into account the formulas (see Fig. 1 as well),

$$\begin{aligned}
\vec{e}_R &\equiv \vec{R}/R = \vec{e}_r\sin\theta + \vec{e}_z\cos\theta,\\
\vec{n}' &= \vec{e}_{r'}\cos\alpha + \vec{e}_{z'}\sin\alpha,\\
\vec{e}_{r'} &= \vec{e}_r\cos\varphi' + \vec{e}_\varphi\sin\varphi',\\
\vec{e}_{\varphi'} &= -\vec{e}_r\sin\varphi' + \vec{e}_\varphi\cos\varphi', \quad \vec{e}_{z'} = \vec{e}_z,\\
\vec{R}' &= r'\vec{e}_{r'} + z'\vec{e}_{z'} = r'\cos\varphi'\vec{e}_r + r'\sin\varphi'\vec{e}_\varphi + z'\vec{e}_z,\\
d\Sigma' &= r'd\varphi'd\xi',\\
r' &= \xi'\sin\alpha,\\
z' &= -\xi'\cos\alpha,
\end{aligned} \tag{12}$$

the integrals (5) can be written in the following form:

$$\begin{Bmatrix} E_r^{(h)} \\ E_\varphi^{(h)} \\ E_z^{(h)} \end{Bmatrix} = \frac{ike^{ikR}\sin\alpha}{2\pi R}\int_{\xi_1}^{\xi_2} d\xi' \int_0^\pi d\varphi' \xi' H_{\varphi'}(\vec{R}') \begin{Bmatrix} -\sin\alpha\cos\varphi' - \sin\theta\,\Phi(\alpha,\theta,\varphi') \\ -\sin\alpha\sin\varphi' \\ \cos\alpha - \cos\theta\,\Phi(\alpha,\theta,\varphi') \end{Bmatrix} e^{ik\xi'\Phi(\alpha,\theta,\varphi')}, \tag{13a}$$

$$\begin{Bmatrix} E_r^{(e)} \\ E_\varphi^{(e)} \\ E_z^{(e)} \end{Bmatrix} = \frac{ike^{ikR}\sin\alpha}{2\pi R}\int_{\xi_1}^{\xi_2} d\xi' \int_0^\pi d\varphi' \xi' E_{\xi'}(\vec{R}') \begin{Bmatrix} -\cos\theta\cos\varphi' \\ -\cos\theta\sin\varphi' \\ \sin\theta\cos\varphi' \end{Bmatrix} e^{ik\xi'\Phi(\alpha,\theta,\varphi')}, \tag{13b}$$

where $\Phi(\alpha,\theta,\varphi') = \cos\alpha\cos\theta - \sin\alpha\sin\theta\cos\varphi'$, $\xi_1 = a/\sin\alpha$, $\xi_2 = d + a/\sin\alpha$.

Using the table integrals [20]

$$\int_0^\pi e^{ia\cos\varphi} d\varphi = \pi J_0(a), \quad \int_0^\pi \cos\varphi e^{ia\cos\varphi} d\varphi = \pi i J_1(a), \quad \int_0^\pi \sin\varphi e^{ia\cos\varphi} d\varphi = 0, \tag{14}$$

where $J_{0,1}(a)$ are Bessel functions, one can obtain the following expressions for nonzero components:

$$\begin{Bmatrix} E_r^{(h)} \\ E_z^{(h)} \end{Bmatrix} = \frac{ke^{ikR}\sin\alpha}{2R}\begin{Bmatrix} -\cos\theta \\ \sin\theta \end{Bmatrix}\int_{\xi_1}^{\xi_2} \xi' H_{\varphi'}(\vec{R}')[\sin\alpha\cos\theta J_1(\chi_s') + i\cos\alpha\sin\theta J_0(\chi_s')]e^{i\chi_c'} d\xi', \tag{15a}$$

$$\begin{Bmatrix} E_r^{(e)} \\ E_z^{(e)} \end{Bmatrix} = \frac{ke^{ikR}\sin\alpha}{2R}\begin{Bmatrix} -\cos\theta \\ \sin\theta \end{Bmatrix}\int_{\xi_1}^{\xi_2} \xi' E_{\xi'}(\vec{R}') J_1(\chi_s') e^{i\chi_c'} d\xi', \tag{15b}$$

$$E_\varphi^{(h)} = E_\varphi^{(e)} = 0, \tag{15c}$$

where $\chi_s' = k\xi'\sin\alpha\sin\theta$, $\chi_c' = k\xi'\cos\alpha\cos\theta$.

Taking into account that $E_R = E_r\sin\theta + E_z\cos\theta$, $E_\theta = E_r\cos\theta - E_z\sin\theta$ one can obtain that $E_R^{(h)} = E_R^{(e)} = 0$, i.e., the radiation field has only θ projection, which is expected since this field is a transversal wave. Calculating this component, we take into account that the field on the aperture is practically a plane wave (in our approximation). Therefore,

$$E_{\xi'} = H_{\varphi'}\cos\theta_t, \quad E_{\zeta'} = H_{\varphi'}\sin\theta_t. \tag{16}$$

Using the first of these relations we obtain the following final result for the total radiation field:

$$E_\theta = -\frac{ke^{ikR}\sin\alpha}{2R}\int_{\xi_1}^{\xi_2} \xi' H_{\varphi'}(\vec{R}')\, e^{i\chi_c'}[(\cos\theta_t + \sin\alpha\cos\theta) \times J_1(\chi_s') + i\cos\alpha\sin\theta J_0(\chi_s')]d\xi'. \tag{17}$$

Recall that the magnetic field $H_{\varphi'}(\vec{R}')$ on the cone surface is determined by the formula (10).

IV. ANALYSIS OF RESULTS

The integral (17) can be analytically calculated if the condition $k\xi'\sin\alpha\sin\theta \gg 1$ is fulfilled for the greater part of the aperture. Since $kd \gg 1$ and $\xi' \sim d$ within the essential part of the integration area, this condition is reduced to the inequality $kd\sin\alpha\sin\theta \gg 1$, that is,

$$\theta \gg (kd\sin\alpha)^{-1}. \tag{18}$$

Thus, we exclude from consideration only small angles θ. Under such conditions one can use asymptotes of Bessel functions for (17). Taking into account (10) as well, we see that the integrand of (17) is proportional to the function $\exp(ip\xi')$, where p is some constant. As a result, under condition (18) the radiation field can be written in the following approximate form:

$$E_\theta \approx -\frac{q\,\eta T_v\sqrt[4]{n^2\beta^2-1}\,\exp(ikR)}{2\pi cR\sqrt{\beta\sin\theta}} \times \left\{ i\frac{\sin(\theta+\alpha)+\cos(\theta_t)}{w_+}\sin\left(\frac{kw_+d}{2}\right)e^{ikw_+\xi_0} + \frac{\sin(\theta-\alpha)-\cos(\theta_t)}{w_-}\sin\left(\frac{kw_-d}{2}\right)e^{ikw_-\xi_0}\right\}, \tag{19}$$

where $w_\pm = \cos(\theta \pm \alpha) - \sin\theta_t$, $\xi_0 = d/2 + a/\sin\alpha$ is the position of the aperture center.

Using (19) one can obtain the direction of maximal radiation. The function $w_+^{-1}\sin(kw_+d/2)$ has the main maximums at $w_+ = 0$, that is, at $\theta + \alpha = \pm\frac{\pi}{2} \mp \theta_t$. However, the factor $\sin(\theta+\alpha) + \cos(\theta_t)$ is zero if $\theta + \alpha = -\frac{\pi}{2} + \theta_t$. Therefore, the first summand in (19) has the main maximum at $\theta = \theta_+ = \frac{\pi}{2} - \theta_t - \alpha$ only. One can see that this is the direction of the ray refracted at the generatrix which is the nearest to the observation point [Fig. 2(a)]. Analogously, the second summand in (19) has the main maximum at $\theta = \theta_- = \theta_t + \alpha - \frac{\pi}{2}$. This is the direction of the ray refracted at the generatrix which is the furthest to the observation point [Fig. 2(b)]. Note that the area of our consideration is bounded by the angle range $0 < \theta < \pi - \alpha$. It is easy to see that the maximum $\theta = \theta_+$ takes place if $-\frac{\pi}{2} < \theta_t < \frac{\pi}{2} - \alpha$; it can be located in all ranges under consideration because $0 < \theta_+ < \pi - \alpha$.

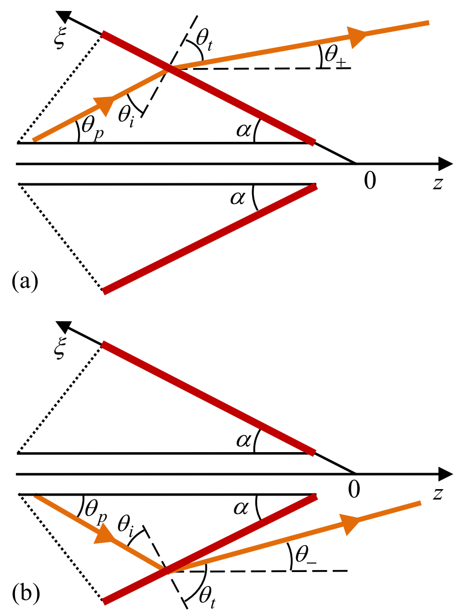

FIG. 2. (a) The case when $-\frac{\pi}{2} < \theta_t < \frac{\pi}{2} - \alpha$ and $\theta_{\max} = \theta_+$; maximum radiation direction $\theta_{\max}$ is the direction of the ray refracted at the cone generatrix nearest to the observation point. (b) The case when $\frac{\pi}{2} - \alpha < \theta_t < \frac{\pi}{2}$ and $\theta_{\max} = \theta_-$; the maximum radiation direction $\theta_{\max}$ is the direction of the ray refracted at the most distant generatrix with respect to the observation point.

The maximum $\theta = \theta_-$ takes place if $\frac{\pi}{2} - \alpha < \theta_t < \frac{\pi}{2}$; it can be located only in the limited range because $0 < \theta_- < \alpha$. One can see that conditions of existence of maximums $\theta_\pm$ are not superposed (there is only one main maximum for given problem parameters). This direction is the direction of the wave refracted either on the nearest generatrix or on the furthest one.

As a result, we obtain the following direction of maximal radiation:

$$\theta_{\max} = \begin{Bmatrix} \theta_+ & \text{for } \theta_t < \pi/2 - \alpha \\ \theta_- & \text{for } \theta_t > \pi/2 - \alpha \end{Bmatrix} = \left| \frac{\pi}{2} - \theta_t - \alpha \right|. \quad (20)$$

The value of the field in this main maximum is

$$E_{\theta\,\max} = \begin{Bmatrix} -i & \text{for } \theta_t < \pi/2 - \alpha \\ 1 & \text{for } \theta_t > \pi/2 - \alpha \end{Bmatrix} \frac{q\eta T_v \sqrt[4]{n^2\beta^2 - 1}\,\cos(\theta_t)}{2\pi\sqrt{\beta}\,|\cos(\alpha + \theta_t)|} \times \frac{kd e^{ikR}}{cR}. \quad (21)$$

The angular half-width of the main lobe of the radiation pattern $\delta\theta_\pm$ is determined by one of the conditions $|kw_\pm d/2| = \pi$, i.e.,

$$|\cos(\theta_\pm + \delta\theta_\pm \pm \alpha) - \sin\theta_t| = \frac{2\pi}{kd}. \quad (22)$$

Using the linear term of the Taylor series, one can obtain the following estimation:

$$\delta\theta_\pm \approx \frac{2\pi}{kd\cos\theta_t}. \quad (23)$$

One can see that the field amplitude in maximum $|E_{\theta\,\max}|$ (21) and the angular width (23) are the same for both "the ray" from the nearest generatrix (+) and "the ray" from the furthest generatrix (−). This is explained by the fact that the target dimension is small in comparison with the distance to the observation point, that is, the distances from the nearest and furthest borders are practically the same.

Note that accordingly to (20), $\theta_{\max} = \theta_+ = \theta_- = 0$ for

$$\theta_t = \frac{\pi}{2} - \alpha, \quad \text{or} \quad \sin\theta_t = \cos\alpha. \quad (24)$$

However, this situation is beyond the condition (18). Moreover, the formula (17) shows that the radiation is absent in the direction $\theta = 0$. In this case it is interesting to obtain the approximation for the field at small angles,

$$\theta \sim (kd\sin\alpha)^{-1} \ll 1. \quad (25)$$

Taking into account that $\sin\theta_t = n\sin\theta_i = n\cos(\alpha + \theta_p)$ and $\cos\theta_p = 1/(n\beta)$, the condition (24) can be written as

$$(1-\beta)\cos\alpha = \sqrt{n^2\beta^2 - 1}\,\sin\alpha. \quad (26)$$

Using (24)–(26) one can write the expression (17) in the following approximate form:

$$E_\theta = -\frac{T_v q\eta\sqrt{s}}{\sqrt{2\pi k}(\sin\theta)^{3/2}} \frac{e^{ikR - i\pi/4}}{cR} \int_{\psi_1}^{\psi_2} \sqrt{\psi} J_1(\psi)\, d\psi, \quad (27)$$

where $\psi_{1,2} = k\xi_{1,2}\sin\alpha\sin\theta$. The integral (27) can be calculated by decomposition of $J_1(\psi)$ into a standard infinite series [21], where we obtain

$$E_\theta = -\frac{T_v q\eta\sqrt[4]{n^2\beta^2 - 1}}{\sqrt{2\pi\beta}} F(\chi)(kd\sin\alpha)^{3/2} \frac{e^{ikR - i\pi/4}}{cR}, \quad (28)$$

where $\chi = \frac{kd\sin\alpha\sin\theta}{2}$,

$$F(\chi) = \sum_{m=0}^{\infty} \frac{(-1)^m \chi^{2m+1}}{m!(m+1)!(2m+5/2)}. \quad (29)$$

To determine the main radiation maximum, it is sufficient to take into account several terms in series (29) and find $\chi = \chi_{\max}$ from the equation $dF/d\chi = 0$. With three terms being taken into account, the solution χ_0 can be found rigorously. Perturbation χ_1 due to the fourth term can be found using the iteration method. After simple calculations we obtain

$$\chi_0 = \sqrt{\frac{13 - \sqrt{13}}{5}} \approx 1.37, \quad \chi_1 \approx -0.09, \quad (30)$$

$\chi_{\max} \approx \chi_0 + \chi_1 \approx 1.28$. The direction of maximum is determined as follows:

$$\theta_{\max} \approx \frac{2\chi_{\max}}{kd\sin\alpha} \approx \frac{2.56}{kd\sin\alpha}. \quad (31)$$

The maximum field is

$$|E_{\theta\,\max}| \approx 0.127\,\frac{T_v q\eta\sqrt[4]{n^2\beta^2 - 1}(kd\sin\alpha)^{3/2}}{cR\sqrt{\beta}}. \quad (32)$$

This approximation is confirmed by numerical calculation of the integral (17).

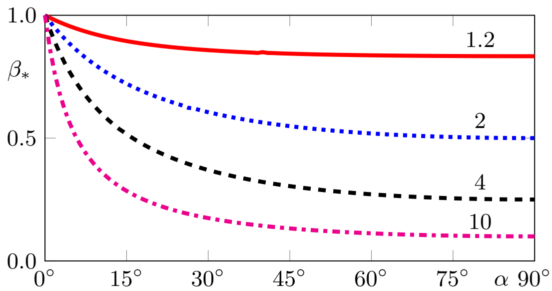

FIG. 3. Dependences of $\beta_*(\alpha)$ for different values of the refractive index $n = \sqrt{\varepsilon\mu}$ indicated near the curves.

Note that the value (32) is essentially larger than $|E_{\theta\,\max}|$ following from (21): their relation is of the order of $\sqrt{kd} \gg 1$. Thus, in this situation, we have powerful radiation directed at a small angle with respect to the z axis (almost "forward" radiation). This effect can be named the "Cherenkov spotlight." The phenomenon is of special interest because one can reach an essential amplification of radiation choosing problem parameters.

Solving Eq. (26) with respect to α, or n, or β one can obtain the following expressions, respectively:

$$\alpha = \alpha_* = \arctan\left(\frac{1-\beta}{\sqrt{n^2\beta^2 - 1}}\right),$$
$$n = n_* = \frac{\sqrt{1-\beta(2-\beta)\cos^2\alpha}}{\beta\sin\alpha}, \qquad (33)$$
$$\beta = \beta_* = \frac{\sin\alpha\sqrt{n^2 - \cos^2\alpha} - \cos^2\alpha}{n^2\sin^2\alpha - \cos^2\alpha}.$$

Dependencies of $\beta_*(\alpha)$ on different values of the refractive index n are shown in Fig. 3. One can see that the function $\beta_*(\alpha, n)$ monotonously decreases with both arguments. The effect of the Cherenkov spotlight is not practically reachable for ultrarelativistic bunches since $\alpha_* \to 0$ for $\beta \to 1$. As a rule, for $\alpha > 45^0$ the value of β_* is close enough to the Cherenkov threshold $1/n$. However, β_* can essentially differ from $1/n$ for narrow conical targets.

V. NUMERICAL CALCULATIONS

It is convenient to characterize the radiation with the help of frequency-angular density of the radiation power. The energy flow density is equal to $\vec{S} = \frac{c}{4\pi}[\vec{\mathbf{E}}, \vec{\mathbf{H}}]$, where $\{\vec{\mathbf{E}}, \vec{\mathbf{H}}\} = \int_{-\infty}^{\infty}\{\vec{E}, \vec{H}\}e^{-i\omega t}d\omega$ are components of a real physical field. One can show [7] that $\vec{S} = \vec{e}_R\int_0^{\infty}\sigma d\omega$, where $\sigma = c|\vec{E}|^2$ is a spectral density of the Pointing vector. Therefore, the frequency-angular density of the radiation power is

$$\frac{d^2W}{d\omega d\Omega} = R^2\sigma = cR^2|\vec{E}|^2. \qquad (34)$$

Some typical dependencies of this value on the angle θ are shown in Fig. 4 for three values of the cone angle α and different values of the charge velocity. Computations have been performed by the formulas (17) and (34). One can see

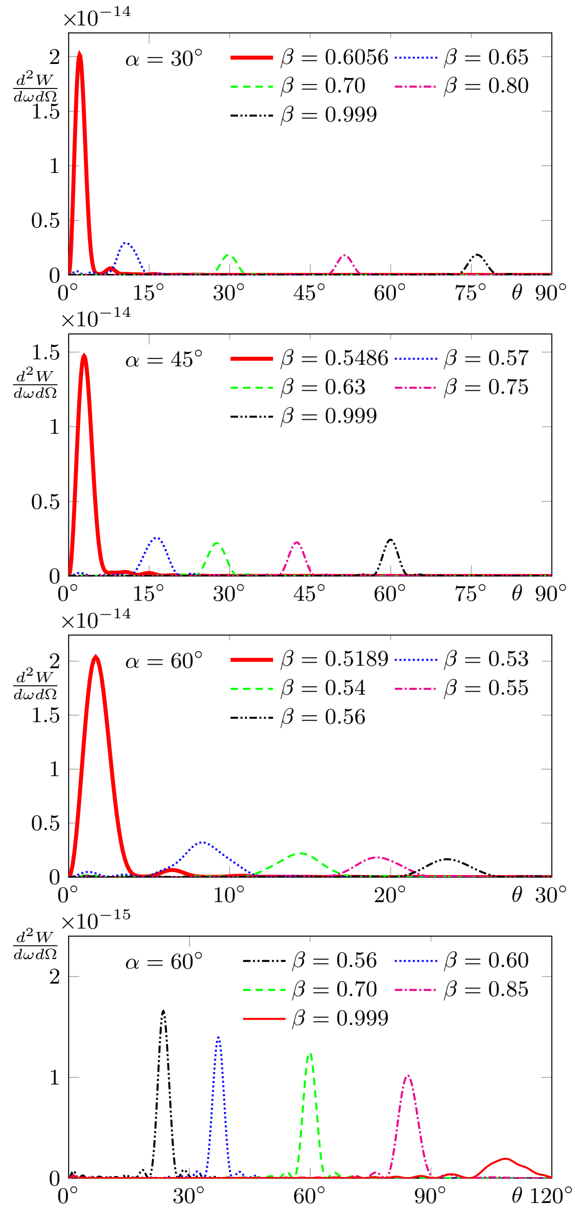

FIG. 4. Frequency-angular density of the radiation power (W s/rad^2) depending on θ for different values of β (indicated in the legend) and α (indicated in the corner of each plot). The case $\alpha = 60^\circ$ is shown in two plots because of the large difference in magnitudes. Problem parameters: $q = 1$ nC, $\varepsilon = 4$, $\mu = 1$, $d\omega/c = 100$, $a\omega/c = 1$.

that the angle of maximal radiation increases with the charge velocity that is explained by increase of the Cherenkov angle $\theta_p = \arccos[1/(n\beta)]$, leading to a decrease of the incident and refraction angles.

One can make sure that approximate results (20), (21), and (23) are true for all curves presented in Fig. 4 excluding the bold red curves. The bold red curves correspond to the case when $\theta_t = \frac{\pi}{2} - \alpha$, that is, the effect of the "Cherenkov spotlight" takes place. This situation is well described by formulas (28), (29), (31), and (32). One can see that the main maximum in this case is much larger than those for the other curves.

VI. CONCLUSION

We have analyzed the radiation from the charge which moves along the symmetry axis of a dielectric cone having a vacuum channel. The method applied uses the solution of a certain "etalon" problem (without external borders of the object), ray-optics laws on the aperture, and Stratton-Chu formulas for the field in the outer area. The field in the far-field area (Fraunhofer area) has been studied. It has been shown that, as a rule, maximal radiation is generated in the direction of the ray refracted at the nearest or furthest generatrix of the cone. However, when this ray is parallel to the symmetry axis, the maximal radiation is generated at some small (but nonzero) angle with respect to this axis. This angle is inversely proportional to the sine of the apex angle of the cone and the length of the cone generatrix. The field maximum in this situation is much larger than that in all other cases. Such an effect can be named the "Cherenkov spotlight" phenomenon.

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APPENDIX: ON VERIFICATION OF THE APERTURE METHOD

The aperture method developed in our papers [8–12] was repeatedly tested using comparison with simulations in the COMSOL MULTIPHYSICS software package [11,12]. Nevertheless, it is useful to consider this issue once again.

Figure 5 shows the spectral-angular density of the radiation energy as a function of the angle θ for $\alpha = 45^\circ$ and for three values of the charge velocity (the same parameters as in the second plot in Fig. 2 are used). As one can see, in all cases there is a good coincidence between the results of our algorithm and the results of COMSOL MULTIPHYSICS simulations in the area of the main maximum. This coincidence is almost ideal for the second and the third plots, when the maximum is in the region of large angles, but somewhat worse (<10% for the energetic characteristic and <5% for the field value) for the first plot, when the maximum is close to the structure axis.

Outside the region of the main maximum this coincidence may be worse, especially in the region of small angles. This fact can be explained by two effects: one of them is re-reflected waves and the another is the diffraction radiation. In Fig. 5 this discrepancy is noticeable at relatively small angles. However, in any case, we can say that the aperture method well describes the radiation field in the most important area, i.e., in the vicinity of the main maximum, even if this maximum lies at small angles.

The small contribution of re-reflected waves can be explained if we consider at least the wave of the first order

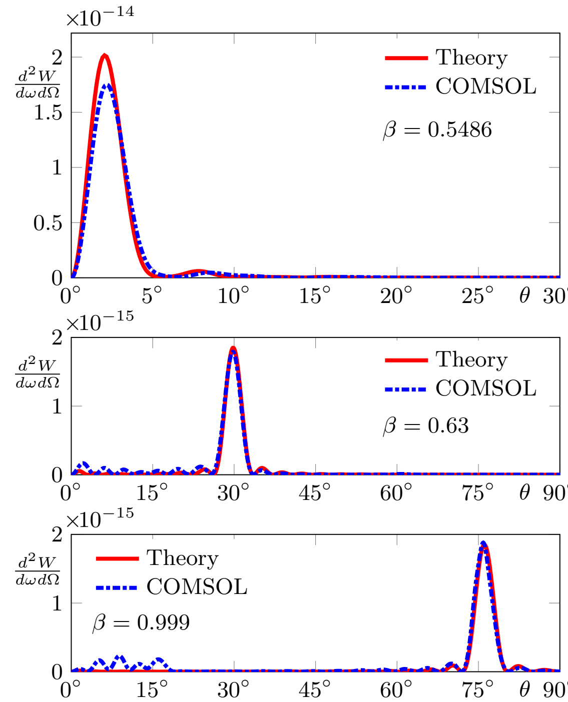

FIG. 5. Comparison of proposed aperture method with COMSOL MULTIPHYSICS simulations for the value of frequency-angular density of the radiation power (W s/rad^2). Problem parameters are as in Fig. 4, cone angle $\alpha = 45^\circ$.

(which is reflected once from the opposite wall of the object). Its amplitude has an additional factor equal to the product of the reflection and transmission coefficients. Considering the fact that the spectral-angular energy density contains the square of the amplitude, it can be shown that the contribution of this wave does not exceed a few percents compared to the contribution of the main wave calculated above. Such an estimation of the error of the method corresponds to the results of comparison shown in Fig. 5. By the way, we note that, if necessary, the method under consideration can be refined by considering re-reflected waves (however, this procedure is very cumbersome).

Note as well that Cherenkov radiation prevails over diffraction radiation, since it is formed over the large part of the trajectory of the charge inside the cone, while diffraction radiation has a much smaller formation zone (near the "nose"). This is mathematically expressed by the fact that expressions (21) and (32) include the large parameter kd, either in the power of 1 or 3/2 (in the "spotlight" mode). Of course, there is no such large parameter for the field of the diffraction radiation, since it is formed in some small neighborhood of the inhomogeneity [22].

In conclusion, we emphasize that the applied method does not claim to describe the radiation field for arbitrary angles of observation. However, it correctly describes the main effect, which is the exit of the main (nonreflected) wave of Cherenkov radiation from the dielectric object.

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