Coherence measure: Logarithmic coherence number

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(Received 20 November 2018; published 28 February 2019)

We introduce a measure of coherence which is extended from the coherence rank via the standard convex roof construction; we call it the logarithmic coherence number. This approach is parallel to the Schmidt measure in entanglement theory. We study some interesting properties of the logarithmic coherence number and show that this quantifier can be considered as a proper coherence measure. We find that the logarithmic coherence number can be calculated exactly for a large class of mixed states. We also discuss the relationships between the logarithmic coherence number and other coherence quantifiers, e.g., the relative entropy of coherence, the l_1-norm coherence, and average fidelity coherence. We give the relationship between coherence and entanglement in a bipartite system, and our results can be generalized to multipartite settings. Finally, we give that the creation of entanglement with bipartite incoherent operations is bounded by the logarithmic coherence number of the initial system during the process.

DOI: 10.1103/PhysRevA.99.022340

I. INTRODUCTION

The fundamental quality that distinguishes quantum states from classical states is quantum coherence, which is the most basic characteristic of quantum mechanics. Quantum coherence plays an important role in the study of quantum information and quantum multipartite systems. Baumgratz *et al.* proposed a theoretical framework for quantitative study of quantum coherence from the perspective of resource theory [1]. Various ways have been presented to develop the resource-theoretic framework for understanding quantum coherence; we refer to [2,3] for more discussions of resource theory of coherence.

Analogously to the Schmidt rank in entanglement theory [4,5], Killoran *et al.* presented a framework for the conversion of nonclassicality (including coherence) into entanglement; they introduced the coherence rank [6]. A concept related to the coherence rank was also discussed by Levi and Mintert [7]. Soon afterwards, Chin introduced a discrete coherence monotone named the coherence number, which is a generalization of the coherence rank to mixed states [8,9]. Regula *et al.* also discussed coherence number of mixed states; they presented a general formalism for the conversion of nonclassicality into multipartite entanglement [10]. Theurer *et al.* employed a natural generalization of the coherent rank to superposition with respect to a finite number of linear independent basis [11]. The coherence number is proved to be a discrete coherence monotone but it is not a proper coherence measure because it does not satisfy convexity [9,12,13]. To resolve this issue, in this paper we try to extend the coherence rank to mixed states via the standard convex roof construction; this approach is parallel to the Schmidt measure in [5]. We can prove that it is not only a coherence monotone but also a proper coherence measure.

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The paper is organized as follows. In Sec. II we review some basic concepts about the resource theory of coherence. In Sec. III we discuss the coherence rank and give a property about it. In Sec. IV, we introduce a coherence measure, which is the so-called the logarithmic coherence number, and give some interesting properties. In Sec. V we discuss the relationships between the logarithmic coherence number and other coherence quantifiers, and give some interesting relations. In Sec. VI we focus on the relationships between coherence and entanglement in bipartite and multipartite settings. In Sec. VII we discuss the interplay between coherence consumption and creation of entanglement. We summarizes our results in Sec. VIII.

II. BASIC CONCEPTS OF COHERENCE MEASURE

We introduce some concepts about coherence measure which can be used for our main results [1–3]. Given a d-dimension Hilbert space $\mathcal{H}$ with a fixed orthogonal basis $\mathcal{O}=\{|i\rangle\}_{i=0}^{d-1}$, we denote the set of all density operators acting on $\mathcal{H}$ by $\mathcal{D}(\mathcal{H})$. The density operators which are diagonal in this fixed basis are called incoherent; we denote the set of all incoherent states by $\mathcal{I}$, and $\mathcal{I}\subset\mathcal{D}(\mathcal{H})$. Any incoherent state δ is of the form

$$\delta=\sum_{i=0}^{d-1}\delta_i|i\rangle\langle i|, \tag{1}$$

where δ_i are probability distribution. Any state which cannot be written in the above form is defined as a coherent state, which means the coherence is basis dependent.

The incoherent operation is to map the incoherent states to incoherent states. The definitions of incoherent operations are not unique and different choices [2]. In this paper, we only consider the incoherent operation in [1]. The incoherent operation (IO) is a completely positive and trace preserving

(CPTP) map Λ that admit a Kraus operator representation

$$\Lambda(\rho) = \sum_n K_n \rho K_n^\dagger, \tag{2}$$

where all the Kraus operators K_n must satisfy $K_n \mathcal{I} K_n^\dagger \subseteq \mathcal{I}$ with $\sum_n K_n^\dagger K_n = I$. In general, the Kraus operator can always be represented as

$$K_n = \sum_i c_i |f(i)\rangle\langle i|, \tag{3}$$

where f is a function in the index set and $c_i \in [0, 1]$ [14].

Baumgratz *et al.* proposed that any proper measure of the coherence $\mathcal{C}$ must satisfy the following conditions [1]:

(C1) *Non-negativity.* $\mathcal{C}(\rho) \geqslant 0$ for all quantum states ρ, and $\mathcal{C}(\rho) = 0$ if and only if ρ is incoherent.

(C2) *Monotonicity.* $\mathcal{C}(\rho)$ is unincreasing under incoherent operation Λ, i.e., $\mathcal{C}(\rho) \geqslant \mathcal{C}[\Lambda(\rho)]$.

(C3) *Strong monotonicity.* $\mathcal{C}(\rho)$ does not increase on average under selective incoherent operations, i.e., $\sum_n q_n \mathcal{C}(\rho_n) \leqslant \mathcal{C}(\rho)$, where $\rho_n = K_n \rho K_n^\dagger / q_n$, and $q_n = \mathrm{Tr}(K_n \rho K_n^\dagger)$.

(C4) *Convexity.* $\mathcal{C}(\rho)$ is a convex function of quantum states, i.e., $\sum_i p_i \mathcal{C}(\rho_i) \geqslant \mathcal{C}(\sum_i p_i \rho_i)$, for any ensemble $\{p_i, \rho_i\}$.

Following standard notions of entanglement theory, we call a quantifier $\mathcal{C}$ which fulfills conditions (C1) and either condition (C2) or (C3) (or both) a coherence monotone. A quantifier $\mathcal{C}$ is further called a coherence measure if it satisfies the four conditions: (C1)–(C4). We also know that conditions (C3) and (C4) automatically imply condition (C2) [2].

III. COHERENCE RANK

For a pure state on Hilbert space $\mathcal{H}$ with the fixed orthogonal basis $\mathcal{O}$, one can define the coherence rank

$$R_C(|\psi\rangle) = \min \left\{ |\hat{\mathcal{O}}| \, \Big| \, |\psi\rangle = \sum_{|j\rangle \in \hat{\mathcal{O}}} \lambda_j |j\rangle, \hat{\mathcal{O}} \subseteq \mathcal{O} \right\}, \tag{4}$$

where λ_j are nonzero complex coefficients. ■

We note that the coherence rank given in Eq. (4) characterizes the minimal number of the incoherent states in the fixed orthogonal basis $\mathcal{O}$ in such a decomposition of $|\psi\rangle$. This is also equivalent to the fact that the coherence rank $R_C(|\psi\rangle) = k$ if exactly k of the coefficients λ_j are nonzero. Thus we say that the definition of the coherence rank given in Eq. (4) is equivalent to the definition introduced in Refs. [6,7,9–11]. Clearly, we have $1 \leqslant R_C(|\psi\rangle) \leqslant d$ and all coherent pure states should have $R_C(|\psi\rangle) \geqslant 2$. We know that the coherence rank is nonincreasing under incoherent operations Λ, that is,

$$R_C[\Lambda(|\psi\rangle)] \leqslant R_C(|\psi\rangle). \tag{5}$$

In particular, following the results in [6,14], we know that there exists a unitary incoherent operation U_{in} on a pure state $|\psi\rangle$ such that the coherence rank of $U_{\text{in}}|\psi\rangle$ is equal to the coherence rank of $|\psi\rangle$, i.e.,

$$R_C(U_{\text{in}}|\psi\rangle) = R_C(|\psi\rangle), \tag{6}$$

where $U_{\text{in}} = \sum_j e^{i\theta_j} |j\rangle\langle j|$ with some phases θ_j. Therefore, we say that the coherence rank is a coherence monotone.

We also consider the coherence rank of superposition of two coherent states. The following result will give the lower and upper bounds of the coherence of superposition.

Proposition 1. Let $|\phi\rangle = a|\psi\rangle + b|\varphi\rangle$ with $|a|^2 + |b|^2 = 1$, we have

$$|R_C(|\psi\rangle) - R_C(|\varphi\rangle)| \leqslant R_C(|\phi\rangle) \leqslant R_C(|\psi\rangle) + R_C(|\varphi\rangle). \tag{7}$$

Proof. By the definition of coherence rank, there exist two sets $\hat{\mathcal{O}}_\psi$ and $\hat{\mathcal{O}}_\varphi$ such that $R_C(|\psi\rangle) = |\hat{\mathcal{O}}_\psi|$, $R_C(|\varphi\rangle) = |\hat{\mathcal{O}}_\varphi|$, and one has

$$|\psi\rangle = \sum_{|j\rangle \in \hat{\mathcal{O}}_\psi} \psi_j |j\rangle, \quad |\varphi\rangle = \sum_{|k\rangle \in \hat{\mathcal{O}}_\varphi} \varphi_k |k\rangle. \tag{8}$$

Then, we will consider three cases as follows.

Case 1. If $\hat{\mathcal{O}}_\psi \perp \hat{\mathcal{O}}_\varphi$, by definition, we directly obtain

$$R_C(\phi\rangle) = R_C(|\psi\rangle) + R_C(|\varphi\rangle). \tag{9}$$

Case 2. If $\hat{\mathcal{O}}_\psi \cap \hat{\mathcal{O}}_\varphi \neq \varnothing$, without loss of generality, we take $\tilde{\mathcal{O}} = \hat{\mathcal{O}}_\psi \cap \hat{\mathcal{O}}_\varphi$, and

$$|\phi\rangle = a \sum_{|j\rangle \in \hat{\mathcal{O}}_\psi \setminus \tilde{\mathcal{O}}} \psi_j |j\rangle + \sum_{|j\rangle \in \tilde{\mathcal{O}}} (a\psi_j + b\varphi_j)|j\rangle + b \sum_{|k\rangle \in \hat{\mathcal{O}}_\varphi \setminus \tilde{\mathcal{O}}} \varphi_k |k\rangle. \tag{10}$$

Then, we have

$$\begin{aligned} R_C(|\phi\rangle) &\leqslant |\hat{\mathcal{O}}_\psi \setminus \tilde{\mathcal{O}}| + |\hat{\mathcal{O}}_\varphi \setminus \tilde{\mathcal{O}}| + |\tilde{\mathcal{O}}| \\ &= |\hat{\mathcal{O}}_\psi| + |\hat{\mathcal{O}}_\varphi| - |\tilde{\mathcal{O}}| \\ &\leqslant R_C(|\psi\rangle) + R_C(|\varphi\rangle). \end{aligned} \tag{11}$$

Case 3. If $\hat{\mathcal{O}}_\psi \subseteq \hat{\mathcal{O}}_\varphi$, then we have

$$|\phi\rangle = \sum_{|j\rangle \in \hat{\mathcal{O}}_\psi} (a\psi_j + b\varphi_j)|j\rangle + b \sum_{|k\rangle \in \hat{\mathcal{O}}_\varphi \setminus \hat{\mathcal{O}}_\psi} \varphi_k |k\rangle. \tag{12}$$

By the definition, we obtain

$$R_C(|\phi\rangle) \geqslant R_C(|\varphi\rangle) - R_C(|\psi\rangle). \tag{13}$$

Similarly, if $\hat{\mathcal{O}}_\varphi \subseteq \hat{\mathcal{O}}_\psi$, we have

$$R_C(|\phi\rangle) \geqslant R_C(|\psi\rangle) - R_C(|\varphi\rangle). \tag{14}$$

Thus, we obtain our desired result. ■

The coherence rank has been generalized to mixed states in [6,9,10]; it is the so-called coherence number, which is defined as

$$R_C(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \max_i [R_C(|\psi_i\rangle)]. \tag{15}$$

The coherence number is the smallest possible maximal coherence rank in any decomposition of the mixed states, and for pure states the coherence rank equals the coherence number. The coherence number only satisfies conditions (C1), (C2), and (C3), but it does not satisfy condition (C4) [9], so it is only a coherence monotone. In the following section, we apply the standard convex roof construction to the mixed states.

IV. LOGARITHMIC COHERENCE NUMBER

In this section, we can define logarithmic coherence rank, the same way as for the Schmidt rank in [5]. Note that Theurer *et al.* used this approach to describe the superposition in [11].

Definition 2. For any pure state $|\psi\rangle$, the logarithmic coherence rank is defined as

$$\mathcal{L}_C(|\psi\rangle) = \log_2 R_C(|\psi\rangle). \tag{16}$$

Obviously, the logarithmic coherence rank inherits some properties of coherence rank. The logarithmic coherence rank is non-negative, that is, $\mathcal{L}_C(|\psi\rangle) \geqslant 0$ for any pure state $|\psi\rangle$. In particular, for the maximally coherent states

$$|\psi_M\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{i\theta_j} |j\rangle, \tag{17}$$

where $\theta_j \in [0, 2\pi)$, we have

$$\mathcal{L}_C(|\psi_M\rangle) = \log_2 d. \tag{18}$$

In addition, we find that the logarithmic coherence rank is also monotone, unitarily invariant, and so on. The logarithmic coherence rank can be extended to mixed states by the standard convex roof construction.

Definition 3. For any mixed state ρ, the logarithmic coherence number is defined as

$$\mathcal{L}_C(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i \mathcal{L}_C(|\psi_i\rangle), \tag{19}$$

where the minimum is taken over all pure state decompositions of $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$.

Note that the minimization is taken by the average over all pure states $|\psi_i\rangle$ rather than a maximization, because the latter is only a coherence monotone [12,13,15]. In subsequent paragraphs we will show that the logarithmic coherence number is a proper coherence measure in the sense of Refs. [1,2].

Proposition 4. The logarithmic coherence number $\mathcal{L}_C$ is a coherence measure which satisfies the conditions (C1)–(C4).

Proof. Obviously, condition (C1) follows immediately from the definition.

To show that $\mathcal{L}_C$ satisfies condition (C3), let $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ be the optimal decomposition of ρ belonging to the minimum in Eq. (19), and we take the mark [11] and define

$$|\hat{\psi}_{i,n}\rangle = \frac{K_n |\psi_i\rangle}{\sqrt{q_n}}, \tag{20}$$

where $q_n = \mathrm{Tr}(K_n^\dagger K_n \rho)$, and K_n are incoherent Kraus operators. Then, every final state ρ_n in an incoherent Kraus operator K_n can be represented as

$$\rho_n = \frac{K_n \rho K_n^\dagger}{q_n} = \sum_i p_i |\hat{\psi}_{i,n}\rangle\langle\hat{\psi}_{i,n}|. \tag{21}$$

Since the coherence rank can never increase under the action of an incoherent Kraus operator, then we have

$$\begin{aligned} \mathcal{L}_C(\rho_n) &\leqslant \sum_i p_i \mathcal{L}_C(|\hat{\psi}_{i,n}\rangle) \\ &\leqslant \sum_i p_i \mathcal{L}_C(|\psi_i\rangle) \\ &= \mathcal{L}_C(\rho). \end{aligned} \tag{22}$$

Thus, we have

$$\sum_n q_n \mathcal{L}_C(\rho_n) \leqslant \mathcal{L}_C(\rho). \tag{23}$$

To show (C4) we take

$$\rho = \lambda_1 \rho_1 + \lambda_2 \rho_2, \tag{24}$$

where $\lambda_1, \lambda_2 \in [0, 1]$. Let $\rho_1 = \sum_j \mu_j |\phi_j\rangle\langle\phi_j|$ and $\rho_2 = \sum_k \eta_k |\varphi_k\rangle\langle\varphi_k|$ be the two decompositions for which the respective minima in Eq. (19) are attained. Then the convex combinations $\lambda_1 \sum_j \mu_j |\phi_j\rangle\langle\phi_j| + \lambda_2 \sum_k \eta_k |\varphi_k\rangle\langle\varphi_k|$ are a valid decomposition of ρ, but it is not necessarily the optimal one. Thus, we have

$$\begin{aligned} \mathcal{L}_C(\lambda_1\rho_1 + \lambda_2\rho_2) &\leqslant \lambda_1 \sum_j \mu_j \mathcal{L}_C(|\phi_j\rangle) + \lambda_2 \sum_k \eta_k \mathcal{L}_C(|\varphi_k\rangle) \\ &= \lambda_1 \mathcal{L}_C(\rho_1) + \lambda_2 \mathcal{L}_C(\rho_2). \end{aligned} \tag{25}$$

We know that the condition (C2) can be derived from conditions (C3) and (C4), so we say the logarithmic coherence number $\mathcal{L}_C$ satisfies conditions (C1)–(C4). ■

This shows that the logarithmic coherence number can indeed be used as a coherence measure quantifying the coherence of a quantum system. Not just these nice properties, we find that the logarithmic coherence number is additive as follows.

Proposition 5. The logarithmic coherence number $\mathcal{L}_C$ is additive.

Proof. Let us consider the case of pure states first. From the definition of the coherence rank, we have

$$R_C(|\psi_1\rangle \otimes |\psi_2\rangle) = R_C(|\psi_1\rangle) R_C(|\psi_2\rangle). \tag{26}$$

Thus, we obtain

$$\begin{aligned} \mathcal{L}_C(|\psi_1\rangle \otimes |\psi_2\rangle) &= \log_2(R_C(|\psi_1\rangle) R_C(|\psi_2\rangle) \\ &= \mathcal{L}_C(|\psi_1\rangle) + \mathcal{L}_C|\psi_2\rangle). \end{aligned} \tag{27}$$

Then we consider the case of mixed states. Without loss of generality, the pure state decomposition of $\rho \otimes \sigma$ is of the form

$$\rho \otimes \sigma = \sum_a p_a |\psi_a\rangle\langle\psi_a| \otimes \sum_b p_b |\phi_b\rangle\langle\phi_b|. \tag{28}$$

Then we have

$$\begin{aligned} \mathcal{L}_C(\rho \otimes \sigma) &= \min \sum_{a,b} p_a p_b \mathcal{L}_C(|\psi_a\rangle \otimes |\phi_b\rangle) \\ &= \min \sum_a p_a \mathcal{L}_C(|\psi_a\rangle) \otimes \min \sum_b p_b \mathcal{L}_C(|\phi_b\rangle) \\ &= \mathcal{L}_C(\rho) + \mathcal{L}_C(\sigma). \end{aligned} \tag{29}$$

This completes the proof of the proposition. ■

From this result, for n copies of the same state $|\psi\rangle$, we have

$$\mathcal{L}_C(|\psi\rangle^{\otimes n}) = n\mathcal{L}_C(|\psi\rangle). \tag{30}$$

In particular, let δ be an incoherent state where we then have

$$\mathcal{L}_C(\delta^{\otimes n} \otimes |\psi\rangle\langle\psi|^{\otimes n}) = n\mathcal{L}_C(|\psi\rangle). \tag{31}$$

If the states $|\psi_1\rangle$ and $|\psi_2\rangle$ satisfy $||\psi_1\rangle - |\psi_2\rangle| < \varepsilon$, we may ask whether the logarithmic coherence number also satisfies $|\mathcal{L}_C(|\psi_1\rangle) - \mathcal{L}_C(|\psi_2\rangle)| < \varepsilon$, where $|\cdot|$ is trace distance [16]. Let

$$|\psi_1\rangle = \sqrt{1-\varepsilon}|0\rangle + \sqrt{\frac{\varepsilon}{d-1}}\sum_{i=1}^{d-1}|i\rangle, \tag{32}$$

and $|\psi_2\rangle = |0\rangle$. When $\varepsilon \to 0$, it means $|\psi_1\rangle \to |\psi_2\rangle$, but we know that

$$|\mathcal{L}_C(|\psi_1\rangle) - \mathcal{L}_C(|\psi_2\rangle)| = \log_2 d. \tag{33}$$

Thus, we claim that the logarithmic coherence number is not continuous.

Although we define the coherence measure of a mixed state via a minimization over all possible realizations of the state, it can be calculated exactly for some states. In order to calculate the logarithmic coherence number of a mixed state, the minimization over decompositions of the state is necessary. The value of $\mathcal{L}_C$ can be fully evaluated for some states. We first consider a family of noisy maximally coherent states

$$\rho_\lambda = \lambda|\psi_M\rangle\langle\psi_M| + (1-\lambda)\frac{I}{d} \tag{34}$$

where $\lambda \in (0, 1)$. Without loss of generality, the identity operator I can be represented with the pure states $|\psi_i\rangle$ as

$$I = \sum_i \alpha_i|\psi_i\rangle\langle\psi_i|, \tag{35}$$

where $\alpha_i \geqslant 0$. Then, the pure state decomposition of ρ_λ is of the form

$$\rho_\lambda = \lambda|\psi_M\rangle\langle\psi_M| + \frac{1-\lambda}{d}\sum_i \alpha_i|\psi_i\rangle\langle\psi_i|. \tag{36}$$

Using the definition (19), we get

$$\mathcal{L}_C(\rho_\lambda) = \min_{\{\alpha_i,|\psi_i\rangle\}}\left[\lambda\log_2 d + \frac{1-\lambda}{d}\sum_i \alpha_i\mathcal{L}_C(|\psi_i\rangle)\right]. \tag{37}$$

Minimizing the right-hand side of Eq. (40) over all pure state decompositions, we immediately see that the minimum is achieved for every i, $\mathcal{L}_C(|\psi_i\rangle) = 0$. Thus, we obtain a closed expression of the logarithmic coherence number for the state ρ_p, i.e.,

$$\mathcal{L}_C(\rho_\lambda) = \lambda\log_2 d. \tag{38}$$

We then consider a class of more general mixed states, the pseudomixed state, which is of the form

$$\rho_p = p|\psi\rangle\langle\psi| + (1-p)\delta, \tag{39}$$

where $p \in (0, 1)$, the state $|\psi\rangle$ is any coherent state, and $\delta \in \mathcal{I}$.

From definition (19), we claim that the minimum over all pure state decompositions reduced to all incoherent states, and then we have

$$\begin{aligned}\mathcal{L}_C(\rho_p) &= \min_{\delta\in\mathcal{I}}[p\mathcal{L}_C(|\psi\rangle) + (1-p)\mathcal{L}_C(\delta)]\\ &= p\mathcal{L}_C(|\psi\rangle).\end{aligned} \tag{40}$$

Note that the state (34) is a special case of the pseudomixed state (34). Even so, in the following section we find that there is a considerable difference from the l_1 norm of coherence.

V. THE RELATIONS WITH THE OTHER COHERENCE QUANTIFIERS

In this section, we will discuss the relationships between the logarithmic coherence number and other coherence quantifiers, e.g., the relative entropy of coherence, the l_1-norm coherence, and average fidelity coherence. The definitions of the first two quantifiers are original defined in [1]; the latter we define here.

A. The relative entropy of coherence

In the framework resource theory of coherence, we know that one of the interesting coherence measures is the relative entropy of coherence [1], which is defined as

$$\mathcal{C}_r(\rho) = \min_{\delta\in\mathcal{I}} S(\rho||\delta), \tag{41}$$

where $S(x||y) = \mathrm{Tr}(x\log_2 x - x\log_2 y)$ is quantum relative entropy [16].

The relative entropy of coherence fulfills conditions C1–C4; it is a proper coherence measure [1]. The relative entropy of coherence can be also interpreted as the minimal amount of noise required for fully decohering states [17] and has been applied in many fields [2,12,14,15,17–26]. We know that the relative entropy of coherence has a closed expression [1], that is,

$$\mathcal{C}_r(\rho) = S[\Delta(\rho)] - S(\rho), \tag{42}$$

where $\Delta(\rho) = \sum_i\langle i|\rho|i\rangle|i\rangle\langle i|$ is a completely dephasing operation [1,2]. Then the following result shows that the relative entropy of coherence is upper bounded by the logarithmic coherence number.

Proposition 6. For any mixed state ρ, we have

$$\mathcal{C}_r(\rho) \leqslant \mathcal{L}_C(\rho). \tag{43}$$

Proof. We first consider pure states. Without loss of generality we suppose that pure state $|\psi\rangle$ with $R_C(|\psi\rangle) = r$ and let

$$|\psi\rangle = \sum_{i=1}^{r}\lambda_i|i\rangle. \tag{44}$$

Since we have

$$S[\Delta(|\psi\rangle\langle\psi|)] = -\sum_{i=1}^{r}\lambda_i^2\log_2\lambda_i^2 \leqslant \log_2 r, \tag{45}$$

for any pure state $|\psi\rangle$, we obtain

$$\mathcal{C}_r(|\psi\rangle) \leqslant \mathcal{L}_C(|\psi\rangle). \tag{46}$$

Next, we can generalize this result to the mixed states. Let $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ be the optimal decomposition of ρ belonging to the minimum in Eq. (19). Then from the convexity of the relative entropy of coherence, we have

$$\begin{aligned}\mathcal{C}_r(\rho) &\leqslant \sum_i p_i \mathcal{C}_r(|\psi_i\rangle)\\ &\leqslant \sum_i p_i \mathcal{L}_C(|\psi_i\rangle)\\ &= \mathcal{L}_C(\rho).\end{aligned} \tag{47}$$

This completes the proof of the proposition. ■

Intuitively, the logarithmic coherence number can give a tighter upper bound of the relative entropy of coherence. This also gives a nice operational expression for the logarithmic coherence number.

B. The l_1-norm coherence

The l_1-norm coherence is another interesting coherence measure [1], which is formally defined as

$$\mathcal{C}_{l_1}(\rho) = \min_{\delta} ||\rho - \delta||_{l_1} = \sum_{i\neq j} |\rho_{ij}|. \tag{48}$$

The l_1-norm coherence captures the simple intuitive idea that on the level of density matrix description of quantum states, superposition coherence corresponds to off-diagonal matrix elements with respect to the fixed basis [27] and an operational interpretation is given in this reference.

For any pure state, we can give a relation between the logarithmic coherence number and the l_1-norm coherence.

Proposition 7. For any pure state $|\psi\rangle$, we have

$$\log_2(\mathcal{C}_{l_1}(|\psi\rangle) + 1) \leqslant \mathcal{L}_C(|\psi\rangle). \tag{49}$$

Proof. We suppose that the pure state $|\psi\rangle$ with $R_C(|\psi\rangle) = r$, from Eq. (44). We then have

$$\mathcal{C}_{l_1}(|\psi\rangle) + 1 = \left(\sum_{i=1}^{r} |\lambda_i|\right)^2. \tag{50}$$

Using Lagrange multipliers to implement the constraint $\sum_{i=1}^{r} |\lambda_i|^2 = 1$, we can obtain

$$\left(\sum_{i=1}^{r} |\lambda_i|\right)^2 \leqslant r. \tag{51}$$

Combining Eq. (50) with Eq. (51), by taking the logarithm, we obtain the desired result. ■

From the proof, we find that if the pure state $|\psi\rangle$ with the coherence rank $R_C(|\psi\rangle) = r$, then the l_1-norm coherence is upper bounded by the coherence rank, i.e.,

$$\mathcal{C}_{l_1}(|\psi_M\rangle) \leqslant R_C(|\psi_M\rangle) - 1. \tag{52}$$

In particular, if we require $R_C(|\psi\rangle) > 1$ (or $\mathcal{L}_C(|\psi\rangle) > 0$), then the upper bound in Eq. (49) is saturated if and only if the states are the maximally coherent states $|\psi_M\rangle$, and we have

$$\log_2(\mathcal{C}_{l_1}(|\psi_M\rangle) + 1) = \mathcal{L}_C(|\psi_M\rangle). \tag{53}$$

For the mixed states, the situation becomes complicated, and it is hard to get a clear relation. From the inspiration in [13,27], we first consider the qubit system. The following result gives a relation between the logarithmic coherence number and the l_1-norm coherence.

Proposition 8. All qubit states ρ with a given l_1-norm coherence $\mathcal{C}_{l_1}(\rho) = 2b$ satisfy

$$\mathcal{L}_C(\rho) \leqslant \log_2(\mathcal{C}_{l_1}(\rho) + 1). \tag{54}$$

Proof. From the proof of Proposition 1 in Ref. [27], let

$$\rho = \begin{pmatrix} a & b \\ b & 1-a \end{pmatrix} \tag{55}$$

be a state with given l_1-norm coherence $2b > 0$, where $0 < b \leqslant \frac{1}{2}$ and

$$\frac{1 - \sqrt{1 - 4b^2}}{2} \leqslant a \leqslant \frac{1 + \sqrt{1 - 4b^2}}{2}. \tag{56}$$

In fact, we can always rewrite the statement above as

$$\rho = 2b|+\rangle\langle+| + (a-b)|0\rangle\langle 0| + (1-a-b)|1\rangle\langle 1|. \tag{57}$$

Thus, for a given fixed b, using the definition of the logarithmic coherence number, we obtain

$$\mathcal{L}_C(\rho) = 2b \leqslant \log_2(2b+1) = \log_2[\mathcal{C}_{l_1}(\rho) + 1]. \tag{58}$$

This completes the proof the proposition. ■

Note that for general mixed states on $\mathcal{H}$ with the dimension $d > 2$, it is difficult to obtain sharp interrelations between them. We take two examples to expound this situation. Let us first consider the state ρ_λ. It is easy to see that its l_1-norm coherence is

$$\mathcal{C}_{l_1}(\rho_\lambda) = (d-1)\lambda. \tag{59}$$

Using Eq. (38), we obtain

$$\mathcal{L}_C(\rho_\lambda) \leqslant \log_2[\mathcal{C}_{l_1}(\rho_\lambda) + 1]. \tag{60}$$

By the result [27] and Proposition 6, we obtain

$$\mathcal{C}_r(\rho) \leqslant \mathcal{L}_C(\rho_\lambda) \leqslant \log_2[\mathcal{C}_{l_1}(\rho_\lambda) + 1]. \tag{61}$$

This shows that the logarithmic coherence number is a tighter upper bound of the relative entropy of coherence, and it also gives another lower bound of the l_1-norm coherence.

Next, we consider the pseudomixed state; without loss of generality, we suppose that $R_C(|\psi\rangle) = r > 2$. Using Eq. (52) together with the definition of the l_1-norm coherence, we obtain

$$\mathcal{C}_{l_1}(\rho_p) + 1 = p\mathcal{C}_{l_1}(|\psi\rangle) + 1 \leqslant p(r-1) + 1. \tag{62}$$

It is easy to see that the inequality $\log_2[p(r-1)+1] \geqslant p\log_2 r$ is true, but we cannot give the following relation:

$$\mathcal{L}_C(\rho_p) \leqslant \log_2[\mathcal{C}_{l_1}(\rho_p) + 1]. \tag{63}$$

This is because we know that the quantity $p(r-1)+1$ is very coarse bound in Eq. (62). If we use the pure state (32) to replace the pure state in the pseudomixed state and take $\epsilon = \frac{1}{d-2}$, we can obtain

$$\mathcal{C}_{l_1}(\rho_p) = \left(1 + \frac{2\sqrt{(d-1)(d-3)}}{d-2}\right)p < 3p. \tag{64}$$

When $d > 16$, we have

$$\log_2[\mathcal{C}_{l_1}(\rho_p) + 1] < \log_2(3p+1) < \mathcal{L}_C(\rho_p). \tag{65}$$

But we also know that the logarithmic coherence number is possibly too high compared to $\log_2[\mathcal{C}_{l_1}(\rho_p) + 1]$ such that the logarithmic coherence number $\mathcal{L}_C(\rho_p)$ is not a better upper bound for the function $\log_2[\mathcal{C}_{l_1}(\rho_p)+1]$. Then, if the equations (65) hold, from the result in [27], we obtain

$$\mathcal{C}_r(\rho_p) \leqslant \log_2[\mathcal{C}_{l_1}(\rho_p)+1] < \mathcal{L}_C(\rho_p). \quad (66)$$

This shows that there is not a fixed order relation between the logarithmic coherence number and the l_1-norm coherence for all mixed states.

C. Average fidelity coherence

The ensemble notion of a quantum source leads to a definition of ensemble average fidelity, which captures the idea that the source is well preserved under the action of a noisy channel [16]. This is one of our motivations for considering the average fidelity coherence. In addition, we know that any state $|\psi\rangle$ with a distillable coherence of c_1 cobits can be asymptotically converted into any other state $|\phi\rangle$ with a distillable coherence of c_2 cobits at a rate c_1/c_2 [14]. In the one-shot scenario, the authors use the fidelity between the resource states and the maximally coherent states to describe the distillable conditions [15,22–25]. From these settings and the result in [4], we give the definition of the average fidelity coherence.

Definition 9. Suppose that the state ρ depends on the pure ensemble $\{p_i, |\psi_i\rangle\}$, i.e., $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$, the average fidelity coherence is then defined as

$$\overline{F}(\rho) = \sum_i p_i F(|\psi_i\rangle), \quad (67)$$

where

$$F(|\psi_i\rangle) = -\log_2 F(|\psi_i\rangle, |\psi_M\rangle)^2, \quad (68)$$

where $F(x,y) = \mathrm{Tr}\sqrt{\sqrt{x}y\sqrt{x}}$ is fidelity [16].

Note that this quantifier is not a coherence quantifier, and it is also not different from the min-relative entropy [28–30]. Obviously, $0 \leqslant \overline{F}(\rho) \leqslant \log_2 d$, and provided $\overline{F}(\rho) = 0$ if and only if for every i, the state $|\psi_i\rangle$ is a maximally coherent state. For all incoherent states, we have $\overline{F}(\delta) = \log_2 d$. This is to say, although the average fidelity coherence $\overline{F}$ is not a coherence quantifier, it is viewed as a coherence witness. We say that if the state ρ satisfies $\overline{F}(\rho) \neq \log_2 d$, then the state is coherent. We can give an uncertainly relation between the logarithm coherence number and the average fidelity coherence as follows:

Proposition 10. For any quantum state ρ, we have

$$\log_2 d \leqslant \overline{F}(\rho) + \mathcal{L}_C(\rho). \quad (69)$$

Proof. For any pure state $|\psi\rangle$ with coherent rank r, we have

$$\langle\psi_M|\psi\rangle = \frac{1}{\sqrt{d}}\sum_j^r \lambda_j e^{i(\omega_j-\theta_j)}, \quad (70)$$

where $|\psi\rangle = \sum_{j=1}^r \lambda_j e^{i\omega}|j\rangle$ with real numbers λ_j and $\omega \in [0, 2\pi)$. It follows that

$$|\langle\psi_M|\psi\rangle|^2 \leqslant \frac{1}{d}\left(\sum_{j=1}^r \lambda_j\right)^2. \quad (71)$$

Using Lagrange multipliers to implement the constraint $\sum_j |\lambda_j|^2 = 1$, we can obtain that

$$\frac{1}{d}\left(\sum_{j=1}^r \lambda_j\right)^2 \leqslant \frac{r}{d}. \quad (72)$$

Thus, we have

$$F(|\psi\rangle) \geqslant \log_2 d - \log_2 r. \quad (73)$$

Next, we consider the mixed states. Let $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ be a pure state decomposition for which the minimum in Eq. (19) is attained, and we denote $r_i = R_C(|\psi_i\rangle)$. Then we have

$$\begin{aligned}\overline{F}(\rho) &= \sum_i p_i F(|\psi_i\rangle)\\ &\geqslant \log_2 d - \sum_j p_i \log_2 r_i\\ &= \log_2 d - \mathcal{L}_C(\rho). \end{aligned} \quad (74)$$

This completes the proof the proposition.

This shows that the smaller the logarithmic coherence rank, the larger the average fidelity coherence, that is to say, the amount of coherence distilled by some operations is possible to decrease. ■

VI. MULTIPARTITE SCENARIO

Let $\mathcal{H}^S$ and $\mathcal{H}^A$ be two d-dimensional Hilbert spaces and $\mathcal{H}^A$ be the Hilbert space of an ancillary system with $\mathcal{H}^S \cong \mathcal{H}^A$. Without loss of generality, we take the orthogonal basis $\{|i\rangle\}_{i=0}^{d-1}$ and $\{|j\rangle\}_{j=0}^{d-1}$ as two fixed bases on $\mathcal{H}^S$ and $\mathcal{H}^A$, respectively. Then their tensor product $\{|i\rangle\otimes|j\rangle\}$ can be viewed as an incoherent basis for compound system SA. Thus, the corresponding logarithmic coherence rank and the logarithmic coherence number can be defined as in (16) and (19). We are particularly interested in the relationship between the total coherence and coherence contained in each individual subsystem. In the following proposition, we prove that the logarithmic coherence number in the bipartite quantum states is no less than the sum between two subsystems. This relation can be viewed as the superadditivity for the logarithm coherence number.

Proposition 11. For any bipartite quantum state ρ^{SA} on SA, we have

$$\mathcal{L}_C(\rho^S) + \mathcal{L}_C(\rho^A) \leqslant \mathcal{L}_C(\rho^{SA}), \quad (75)$$

where ρ^S and ρ^A are reduced states on S and A, respectively.

Proof. First, we consider the case of pure states. Let

$$|\psi^{SA}\rangle = \sum_{i=0}^{r_S-1}\sum_{j=0}^{r_A-1} a_{ij}|i^S\rangle|j^A\rangle \quad (76)$$

be the optimal decomposition of $|\psi^{SA}\rangle$ belonging to the minimum in Eq. (16). It follows then that

$$R_C(|\psi^{SA}\rangle) = r_S \times r_A. \quad (77)$$

Further, the matrix M of complex numbers a_{ij} can be represented as

$$M = \begin{pmatrix} N_{r_S\times r_A} & O \\ O & O \end{pmatrix}, \tag{78}$$

where $N_{r_S\times r_A} = (a_{ij})_{r_S\times r_A}$, and O are zero matrices. Using the singular value decomposition, $M = U\Sigma V$, where Σ is a diagonal matrix with non-negative elements λ_m, which are the singular values of M, and U and V are unitary matrices. Thus, it is always possible to write $|\psi^{SA}\rangle$ in the following way,

$$|\psi^{SA}\rangle = \sum_{m=0}^{r-1} \lambda_m |m^S\rangle|m^A\rangle, \tag{79}$$

where r is the Schmidt number of the state $|\psi^{SA}\rangle$, and

$$|m^S\rangle = \sum_{i=0}^{r_S-1} u_{im}|i^S\rangle, \quad |m^A\rangle = \sum_{j=0}^{r_A-1} v_{mj}|j^A\rangle. \tag{80}$$

Here the complex numbers u_{im} and v_{mj} are matrix elements of unitary matrices U and V. It is easy to see that the coherence rank of states $|m^S\rangle$ and $|m^A\rangle$ cannot exceed the numbers r_S and r_A, respectively. This means that for every m the following inequalities hold:

$$\mathcal{L}_C(|m^S\rangle) \leqslant \log_2 r_S, \quad \mathcal{L}_C(|m^A\rangle) \leqslant \log_2 r_A. \tag{81}$$

For the subsystem S, we know that $\rho^S = \sum_m \lambda_m^2 |m\rangle^S\langle m|$ is a valid decomposition of ρ^S. Then we obtain

$$\begin{aligned}\mathcal{L}_C(\rho^S) &\leqslant \sum_m \lambda_m^2 \mathcal{L}_C(|m^S\rangle) \\ &\leqslant \sum_m \lambda_m^2 \log_2 r_S \\ &= \log_2 r_S,\end{aligned} \tag{82}$$

and similarly, that

$$\mathcal{L}_C(\rho^A) \leqslant \log_2 r_A. \tag{83}$$

The above inequalities together with Eq. (77) imply the following inequality:

$$\mathcal{L}_C(\rho^S) + \mathcal{L}_C(\rho^A) \leqslant \mathcal{L}_C(|\psi^{SA}\rangle). \tag{84}$$

For any mixed state ρ^{SA}, let $\rho^{SA} = \sum_i p_i|\psi_i\rangle^{SA}\langle\psi_i|$ be the optimal decomposition of $|\psi^{SA}\rangle$ belonging to the minimum in Eq. (19), where we then have

$$\mathcal{L}_C(\rho^{SA}) = \sum_i p_i \mathcal{L}_C(|\psi_i^{SA}\rangle). \tag{85}$$

Combining Eqs. (84) and (85), we obtain

$$\begin{aligned}\mathcal{L}_C(\rho^{SA}) &= \sum_i p_i \mathcal{L}_C(|\psi_i^{SA}\rangle) \\ &\geqslant \sum_i p_i \mathcal{L}_C(\rho_i^S) + \sum_i p_i \mathcal{L}_C(\rho_i^A) \\ &\geqslant \mathcal{L}_C(\rho^S) + \mathcal{L}_C(\rho^A).\end{aligned} \tag{86}$$

This completes the proof of the proposition. ■

From the proof of the proposition, we immediately see that the Schmidt number r does not exceed the numbers r_S and r_A, i.e.,

$$r \leqslant \min\{r_S, r_A\}. \tag{87}$$

Thus, we can obtain an interesting relation between entanglement and coherence as follows:

$$\max\{\mathcal{L}_C(\rho^S), \mathcal{L}_C(\rho^A)\} + \mathcal{L}_E(|\psi^{SA}\rangle) \leqslant \mathcal{L}_C(|\psi^{SA}\rangle), \tag{88}$$

where $\mathcal{L}_E(|\psi^{SA}\rangle)$ is the Schmidt number, which is defined in [5], and $\mathcal{L}_E(|\psi^{SA}\rangle) = \log_2 r$. Note that the equality in the above inequality holds if and only if the matrix M is a diagonal matrix.

This relation shows that the sum between the entanglement and coherence contained in one subsystem cannot be more than the total coherence. This relation can be generalized to the mixed states, so for any bipartite mixed state ρ^{SA}, we have

$$\max\{\mathcal{L}_C(\rho^S), \mathcal{L}_C(\rho^A)\} + \mathcal{L}_E(\rho^{SA}) \leqslant \mathcal{L}_C(\rho^{SA}). \tag{89}$$

Here, $\mathcal{L}_E(\rho^{SA})$ is the Schmidt number of a mixed state, which is defined as [5]

$$\mathcal{L}_E(\rho^{SA}) = \min_{\{p_i,|\psi_i^{SA}\rangle\}} \sum_i p_i \mathcal{L}_E(|\psi_i^{SA}\rangle), \tag{90}$$

where the minimum is taken over all pure state decompositions of $\rho^{SA} = \sum_i p_i|\psi_i\rangle^{SA}\langle\psi_i|$.

In fact, our results (75) and (89) are also generalized to the multipartite setting. Let $\rho^{SA_1\ldots A_N}$ be an $N+1$-partite state. Then by the repeated use of superadditivity, we have

$$\mathcal{L}_C(\rho^S) + \sum_{i=1}^N \mathcal{L}_C(\rho^{A_i}) \leqslant \mathcal{L}_C(\rho^{A_1\ldots A_N}). \tag{91}$$

Combining Eqs. (89) and (91), we have

$$\mathcal{L}_E(\rho^{S|A_1\ldots A_N}) + \sum_{i=1}^N \mathcal{L}_C(\rho^{A_i}) \leqslant \mathcal{L}_C(\rho^{A_1\ldots A_N}), \tag{92}$$

where $\mathcal{L}_E(\rho^{S|A_1\ldots A_N})$ is the Schmidt number with the bipartite cut $S|A_1\ldots A_N$.

Finally, it is interesting to compare the logarithmic coherence number with the Schmidt number. We consider a quantum-incoherent state which has the following form:

$$\chi^{SA} = \sum_i p_i|i\rangle^S\langle i| \otimes \rho_i^A, \tag{93}$$

where ρ_i^A are arbitrary quantum states on A, and the states $|i^S\rangle$ belong to the local incoherent basis of S [20]. For any quantum-incoherent state, we can easily obtain that the Schmidt number is zero, i.e.,

$$\mathcal{L}_E(\rho^{SA}) = 0. \tag{94}$$

Meanwhile, we can obtain the following relation, i.e.,

$$\mathcal{L}_C(\chi^{SA}) \leqslant \sum_i p_i \mathcal{L}_C(\rho_i^A). \tag{95}$$

We note that the minimum in $\mathcal{L}_C(\chi^{SA})$ depends only on the pure decomposition of ρ_i^A, so without loss of generality, let

$\chi^{SA} = \sum_{ij} p_i q_j |i\rangle^S\langle i| \otimes |\psi_{ij}\rangle^A\langle\psi_{ij}|$ be the optimal decomposition of χ^{SA} belonging to the minimum in Eq. (19). We then have

$$\begin{aligned}\mathcal{L}_C(\chi^{SA}) &= \sum_{ij} p_i q_j \mathcal{L}_C(|\psi_{ij}\rangle^A\langle\psi_{ij}|)\\ &= \sum_i p_i \sum_j q_j \mathcal{L}_C(|\psi_{ij}\rangle^A\langle\psi_{ij}|)\\ &\geqslant \sum_i p_i \mathcal{L}_C\left(\sum_j q_j |\psi_{ij}\rangle^A\langle\psi_{ij}|\right)\\ &= \sum_i p_i \mathcal{L}_C(\rho_i^A). \end{aligned} \tag{96}$$

Combining Eqs. (95) and (96), we have

$$\mathcal{L}_C(\chi^{SA}) = \sum_i p_i \mathcal{L}_C(\rho_i^A). \tag{97}$$

VII. CONVERTING COHERENCE TO ENTANGLEMENT

In this section, using the logarithmic coherence number, we discuss the relation between the coherence of a mixed state ρ^S in an initial system S with the entanglement generated from ρ^S by attaching an ancilla system A and taking an incoherent operation Λ^{SA} on the bipartite system SA. Based on different measures, some authors have been investigated as well [6,10,31,32].

Proposition 12. The entanglement generated from a state ρ^S via an incoherent operation Λ^{SA} is bounded above by the logarithmic coherence number, i.e.,

$$\mathcal{L}_C(\rho^S) \geqslant \mathcal{L}_E(\Lambda^{SA}(\rho^S \otimes |0\rangle^A\langle 0|)). \tag{98}$$

Proof. Let $|0\rangle\langle 0|^A$ be an incoherent state on A. Then we have

$$\begin{aligned}\mathcal{L}_C(\rho^S) &= \mathcal{L}_C(\rho^S \otimes |0\rangle\langle 0|^A)\\ &\geqslant \mathcal{L}_C[\Lambda^{SA}(\rho^S \otimes |0\rangle\langle 0|^A)]\\ &= \sum_k \lambda_k \mathcal{L}_C(|\phi\rangle^{SA})\\ &\geqslant \sum_k \lambda_k \mathcal{L}_E(|\phi\rangle^{SA})\\ &= \mathcal{L}_E[\Lambda^{SA}(\rho^S \otimes |0\rangle\langle 0|^A)], \end{aligned} \tag{99}$$

where the second equality comes from the fact that $\Lambda^{SA}(\rho^S \otimes |0\rangle\langle 0|^A) = \sum_k \lambda_k |\phi_k\rangle^{SA}\langle\phi_k|$ is an optimal pure states decomposition of $\Lambda^{SA}(\rho^S \otimes |0\rangle\langle 0|^A)$ belonging to the minimum in Eq. (19), and the second inequality depends on the fact that the coherence rank is greater than or equal to the Schmidt rank. ■

From the results in [6,9,10,31], we know that a unitary operation which makes the coherence rank and the Schmidt number equal is given by

$$U = \sum_{i=0}^{d-1}\sum_{j=i}^{d-1} |i\rangle^S\langle i| \otimes |i \oplus (j-1)\rangle^A\langle j|, \tag{100}$$

where $\oplus$ means an addition modula d. Let $|\psi^S\rangle = \sum_i \lambda_i |i^S\rangle$ be a pure state on S; then the unitary operation can map the state $|\psi^S\rangle \otimes |0^A\rangle$ to the state

$$U(|\psi^S\rangle \otimes |0^A\rangle) = \sum_i \lambda_i |i^S\rangle |i^A\rangle. \tag{101}$$

Then we easily obtain

$$\mathcal{L}_C(|\psi^S\rangle) = \mathcal{L}_E[U(|\psi^S\rangle \otimes |0^A\rangle)]. \tag{102}$$

Similar to the result in [10], we can extend the above result to the general case of mixed states as follows.

Proposition 13. There exists an isometry $W : \mathcal{H}^S \to \mathcal{H}^S \otimes \mathcal{H}^A$ such that for any state ρ^S on S, we have

$$\mathcal{L}_C(\rho^S) = \mathcal{L}_E(W\rho^S W^\dagger). \tag{103}$$

Proof. Let $\{|i\rangle\}$ be an orthonormal basis and $|0\rangle$ be any state in $\mathcal{H}^A$; from the result [33], one can define

$$W = \sum_i K_i \otimes |i\rangle\langle 0|, \tag{104}$$

where $W^\dagger W = I \otimes |0\rangle\langle 0|$, and there exists a unitary operation U such that $W = U(I \otimes |0\rangle\langle 0|)$. In particular, we take the unitary operation given in Eq. (100). Let $\rho = \sum_i \lambda_i^* |\psi_i^*\rangle\langle\psi_i^*|$ be a decomposition for which the minima in Eq. (19) is attained. Since the operation $I \otimes |0\rangle\langle 0|$ does not effect the Schmidt number, for any state $|\psi_i^*\rangle$, using Eq. (102), we have

$$\mathcal{L}_C(|\psi_i^*\rangle) = \mathcal{L}_E(W|\psi_i^*\rangle). \tag{105}$$

We know that there exists a one-to-one correspondence between the pure states decompositions of ρ and the decompositions of $\rho' = W\rho W^\dagger$ for given W. Then we obtain $\{\lambda_i^*, W|\psi_i^*\rangle\}$ and will form an optimal pure state decomposition of ρ', and

$$\begin{aligned}\mathcal{L}_C(\rho) &= \sum_i \lambda_i^* \mathcal{L}_C(|\psi_i^*\rangle)\\ &= \sum_i \lambda_i^* \mathcal{L}_E(W|\psi_i^*\rangle)\\ &= \mathcal{L}_E(W\rho^S W^\dagger). \end{aligned} \tag{106}$$

This completes the proof of the proposition. ■

VIII. CONCLUSIONS

We have introduced a measure of coherence, the logarithmic coherence number, which is generalized from the Schmidt measure and coherence rank. We have shown that the logarithmic coherence number is a proper coherence measure. We have also proved the logarithmic coherence number is additive but not continuous. In particular, we have found that the logarithmic coherence number is computable for a large class of states. We have shown that the logarithmic coherence number is a better upper bound for the relative entropy of coherence; this gave an operational expression for the logarithmic coherence number. At the same time, we also discussed the relationships between the logarithmic coherence number and the l_1-norm coherence. By introducing average fidelity coherence, we obtained an uncertainty relation between them. We have shown that the logarithmic coherence number satisfies the superadditivity and obtained the relationship between coherence and entanglement via our presented measures. The results can also be extended to multipartite settings. We have shown that the creation of entanglement with bipartite incoherent operations is bounded by the logarithmic coherence number of the initial system during the process. Some interesting results are given. We hope this measure of coherence will improve the understanding of quantum resource theory.

ACKNOWLEDGMENTS

Z.X. is supported by the National Natural Science Foundation of China (Grants No. 61671280, No. 11531009, and No. 11771009), and by the Natural Science Basic Research Plan in Shaanxi Province of China (No. 2017KJXX-92), and by the Fundamental Research Funds for the Central Universities (GK201902007), and by the Funded Projects for the Academic Leaders and Academic Backbones, Shaanxi Normal University (No. 16QNGG013).

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