Local unambiguous discrimination of symmetric ternary states

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We investigate unambiguous discrimination between given quantum states with a sequential measurement, which is restricted to local measurements and one-way classical communication. If the given states are binary or each of their individual state spaces is two dimensional, then it is in some cases known whether a sequential measurement achieves a globally optimal unambiguous measurement. In contrast, for more than two states each of whose individual systems is more than two dimensional, the problem becomes extremely complicated. This paper focuses on symmetric ternary separable pure states each of whose individual systems is three dimensional, which include phase shift keyed (PSK) optical coherent states and a lifted version of "double trine" states. We provide a necessary and sufficient condition for an optimal sequential measurement to be globally optimal for the bipartite case. A sufficient condition of global optimality for multipartite states is also presented. One can easily judge whether these conditions hold for given states. Some examples are given, which demonstrate that, despite the restriction to local measurements and one-way classical communication, a sequential measurement can be globally optimal in quite a few cases.

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I. INTRODUCTION

Discrimination between quantum states as accurately as possible is a fundamental issue in quantum information theory. It is a well-known property of quantum theory that perfect discrimination among nonorthogonal quantum states is impossible. Then, given a finite set of nonorthogonal quantum states, we need to find an optimal measurement with respect to a reasonable criterion. Unambiguous discrimination is one of the most common strategies to distinguish between quantum states [1–3]. An unambiguous measurement achieves error-free (i.e., unambiguous) discrimination at the expense of allowing for a certain rate of inconclusive results. Finding an unambiguous measurement that maximizes the average success probability for various quantum states has been widely investigated (e.g., [4–12]).

When given quantum states are shared between two or more systems, measurement strategies can be classified into two types: global and local. A local measurement is performed by a series of individual measurements on the subsystems combined with classical communication. In particular, sequential measurements, in which the classical communication is one-way only, have been widely investigated under several optimality criteria (e.g., [13–22]). Although the performance of an optimal sequential measurement is often strictly less than that of an optimal global measurement even if given states are not entangled, a sequential measurement has the advantage of being relatively easy to implement with current technology. As an example of a realizable sequential measurement for optical coherent states, a receiver based on a combination of a photon detector and a feedback circuit, which we call a Dolinar-like receiver, has been proposed [23] and experimentally demonstrated [24]. Also, unambiguous discrimination using Dolinar-like receivers has been studied [25–27].

Several studies on optimal unambiguous sequential measurements have also been carried out [28–31]. For binary pure states with any prior probabilities, it has been shown that an optimal unambiguous sequential measurement can achieve the performance of an optimal global measurement [28,29]. For the sake of brevity, we say that a sequential measurement can be globally optimal. As for more than two states, in the case in which each of the individual systems is two dimensional, whether a sequential measurement can be globally optimal has been clarified for several cases [30,31]. However, in the case in which individual systems are more than two dimensional, the problem becomes extremely complicated. Due to the restriction of local measurements and one-way classical communication, it would not be surprising if a sequential measurement cannot be globally optimal except for some special cases. It is worth mentioning that, according to Ref. [32], in the case of a minimum-error measurement, which maximizes the average success probability but sometimes returns an incorrect answer, an optimal sequential measurement does not seem to be globally optimal for any ternary phase shift keyed (PSK) optical coherent states.

In this paper we focus on symmetric ternary separable pure states each of whose individual systems is three dimensional. These states include PSK optical coherent states, which have equally spaced phases at constant amplitude, and a lifted version of "double trine" states [33]. We provide a necessary and sufficient condition that a sequential measurement can be globally optimal for the bipartite case, using which one can easily judge whether global optimality is achieved by a sequential measurement for given states. We use the convex optimization approach reported in Ref. [34] to derive the condition. We also give a sufficient condition of global optimality for the multipartite case. Some examples of symmetric ternary separable pure states are presented, which show that a sequential measurement can be globally optimal in quite a few cases. One of the examples shows that the problem of whether a sequential measurement for bipartite ternary PSK optical coherent states can be globally optimal is completely solved analytically, while its minimum-error measurement version has been solved only numerically [32]. Moreover, we show that a Dolinar-like receiver for any ternary PSK optical coherent states cannot be a globally optimal unambiguous measurement.

The paper is organized as follows. In Sec. II we formulate the problem of finding an optimal unambiguous measurement and its sequential-measurement version as convex programming problems. In Sec. III we present our main theorem. Using this theorem, we derive a necessary and sufficient condition for an optimal sequential measurement for bipartite symmetric ternary separable pure states to be globally optimal. A sufficient condition of global optimality for multipartite symmetric ternary separable pure states is also derived. In Sec. IV we provide some examples to demonstrate the usefulness of our results. Finally, we prove the main theorem in Sec. V.

II. OPTIMAL UNAMBIGUOUS SEQUENTIAL MEASUREMENTS

In this section we first provide an optimization problem of finding optimal unambiguous measurements. Then, we discuss a sequential-measurement version of the optimization problem. We also provide a necessary and sufficient condition for an optimal sequential measurement to be globally optimal. Note that this condition is quite general but requires extra effort to decide whether global optimality is achieved by a sequential measurement for given quantum states. In Sec. III we will use this condition to derive a formula that is directly applicable to symmetric ternary separable pure states.

A. Problem of finding optimal unambiguous measurements

We here consider unambiguous measurements without restriction to sequential measurements. Consider a quantum system prepared in one of R quantum states represented by density operators $\{\tilde{\rho}_r\}_{r\in\mathcal{I}_R}$ on a complex Hilbert space $\mathcal{H}$, where $\mathcal{I}_R := \{0, 1, \ldots, R-1\}$. The density operator $\tilde{\rho}_r$ satisfies $\tilde{\rho}_r \geqslant 0$ and $\mathrm{Tr}\,\tilde{\rho}_r = 1$, where $\hat{A} \geqslant 0$ denotes that $\hat{A}$ is positive semidefinite (similarly, $\hat{A} \geqslant \hat{B}$ denotes $\hat{A} - \hat{B} \geqslant 0$). To unambiguously discriminate the R states, we can consider a measurement represented by a positive-operator-valued measure (POVM), $\hat{\Pi} := \{\hat{\Pi}_r\}_{r=0}^R$, consisting of $R+1$ detection operators, on $\mathcal{H}$, where $\hat{\Pi}_r$ satisfies $\hat{\Pi}_r \geqslant 0$ and $\sum_{r=0}^R \hat{\Pi}_r = \hat{1}$ ($\hat{1}$ is the identity operator on $\mathcal{H}$). The detection operator $\hat{\Pi}_r$ with $r < R$ corresponds to the identification of the state $\tilde{\rho}_r$, while $\hat{\Pi}_R$ corresponds to the inconclusive answer. Any unambiguous measurement $\hat{\Pi}$ satisfies $\mathrm{Tr}(\tilde{\rho}_r\hat{\Pi}_k) = 0$ for any $k \in \mathcal{I}_R\backslash\{r\}$, where $\backslash$ denotes set difference. Given possible states $\{\tilde{\rho}_r\}$ and their prior probabilities $\{\xi_r\}$, we want to find an unambiguous measurement maximizing the average success probability, which we call an optimal unambiguous measurement or just an optimal measurement for short. Reference [6] shows that the problem of finding an optimal measurement can be formulated as a semidefinite programming problem, which is a special case of a convex programming problem. For analytical convenience, instead of the formulation of Ref. [6], we consider the following semidefinite programming problem:

$$\begin{aligned} \mathrm{P_G}: \quad & \text{maximize} \quad P(\hat{\Pi}) := \lim_{\lambda\to\infty} \sum_{r=0}^{R-1} \mathrm{Tr}[(\hat{\rho}_r - \lambda\hat{\nu}_r)\hat{\Pi}_r] \\ & \text{subject to} \quad \hat{\Pi} : \text{POVM}, \end{aligned} \tag{1}$$

where $\hat{\rho}_r := \xi_r\tilde{\rho}_r$ and $\hat{\nu}_r := \sum_{k\in\mathcal{I}_R\backslash\{r\}} \hat{\rho}_k$. Since $P(\hat{\Pi}) = -\infty$ holds if there exists $r \in \mathcal{I}_R$ such that $\mathrm{Tr}(\hat{\nu}_r\hat{\Pi}_r) \neq 0$ (i.e., $\hat{\Pi}$ is not an unambiguous measurement), any optimal solution to problem $\mathrm{P_G}$ is guaranteed to be an unambiguous measurement. The optimal value, which is the average success probability of an optimal measurement, is larger than zero if and only if at least one of the operators $\hat{\rho}_r$ has a nonzero overlap with the kernels of $\hat{\nu}_r$ [35].

The dual problem to problem $\mathrm{P_G}$ can be written as[1]

$$\begin{aligned} \mathrm{DP_G}: \quad & \text{minimize} \quad \mathrm{Tr}\,\hat{Z}_\mathrm{G} \\ & \text{subject to} \quad \hat{\Gamma}_\mathrm{G}(r, \hat{Z}_\mathrm{G}) \geqslant 0\ (\forall r \in \mathcal{I}_R), \end{aligned} \tag{2}$$

where

$$\hat{\Gamma}_\mathrm{G}(r, \hat{Z}_\mathrm{G}) := \lim_{\lambda\to\infty} \hat{Z}_\mathrm{G} - \hat{\rho}_r + \lambda\hat{\nu}_r \tag{3}$$

and $\hat{Z}_\mathrm{G}$ is a (bounded) positive semidefinite operator on $\mathcal{H}$. The optimal values of problems $\mathrm{P_G}$ and $\mathrm{DP_G}$ are the same. Note that $\hat{\Gamma}_\mathrm{G}(r, \hat{Z}_\mathrm{G})$ is obviously unbounded. For an unbounded operator $\hat{A}$, $\hat{A} \geqslant 0$ holds if and only if $\langle c|\hat{A}|c\rangle \geqslant 0$ holds for any vector $|c\rangle$ in the domain of $\hat{A}$. Since the domain of $\hat{\Gamma}_\mathrm{G}(r, \hat{Z}_\mathrm{G})$ is $\mathrm{Ker}\,\hat{\nu}_r$, $\hat{\Gamma}_\mathrm{G}(r, \hat{Z}_\mathrm{G}) \geqslant 0$ is equivalent to the following:

$$\langle c|(\hat{Z}_\mathrm{G} - \hat{\rho}_r)|c\rangle \geqslant 0, \quad \forall|c\rangle \in \mathrm{Ker}\,\hat{\nu}_r. \tag{4}$$

B. Problem of finding optimal unambiguous sequential measurements

Now, let us assume that $\mathcal{H}$ is a bipartite Hilbert space, $\mathcal{H} = \mathcal{H}_\mathrm{A} \otimes \mathcal{H}_\mathrm{B}$, and let us restrict our attention to a sequential measurement from Alice to Bob. In a sequential measurement, Alice performs a measurement on $\mathcal{H}_\mathrm{A}$ and communicates her result to Bob. Then he performs a measurement on $\mathcal{H}_\mathrm{B}$, which can depend on Alice's outcomes, and obtains the final measurement result. This sequential measurement can be considered from a different point of view [32]. Let ω be an index associated with Bob's measurement $\hat{B}^{(\omega)} := \{\hat{B}_r^{(\omega)}\}_{r=0}^R$, and Ω be the entire set of indices ω. Alice performs a measurement $\hat{A} := \{\hat{A}(\omega)\}_{\omega\in\Omega}$, with continuous outcomes, and sends the result $\omega \in \Omega$ to Bob. Then he performs the corresponding measurement $\hat{B}^{(\omega)}$, which is uniquely determined by the result ω. This sequential measurement is denoted as $\hat{\Pi}^{(\hat{A})} := \{\hat{\Pi}_r^{(\hat{A})}\}_{r=0}^R$ with

$$\hat{\Pi}_r^{(\hat{A})} := \int_\Omega \hat{A}(d\omega) \otimes \hat{B}_r^{(\omega)}, \tag{5}$$

[1] One can obtain this problem from Eq. (12) in Ref. [36] with $M = R+1$, $J = 0$, $\hat{c}_m = \hat{\rho}_m - \lambda\hat{\nu}_m$ $(m < R)$, $\hat{c}_R = 0$, and $\lambda \to \infty$.

which is uniquely determined by Alice's POVM $\hat{A}$.

The problem of finding an unambiguous sequential measurement maximizing the average success probability, which we call an optimal unambiguous sequential measurement or just an optimal sequential measurement, can be formulated as the following optimization problem:

$$\begin{aligned} \mathrm{P}: \quad & \text{maximize} \quad P[\hat{\Pi}^{(\hat{A})}] \\ & \text{subject to} \quad \hat{A} \in \mathcal{M}_{\mathrm{A}}, \end{aligned} \tag{6}$$

with Alice's POVM $\hat{A}$, where $\mathcal{M}_{\mathrm{A}}$ is the entire set of Alice's continuous measurements $\{\hat{A}(\omega)\}_{\omega\in\Omega}$. Compared to problem $\mathrm{P_G}$, this problem restricts $\hat{\Pi}$ to the form $\hat{\Pi} = \hat{\Pi}^{(\hat{A})}$. We can easily see that this problem is a convex programming problem and obtain the following dual problem [34]:

$$\begin{aligned} \mathrm{DP}: \quad & \text{minimize} \quad \mathrm{Tr}\,\hat{X} \\ & \text{subject to} \quad \hat{\Gamma}(\omega;\hat{X}) \geqslant 0\,(\forall\omega\in\Omega), \end{aligned} \tag{7}$$

with a Hermitian operator $\hat{X}$, where

$$\hat{\Gamma}(\omega;\hat{X}) := \lim_{\lambda\to\infty} \hat{X} - \sum_{r=0}^{R-1} \mathrm{Tr_B}\big[(\hat{\rho}_r - \lambda\hat{\nu}_r)\hat{B}_r^{(\omega)}\big]. \tag{8}$$

$\mathrm{Tr_B}$ is the partial trace over $\mathcal{H}_{\mathrm{B}}$. The optimal values of problems P and DP are also the same. Note that $\hat{\Gamma}(\omega;\hat{X}) \geqslant 0$ is equivalent to the following:

$$\langle c|\hat{X} - \sum_{r=0}^{R-1} \mathrm{Tr_B}\big[\hat{\rho}_r\hat{B}_r^{(\omega)}\big]\,|c\rangle \geqslant 0,$$

$$\forall|c\rangle \in \mathrm{Ker} \sum_{r=0}^{R-1} \mathrm{Tr_B}\big[\hat{\nu}_r\hat{B}_r^{(\omega)}\big]. \tag{9}$$

C. Condition for sequential measurement to be globally optimal

Let $\hat{Z}_{\mathrm{G}}^{\star}$ be an optimal solution to problem $\mathrm{P_G}$ and $\hat{X}_{\mathrm{G}}^{\star} := \mathrm{Tr_B}\,\hat{Z}_{\mathrm{G}}^{\star}$. Also, let

$$\hat{\Gamma}^{\star}(\omega) := \hat{\Gamma}(\omega;\hat{X}_{\mathrm{G}}^{\star}). \tag{10}$$

We now want to know whether a sequential measurement can be globally optimal, i.e., whether an optimal solution to problem P is also optimal to problem $\mathrm{P_G}$. To this end, we utilize the following lemma.

Lemma 1. A sequential measurement $\hat{\Pi}^{(\hat{A})}$ ($\hat{A} \in \mathcal{M}_{\mathrm{A}}$) is an optimal unambiguous measurement if and only if it satisfies

$$\hat{\Gamma}^{\star}(\omega)\hat{A}(\omega) = 0, \quad \forall\omega\in\Omega. \tag{11}$$

Proof. Assume that $\hat{X}_{\mathrm{G}}^{\star}$ is a feasible solution to problem DP, i.e., $\hat{\Gamma}^{\star}(\omega) \geqslant 0$ holds for any $\omega\in\Omega$. It is known that $\hat{\Pi}^{(\hat{A})}$ and $\hat{X}$ are, respectively, optimal solutions to problems P and DP if and only if $\hat{\Gamma}(\omega;\hat{X}) \geqslant 0$ and $\hat{\Gamma}(\omega;\hat{X})\hat{A}(\omega) = 0$ hold for any $\omega\in\Omega$ (see Theorem 2 of Ref. [34]).[2] Thus, $\hat{\Pi}^{(\hat{A})}$ and $\hat{X}_{\mathrm{G}}^{\star}$ are, respectively, optimal solutions to problems P and DP if and only if Eq. (11) holds. If Eq. (11) holds, then, since $P[\hat{\Pi}^{(\hat{A})}] = \mathrm{Tr}\,\hat{X}_{\mathrm{G}}^{\star} = \mathrm{Tr}\,\hat{Z}_{\mathrm{G}}^{\star}$ is equal to the optimal value of problem $\mathrm{P_G}$, $\hat{\Pi}^{(\hat{A})}$ is globally optimal. Therefore, to prove this lemma, it suffices to show that $\hat{X}_{\mathrm{G}}^{\star}$ is a feasible solution to problem DP.

Multiplying $[\hat{B}_r^{(\omega)}]^{1/2}$ on both sides of the constraint of problem $\mathrm{DP_G}$ and taking the partial trace over $\mathcal{H}_{\mathrm{B}}$ gives

$$\lim_{\lambda\to\infty} \mathrm{Tr_B}\big[\hat{Z}_{\mathrm{G}}^{\star}\hat{B}_r^{(\omega)}\big] - \mathrm{Tr_B}\big[(\hat{\rho}_r - \lambda\hat{\nu}_r)\hat{B}_r^{(\omega)}\big] \geqslant 0. \tag{12}$$

Therefore, we have

$$\lim_{\lambda\to\infty} \sum_{r=0}^{R-1} \mathrm{Tr_B}\big[\hat{Z}_{\mathrm{G}}^{\star}\hat{B}_r^{(\omega)}\big] - \sum_{r=0}^{R-1} \mathrm{Tr_B}\big[(\hat{\rho}_r - \lambda\hat{\nu}_r)\hat{B}_r^{(\omega)}\big] \geqslant 0. \tag{13}$$

Also, from $\hat{X}_{\mathrm{G}}^{\star} = \mathrm{Tr_B}\,\hat{Z}_{\mathrm{G}}^{\star}$, we have

$$\hat{X}_{\mathrm{G}}^{\star} = \sum_{r=0}^{R} \mathrm{Tr_B}\big[\hat{Z}_{\mathrm{G}}^{\star}\hat{B}_r^{(\omega)}\big] \geqslant \sum_{r=0}^{R-1} \mathrm{Tr_B}\big[\hat{Z}_{\mathrm{G}}^{\star}\hat{B}_r^{(\omega)}\big], \tag{14}$$

From these equations and Eq. (8), $\hat{\Gamma}^{\star}(\omega) \geqslant 0$ holds for any $\omega\in\Omega$, and thus $\hat{X}_{\mathrm{G}}^{\star}$ is a feasible solution to problem DP. ■

We will further investigate Alice's POVM $\hat{A}$ satisfying Eq. (11). Let

$$\mathcal{K}_\omega := \mathrm{Ker} \sum_{r=0}^{R-1} \mathrm{Tr_B}\big[\hat{\nu}_r\hat{B}_r^{(\omega)}\big]. \tag{15}$$

Let us consider $|\gamma\rangle \in \mathrm{supp}\,\hat{A}(\omega)$. Suppose that Eq. (11) holds; then, from Eqs. (8) and (15), $|\gamma\rangle \in \mathcal{K}_\omega$ and $\hat{P}_\omega[\hat{X}_{\mathrm{G}}^{\star} - \sum_{r=0}^{R-1}\mathrm{Tr_B}[\hat{\rho}_r\hat{B}_r^{(\omega)}]]|\gamma\rangle = 0$ hold, where $\hat{P}_\omega$ is the projection operator onto $\mathcal{K}_\omega$. Conversely, if these two equations hold for any $|\gamma\rangle \in \mathrm{supp}\,\hat{A}(\omega)$, then Eq. (11) holds. Therefore, Eq. (11) is equivalent to the following equations:

$$\mathrm{supp}\,\hat{A}(\omega) \subseteq \mathcal{K}_\omega,$$

$$\hat{P}_\omega\left[\hat{X}_{\mathrm{G}}^{\star} - \sum_{r=0}^{R-1} \mathrm{Tr_B}\big[\hat{\rho}_r\hat{B}_r^{(\omega)}\big]\right]\hat{A}(\omega) = 0. \tag{16}$$

Let us consider the case in which each state $\hat{\rho}_r$ is separable, i.e., it is in the form of

$$\hat{\rho}_r = \xi_r\hat{a}_r \otimes \hat{b}_r, \tag{17}$$

where $\hat{a}_r$ and $\hat{b}_r$ are, respectively, density operators on $\mathcal{H}_{\mathrm{A}}$ and $\mathcal{H}_{\mathrm{B}}$. Then Eq. (8) reduces to

$$\hat{\Gamma}(\omega;\hat{X}) = \lim_{\lambda\to\infty} \hat{X} - \sum_{r=0}^{R-1} p_r^{(\omega)}\xi_r\hat{a}_r + \lambda\sum_{r=0}^{R-1} e_r^{(\omega)}\xi_r\hat{a}_r, \tag{18}$$

where $p_r^{(\omega)} := \mathrm{Tr}[\hat{b}_r\hat{B}_r^{(\omega)}]$ is the probability of Bob correctly identifying the state $\hat{b}_r$ and $e_r^{(\omega)} := \sum_{k\in\mathcal{I}_R\setminus\{r\}} \mathrm{Tr}[\hat{b}_r\hat{B}_k^{(\omega)}]$ is the probability of Bob misidentifying the state $\hat{b}_r$. Also, it follows from $\mathcal{K}_\omega = \mathrm{Ker}\sum_{r=0}^{R-1} e_r^{(\omega)}\xi_r\hat{a}_r$ that the first line of Eq. (16) can be expressed as

$$\mathrm{Tr}[\hat{a}_r\hat{A}(\omega)] = 0, \quad \forall r \notin T^{(\omega)}, \tag{19}$$

where $T^{(\omega)}$ is the entire set of indices $r\in\mathcal{I}_R$ such that Bob's measurement never gives incorrect results, i.e.,

$$T^{(\omega)} := \big\{r\in\mathcal{I}_R : e_r^{(\omega)} = 0\big\}. \tag{20}$$

[2] We here consider the case of $M = R+1$, $J = 0$, $\hat{c}_m = \hat{\rho}_m - \lambda\hat{\nu}_m$ $(m < R)$, $\hat{c}_R = 0$, and $\lambda\to\infty$.

Equation (19) implies that, for any $r \in \mathcal{I}_R$ and $\omega \in \Omega$ such that the state $\hat{b}_r$ will be incorrectly identified by Bob's measurement $\hat{B}^{(\omega)}$ (i.e., $e_r^{(\omega)} \neq 0$), Alice's outcome must not be ω for the state $\hat{a}_r$ (i.e., $\mathrm{Tr}[\hat{a}_r\hat{A}(\omega)] = 0$). Thus, Eq. (19) ensures that the measurement $\hat{\Pi}^{(\hat{A})}$ never gives erroneous results.

III. SEQUENTIAL MEASUREMENTS FOR SYMMETRIC TERNARY SEPARABLE PURE STATES

Lemma 1 is useful in determining whether a sequential measurement can be globally optimal. Concretely, it is possible to decide whether a sequential measurement can be globally optimal by examining whether there exists $\hat{A} \in \mathcal{M}_\mathrm{A}$ satisfying Eq. (11). However, in general, it is quite difficult to examine this for all continuous values $\omega \in \Omega$. In this section we consider sequential measurements for bipartite symmetric ternary separable pure states and derive a formula that can directly determine whether a sequential measurement can be globally optimal. Extending our results to the multipartite case enables us to obtain a sufficient condition that a sequential measurement can be globally optimal.

A. Main results

Let us consider bipartite ternary separable pure states $\{|\Psi_r\rangle := |a_r\rangle \otimes |b_r\rangle\}_{r=0}^2$, which are the special case of Eq. (17) with $\hat{a}_r = |a_r\rangle\langle a_r|$ and $\hat{b}_r = |b_r\rangle\langle b_r|$. Assume that $\{|a_r\rangle\}$ and $\{|b_r\rangle\}$, respectively, span three-dimensional Hilbert spaces $\mathcal{H}_\mathrm{A}$ and $\mathcal{H}_\mathrm{B}$. Note that unambiguous discrimination is possible if and only if the pure states are linearly independent [37]; there exist unambiguous measurements for the partial states $\{|a_r\rangle\}$ and $\{|b_r\rangle\}$. Also, assume that $\{|\Psi_r\rangle\}$ is symmetric in the following sense: the prior probabilities are equal (i.e., $\xi_r = 1/3$) and there exist unitary operators $\hat{V}_\mathrm{A}$ on $\mathcal{H}_\mathrm{A}$ and $\hat{V}_\mathrm{B}$ on $\mathcal{H}_\mathrm{B}$ satisfying

$$|a_{r\oplus 1}\rangle = \hat{V}_\mathrm{A}|a_r\rangle, \quad |b_{r\oplus 1}\rangle = \hat{V}_\mathrm{B}|b_r\rangle, \tag{21}$$

where $\oplus$ denotes addition modulo 3. These states are characterized by the inner products $K_\mathrm{A} := \langle a_0|a_1\rangle$ and $K_\mathrm{B} := \langle b_0|b_1\rangle$, which are generally complex values. For any $r \in \mathcal{I}_3$, we have

$$\langle a_r|a_{r\oplus 1}\rangle = K_\mathrm{A}, \quad \langle b_r|b_{r\oplus 1}\rangle = K_\mathrm{B}. \tag{22}$$

$|a_r\rangle$ and/or $|b_r\rangle$ can be PSK optical coherent states, pulse position modulated (PPM) optical coherent states, and lifted trine states [33]. If $\{|a_r\rangle\}$ or $\{|b_r\rangle\}$ is mutually orthogonal (i.e., $K_\mathrm{A} = 0$ or $K_\mathrm{B} = 0$), then an optimal sequential measurement perfectly discriminates $\{|\Psi_r\rangle\}$, and thus is globally optimal. So, assume that neither $\{|a_r\rangle\}$ nor $\{|b_r\rangle\}$ is mutually orthogonal.

We shall present a theorem that can be used to determine whether a sequential measurement can be globally optimal for given bipartite symmetric ternary separable pure states. Let us consider the following set with seven elements:

$$\Omega^\star := \{\omega_{1,j}, \omega_{2,j}, \omega_3 : j \in \mathcal{I}_3\}, \tag{23}$$

where

(1) $\hat{B}^{(\omega_{1,j})}$ is the measurement that always returns j, i.e., $\hat{B}_r^{(\omega_{1,j})} = \delta_{r,j}\hat{1}_\mathrm{B}$, where $\delta_{r,j}$ is the Kronecker delta and $\hat{1}_\mathrm{B}$ is the identity operator on $\mathcal{H}_\mathrm{B}$.

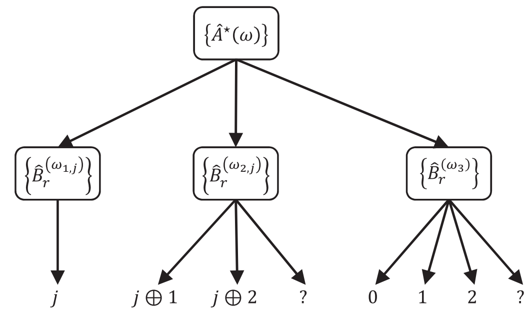

FIG. 1. Schematic diagram of an optimal sequential measurement $\hat{\Pi}^{(\hat{A}^\star)}$.

(2) $\hat{B}^{(\omega_{2,j})}$ is an optimal unambiguous measurement for binary states $\{|b_{j\oplus 1}\rangle, |b_{j\oplus 2}\rangle\}$ with equal prior probabilities of 1/2.

(3) $\hat{B}^{(\omega_3)}$ is an optimal unambiguous measurement for ternary states $\{|b_r\rangle\}_{r=0}^2$ with equal prior probabilities of 1/3.

For simpler notation, we write ω_k for $\omega_{k,0}$ for each $k \in \{1, 2\}$.

When a sequential measurement can be globally optimal, there can exist a large (or even infinite) number of optimal sequential measurements. However, as we shall show in the following theorem, if a sequential measurement can be globally optimal, then there always exists an optimal sequential measurement in which Alice never returns an index ω with $\omega \notin \Omega^\star$ (proof in Sec. V).

Theorem 2. Suppose that, for bipartite symmetric ternary separable pure states $\{|\Psi_r\rangle := |a_r\rangle \otimes |b_r\rangle\}_{r=0}^2$, a sequential measurement can be globally optimal. Also, suppose that $\{|a_r\rangle\}$ and $\{|b_r\rangle\}$ respectively span three-dimensional Hilbert spaces. Then, there exists an optimal sequential measurement $\hat{\Pi}^{(\hat{A}^\star)}$ with $\hat{A}^\star \in \mathcal{M}_\mathrm{A}$ such that

$$\hat{A}^\star(\omega) = 0, \quad \forall \omega \notin \Omega^\star. \tag{24}$$

The measurement $\hat{\Pi}^{(\hat{A}^\star)}$ is schematically illustrated in Fig. 1. Due to the definition of $\omega_{1,j}, \omega_{2,j}, \omega_3 \in \Omega^\star$, $T^{(\omega)}$, defined by Eq. (20), satisfies $T^{(\omega_{1,j})} = \{j\}$, $T^{(\omega_{2,j})} = \{j \oplus 1, j \oplus 2\}$, and $T^{(\omega_3)} = \{0, 1, 2\}$. From Eq. (19), $\mathrm{Tr}[\hat{a}_r\hat{A}^\star(\omega_{1,j})] = 0$ must hold for any distinct $r, j \in \mathcal{I}_3$. Thus, if Alice returns the index $\omega_{1,j}$, then the given state must be $|\Psi_j\rangle$. (In this case, the given state is uniquely determined before Bob performs the measurement.) Also, from Eq. (19), $\mathrm{Tr}[\hat{a}_j\hat{A}^\star(\omega_{2,j})] = 0$ holds for any $j \in \mathcal{I}_3$, which indicates that if Alice returns the index $\omega_{2,j}$, then the state $|\Psi_j\rangle$ is unambiguously filtered out. In this case, Alice's measurement result does not indicate which of the two states $|\Psi_{j\oplus 1}\rangle$ and $|\Psi_{j\oplus 2}\rangle$ is given. If Alice returns the index ω_3, then Alice's result provides no information about the given state.

Using Theorem 2, we can derive a simple formula for determining whether a sequential measurement can be globally optimal. Before we state this formula, we shall give some preliminaries. Let $\tau := \exp(i2\pi/3)$, where $i := \sqrt{-1}$. Also, let $|\phi_n\rangle$ and $|\phi'_n\rangle$, respectively, denote the normalized eigenvectors corresponding to the eigenvalues τ^n $(n \in \mathcal{I}_3)$ of $\hat{V}_\mathrm{A}$ and $\hat{V}_\mathrm{B}$. Moreover, let

$$x_n := |\langle\phi_n|a_0\rangle|, \quad y_n := |\langle\phi'_n|b_0\rangle|. \tag{25}$$

Note that $x_n, y_n > 0$ holds for any $n \in \mathcal{I}_3$. By selecting appropriate global phases of $|a_r\rangle$ and $|b_r\rangle$ and permuting $|\Psi_1\rangle$ and $|\Psi_2\rangle$ as a preprocessing step if necessary, we may assume

$$x_0 > x_2, \quad x_1 \geqslant x_2, \quad y_0 \geqslant y_1 \geqslant y_2, \quad y_0 \neq y_2. \tag{26}$$

We keep the same notation $\{|a_r\rangle\}$ and $\{|b_r\rangle\}$ before and after the preprocessing step. Also, by selecting global phases of $|\phi_n\rangle$ and $|\phi'_n\rangle$ such that $\langle\phi_n|a_0\rangle$ and $\langle\phi'_n|b_0\rangle$ are positive real numbers, $|a_r\rangle$ and $|b_r\rangle$ are written as

$$|a_r\rangle = \sum_{n=0}^{2} x_n \tau^{rn}|\phi_n\rangle, \quad |b_r\rangle = \sum_{n=0}^{2} y_n \tau^{rn}|\phi'_n\rangle. \tag{27}$$

x_n and y_n are uniquely determined by K_A and K_B. Let $K'_A := \langle a_0|a_1\rangle$ and $K'_B := \langle b_0|b_1\rangle$, where $|a_r\rangle$ and $|b_r\rangle$ are preprocessed vectors, which is expressed by Eqs. (26) and (27); then we have

$$x_n = \sqrt{\frac{1 + \tau^{2n}K'_A + \tau^n (K'_A)^*}{3}}, \quad y_n = \sqrt{\frac{1 + \tau^{2n}K'_B + \tau^n (K'_B)^*}{3}}, \tag{28}$$

where * designates complex conjugate. Note that since K_A and K'_A are, respectively, defined as $\langle a_0|a_1\rangle$ with the unpreprocessed and preprocessed vectors $\{|a_r\rangle\}$, $K_A \neq K'_A$ generally holds (the same is true for K_B). However, we can easily see that $|K_A| = |K'_A|$ and $|K_B| = |K'_B|$ always hold. Let

$$\eta := \tfrac{1}{3}(1 - |K'_B|). \tag{29}$$

$3\eta = 1 - |K'_B|$ equals the average success probability of the optimal unambiguous measurement $\hat{B}^{(\omega_2)}$ for binary states $\{|b_1\rangle, |b_2\rangle\}$ with equal prior probabilities of 1/2. From Eq. (28) we have $|K'_B|^2 = 1 - 3\chi$, where

$$\chi := y_0^2 y_1^2 + y_1^2 y_2^2 + y_2^2 y_0^2. \tag{30}$$

Thus, η is expressed by

$$\eta = \tfrac{1}{3}(1 - \sqrt{1 - 3\chi}). \tag{31}$$

We get the following corollary (proof in Appendix A).

Corollary 3. For bipartite symmetric ternary separable pure states $\{|\Psi_r\rangle := |a_r\rangle \otimes |b_r\rangle\}_{r=0}^2$ expressed by Eqs. (26) and (27), the following two statements are equivalent.

(1) A sequential measurement can be globally optimal.

(2) Either $y_1 = y_2$ or

$$x_2 z_0 - x_1 z_1 \geqslant 0, \quad \sum_{k=0}^{2} x_k^2 \left(z_{1\ominus k}^{-2} - z_{3\ominus k}^{-2}\right) \geqslant 0 \tag{32}$$

holds, where $z_k := y_k^2 - \eta$ and $\ominus$ denotes subtraction modulo 3.

Using this corollary, we can easily judge whether a sequential measurement can be globally optimal for bipartite symmetric ternary separable pure states.

B. Extension to multipartite states

We can extend the above results to multipartite states. As a simple example, we consider tripartite symmetric ternary separable pure states $\{|\Psi_r\rangle = |a_r\rangle \otimes |b_r\rangle \otimes |c_r\rangle\}_{r=0}^2$, which have equal prior probabilities. There exist unitary operators $\hat{V}_A$, $\hat{V}_B$, and $\hat{V}_C$ on $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}_C$ satisfying Eq. (21) and $|c_{r\oplus 1}\rangle = \hat{V}_C|c_r\rangle$. Here, let us consider the composite system of $\mathcal{H}_B$ and $\mathcal{H}_C$, $\mathcal{H}_{BC} := \mathcal{H}_B \otimes \mathcal{H}_C$, and interpret these states as bipartite states $\{|\Psi_r\rangle = |a_r\rangle \otimes |B_r\rangle\}_{r=0}^2$, where $|B_r\rangle := |b_r\rangle \otimes |c_r\rangle \in \mathcal{H}_{BC}$. It is obvious that if a sequential measurement can be globally optimal for the tripartite states, then it is also true for the bipartite states. Assume that it is true for the bipartite states; then, from Theorem 2, there exists a sequential measurement $\hat{\Pi}^{(\hat{A}^\star)}$ satisfying Eq. (24), which is globally optimal. Also, it follows that $\hat{\Pi}^{(\hat{A}^\star)}$ can be realized by a sequential measurement on the tripartite system $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ if and only if, for any $\omega \in \Omega^\star$, the measurement $\hat{B}^{(\omega)}$ on $\mathcal{H}_{BC}$ can be realized by a sequential measurement on $\mathcal{H}_B \otimes \mathcal{H}_C$. $\hat{B}^{(\omega_{1,j})} = \{\hat{B}_r^{(\omega_{1,j})} = \delta_{r,j}\}_r$ can obviously be realized by a sequential measurement. Also, it is known that a globally optimal measurement for any bipartite binary pure states can be realized by a sequential measurement [28,29], and thus $\hat{B}^{(\omega_{2,j})}$ can also be realized by a sequential one. Therefore, $\hat{\Pi}^{(\hat{A}^\star)}$ can be realized by a sequential measurement on the tripartite system if and only if $\hat{B}^{(\omega_3)}$ can be realized by a sequential measurement. Since $\hat{B}^{(\omega_3)}$ is globally optimal for the bipartite symmetric ternary separable pure states $\{|b_r\rangle \otimes |c_r\rangle\}_{r=0}^2$, $\hat{B}^{(\omega_3)}$ can be realized by a sequential measurement if and only if a sequential measurement for $\{|b_r\rangle \otimes |c_r\rangle\}_r$ can be globally optimal. We can summarize the above discussion as follows: if a sequential measurement can be globally optimal for each of the two sets of states $\{|a_r\rangle \otimes |B_r\rangle\}_r$ and $\{|b_r\rangle \otimes |c_r\rangle\}_r$, then the same is true for the tripartite states $\{|\Psi_r\rangle\}$.

Repeating the above arguments, we can extend it to more than three-partite system, as stated in the following corollary.

Corollary 4. Let us consider N-partite ternary pure states $\{|\Psi_r\rangle := |\psi_r^{(0)}\rangle \otimes |\psi_r^{(1)}\rangle \otimes \cdots \otimes |\psi_r^{(N-1)}\rangle\}_{r=0}^2$ with equal prior probabilities, where $N \geqslant 3$. Suppose that $\{|\Psi_r\rangle\}$ are symmetric, i.e., for any $n \in \mathcal{I}_N$, there exists a unitary operator $\hat{V}^{(n)}$ satisfying $|\psi_{r\oplus 1}^{(n)}\rangle = \hat{V}^{(n)}|\psi_r^{(n)}\rangle$. Let $|b_r^{(n)}\rangle := |\psi_r^{(n+1)}\rangle \otimes \cdots \otimes |\psi_r^{(N-1)}\rangle$ $(n \in \mathcal{I}_{N-1})$. If for any $n \in \mathcal{I}_{N-1}$, a sequential measurement can be globally optimal for bipartite states $\{|\psi_r^{(n)}\rangle \otimes |b_r^{(n)}\rangle\}_{r=0}^2$ with equal prior probabilities, then the same is true for $\{|\Psi_r\rangle\}$.

By using Corollary 3, one can easily judge whether a sequential measurement for the bipartite states $\{|\psi_r^{(n)}\rangle \otimes |b_r^{(n)}\rangle\}_{r=0}^2$ can be globally optimal. Note that the above sufficient condition may not be necessary. For example, let us again consider the tripartite states $\{|a_r\rangle \otimes |b_r\rangle \otimes |c_r\rangle\}_r$. For an optimal sequential measurement for these states to be globally optimal, it is sufficient that there exists a globally optimal sequential measurement $\hat{\Pi}^{(\hat{A})}$ for the bipartite states $\{|a_r\rangle \otimes |B_r\rangle\}_r$ such that the measurement $\hat{B}^{(\omega)}$ can be realized by a sequential measurement on the bipartite system $\mathcal{H}_B \otimes \mathcal{H}_C$ for any ω with $\hat{A}(\omega) \neq 0$, where A can be different from $\hat{A}^\star$.

IV. EXAMPLES

In this section we present some examples of symmetric ternary separable pure states in which a sequential measurement can be globally optimal. In Secs. IV A and IV B we

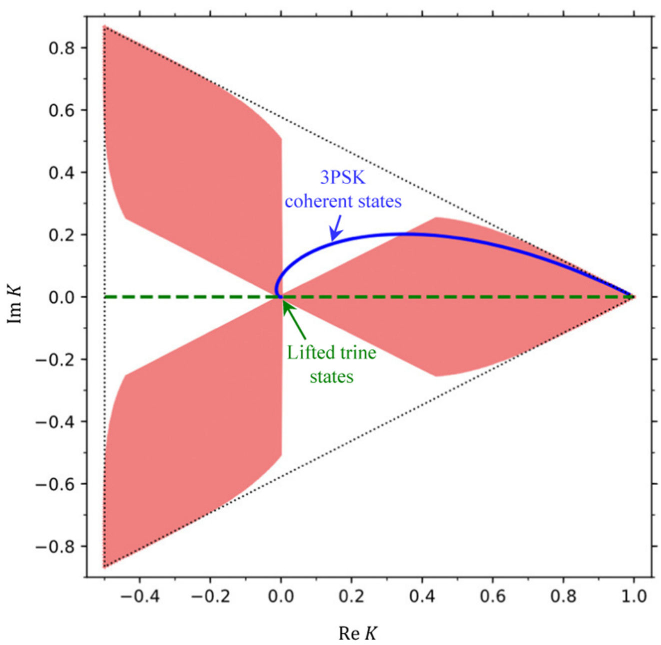

FIG. 2. The region of the complex plane where a sequential measurement for symmetric ternary separable pure states with $K_\mathrm{A} = K_\mathrm{B} =: K$ can be globally optimal.

consider the bipartite case. In Sec. IV C we consider the multipartite case.

A. Case of $K_\mathrm{A} = K_\mathrm{B}$

We first give some examples of bipartite states $\{|\Psi_r\rangle := |a_r\rangle \otimes |b_r\rangle\}_r$ with $K_\mathrm{A} = K_\mathrm{B} =: K$. Note that x_n and y_n of Eq. (28) are the same for each $n \in \mathcal{I}_3$, and thus $x_0 \geqslant x_1$ holds from $y_0 \geqslant y_1$.

The region of the complex plane where a sequential measurement for the states $\{|\Psi_r\rangle\}$ with $K_\mathrm{A} = K_\mathrm{B} = K$ can be globally optimal is shown in red in Fig. 2. This region is easily obtained from Corollary 3. The horizontal and vertical directions are the real and imaginary axes, respectively. The region of all possible K is represented as the dotted equilateral triangle. This figure implies that, at least in the case of $K_\mathrm{A} = K_\mathrm{B}$, a sequential measurement can be globally optimal in quite a few cases.

As a concrete example, let us consider the symmetric ternary separable pure states in which $\{|a_r\rangle\}$ and $\{|b_r\rangle\}$ are the lifted trine states $\{|\mathrm{L}_r\rangle\}_{r=0}^2$, which are expressed by [33]

$$|\mathrm{L}_r\rangle = \sqrt{1-g}\left(\cos\frac{2\pi r}{3}|u_0\rangle + \sin\frac{2\pi r}{3}|u_1\rangle\right) + \sqrt{g}|u_2\rangle, \tag{33}$$

where $\{|u_n\rangle\}_{n=0}^2$ is an orthonormal basis. The real parameter g is in the range $0 < g < 1$. Equation (33) gives $K = (3g-1)/2$, and thus K is real in the range $-1/2 < K < 1$. It follows that the states $\{|\Psi_r\rangle = |\mathrm{L}_r\rangle \otimes |\mathrm{L}_r\rangle\}_r$ are also regarded as lifted trine states. The region of possible values of K is shown in the dashed green line in Fig. 2. From this figure, a sequential measurement for $\{|\Psi_r\rangle\}$ can be globally optimal if and only if $K \geqslant 0$ (i.e., $g \geqslant 1/3$).

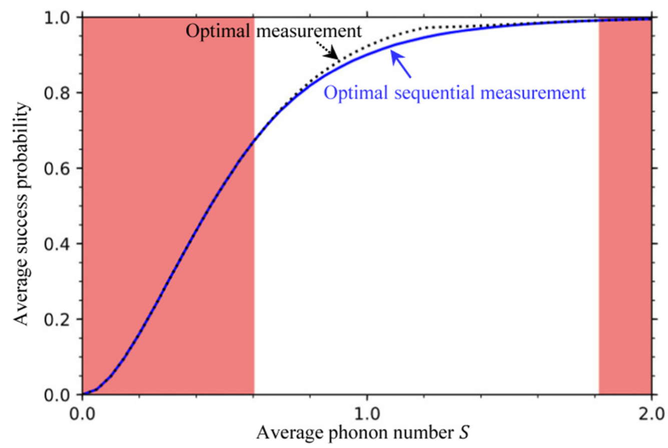

FIG. 3. The average success probabilities of an optimal sequential measurement and an optimal measurement for the ternary PSK optical coherent states $\{|\alpha_r\rangle \otimes |\alpha_r\rangle\}_r$, where $\{|\alpha_r\rangle\}$ are the ternary PSK optical coherent states with an average photon number S.

Another example is the states in which $\{|a_r\rangle\}$ and $\{|b_r\rangle\}$ are the ternary PSK optical coherent states $\{|\alpha_r\rangle\}_{r=0}^2$, which have equally spaced phases of $\{0, 2\pi/3, 4\pi/3\}$ at constant amplitude. $|\alpha_r\rangle$ is a normalized eigenvector of the photon annihilation operator with the eigenvalue $\alpha_r := \sqrt{S}\tau^r$, and $S = |\alpha_r|^2$ is the average photon number of $|\alpha_r\rangle$. The phase of the state $|\alpha_r\rangle$ is $\arg \alpha_r = 2\pi r/3$. In this case, the states $\{|\Psi_r\rangle = |\alpha_r\rangle \otimes |\alpha_r\rangle\}_r$ are also regarded as the ternary PSK optical coherent states with the average photon number $2S$. The coherent state $|\alpha_r\rangle$ is written as

$$|\alpha_r\rangle = e^{-\frac{S}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{34}$$

where $|n\rangle$ is the photon number state with n photons. This gives

$$K = \langle\alpha_0|\alpha_1\rangle = e^{-\frac{3}{2}S} e^{i\frac{\sqrt{3}}{2}S}. \tag{35}$$

The solid blue line in Fig. 2 shows the region of possible values of K. It follows from Eq. (35) that $\arg K = \frac{\sqrt{3}}{2}S$ is proportional to S. From Fig. 2, a necessary and sufficient condition that a sequential measurement for $\{|\Psi_r\rangle\}$ can be globally optimal is $2\pi k/3 \leqslant \arg K + \pi/6 \leqslant 2\pi k/3 + \pi/3$, i.e.,

$$\frac{(4k-1)\pi}{3\sqrt{3}} \leqslant S \leqslant \frac{(4k+1)\pi}{3\sqrt{3}}, \quad k \in \{0, 1, 2, \dots\}. \tag{36}$$

The average success probability of an optimal sequential measurement for $\{|\Psi_r\rangle = |\alpha_r\rangle \otimes |\alpha_r\rangle\}_r$ is plotted in the solid blue line in Fig. 3. Also, that of an optimal measurement is shown in the dashed black line. These probabilities can be numerically computed using a modified version of the method given in Ref. [32]. The region of S satisfying Eq. (36), in which a sequential measurement can be globally optimal, is shown in red. It is worth mentioning that, as shown in Fig. 2 of Ref. [32], in the strategy for minimum-error discrimination, an optimal sequential measurement for the ternary PSK optical coherent states is unlikely to be globally optimal, at least when S is small. In the strategy for unambiguous discrimination, a

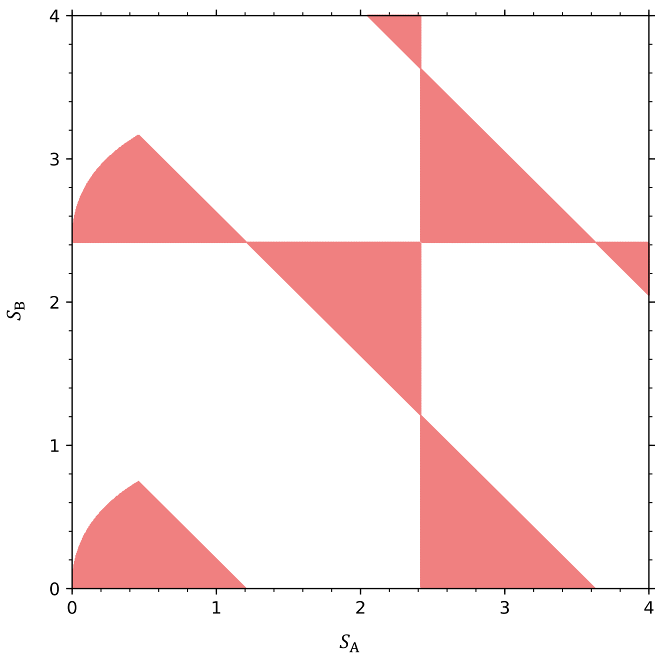

FIG. 4. The region where a sequential measurement for the ternary PSK optical coherent states $\{|a_r\rangle \otimes |b_r\rangle\}_r$ can be globally optimal, where $\{|a_r\rangle\}$ and $\{|b_r\rangle\}$ are the ternary PSK optical coherent states with average photon numbers S_A and S_B, respectively.

sequential measurement can be globally optimal if (and only if) S satisfies Eq. (36).

B. Case of $K_A \neq K_B$

Two concrete examples of symmetric ternary separable pure states with $K_A \neq K_B$ will be given. The first is a set of states $\{|\Psi_r\rangle := |a_r\rangle \otimes |\delta_r\rangle\}_r$, where $\{|\delta_r\rangle\}_{r=0}^2$ are ternary PPM optical coherent states. Ternary PPM optical coherent states have three slots that contain one signal slot (expressed by the optical coherent state $|\alpha\rangle$) and the remaining two nonsignal slots (each expressed by the optical coherent state $|\beta\rangle$ with $|\beta\rangle \neq |\alpha\rangle$, which is usually set to the vacuum state $|0\rangle$). They are written as

$$\begin{aligned} |\delta_0\rangle &:= |\alpha\rangle \otimes |\beta\rangle \otimes |\beta\rangle, \\ |\delta_1\rangle &:= |\beta\rangle \otimes |\alpha\rangle \otimes |\beta\rangle, \\ |\delta_2\rangle &:= |\beta\rangle \otimes |\beta\rangle \otimes |\alpha\rangle. \end{aligned} \tag{37}$$

$\{|a_r\rangle\}$ are not necessarily ternary PPM optical coherent states. From Eq. (37), $K_B = \langle\delta_r|\delta_{r\oplus 1}\rangle = |\langle\alpha|\beta\rangle|^2\langle\beta|\beta\rangle$ holds, and thus K_B is nonnegative real. Therefore, as described in Sec. V C, a sequential measurement can be globally optimal. The same argument can be applied to states $\{|\Psi_r\rangle := |\delta_r\rangle \otimes |b_r\rangle\}_r$.

The second example is the states $\{|\Psi_r\rangle := |a_r\rangle \otimes |b_r\rangle\}_r$, where $\{|a_r\rangle\}$ and $\{|b_r\rangle\}$ are the ternary PSK optical coherent states with average photon numbers S_A and S_B, respectively. The states $\{|\Psi_r\rangle\}$ are also regarded as the ternary PSK optical coherent states with the average photon number $S_A + S_B$. The region of (S_A, S_B) in which a sequential measurement can be globally optimal is shown in red in Fig. 4. We can see that a sequential measurement can be globally optimal in some cases. If S_A (or S_B) is equal to $4\pi k/(3\sqrt{3}) \approx 2.42k$ ($k = 1, 2, \dots$), then, since K_A (or K_B) is nonnegative real, a sequential measurement can be globally optimal.

Let us consider whether the ternary PSK optical coherent states $\{|\alpha_r\rangle\}$ with an average photon number S can be unambiguously discriminated by a Dolinar-like receiver, which consists of continuous photon counting and infinitely fast feedback (e.g., [38]). The performance of this receiver never exceeds that of an optimal sequential measurement for N-partite PSK optical coherent states $\{|\alpha'_r\rangle^{\otimes N}\}$ with $N \to \infty$, where $\{|\alpha'_r\rangle := |\alpha_r/\sqrt{N}\rangle\}_r$ is also the PSK optical coherent states with the average photon number S/N. Note that n identical copies of $|\alpha'_r\rangle$ are regarded as $|\alpha_r\rangle$ whose average photon number is S (i.e., $|\alpha_r\rangle = |\alpha'_r\rangle^{\otimes N}$). We here want to know whether a Dolinar-like receiver can be globally optimal. We consider the bipartite ternary states $\{|\alpha_r\rangle = |a_r\rangle \otimes |b_r\rangle\}_r$, where $|a_r\rangle = |\sqrt{t}\alpha_r\rangle (=|\alpha'_r\rangle^{\otimes tN})$ and $|b_r\rangle = |\sqrt{1-t}\alpha_r\rangle (=|\alpha'_r\rangle^{\otimes(1-t)N})$ with $0 < t < 1$ are optical coherent states with average photon numbers tS and $(1-t)S$, respectively. The average success probability of an optimal sequential measurement for the bipartite states with any t is an upper bound on that of an optimal sequential measurement for N-partite states $\{|\alpha'_r\rangle^{\otimes N}\}$ with $N \to \infty$, and thus is an upper bound on that of a Dolinar-like receiver. We here show that there exists t such that an optimal sequential measurement for the corresponding bipartite states $\{|a_r\rangle \otimes |b_r\rangle\}_r$ is not globally optimal, which means that a Dolinar-like receiver cannot be globally optimal. In the case in which $\langle\alpha_0|\alpha_1\rangle$ is nonnegative real (i.e., $S = 4\pi k/\sqrt{3}$ with $k = 1, 2, \dots$), we choose $t = 1/2$; then, from Eq. (35), $K_A = K_B$ and $\arg K_A = \pi$ holds, and thus a sequential measurement cannot be globally optimal, as already shown in Fig. 2. In the other case, we choose $t \to 0$; formulating preprocessed $\{|a_r\rangle\}$ and $\{|b_r\rangle\}$ in the form of Eqs. (26) and (27), we have that for each $n \in \{1, 2\}$

$$x_n^2 = \frac{1}{3} + \frac{2}{3}e^{-\frac{3tS}{2}} \cos\left[(-1)^n\frac{2\pi}{3} + \frac{\sqrt{3}tS}{2}\right]. \tag{38}$$

Taking the limit of $t \to 0$, we obtain $x_2/x_1 \to 0$. From Corollary 3, it is necessary to satisfy $x_2 z_0 - x_1 z_1 \geqslant 0$ for a sequential measurement to be able to be globally optimal. When $t \to 0$, from $x_2/x_1 \to 0$, $z_1 \to 0$ must hold. However, z_1 converges to a positive number. ($z_1 \to 0$ holds only if $\langle b_0|b_1\rangle$ converges to a nonnegative real number, i.e., $y_1 - y_2 \to 0$; however, $\langle b_0|b_1\rangle$ converges to $\langle\alpha_0|\alpha_1\rangle$, which is not a nonnegative real number.) Therefore, a Dolinar-like receiver cannot be globally optimal for any ternary PSK optical coherent states.

C. Case of multipartite states

As an example of multipartite states, let us address the problem of multiple-copy state discrimination [13,16,39–41]. We again consider N-partite ternary PSK optical coherent states $\{|\alpha'_r\rangle^{\otimes N}\}$ ($|\alpha'_r\rangle := |\alpha_r/\sqrt{N}\rangle$). As described in Sec. IV B, in the limit of $N \to \infty$, a sequential measurement cannot be globally optimal. In this section, we consider N to be finite.

By using Corollaries 3 and 4, we can judge whether a sequential measurement can be globally optimal. The region of the average photon number S of $|\alpha_r\rangle$ for which the sufficient

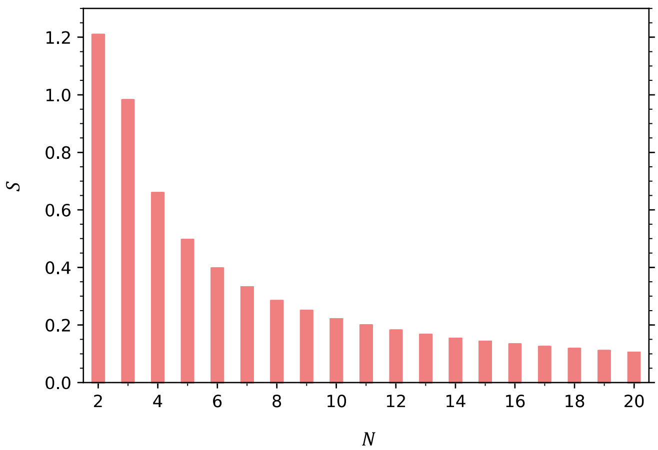

FIG. 5. Sufficient condition for a sequential measurement for the N-partite ternary PSK optical coherent states $\{|\alpha'_r\rangle^{\otimes N}\}$ to be able to be globally optimal. S is the average photon number of $|\alpha'_r\rangle^{\otimes N}$.

condition holds is shown in red in Fig. 5. We here consider the range $S \leqslant 1.3$. We can see in this figure that a sequential measurement can be globally optimal even for large N (such as $N = 20$) if S is sufficiently small (such as $S \leqslant 0.1$).

V. PROOF OF THEOREM 2

We now prove Theorem 2 using Lemma 1. After some preliminaries in Sec. V A, we first obtain $\hat{X}_G^\star$ in Sec. V B. Next, in Sec. V C we consider the case $y_1 = y_2$. After that, we consider the case $y_1 \neq y_2$. A sufficient condition for this theorem to hold and its reformulation are given in Secs. V D and V E, respectively. In Sec. V F we prove that this sufficient condition holds. We should note that the case in which $\{|a_r\rangle\}$ or $\{|b_r\rangle\}$ is mutually orthogonal is trivial, we assume that neither $\{|a_r\rangle\}$ nor $\{|b_r\rangle\}$ is mutually orthogonal.

A. Preliminaries

Assume without loss of generality that $|a_r\rangle$ and $|b_r\rangle$ are expressed as Eq. (27), where x_n and y_n satisfy Eq. (26). Let $K'_A := \langle a_0|a_1\rangle$ and $K'_B := \langle b_0|b_1\rangle$; then, Eq. (28) holds. η and χ are, respectively, defined by Eqs. (29) and (30).

Before giving the proof of Theorem 2, we shall prove some lemmas.

Lemma 5.

$$y_2^2 < \eta < y_1^2 \leqslant y_0^2. \tag{39}$$

Proof. Since $y_1^2 \leqslant y_0^2$ obviously holds, we shall show $y_2^2 < \eta < y_1^2$. We have

$$\begin{aligned}
&(1 - 3y_k^2)^2 - (1 - 3\chi) \\
&= 3(3y_k^4 - 2y_k^2 + \chi) \\
&= 3\left[y_k^4 - 2y_k^2(y_{k\oplus1}^2 + y_{k\oplus2}^2) + y_k^2(y_{k\oplus1}^2 + y_{k\oplus2}^2) + y_{k\oplus1}^2 y_{k\oplus2}^2\right] \\
&= 3(y_k^2 - y_{k\oplus1}^2)(y_k^2 - y_{k\oplus2}^2),
\end{aligned} \tag{40}$$

where the third line follows from $\sum_{n=0}^2 y_n^2 = 1$. Substituting $k = 1$ into Eq. (40) yields $(1 - 3y_1^2)^2 \leqslant 1 - 3\chi$. The equality holds when $y_0 = y_1$. In this case, from $y_2^2 = 1 - 2y_0^2$, we have $1 - 3y_1^2 = y_2^2 - y_0^2 < 0 \leqslant \sqrt{1 - 3\chi}$. Thus, $1 - 3y_1^2 < \sqrt{1 - 3\chi}$ always holds. Therefore,

$$y_1^2 > \frac{1}{3}(1 - \sqrt{1 - 3\chi}) = \eta, \tag{41}$$

where the equality follows from Eq. (31). In the same way, substituting $k = 2$ into Eq. (40) yields $1 - 3y_2^2 > \sqrt{1 - 3\chi}$, which gives $\eta > y_2^2$. ■

Let

$$u_n(q) := \left(y_n^2 - q\right)^{-1}, \quad \forall q \leqslant \eta, q \neq y_2^2. \tag{42}$$

Note that, from Lemma 5, $q \neq y_0^2$ and $q \neq y_1^2$ hold for any $q \leqslant \eta$.

Lemma 6. For any $q \leqslant \eta$ with $q \neq y_2^2$, we have

$$u_0^2(q) \leqslant u_1^2(q) \leqslant u_2^2(q). \tag{43}$$

Proof. Since $q \leqslant \eta < y_1^2 \leqslant y_0^2$ holds from Lemma 5, $u_1(q) \geqslant u_0(q) > 0$ holds, which gives $u_1^2(q) \geqslant u_0^2(q)$. Thus, it suffices to prove $u_2^2(q) \geqslant u_1^2(q)$. In the case of $q < y_2^2$, from $u_2(q) > u_1(q) > 0$, this is obvious. Let us consider the case of $q > y_2^2$. Since $u_2(q) < 0$ holds, it suffices to show $u_2(q) + u_1(q) \leqslant 0$. Let $\tilde{u}_k := u_k(\eta)$; then we have

$$\begin{aligned}
\tilde{u}_2 + \tilde{u}_1 &\leqslant \tilde{u}_2 + \tilde{u}_1 + \tilde{u}_0 \\
&= \tilde{u}_2\tilde{u}_1\tilde{u}_0[(\tilde{u}_0\tilde{u}_1)^{-1} + (\tilde{u}_1\tilde{u}_2)^{-1} + (\tilde{u}_2\tilde{u}_0)^{-1}] \\
&= \tilde{u}_2\tilde{u}_1\tilde{u}_0(3\eta^2 - 2\eta + \chi) \\
&= 0,
\end{aligned} \tag{44}$$

where χ is defined by Eq. (30). The last line follows from Eq. (31), i.e., $3\eta^2 - 2\eta + \chi = 0$. Since $u_2(q) \leqslant \tilde{u}_2$ and $u_1(q) \leqslant \tilde{u}_1$ hold, $u_2(q) + u_1(q) \leqslant \tilde{u}_2 + \tilde{u}_1 \leqslant 0$ holds. ■

Lemma 7. Let $\{|s_r\rangle\}_{r=0}^2$ be the ternary pure states expressed by

$$|s_r\rangle := \sum_{n=0}^{2} \varsigma_n \tau^{rn} |\varphi_n\rangle, \tag{45}$$

where $\{\varsigma_n\}_{n=0}^2$ are positive real numbers and $\{|\varphi_n\rangle\}_{n=0}^2$ is an orthonormal basis. Also, let $n_{\min}$ be an element of $\mathrm{argmin}_{n\in\mathcal{I}_3}\, \varsigma_n$. We consider problem $\mathrm{P_G}$ with $\tilde{\rho}_r := |s_r\rangle\langle s_r|$ and $\xi_r := 1/3$. Then

$$\hat{Z}_G^\bullet := 3\varsigma_{n_{\min}}^2 |\varphi_{n_{\min}}\rangle\langle\varphi_{n_{\min}}| \tag{46}$$

is an optimal solution to its dual problem (i.e., problem $\mathrm{DP_G}$).

Proof. We recall that $\hat{\rho}_r := \xi_r\tilde{\rho}_r = \frac{1}{3}|s_r\rangle\langle s_r|$ and $\hat{\nu}_r := \sum_{k\in\mathcal{I}_3\setminus\{r\}} \hat{\rho}_k$. Let $\hat{V} := \sum_{n=0}^2 \tau^n |\varphi_n\rangle\langle\varphi_n|$; then, $\hat{\rho}_{r\oplus1} = \hat{V}\hat{\rho}_r\hat{V}^\dagger$ holds, where † denotes conjugate transpose.

First, we show that $\hat{Z}_G^\bullet$ is feasible. $\hat{Z}_G^\bullet \geqslant 0$ obviously holds. We can easily see that $|c\rangle \in \mathrm{Ker}\, \hat{\nu}_r$ (i.e., $\langle c|s_{r\oplus1}\rangle = \langle c|s_{r\oplus2}\rangle = 0$) holds if and only if $|c\rangle \propto |s_r^\perp\rangle$ holds, where

$$|s_r^\perp\rangle := \sum_{n=0}^{2} \varsigma_n^{-1} \tau^{rn} |\varphi_n\rangle. \tag{47}$$

We have

$$\begin{aligned}
\langle s_r^\perp|(\hat{Z}_G^\bullet - \hat{\rho}_r)|s_r^\perp\rangle &= \langle s_r^\perp|\hat{Z}_G^\bullet|s_r^\perp\rangle - \langle s_r^\perp|\hat{\rho}_r|s_r^\perp\rangle \\
&= 3 - 3 = 0.
\end{aligned} \tag{48}$$

Thus, Eq. (4) with $\hat{Z}_G = \hat{Z}_G^\bullet$ holds, i.e., $\hat{Z}_G^\bullet$ is feasible.

Next, we show that $\hat{Z}_G^\bullet$ is optimal. Let $\hat{Z}_G$ be a feasible solution to problem DP_G. Due to the symmetry of the states, we assume without loss of generality that $\hat{Z}_G = \hat{V}\hat{Z}_G\hat{V}^\dagger$ holds. (Indeed, for any feasible solution $\hat{Z}'_G$, $\hat{Z}_G := \frac{1}{3}\sum_{n=0}^{2}\hat{V}^n\hat{Z}'_G(\hat{V}^\dagger)^n$ is a feasible solution that satisfies $\mathrm{Tr}\,\hat{Z}_G = \mathrm{Tr}\,\hat{Z}'_G$ and $\hat{Z}_G = \hat{V}\hat{Z}_G\hat{V}^\dagger$.) Thus, $\hat{Z}_G$ is expressed in the form of $\hat{Z}_G = \sum_{n=0}^{2}\theta_n|\varphi_n\rangle\langle\varphi_n|$, where θ_n is nonnegative real. Since $\hat{Z}_G$ is feasible, we have

$$0 \leqslant \langle s_r^\perp|(\hat{Z}_G - \hat{\rho}_r)|s_r^\perp\rangle = \sum_{n=0}^{2}\varsigma_n^{-2}\theta_n - 3, \tag{49}$$

which yields

$$\mathrm{Tr}\,\hat{Z}_G = \varsigma_{n_{\min}}^2\sum_{n=0}^{2}\varsigma_{n_{\min}}^{-2}\theta_n \geqslant \varsigma_{n_{\min}}^2\sum_{n=0}^{2}\varsigma_n^{-2}\theta_n \geqslant 3\varsigma_{n_{\min}}^2 = \mathrm{Tr}\,\hat{Z}_G^\bullet. \tag{50}$$

Therefore, $\hat{Z}_G^\bullet$ is optimal. ■

Note that although $\hat{\Gamma}_G(r, \hat{Z}_G^\bullet) \geqslant 0$ holds, $\hat{Z}_G^\bullet - \hat{\rho}_r + \lambda\hat{\upsilon}_r \geqslant 0$ does not hold for any finite positive real number λ. However, taking the limit $\lambda \to \infty$, $\langle t|(\hat{Z}_G^\bullet - \hat{\rho}_r + \lambda\hat{\upsilon}_r)|t\rangle$ becomes nonnegative for any vector $|t\rangle$. [More precisely, $\langle t|(\hat{Z}_G^\bullet - \hat{\rho}_r + \lambda\hat{\upsilon}_r)|t\rangle \to 0$ holds if $|t\rangle \propto |s_r^\perp\rangle$; otherwise, $\langle t|(\hat{Z}_G^\bullet - \hat{\rho}_r + \lambda\hat{\upsilon}_r)|t\rangle \to \infty$ holds].

B. Derivation of $\hat{X}_G^\star$

Let us consider $|a_r\rangle$ and $|b_r\rangle$ in the form of Eqs. (26) and (27). A simple calculation gives

$$|\Psi_r\rangle := |a_r\rangle \otimes |b_r\rangle = \sum_{n=0}^{2}\tilde{x}_n\tau^{rn}|\tilde{\phi}_n\rangle, \tag{51}$$

where

$$|\tilde{\phi}_n\rangle := \frac{1}{\tilde{x}_n}\sum_{k=0}^{2}x_k y_{n\ominus k}|\phi_k\rangle \otimes |\phi'_{n\ominus k}\rangle,$$
$$\tilde{x}_n := \sqrt{\sum_{k=0}^{2}x_k^2 y_{n\ominus k}^2}. \tag{52}$$

Obviously $\{|\tilde{\phi}_n\rangle\}_{n=0}^2$ is an orthonormal basis and $\tilde{x}_n$ is positive real. We have

$$\begin{aligned}\tilde{x}_1^2 - \tilde{x}_2^2 &= x_0^2y_1^2 + x_1^2y_0^2 + x_2^2y_2^2 - x_0^2y_2^2 - x_1^2y_1^2 - x_2^2y_0^2\\ &= (x_0^2 - x_2^2)(y_1^2 - y_2^2) + (x_1^2 - x_2^2)(y_0^2 - y_1^2)\\ &\geqslant 0,\end{aligned} \tag{53}$$

where the inequality follows from Eq. (26). Thus, $\tilde{x}_1 \geqslant \tilde{x}_2$ holds. Also, we have

$$\begin{aligned}\tilde{x}_0^2 - \tilde{x}_2^2 &= x_0^2y_0^2 + x_1^2y_2^2 + x_2^2y_1^2 - x_0^2y_2^2 - x_1^2y_1^2 - x_2^2y_0^2\\ &= (x_0^2 - x_1^2)(y_1^2 - y_2^2) + (x_0^2 - x_2^2)(y_0^2 - y_1^2).\end{aligned} \tag{54}$$

Although $x_0 \geqslant x_2$ and $y_0 \geqslant y_1 \geqslant y_2$ hold from Eq. (26), $x_0 \geqslant x_1$ does not always hold, which implies that whether $\tilde{x}_0 \geqslant \tilde{x}_2$ or not depends on given states. Let

$$[\upsilon_0, \upsilon_1, \upsilon_2] := \begin{cases}[2, 1, 0], & \tilde{x}_0 \geqslant \tilde{x}_2,\\ [0, 2, 1], & \text{otherwise};\end{cases} \tag{55}$$

then, $\upsilon_0 \in \mathrm{argmin}_{n\in\mathcal{I}_3}\tilde{x}_n$ holds. Thus, from Lemma 7 with $|s_r\rangle = |\Psi_r\rangle$, $\varsigma_n = \tilde{x}_n$, and $|\varphi_n\rangle = |\tilde{\phi}_n\rangle$,

$$\hat{Z}_G^\star := 3\tilde{x}_{\upsilon_0}^2|\tilde{\phi}_{\upsilon_0}\rangle\langle\tilde{\phi}_{\upsilon_0}| \tag{56}$$

is an optimal solution to problem DP_G. Therefore, we have

$$\hat{X}_G^\star = \mathrm{Tr}_B\,\hat{Z}_G^\star = 3\sum_{n=0}^{2}x_n^2y_{\upsilon_n}^2|\phi_n\rangle\langle\phi_n|. \tag{57}$$

C. Case of $y_1 = y_2$

We here show that, in the case of $y_1 = y_2$ (i.e., K'_B is positive real), there exists a globally optimal sequential measurement $\hat{\Pi}^{(\hat{A}^\star)}$ satisfying Eq. (24). Let

$$\hat{A}^\star(\omega) := \begin{cases}\hat{A}_r, & \omega = \omega_{1,r}\ (r \in \mathcal{I}_3),\\ \hat{A}_3, & \omega = \omega_3,\\ 0, & \text{otherwise},\end{cases} \tag{58}$$

where $\{\hat{A}_r\}_{r=0}^3$ is an optimal unambiguous measurement for $\{|a_r\rangle\}$ with equal prior probabilities. Obviously $\hat{A}^\star$ is in $\mathcal{M}_A$ and satisfies Eq. (24). It follows that the sequential measurement $\hat{\Pi}^{(\hat{A}^\star)}$ can be interpreted as follows: Alice and Bob, respectively, perform optimal measurements for $\{|a_r\rangle\}$ and $\{|b_r\rangle\}$ with equal prior probabilities and get the results r_A and r_B. $\hat{\Pi}^{(\hat{A}^\star)}$ returns r_A if $r_A \in \mathcal{I}_3$, r_B if $r_B \in \mathcal{I}_3$, and $r = 3$ otherwise. Note that $r_A = r_B$ holds whenever r_A and r_B are in $\mathcal{I}_3$.

Let P_A and P_B be, respectively, the average success probabilities of optimal measurements for $\{|a_r\rangle\}$ and $\{|b_r\rangle\}$ with equal prior probabilities. Lemma 7 with $|s_r\rangle = |a_r\rangle$, $\varsigma_n = x_n$, and $|\varphi_n\rangle = |\phi_n\rangle$ gives $\hat{Z}_G^\bullet = 3x_2^2|\phi_2\rangle\langle\phi_2|$, and thus $P_A = \mathrm{Tr}\,\hat{Z}_G^\bullet = 3x_2^2$ holds. We obtain $P_B = 3y_2^2$ in the same way. Thus, the average success probability of $\hat{\Pi}^{(\hat{A}^\star)}$ is

$$\begin{aligned}P[\hat{\Pi}^{(\hat{A}^\star)}] &= 1 - (1 - P_A)(1 - P_B)\\ &= 3(x_2^2 + y_2^2 - 3x_2^2y_2^2)\\ &= 3[(1 - x_2^2)y_2^2 + x_2^2(1 - 2y_2^2)]\\ &= 3\sum_{k=0}^{2}x_k^2y_{2-k}^2\\ &= 3\tilde{x}_2^2 = \mathrm{Tr}\,\hat{Z}_G^\star,\end{aligned} \tag{59}$$

where the last line follows from Eq. (57) and the fact that $\tilde{x}_0 \geqslant \tilde{x}_2$ (i.e., $\upsilon_0 = 2$) holds when $y_1 = y_2$. Thus, $\hat{\Pi}^{(\hat{A}^\star)}$ is globally optimal.

We should note that the same discussion is applicable to the case of $x_1 = x_2$ (i.e., K'_A is positive real); in this case, there also exists a globally optimal sequential measurement $\hat{\Pi}^{(\hat{A}^\star)}$ satisfying Eq. (24).

D. Sufficient condition for Theorem 2

Since we have already proved the theorem in the case of $y_1 = y_2$, in what follows, we only consider the case $y_1 \neq y_2$. (We do not have to assume $x_1 \neq x_2$; the following proof is also valid for $x_1 = x_2$.) Substituting Eqs. (8) and (57) into Eq. (10),

$\hat{\Gamma}^\star(\omega)$ with $\omega \in \Omega^\star$ is written by

$$\hat{\Gamma}^\star(\omega) = \begin{cases} \lim_{\lambda\to\infty} \hat{X}_G^\star + \lambda\hat{\Psi} - \left(\lambda + \frac{1}{3}\right)|a_j\rangle\langle a_j|, & \omega = \omega_{1,j}, \\ \lim_{\lambda\to\infty} \hat{X}_G^\star - \eta\hat{\Psi} + \lambda|a_j\rangle\langle a_j|, & \omega = \omega_{2,j}, \\ \hat{X}_G^\star - y_2^2\hat{\Psi}, & \omega = \omega_3, \end{cases} \tag{60}$$

where

$$\hat{\Psi} := \sum_{k=0}^{2} |a_k\rangle\langle a_k|. \tag{61}$$

In obtaining Eq. (60), we use the fact that $p_j^{(\omega_{1,j})} = 1$, $p_{j\oplus 1}^{(\omega_{2,j})} = p_{j\oplus 2}^{(\omega_{2,j})} = 3\eta$, and $p_r^{(\omega_3)} = 3y_2^2$. After some algebra using Eq. (60), we can see that, for any $\omega \in \Omega^\star$, $\hat{\Gamma}^\star(\omega)|\pi\rangle = 0$ holds if and only if $|\pi\rangle \propto |\pi_\omega^\star\rangle$ holds, where $|\pi_\omega^\star\rangle \in \mathcal{H}_A$ ($\omega \in \Omega^\star$) is the normal vector defined as

$$|\pi_\omega^\star\rangle := \begin{cases} C_1\hat{V}_A^j \sum_{n=0}^{2} x_n^{-1}|\phi_n\rangle, & \omega = \omega_{1,j}, \\ C_2\hat{V}_A^j \sum_{n=0}^{2} x_n^{-1}z_{\upsilon_n}^{-1}|\phi_n\rangle, & \omega = \omega_{2,j}, \\ |\phi_{\upsilon_2}\rangle, & \omega = \omega_3 \end{cases} \tag{62}$$

and C_1 and C_2 are normalization constants. Thus, it follows that $\hat{A}^\star$ satisfies Eqs. (24) and (11) with $\hat{A} = \hat{A}^\star$ if only if $\hat{A}^\star$ is expressed as

$$\hat{A}^\star(\omega) = \begin{cases} \kappa_\omega^\star|\pi_\omega^\star\rangle\langle\pi_\omega^\star|, & \omega \in \Omega^\star, \\ 0, & \text{otherwise}, \end{cases} \tag{63}$$

where, for each $\omega \in \Omega^\star$, $\kappa_\omega^\star$ is a nonnegative real number. Therefore, from Lemma 1, to prove that $\hat{\Pi}^{(\hat{A}^\star)}$ is an optimal measurement, it suffices to show that there exists Alice's POVM $\hat{A}^\star$ (i.e., $\hat{A}^\star \in \mathcal{M}_A$) in the form of Eq. (63).

Due to the symmetry of the states, there exists an optimal solution to problem P, $\hat{A}$, that is symmetric in the following sense: for any $\omega \in \Omega$, $\hat{A}(\omega^\diamond) = \hat{V}_A\hat{A}(\omega)\hat{V}_A^\dagger$ and $\hat{A}(\omega^{\diamond\diamond}) = \hat{V}_A^\dagger\hat{A}(\omega)\hat{V}_A$ hold, where the operator $^\diamond$ is defined by $\hat{B}_r^{(\omega^\diamond)} = \hat{V}_B\hat{B}_{r\ominus 1}^{(\omega)}\hat{V}_B^\dagger$ for any $r \in \mathcal{I}_3$. It follows that $\hat{B}_r^{(\omega^{\diamond\diamond})} = \hat{V}_B^\dagger\hat{B}_{r\oplus 1}^{(\omega)}\hat{V}_B$ holds for any $r \in \mathcal{I}_3$. Note that we can easily verify that $\omega_k^\diamond = \omega_{k,1}$ and $\omega_k^{\diamond\diamond} = \omega_{k,2}$ hold for any $k \in \{1, 2\}$; we assume without loss of generality that $\hat{A}^\star(\omega_{k,1}) = \hat{A}^\star(\omega_k^\diamond) = \hat{V}_A\hat{A}^\star(\omega_k)\hat{V}_A^\dagger$ and $\hat{A}^\star(\omega_{k,2}) = \hat{A}^\star(\omega_k^{\diamond\diamond}) = \hat{V}_A^\dagger\hat{A}^\star(\omega_k)\hat{V}_A$, which indicates $\kappa_{\omega_{k,j}}^\star = \kappa_{\omega_k}^\star$ for any $k \in \{1, 2\}$ and $j \in \mathcal{I}_3$.

Let

$$\hat{S}(\hat{T}) := \frac{1}{3}\sum_{j=0}^{2} \hat{V}_A^j\hat{T}\left(\hat{V}_A^j\right)^\dagger, \tag{64}$$

where $\hat{T}$ is a positive semidefinite operator on $\mathcal{H}_A$. $\hat{S}(\hat{T})$ is a positive semidefinite operator on $\mathcal{H}_A$ satisfying $\mathrm{Tr}[\hat{S}(\hat{T})] = \mathrm{Tr}\,\hat{T}$ and commuting with $\hat{V}_A$. For notational simplicity we denote $\hat{S}[\hat{A}(\omega)]$ by $\hat{S}(\omega)$. Due to the symmetry of $\hat{A}$, $\hat{S}(\omega) = \hat{S}(\omega^\diamond) = \hat{S}(\omega^{\diamond\diamond})$ holds for any $\{\omega, \omega^\diamond, \omega^{\diamond\diamond}\}$ satisfying $\hat{B}_r^{(\omega^\diamond)} = \hat{V}_B\hat{B}_{r\ominus 1}^{(\omega)}\hat{V}_B^\dagger$ and $\hat{B}_r^{(\omega^{\diamond\diamond})} = \hat{V}_B^\dagger\hat{B}_{r\oplus 1}^{(\omega)}\hat{V}_B$ ($\forall r \in \mathcal{I}_3$). Let

$$\hat{E}_k^\star := \hat{S}\left(|\pi_{\omega_k}^\star\rangle\langle\pi_{\omega_k}^\star|\right), \quad k \in \{1, 2, 3\}. \tag{65}$$

Since, from Eq. (62), $|\pi_{\omega_{k,j}}^\star\rangle = \hat{V}_A^j|\pi_{\omega_k}^\star\rangle$ and $|\pi_{\omega_3}^\star\rangle = \hat{V}_A^j|\pi_{\omega_3}^\star\rangle$ hold for any $k \in \{1, 2\}$ and $j \in \mathcal{I}_3$, Eq. (65) gives

$$\sum_{j=0}^{2} |\pi_{\omega_{k,j}}^\star\rangle\langle\pi_{\omega_{k,j}}^\star| = 3\hat{E}_k^\star, \quad k \in \{1, 2\},$$
$$|\pi_{\omega_3}^\star\rangle\langle\pi_{\omega_3}^\star| = \hat{E}_3^\star. \tag{66}$$

Here, assume that $\hat{S}(\omega)$ can be expressed as

$$\hat{S}(\omega) = \sum_{k=1}^{3} w_{\omega,k}\hat{E}_k^\star, \quad \forall\omega \in \Omega_+,$$
$$w_{\omega,k} \geqslant 0, \quad \forall\omega \in \Omega_+, k \in \{1, 2, 3\}, \tag{67}$$

where

$$\Omega_+ := \{\omega \in \Omega : \hat{A}(\omega) \neq 0\} \tag{68}$$

and $w_{\omega,k}$ is a weight. Let us choose

$$\kappa_\omega^\star = \begin{cases} \frac{w_k^\star}{3}, & \omega = \omega_{k,j}\ (k \in \{1, 2\}), \\ w_3^\star, & \omega = \omega_3, \end{cases}$$
$$w_k^\star := \int_{\Omega_+} w_{\omega,k}d\omega; \tag{69}$$

then, from Eq. (63), we have

$$\int_\Omega \hat{A}^\star(d\omega) = \sum_{k=1}^{3} w_k^\star\hat{E}_k^\star = \int_{\Omega_+}\sum_{k=1}^{3} w_{\omega,k}\hat{E}_k^\star d\omega$$
$$= \int_{\Omega_+}\hat{S}(d\omega) = \int_\Omega \hat{A}(d\omega) = \hat{1}_A, \tag{70}$$

where $\hat{1}_A$ is the identity operator on $\mathcal{H}_A$. The first equation follows from Eq. (66). Equation (70) yields $\hat{A}^\star \in \mathcal{M}_A$. Therefore, to prove Theorem 2, it suffices to prove Eq. (67).

E. Reformulation of Eq. (67)

For convenience of analysis, we shall reformulate the sufficient condition given by Eq. (67). For any positive semidefinite operator $\hat{T} \neq 0$, $s_n(\hat{T})$ is defined as follows:

$$s_n(\hat{T}) := \langle\phi_n|\frac{\hat{S}(\hat{T})}{\mathrm{Tr}[\hat{S}(\hat{T})]}|\phi_n\rangle. \tag{71}$$

From $\sum_{n=0}^{2}\langle\phi_n|\hat{S}(\hat{T})|\phi_n\rangle = \mathrm{Tr}[\hat{S}(\hat{T})]$, $\sum_{n=0}^{2} s_n(\hat{T}) = 1$ holds. Let us consider the following point:

$$s(\hat{T}) := \left[s_{\upsilon_1}(\hat{T}), s_{\upsilon_0}(\hat{T})\right], \tag{72}$$

which is in a two-dimensional space (we call it the S plane). Since $s_n(\hat{T}) \geqslant 0$ holds from Eq. (71), each $s(\hat{T})$ is in the first quadrant of the S plane. We can easily verify that the point $s(\hat{T})$ has a one-to-one correspondence with $\hat{S}(\hat{T})/\mathrm{Tr}[\hat{S}(\hat{T})]$. Let

$$e_k^\star := s\left(|\pi_{\omega_k}^\star\rangle\langle\pi_{\omega_k}^\star|\right), \quad k \in \{1, 2, 3\}, \tag{73}$$

which is the point in the S plane that corresponds to $\hat{E}_k^\star$ defined by Eq. (65). $e_3^\star = [0, 0]$ holds from Eq. (62). Also, let $\mathcal{T}^\star$ be the triangle formed by $e_1^\star$, $e_2^\star$, and $e_3^\star$. Note that $\mathcal{T}^\star$ may degenerate to a straight line segment in special cases. For simplicity,

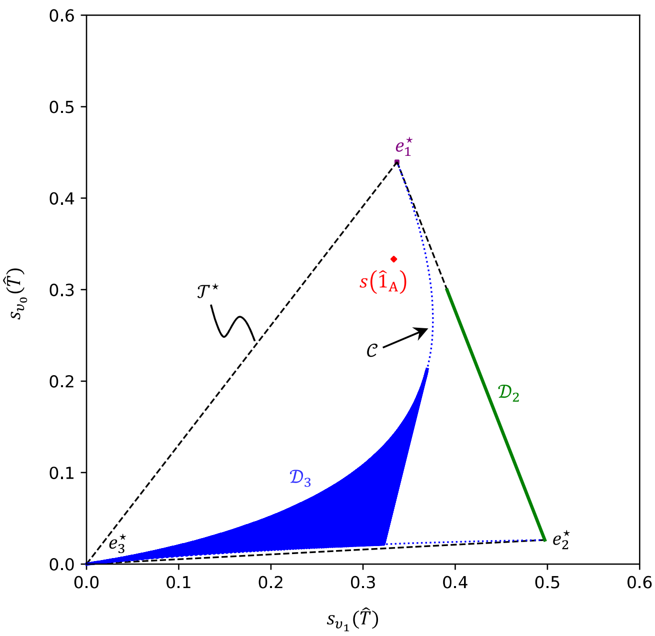

FIG. 6. S-plane representation in the case of $K'_A = K'_B = 0.2\exp(i\pi/10)$. $e_1^\star$ (purple), $\mathcal{D}_2$ (green), and $\mathcal{D}_3$ (blue), respectively, show the entire sets of points $s(\omega)$ with $|T^{(\omega)}| = 1$, 2, and 3. The dashed line represents the triangle $\mathcal{T}^\star$.

we denote $s_n(\omega) := s_n[\hat{A}(\omega)]$ and $s(\omega) := s[\hat{A}(\omega)]$ $(\omega \in \Omega_+)$. From the first line of Eq. (67), we have

$$s(\omega) = \frac{1}{\mathrm{Tr}[\hat{S}(\omega)]} \sum_{k=1}^{3} w_{\omega,k} e_k^\star. \tag{74}$$

Thus, it follows that Eq. (67) is equivalent to the following:

$$s(\omega) \in \mathcal{T}^\star, \quad \forall \omega \in \Omega_+. \tag{75}$$

Figure 6 shows the S-plane representation in the case of $K'_A = K'_B = 0.2\exp(i\pi/10)$. The entire sets of points $s(\omega)$ $(\omega \in \Omega_+)$ with $|T^{(\omega)}| = 2$ and 3, denoted by $\mathcal{D}_2$ and $\mathcal{D}_3$, are depicted by the green and blue regions in this figure, respectively. Also, $s(\omega) = e_1^\star$ holds when ω satisfies $|T^{(\omega)}| = 1$. Indeed, in this case, since $T^{(\omega)} = \{j\}$ holds for certain $j \in \mathcal{I}_3$, we can easily see that $\hat{A}(\omega) \propto |\pi^\star_{\omega_{1,j}}\rangle\langle\pi^\star_{\omega_{1,j}}|$ must hold from Eq. (19), which gives $s(\omega) = e_1^\star$. The triangle $\mathcal{T}^\star$ is also shown in the dashed line in Fig. 6. One can see that $e_1^\star$, $\mathcal{D}_2$, and $\mathcal{D}_3$ are all included in $\mathcal{T}^\star$. Note that we can show, under the assumption that Theorem 2 holds, that a sequential measurement can be globally optimal if and only if $s(\hat{1}_A)(= [1/3, 1/3]) \in \mathcal{T}^\star$ holds (see Appendix B). In the case shown in Fig. 6, $s(\hat{1}_A)$ is in $\mathcal{T}^\star$, and thus a sequential measurement can be globally optimal.

F. Proof of Eq. (75)

We shall prove that Eq. (75), which is a sufficient condition of Theorem 2, holds. $\hat{\Gamma}^\star(\omega)$ of Eq. (10) can be rewritten as the following form:

$$\hat{\Gamma}^\star(\omega) = \hat{X}_G^\star - \sum_{r=0}^{2} \mu_r^{(\omega)} |a_r\rangle\langle a_r|. \tag{76}$$

From Eq. (18), $\mu_r^{(\omega)} = -\infty$ holds if $e_r^{(\omega)} \neq 0$; otherwise, $\mu_r^{(\omega)} = p_r^{(\omega)}/3$ holds. It is very hard to show in a naive way that each $s(\omega)$ with $\omega \in \Omega_+$ is included in $\mathcal{T}^\star$. However, we can rather easily show that Eq. (75) holds by considering the following two cases: (1) the case in which at least two of $\{\mu_r^{(\omega)}\}_r$ are the same and (2) the other case in which $\{\mu_r^{(\omega)}\}_r$ are all different.

Case (1): At least two of $\{\mu_r^{(\omega)}\}_r$ are the same

Due to the symmetry of the states, we assume $\mu_1^{(\omega)} = \mu_2^{(\omega)} =: q_\omega$ without loss of generality; then $\hat{\Gamma}^\star(\omega)$ can be expressed as

$$\hat{\Gamma}^\star(\omega) = \hat{X}_G^\star - q_\omega \hat{\Psi} - p|a_0\rangle\langle a_0|, \tag{77}$$

where $\hat{\Psi}$ is defined in Eq. (61) and p is a real number. If $|T^{(\omega)}| = 3$, then $q_\omega = p_1^{(\omega)}/3 = p_2^{(\omega)}/3$ holds to satisfy Eq. (77). Also, in this case, we can easily see that $p_1^{(\omega)} = p_2^{(\omega)} \leqslant p_1^{(\omega_2)} = p_2^{(\omega_2)} = 3\eta$ holds, which gives $0 \leqslant q_\omega \leqslant \eta$. Moreover, $q_\omega = -\infty$ holds if $|T^{(\omega)}| = 1$, and $q_\omega = \eta$ holds if $|T^{(\omega)}| = 2$. Thus, $q_\omega \leqslant \eta$ always holds.

For each $q \leqslant \eta$, let $|\gamma_q\rangle$ be a normal vector satisfying

$$|\gamma_q\rangle \in \mathrm{Ker}\, \hat{\Gamma}^{(1)}(q),$$
$$\hat{\Gamma}^{(1)}(q) := \hat{X}_G^\star - q\hat{\Psi} - p_q|a_0\rangle\langle a_0|, \tag{78}$$

where p_q is a real number determined such that $\mathrm{Ker}\, \hat{\Gamma}^{(1)}(q) \neq \{0\}$. We shall show that $|\gamma_q\rangle$ can be uniquely written, up to a global phase, as

$$|\gamma_q\rangle = \begin{cases} C'_q \sum_{n=0}^{2} \frac{1}{x_n(y_{v_n}^2 - q)} |\phi_n\rangle, & q \neq y_2^2, \\ |\phi_{v_2}\rangle, & \text{otherwise}, \end{cases} \tag{79}$$

where C'_q is a normalization constant. From Eqs. (27), (57), and (61), we have

$$\hat{X}_G^\star - q\hat{\Psi} = 3\sum_{n=0}^{2} x_n^2 (y_{v_n}^2 - q)|\phi_n\rangle\langle\phi_n|. \tag{80}$$

From Lemma 5, $\hat{X}_G^\star - q\hat{\Psi}$ $(q \leqslant \eta)$ is singular if and only if $q = y_2^2$ holds. One can easily see $|\gamma_q\rangle = |\phi_{v_2}\rangle$ when $q = y_2^2$. Note that, in this case, one can define $p_q := 0$. In what follows, we consider the case $q \neq y_2^2$. From Eq. (78), we have

$$(\hat{X}_G^\star - q\hat{\Psi})|\gamma_q\rangle = p_q|a_0\rangle\langle a_0|\gamma_q\rangle \propto |a_0\rangle. \tag{81}$$

Thus, from Eqs. (27) and (80), we have

$$|\gamma_q\rangle \propto (\hat{X}_G^\star - q\hat{\Psi})^{-1}|a_0\rangle \propto \sum_{n=0}^{2} \frac{1}{x_n(y_{v_n}^2 - q)} |\phi_n\rangle. \tag{82}$$

Therefore, $|\gamma_q\rangle$ is expressed by Eq. (79). One can verify $\mathrm{Ker}\, \hat{\Gamma}^{(1)}(q) \neq \{0\}$ by letting $p_q := \langle a_0|(\hat{X}_G^\star - q\Psi)^{-1}|a_0\rangle^{-1}$ for $q < \eta$ and $p_\eta := -\infty$. Note that since $|\gamma_q\rangle$ is unique up to a global phase, $\dim \mathrm{Ker}\, \hat{\Gamma}^{(1)}(q) = 1$ holds.

Since $\hat{\Gamma}^\star(\omega)$ is in the form of Eq. (77) satisfying $\mathrm{Ker}\, \hat{\Gamma}^\star(\omega) \neq \{0\}$, $\hat{\Gamma}^\star(\omega) = \hat{\Gamma}^{(1)}(q_\omega)$ holds. Thus, $\hat{A}(\omega) \propto |\gamma_{q_\omega}\rangle\langle\gamma_{q_\omega}|$ holds. Let C be the set defined as

$$C := \{s(|\gamma_q\rangle\langle\gamma_q|) : q \leqslant \eta\}; \tag{83}$$

then, $s(\omega) = s(|\gamma_{q_\omega}\rangle\langle\gamma_{q_\omega}|) \in C$ holds. In Fig. 6, C is shown in the blue dotted line. Therefore, to prove $s(\omega) \in \mathcal{T}^\star$, it suffices to show $C \subseteq \mathcal{T}^\star$.

In what follows, we shall show $C \subseteq \mathcal{T}^\star$. Since $s(|\gamma_q\rangle\langle\gamma_q|) = e_3^\star \in \mathcal{T}^\star$ holds when $q = y_2^2$, we have only to consider the case of $q \neq y_2^2$. From Eqs. (64), (71), and (79), we have

$$s_k(|\gamma_q\rangle\langle\gamma_q|) \propto \frac{1}{x_k^2\left(y_{\upsilon_k}^2 - q\right)^2} = x_k^{-2}u_{\upsilon_k}^2(q), \tag{84}$$

where $u_k(q)$ is defined by Eq. (42). Substituting $k = \upsilon_n$ into this equation and using $\upsilon_{\upsilon_n} = n$, which follows from Eq. (55), yields

$$s_{\upsilon_n}(|\gamma_q\rangle\langle\gamma_q|) \propto x_{\upsilon_n}^{-2}u_n^2(q). \tag{85}$$

This gives

$$s(|\gamma_q\rangle\langle\gamma_q|) \propto \left[x_{\upsilon_1}^{-2}u_1^2(q), x_{\upsilon_0}^{-2}u_0^2(q)\right]. \tag{86}$$

First, let us consider the case in which the three points $e_1^\star$, $e_2^\star$, and $e_3^\star$ lie on a straight line. From Eq. (62), this case occurs only when $y_0 = y_1$. Since $s(|\gamma_q\rangle\langle\gamma_q|) \propto [x_{\upsilon_1}^{-2}, x_{\upsilon_0}^{-2}]$ holds from Eq. (86), every point in C is on the line joining the origin $e_3^\star$ to the point $e_1^\star$. From Lemma 6 [i.e., $u_0^2(q) = u_1^2(q) \leqslant u_2^2(q)$], we have

$$\begin{aligned} s_{\upsilon_0}(|\gamma_q\rangle\langle\gamma_q|) &= \frac{x_{\upsilon_0}^{-2}u_0^2(q)}{\sum_{k=0}^2 x_{\upsilon_k}^{-2}u_k^2(q)} \\ &\leqslant \frac{x_{\upsilon_0}^{-2}u_0^2(q)}{u_0^2(q)\sum_{k=0}^2 x_{\upsilon_k}^{-2}} = s_{\upsilon_0}(\omega_1). \end{aligned} \tag{87}$$

Thus, $s(|\gamma_q\rangle\langle\gamma_q|)$ is an interior point between $e_1^\star$ and $e_3^\star$. Therefore, $C \subseteq \mathcal{T}^\star$ holds.

Next, let us consider the other case in which $e_1^\star$, $e_2^\star$, and $e_3^\star$ do not lie on a straight line. Let l_{jk} denote the straight line joining $e_j^\star$ and $e_k^\star$. It suffices to prove the following two statements:

(a) C is in the region between the two lines l_{13} and l_{23},

(b) C is in the region between the two lines l_{12} and l_{13}.

First, we prove statement (a). The gradient of the line joining the origin to the point $s(|\gamma_q\rangle\langle\gamma_q|)$ is

$$\zeta(q) := \frac{s_{\upsilon_0}(|\gamma_q\rangle\langle\gamma_q|)}{s_{\upsilon_1}(|\gamma_q\rangle\langle\gamma_q|)} = \frac{x_{\upsilon_1}^2\left(y_1^2 - q\right)^2}{x_{\upsilon_0}^2\left(y_0^2 - q\right)^2}, \tag{88}$$

where the last equality follows from Eq. (85). From Lemma 5, one can easily verify that $\zeta(q)$ monotonically decreases in the range $q \leqslant \eta$, which gives

$$\zeta(-\infty) \geqslant \zeta(q) \geqslant \zeta(\eta), \quad \forall q \leqslant \eta. \tag{89}$$

Also, from $e_1^\star = s(|\gamma_{-\infty}\rangle\langle\gamma_{-\infty}|)$ and $e_2^\star = s(|\gamma_\eta\rangle\langle\gamma_\eta|)$, the gradients of the lines l_{13} and l_{23} are, respectively, $\zeta(-\infty)$ and $\zeta(\eta)$. Therefore, from Eq. (89), statement (a) holds.

Next, we prove statement (b). Let $c(q)$ denote the s_{υ_1} coordinate of the intersection of the s_{υ_1} axis and the line joining the two points $e_1^\star$ and $s(|\gamma_q\rangle\langle\gamma_q|)$ in C. It follows that statement (b) holds if and only if $c(q)$ satisfies

$$0 \leqslant c(q) \leqslant c(\eta), \quad \forall q \leqslant \eta. \tag{90}$$

Since $s(|\gamma_q\rangle\langle\gamma_q|)$ is on the line joining $e_1^\star$ and $[c(q), 0]$, we have that for some real number w,

$$\begin{aligned} s_{\upsilon_1}(|\gamma_q\rangle\langle\gamma_q|) &= w s_{\upsilon_1}(\omega_1) + (1-w)c(q), \\ s_{\upsilon_0}(|\gamma_q\rangle\langle\gamma_q|) &= w s_{\upsilon_0}(\omega_1). \end{aligned} \tag{91}$$

Also, since $\sum_{n=0}^2 s_n(\hat{T}) = 1$ holds for any nonzero positive semidefinite operator $\hat{T}$, we have

$$s_{\upsilon_2}(|\gamma_q\rangle\langle\gamma_q|) = w s_{\upsilon_2}(\omega_1) + (1-w)[1 - c(q)]. \tag{92}$$

After some algebra with Eqs. (91), (92), and (85), we obtain

$$\tilde{c}(q) := \frac{x_{\upsilon_1}^2 c(q)}{x_{\upsilon_2}^2[1 - c(q)]} = \frac{u_1^2(q) - u_0^2(q)}{u_2^2(q) - u_0^2(q)}. \tag{93}$$

It follows from the definition of $\tilde{c}(q)$ that $\tilde{c}(q)$ monotonically increases with $c(q)$. Thus, the statement (b), i.e., Eq. (90), is equivalent to

$$0 \leqslant \tilde{c}(q) \leqslant \tilde{c}(\eta), \quad \forall q \leqslant \eta. \tag{94}$$

From Lemma 6, $\tilde{c}(q) \geqslant 0$ obviously holds. Therefore, we need only show $\tilde{c}(q) \leqslant \tilde{c}(\eta)$.

Differentiating $\tilde{c}(q)$ of Eq. (93) with respect to q gives

$$\begin{aligned} \frac{d\tilde{c}(q)}{dq} &= \frac{2\left[u_1^2(q) - u_0^2(q)\right]}{u_2^2(q) - u_0^2(q)}\{f[u_1(q)] - f[u_2(q)]\}, \\ f(x) &:= \frac{x^2 + u_0(q)x + u_0^2(q)}{x + u_0(q)}, \end{aligned} \tag{95}$$

which implies that $d\tilde{c}(q)/dq \geqslant 0$ is equivalent to $f[u_1(q)] \geqslant f[u_2(q)]$. In the case of $q < y_2^2$, from $0 \leqslant u_1(q) \leqslant u_2(q)$, $f[u_1(q)] \leqslant f[u_2(q)]$ [i.e., $d\tilde{c}(q)/dq \leqslant 0$] holds, which follows from the fact that $f(x)$ monotonically increases in the range $x \geqslant 0$. In the other case of $q > y_2^2$, from Lemma 6 [i.e., $u_2(q) \leqslant -u_1(q) \leqslant -u_0(q)$], $f[u_2(q)] < 0 \leqslant f[u_1(q)]$ [i.e., $d\tilde{c}(q)/dq \geqslant 0$] holds, which follows from the fact that $f(x) < 0$ holds if and only if $x < -u_0(q)$. Therefore, $\tilde{c}(q)$ $(q \leqslant \eta)$ attains its maximum at $q = -\infty$ and/or $q = \eta$, and thus, for the rest, it suffices to show $\tilde{c}(-\infty) \leqslant \tilde{c}(\eta)$.

From Eq. (93) we have

$$\begin{aligned} \tilde{c}(q) &= \frac{\left(y_2^2 - q\right)^2\left[\left(y_0^2 - q\right)^2 - \left(y_1^2 - q\right)^2\right]}{\left(y_1^2 - q\right)^2\left[\left(y_0^2 - q\right)^2 - \left(y_2^2 - q\right)^2\right]} \\ &= \frac{\left(y_0^2 - y_1^2\right)\left(y_2^2 - q\right)^2\left(y_0^2 + y_1^2 - 2q\right)}{\left(y_0^2 - y_2^2\right)\left(y_1^2 - q\right)^2\left(y_0^2 + y_2^2 - 2q\right)}, \end{aligned} \tag{96}$$

which gives

$$\tilde{c}(-\infty) = \frac{y_0^2 - y_1^2}{y_0^2 - y_2^2}. \tag{97}$$

Also, we have

$$\begin{aligned} &\left(y_1^2 - \eta\right)^2\left(y_0^2 + y_2^2 - 2\eta\right) - \left(y_2^2 - \eta\right)^2\left(y_0^2 + y_1^2 - 2\eta\right) \\ &\quad = \left(y_1^2 - y_2^2\right)\left(y_0^2y_1^2 + y_1^2y_2^2 + y_2^2y_0^2 - 2\eta + 3\eta^2\right) \\ &\quad = \left(y_1^2 - y_2^2\right)\left(3\eta^2 - 2\eta + \chi\right) \\ &\quad = 0, \end{aligned} \tag{98}$$

where the second to fourth lines, respectively, follow from $\sum_{k=0}^2 y_k^2 = 1$, Eq. (30), and Eq. (31). Thus, substituting $q = \eta$

into Eq. (96) gives $\tilde{c}(\eta) = (y_0^2 - y_1^2)/(y_0^2 - y_2^2) = \tilde{c}(-\infty)$. Therefore, $C \subseteq \mathcal{T}^\star$ holds.

Case (2): $\{\mu_r^{(\omega)}\}_r$ are all different

To prove $s(\omega) \in \mathcal{T}^\star$, it suffices to show that each $s(\omega)$ is on a straight line segment whose endpoints are in C. Indeed, since $C \subseteq \mathcal{T}^\star$ holds, such a line segment is in the triangle $\mathcal{T}^\star$, and thus $s(\omega) \in \mathcal{T}^\star$ holds in this case.

Let us consider, without loss of generality, $\omega \in \Omega_+$ such that $\mu_2^{(\omega)} < \mu_0^{(\omega)}$ and $\mu_2^{(\omega)} < \mu_1^{(\omega)}$. In order to show that $s(\omega)$ is on a straight line segment whose endpoints are in C, we shall show the two statements: (a) $s(\omega)$ is on a certain straight line segment, and (b) the line segment is part of a straight line segment whose endpoints are in C.

Since we now consider case (2), $|T^{(\omega)}|$ must be 2 or 3. Let

$$\hat{X} := \begin{cases} \hat{X}_G^\star - \mu_2^{(\omega)}\Psi, & |T^{(\omega)}| = 3, \\ \hat{X}_G^\star + \infty|a_2\rangle\langle a_2|, & |T^{(\omega)}| = 2. \end{cases} \quad (99)$$

One can easily see that $\hat{X}$ is a positive definite operator. Let $|\varpi(q)\rangle$ be a normal vector satisfying

$$\begin{aligned} |\varpi(q)\rangle &\in \operatorname{Ker} \hat{\Gamma}^{(2)}(q), \\ \langle a_0|\varpi(q)\rangle &\geqslant 0, \\ \hat{\Gamma}^{(2)}(q) &:= \hat{X} - q|a_1\rangle\langle a_1| - p'_q|a_0\rangle\langle a_0|, \end{aligned} \quad (100)$$

where p'_q is the function of q such that $\operatorname{Ker} \hat{\Gamma}^{(2)}(q) \neq \{0\}$. [We can define such p'_q as $p'_q := \langle a_0|(\hat{X} - q|a_1\rangle\langle a_1|)^{-1}|a_0\rangle^{-1}$. Since $\hat{X} - q|a_1\rangle\langle a_1|$ is positive definite, such p'_q always exists.] p'_q monotonically decreases with q. $\hat{\Gamma}^\star(\omega) = \hat{\Gamma}^{(2)}[\mu_1^{(\omega)} - \mu_2^{(\omega)}]$ holds if $|T^{(\omega)}| = 3$; otherwise, $\hat{\Gamma}^\star(\omega) = \hat{\Gamma}^{(2)}[\mu_1^{(\omega)}]$ holds.

First, we show statement (a). Let $\hat{\Gamma}_0^{(2)} := \hat{\Gamma}^{(2)}(0)$, $\hat{\Gamma}_1^{(2)} := \hat{\Gamma}^{(2)}(p'_0)$, $|\varpi_0\rangle := |\varpi(0)\rangle$, and $|\varpi_1\rangle := |\varpi(p'_0)\rangle$. Note that $p'_q = 0$ holds when $q = p'_0$ (i.e., $p'_{p'_0} = 0$). We shall express $|\varpi(q)\rangle$ in terms of $|\varpi_0\rangle$ and $|\varpi_1\rangle$. For each $k \in \{0, 1\}$, from $\hat{\Gamma}_k^{(2)}|\varpi_k\rangle = 0$, we have

$$|a_k\rangle = \frac{\hat{X}|\varpi_k\rangle}{p'_0\langle a_k|\varpi_k\rangle}. \quad (101)$$

Note that since $\hat{X}$ is positive definite, we have $\hat{X}|\varpi_k\rangle \neq 0$, which yields $p'_0\langle a_k|\varpi_k\rangle \neq 0$.

Substituting Eq. (100) into $\hat{\Gamma}^{(2)}(q)|\varpi(q)\rangle = 0$ and using Eq. (101) yields

$$\begin{aligned} |\varpi(q)\rangle &= \hat{X}^{-1}(p'_q r_0|a_0\rangle + q r_1|a_1\rangle) \\ &= \frac{1}{p'_0}\left(\frac{p'_q r_0}{\langle a_0|\varpi_0\rangle}|\varpi_0\rangle + \frac{q r_1}{\langle a_1|\varpi_1\rangle}|\varpi_1\rangle\right), \end{aligned} \quad (102)$$

where $r_k := \langle a_k|\varpi(q)\rangle$. Premultiplying this equation by $\rangle a_0|$ and some algebra gives

$$\frac{q r_1}{\langle a_1|\varpi_1\rangle} = \frac{(p'_0 - p'_q) r_0}{\langle a_0|\varpi_1\rangle}. \quad (103)$$

Substituting this equation into Eq. (102) gives

$$|\varpi(q)\rangle = \frac{r_0}{p'_0}\left(\frac{p'_q}{\langle a_0|\varpi_0\rangle}|\varpi_0\rangle + \frac{p'_0 - p'_q}{\langle a_0|\varpi_1\rangle}|\varpi_1\rangle\right). \quad (104)$$

Since $r_0 \geqslant 0$ and $\langle a_0|\varpi_k\rangle \geqslant 0$ hold from Eq. (100), it follows from Eq. (104) that $|\varpi(q)\rangle$ is expressed as

$$|\varpi(q)\rangle = c_0|\varpi_0\rangle + c_1|\varpi_1\rangle, \quad (105)$$

with certain nonnegative real numbers c_0 and c_1. Let q_2 be the real number satisfying $p'_{q_2} = q_2$. One can easily verify that, when $q = q_2$, Eq. (105) with $c_0 = c_1 =: c$ holds. Let $|\varpi_2\rangle := |\varpi(q_2)\rangle$.

Due to the symmetry of the states, $\hat{S}(|\varpi_0\rangle\langle\varpi_0|) = \hat{S}(|\varpi_1\rangle\langle\varpi_1|) =: \hat{S}_{\varpi_0}$ holds. Thus, from Eq. (105), we have

$$\hat{S}[|\varpi(q)\rangle\langle\varpi(q)|] = (c_0^2 + c_1^2)\hat{S}_{\varpi_0} + c_0 c_1 \hat{S}', \quad (106)$$

where

$$\hat{S}' := \frac{1}{3}\sum_{j=0}^{2} \hat{V}_A^j(|\varpi_0\rangle\langle\varpi_1| + |\varpi_1\rangle\langle\varpi_0|)(\hat{V}_A^j)^\dagger. \quad (107)$$

Substituting $q = q_2$ into Eq. (106) and letting $\hat{S}_{\varpi_2} := \hat{S}(|\varpi_2\rangle\langle\varpi_2|)$ yields

$$\hat{S}_{\varpi_2} = 2c^2\hat{S}_{\varpi_0} + c^2\hat{S}'. \quad (108)$$

Substituting this into Eq. (106) gives

$$\hat{S}[|\varpi(q)\rangle\langle\varpi(q)|] = c'_0\hat{S}_{\varpi_0} + c'_2\hat{S}_{\varpi_2}, \quad (109)$$

where $c'_0 := (c_0 - c_1)^2$ and $c'_2 = c_0 c_1/c^2$. Note that taking the trace of this gives $c'_0 + c'_2 = 1$ and that $c'_0, c'_2 \geqslant 0$ holds. Also, Eq. (109) gives

$$s[|\varpi(q)\rangle\langle\varpi(q)|] = c'_0 s_{\varpi_0} + c'_2 s_{\varpi_2}, \quad (110)$$

where $s_{\varpi_k} := s(|\varpi_k\rangle\langle\varpi_k|)$ for each $k \in \{0, 2\}$. Therefore, $s[|\varpi(q)\rangle\langle\varpi(q)|]$ is on the straight line segment, denoted by $\mathcal{L}$, whose endpoints are s_{ϖ_0} and s_{ϖ_2}.

Next, we show statement (b). In the case of $q = q_2$, since Eq. (100) with $q = p'_q = q_2$ holds, this is case (1), i.e., at least two of $\{\mu_r^{(\omega)}\}_r$ in Eq. (76) are the same; thus, s_{ϖ_2} is in C. Also, if ω satisfies $|T^{(\omega)}| = 3$, then $q = 0$ is also case (1), and thus s_{ϖ_0} is in C. Therefore, in the case of $|T^{(\omega)}| = 3$, $\mathcal{L}$ is the line segment whose endpoints s_{ϖ_0} and s_{ϖ_2} are in C. In what follows, assume $|T^{(\omega)}| = 2$. We shall show that $\mathcal{L}$ is part of the straight line segment whose endpoints are $e_1^\star = s(\omega_1) \in C$ and $s_{\varpi_2} \in C$. Taking the limit as $q \to -\infty$ in Eq. (100) gives $s[|\varpi(-\infty)\rangle\langle\varpi(-\infty)|] = e_1^\star$. Thus, repeating the above argument with $q \to -\infty$ indicates that $|\varpi(-\infty)\rangle$ is expressed as Eq. (105) with $c_0 > 0$ and $c_1 < 0$, and that Eq. (109) holds with $c'_2 < 0$. Thus, s_{ϖ_0} is an interior point between $e_1^\star$ and s_{ϖ_2}. Therefore, $\mathcal{L}$ is part of the line segment whose endpoints are $e_1^\star$ and s_{ϖ_2}.

The two cases (1) and (2) exhaust all possibilities; thus, from the above arguments, Eq. (75) holds, and thus we complete the proof. ■

VI. CONCLUSION

An unambiguous sequential measurement for bipartite symmetric ternary separable pure states has been investigated. We have shown that a certain type of sequential measurement can always be globally optimal whenever there exists a globally optimal sequential measurement. From this result we have derived a formula that can easily determine whether an optimal sequential measurement is globally optimal. We have presented some examples in which an optimal sequential measurement is globally optimal. In particular, for ternary PSK optical coherent states, a sequential measurement can be globally optimal in some cases, while, in the strategy for minimum-error discrimination, an optimal sequential measurement may never be globally optimal. Moreover, our results have been extended to multipartite states and have given a sufficient condition that a sequential measurement can be globally optimal.

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APPENDIX A: PROOF OF COROLLARY 3

Since, as already described in Sec. V C, a sequential measurement can be globally optimal when $y_1 = y_2$, we only have to consider the case $y_1 \neq y_2$.

(1) ⇒ (2): From the discussion in Sec. V D, there exists an optimal solution $\hat{A}^\star$ to problem P that is expressed by Eq. (63) with $\kappa^\star_{\omega_k}$ $(k \in \{1, 2, 3\})$ independent of $j \in \mathcal{I}_3$. Since $\hat{A}^\star$ is a POVM, we have

$$\sum_{j=0}^{2}[\hat{A}^\star(\omega_{1,j}) + \hat{A}^\star(\omega_{2,j})] + \hat{A}^\star(\omega_3) = \hat{1}_{\rm A}. \quad \text{(A1)}$$

Substituting Eqs. (62) and (63) into Eq. (A1) gives

$$\begin{bmatrix} x_{\upsilon_0}^{-2} & x_{\upsilon_0}^{-2} z_0^{-2} & 0 \\ x_{\upsilon_1}^{-2} & x_{\upsilon_1}^{-2} z_1^{-2} & 0 \\ x_{\upsilon_2}^{-2} & x_{\upsilon_2}^{-2} z_2^{-2} & 1 \end{bmatrix} \begin{bmatrix} 3\kappa^\star_{\omega_1}|C_1|^2 \\ 3\kappa^\star_{\omega_2}|C_2|^2 \\ \kappa^\star_{\omega_3} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{(A2)}$$

where we use $\upsilon_{\upsilon_k} = k$ $(k \in \mathcal{I}_3)$, which follows from Eq. (55). After some algebra, we can see that Eq. (32) must hold if and only if there exists $\kappa^\star_{\omega_k} \geqslant 0$ satisfying Eq. (A2).

(2) ⇒ (1): Let $\kappa^\star_{\omega_k}$ be the solution to Eq. (A2); then, $\hat{A}^\star$ defined by Eq. (63) is a POVM. Since, as already described in Sec. V D, $\hat{\Gamma}^\star(\omega)|\pi^\star_\omega\rangle = 0$ holds for any $\omega \in \Omega^\star$, Eq. (11) with $\hat{A} = \hat{A}^\star$ obviously holds. Therefore, from Lemma 1, the sequential measurement $\hat{\Pi}^{(\hat{A}^\star)}$ is globally optimal. ■

Note that one can obtain an analytical expression of $\hat{A}^\star$ by substituting the solution $\kappa^\star_{\omega_k}$ to Eq. (A2) into Eq. (62).

APPENDIX B: SUPPLEMENT OF THE S PLANE

Under the assumption that Theorem 2 holds, we shall show that $s(\hat{1}_{\rm A}) \in \mathcal{T}^\star$ is a necessary and sufficient condition that a sequential measurement can be globally optimal.

First, we show the necessity. Assume that a sequential measurement can be globally optimal. From Theorem 2, there exists $\hat{A}^\star \in \mathcal{M}_{\rm A}$ satisfying Eq. (24) such that $\hat{\Pi}^{(\hat{A}^\star)}$ is globally optimal. As described in Sec. V D, $\hat{A}^\star$ is expressed by Eq. (63). Thus, let $\kappa'_3 := \kappa^\star_{\omega_3}$ and $\kappa'_k := 3\kappa^\star_{\omega_k}$ for $k \in \{1, 2\}$; then, since $\hat{A}^\star$ is a POVM, we have

$$\sum_{k=1}^{3} \kappa'_k \hat{E}^\star_k = \int_\Omega \hat{A}^\star(d\omega) = \hat{1}_{\rm A}. \quad \text{(B1)}$$

Premultiplying and postmultiplying this equation by $\rangle\phi_n|$ and $|\phi_n\rangle$, respectively, gives

$$\sum_{k=1}^{3} \frac{\kappa'_k}{3} e^\star_k = s(\hat{1}_{\rm A}). \quad \text{(B2)}$$

This indicates that $s(\hat{1}_{\rm A})$ is the weighted sum of $e^\star_k$ with the weights $\kappa'_k/3 \geqslant 0$, and thus $s(\hat{1}_{\rm A}) \in \mathcal{T}^\star$ holds.

Next, we show the sufficiency. The above argument can be applied in the reverse direction. Assume $s(\hat{1}_{\rm A}) \in \mathcal{T}^\star$; then, there exists $\kappa'_k \geqslant 0$ satisfying Eq. (B2). Consider $\hat{A}^\star$ expressed by Eq. (63) with $\kappa^\star_{\omega_3} = \kappa'_3$ and $\kappa^\star_{\omega_k} = \kappa'_k/3$ $(k = \{1, 2\})$. It follows that $\hat{A}^\star$ is a POVM satisfying Eq. (24) and $\hat{\Gamma}^\star(\omega)\hat{A}^\star(\omega) = 0$. Thus, from Lemma 1, $\hat{\Pi}^{(\hat{A}^\star)}$ is globally optimal, and thus a sequential measurement can be globally optimal. ■

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