Fluctuation theorems for continuous quantum measurements and absolute irreversibility

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Fluctuation theorems are relations constraining the out-of-equilibrium fluctuations of thermodynamic quantities like the entropy production that were initially introduced for classical or quantum systems in contact with a thermal bath. Here we show, in the absence of thermal bath, the dynamics of continuously measured quantum systems can also be described by a fluctuation theorem, expressed in terms of a recently introduced arrow of time measure. This theorem captures the emergence of irreversible behavior from microscopic reversibility in continuous quantum measurements. From this relation, we demonstrate that measurement-induced wave-function collapse exhibits absolute irreversibility, such that Jarzynski-like equalities are violated, and that this property is inherent to quantum information acquisition. We apply our results to different continuous measurement schemes on a qubit: dispersive measurement, homodyne, and heterodyne detection of qubit's fluorescence.

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I. INTRODUCTION

The emergence of macroscopic irreversibility from microscopic time-reversal invariant physical laws has been a longstanding issue, well described by the formalism of statistical thermodynamics [1,2]. In this framework, the small system under study follows stochastic trajectories in its phase-space, where the randomness models the uncontrolled forces exerted on the system by its thermal environment. Although these trajectories are microscopically reversible, one direction of time is more probable than the other and an arrow of time emerges for the ensemble of trajectories. In this framework, the thermodynamic variables like the work, the heat, and the entropy production during a process appear as random variables, defined for a single realization (i.e., a single trajectory), whose averages comply with the first and second law of thermodynamics. Furthermore, the fluctuations of these quantities are constrained beyond the second law, as captured by the so-called fluctuation theorems (FTs) [3–5]. In particular, the integral FT can be written under the form $\langle e^{-\sigma(\Gamma)}\rangle = 1$, where $\sigma(\Gamma)$ is the entropy production along a single trajectory Γ. We denote $\langle\cdot\rangle$, the ensemble average over the realizations of the studied process (or equivalently, over the possible trajectories). The stochastic entropy production $\sigma(\Gamma)$, fulfilling the FT is equal to the ratio of the probability of the (forward in time) trajectory Γ and the probability of the time-reversed (or backward in time) trajectory corresponding to Γ. During the last decades, these results have been investigated in the quantum regime where the system and the thermal bath can be quantum systems, allowing the proof of quantum extensions of the FTs [6–16]. Experiments have demonstrated the validity of these FTs in both classical and quantum regimes [17–22].

However, it was shown that the form of the FTs must be modified for special processes [23–32], which are such that some theoretically allowed backward trajectories do not have a forward-in-time counterpart. A canonical example is the free expansion of a single particle gas initially contained in the left half of a box by a wall. The wall is removed at time $t = 0$, letting the gas expand and reach thermal equilibrium in the whole box. The reverse process consists in starting with the gas particle equilibrated in the whole box and reinserting the wall in the middle. Half of the time, after putting back the wall, the gas particle will be found in the right half of the box. However, this configuration is forbidden in the initial state of the gas, and then only the realizations for which the particle is found in the left-hand side after reinserting the wall can be associated to a realization of the direct process [23,25,32,33]. Such phenomenon has been named absolute irreversibility and is a general feature of transformation on system initialized in small regions of their configuration space [25,27]. For such processes, the FTs takes the form $\langle e^{-\sigma(\Gamma)}\rangle = 1 - \lambda$, where $\lambda \in [0, 1]$ is the accumulated probability of the backward trajectories with no forward counterparts. Absolutely irreversible processes exhibit a strictly positive average entropy production, bounded below by $-\ln(1 - \lambda) > 0$. Reversibility, i.e., a zero average entropy production, is impossible for such processes, no matter the speed at which one implements the transformation under study. Similar modifications of the FTs were also demonstrated when the transformation undergone by the system is interrupted by a quantum measurement [27,34].

Recently, stochastic thermodynamics was extended to describe quantum systems undergoing generic quantum maps [35–39] and in particular quantum measurements in the absence of a thermal reservoir [40–46]. Indeed, the latter situation leads to quantum trajectories of the measured system

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that are analogous to the stochastic trajectories in phase space of classical stochastic thermodynamics. The equivalent of the first law and the second law have been derived for generic form of measurements [42], leading to applications such as an engine fueled by the quantum measurement process [47–50].

Here, we focus on the case of a qubit undergoing weak continuous quantum measurements. Such weak measurements do not completely project the qubit's wavefunction on an eigenstate of the measured observable and therefore generate coherent diffusive trajectories of the state of the measured system. They have been studied intensively [51–57] and provide a wide range of applications exploiting their low invasiveness with respect to strong (projective) measurements [58–65], which justifies to extend quantum stochastic thermodynamics to describe them. In Refs. [44,45], a new quantity was introduced to quantify the arrow of time in continuous monitoring. More specifically, this arrow of time measure compares the probability of the observed quantum trajectory as generated in forward or backward-in-time. The underlying idea is that a weak quantum measurement can always be reversed physically, i.e., the measured system can always be brought to the state it was right before the measurement, by another weak measurement [44,45,58,65]. Such backward measurement occurs with a given probability, that can be larger or smaller than the probability of the forward measurement. A correspondence rule to identify the backward measurement corresponding to a given forward measurement was demonstrated in Ref. [45] using the time-reversal symmetry of the dynamical equation, and used to study the average length of the arrow of time for different measurement processes. The equivalent of the second law in this context is that the average arrow of time is indeed positive. Finally, experimental measurement of this arrow of time was recently reported in Ref. [66].

Here we extend the analysis showing that the probability distribution of the arrow of time is constrained by a FT analogous to those previously derived for the entropy produced in contact with a heat bath. We demonstrate that continuous quantum measurements lead to absolutely irreversible dynamics, and that this property is deeply connected to the acquisition of quantum information during collapse of the wavefunction: Just as for the free expansion of a gas, the dynamics of a measured quantum system generates backward trajectories without forward counterparts as soon as the measurement brings new information about the system. We emphasize that the time-reversal rule used here exactly reverses the quantum state dynamics along single trajectories, in contrast with previous proposals, and that the arrow of time is computed without any projective end point measurement, leading to a FT with absolute irreversibility different from its previous appearances [23,26,27], and from other quantum generalizations of FTs in general [6–10,12–15,21,22,40,67]. We apply our results for different measurement schemes on a qubit, highlighting how the arrow of time varies in these different contexts, and investigating the influence of measurement strength.

This article is organized as follows: We first introduce the arrow of time measure for a simple two-outcome weak measurement of the qubit population and then for a general weak measurement performed on a qubit. We then express the FT and its proof. Finally, we apply our formal results to several physical systems.

II. ARROW OF TIME MEASURE

We consider a qubit of Hamiltonian $H_0 = (\hbar\omega_0/2)\sigma_z = (\hbar\omega_0/2)(|e\rangle\langle e| - |g\rangle\langle g|)$, initially in a pure state $|x_0\rangle$ and then weakly measured. To introduce our arrow of time measure, we first consider that the measurement is a weak discrete measurement of the qubit population characterized by the two following Kraus operators $M_k(r)$, associated with outcomes $r \in \{1, -1\}$:

$$M_k(1) = \begin{pmatrix} \sqrt{1-k} & 0 \\ 0 & \sqrt{k} \end{pmatrix},$$
$$M_k(-1) = \begin{pmatrix} \sqrt{k} & 0 \\ 0 & \sqrt{1-k} \end{pmatrix}. \quad (1)$$

This POVM models, for example, a weak polarization measurement using a single photon meter [68]. The parameter $k \in [0, 1/2]$ quantifies the measurement strength ($k = 0$ corresponds to a strong measurement, $k = 1/2$ corresponds to a noninformative measurement). After the measurement, the qubit is in state $|x_1(1)\rangle \propto M_k(1)|x_0\rangle$ (respectively, $|x_1(-1)\rangle \propto M_k(-1)|x_0\rangle$) when outcomes $r = \pm 1$ are obtained. As $M_k(r)M_k(-r)$ is proportional to identity, the forward trajectory $\Gamma_{|x_0,r} \equiv \{x_0, x_1(r)\}$ is reversed (i.e., the qubit follows the backward trajectory $\tilde{\Gamma}_{|x_1(r),r} \equiv \{x_1(r), x_0\}$) when Kraus operator $M_k(-r)$ is applied on $|x_1(r)\rangle$. This reversal of the measurement is stochastic; it requires the result $-r$ is realized, which occurs with probability $P_B[r|x_1(r)] = \|\tilde{M}_k(r)|x_1(r)\rangle\|^2$, where we have denoted $\tilde{M}_k(r) = M_k(-r)$ the backward Kraus operator associated with $M_k(r)$. The state $|x_1(r)\rangle$ is properly re-normalized such that $P_B[r|x_1(r)]$ is a legitimate conditional probability, conditioned on the normalized final state $|x_1(r)\rangle$ and the readout r. A quantitative measure of the arrow of time can then be obtained by comparing the probabilities $P_F[\Gamma_{|x_0,r}] = P_F[r|x_0]$ and $P_B[\tilde{\Gamma}_{|x_1(r),r}] = P_B[r|x_1(r)]$. We define the quantity $\mathcal{Q}_k(\Gamma_{|x_0,r}) = \ln\{P_F[\Gamma_{|x_0,r}]/P_B[\tilde{\Gamma}_{|x_1(r),r}]\}$, here given by $\mathcal{Q}_k(\Gamma_{|x_0,r}) = \ln\{[(r + z_0 - 2kz_0)^2]/[4k(1-k)]\}$, with $z_0 = \langle x_0|\sigma_z|x_0\rangle$. The sign of $\mathcal{Q}_k(\Gamma_{|x_0,r})$ indicates which time-direction of the trajectory—forward or backward—is the most probable [44,45]. Note that $\mathcal{Q}_k(\Gamma_{|x_0,r})$ diverges in the limit $k \to 0$, which is consistent with the fact that an ideal strong measurement has a zero probability to be reversed this way. Interestingly, the average over the measurement outcomes $\langle \mathcal{Q}_k(\Gamma_{|x_0,r})\rangle_r$ is nonnegative for any value of k [see Appendix A], demonstrating that a clear arrow of time emerges in the measurement process despite microscopic reversibility. The initial condition $z_0 = \mp 1$ corresponds to a fixed point of the measurement, leading to deterministic quantum state dynamics independent from the records. Yet, when $k \in [0, 1/2)$, one finds a nonvanishing arrow of time reflecting the probabilistic nature of the weak measurement readout r.

We now want to study weak measurements with continuous outcomes, performed during some finite time $T = Ndt$ on the qubit. The evolution of the qubit follows a quantum trajectory defined by the set of Kraus operators $\{M(r_n)\}_{0 \leqslant n \leqslant N-1}$ associated with elementary outcomes r_n obtained at times $t_n = ndt$. We introduce $\mathbf{r} = \{r_0, ..., r_{N-1}\}$ the measurement record obtained in a single realization of the process which together with the initial state x_0 uniquely defines a quantum trajectory,

$$\Gamma_{|x_0,\mathbf{r}} \equiv \{x_0, x_1(r_0|x_0), x_2(r_1|x_1)...x_N(\mathbf{r})\}, \tag{2}$$

followed by the qubit. We denote $x_N(\mathbf{r}) = x_N[r_{(N-1)}|x_{(N-1)}]$ for brevity. The probability density of the records reads $P_F(\Gamma_{|x_0,\mathbf{r}}) \equiv P_F(\mathbf{r}|x_0) = \|\overleftarrow{\prod}_n M(r_n)|x_0\rangle\|^2$ where the arrow indicates that the operators are ordered from right to left [45,57].

It has been demonstrated in Ref. [45] that the trajectory $\Gamma_{|x_0,\mathbf{r}}$ followed by the qubit when record $\mathbf{r}$ is obtained can be reversed by applying the Kraus operators given by

$$\tilde{M}(r_n) = \theta^{-1} M^{\dagger}(r_n)\theta \tag{3}$$

on the final state $|x_N(\mathbf{r})\rangle$, in reversed order [i.e., starting with $\tilde{M}(r_{N-1})$]. Here θ is the time-reversal operator, which in the case of rank-2 Kraus operators ensures $\tilde{M}(r_n)M(r_n) \propto \mathbb{1}$ [45]. Applying $\tilde{M}(r_n)$ sequentially generates the backward trajectory $\tilde{\Gamma}_{|x_N(\mathbf{r}),\tilde{\mathbf{r}}} \equiv \{x_N(\mathbf{r})\dots x_0\}$, bringing the qubit through the exact same sequence of states, in reversed order, back to $|x_0\rangle$.

We stress that the correspondence rule given in Eq. (3) differs from previous approaches, e.g., defining the backward measurement operator as the adjoint of the forward measurement operator [35–37] and which do not necessary bring back the system through the same sequence of quantum states as $M^{\dagger}(r_n)M(r_n)$ is not always proportional to identity. However, a similar correspondence rule has appeared previously in Ref. [69] and was applied to the transformation of a quantum system in contact with a thermal bath, interrupted by discrete measurements. The particular choice of inverse given in Eq. (3) maps a qubit unitary operator to its inverse unitary operator, returns a complete POVM for unital maps, and preserves the set of dynamical equations for the measurement as discussed in Ref. [45]. Reversing exactly the dynamics is a tighter constraint, and as a consequence the present approach is valid solely when the Kraus operators are invertible (i.e., rank-2 when the system is a single qubit). Interestingly, this method leads to an arrow of time measure particularly well-suited for continuous measurement and zero temperature, two limits in which previous ways of quantifying irreversibility lead to divergences [42,46].

The probability to obtain the sequence of measurement exactly reversing trajectory $\Gamma_{|x_0,\mathbf{r}}$ is finite and equal to $P_B[\tilde{\Gamma}_{|x_N(\mathbf{r}),\tilde{\mathbf{r}}}] \equiv P_B(\tilde{\mathbf{r}}|x_N) = \|\overleftarrow{\prod}_n \tilde{M}(\tilde{r}_n)|x_N\rangle\|^2$, where $\tilde{\mathbf{r}} = \{r_{N-n}\}_{1\leqslant n\leqslant N}$ is the backward record. One can then define for any trajectory $\Gamma_{|x_0,\mathbf{r}}$ the arrow of time measure,

$$\mathcal{Q}(\Gamma_{|x_0,\mathbf{r}}) = \ln\{P_F[\Gamma_{|x_0,\mathbf{r}}]/P_B^{\mathrm{AC}}[\tilde{\Gamma}_{|x_N(\mathbf{r}),\tilde{\mathbf{r}}}]\}. \tag{4}$$

Here, the superscript AC indicates that we consider the *absolutely continuous part of P_B with respect to P_F*, in the sense of Lebesgue's decomposition of probability distributions [70]. In less technical words, $P_B^{\mathrm{AC}}[\tilde{\Gamma}_{|x_N(\mathbf{r}),\tilde{\mathbf{r}}}]$ is equal to $P_B[\tilde{\Gamma}_{|x_N(\mathbf{r}),\tilde{\mathbf{r}}}]$, except when $P_F[\Gamma_{|x_0,\mathbf{r}}]$ vanishes (when a given backward trajectory does not have a forward counterpart), where it is equal to 0. We discuss this prescription later in the letter.

As an example, we review the continuous weak measurement of observable σ_z, which can be implemented exploiting a dispersive coupling between the qubit and a cavity (see Fig. 1). The evolution of the qubit's state between t_n and t_{n+1}

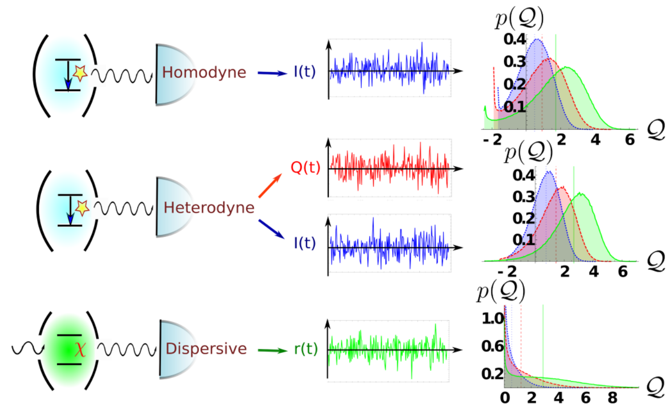

FIG. 1. Three different continuous measurement schemes compared in the manuscript. Top: Homodyne detection of qubit fluorescence [single readout, $I(t)$]. Middle: Heterodyne detection of qubit fluorescence [two readouts, $Q(t)$, $I(t)$]. Bottom: dispersive spin measurement, having a single readout $r(t)$. In each case, we plot an example of measurement record (the amplitude is in arbitrary units), and the probability distribution of the arrow of time measure $\mathcal{Q}$ for different measurement durations $T = 0.5\tau$ (blue, dotted), $T = \tau$ (red, dashed), $T = 2\tau$ (green, joined). The qubit is initialized in the eigenstate of σ_x with eigenvalue 1. We have set $\gamma^{-1} = \tau$. The remarkable shape of $P(\mathcal{Q})$ for the Homodyne (top) and dispersive (bottom) schemes is analytically explained in the appendices.

without Rabi drive is obtained by applying the Kraus operator $M_z(r_n) = (dt/2\pi\tau)^{1/4} e^{-(dt/4\tau)(r_n-\sigma_z)^2}$, with τ the characteristic measurement time, and $r_n \in \mathbb{R}$. After $T = Ndt$, the qubit's state is $|x_N(\mathbf{r}, x_0)\rangle \propto e^{-(dt/4\tau)\sum_n (r_n-\sigma_z)^2}|x_0\rangle$. The Kraus operators generating the backward dynamics are given by $\tilde{M}_z(r_n) = M_z(-r_n)$. We obtain the arrow of time in this case, $\mathcal{Q}_z$ [44],

$$\mathcal{Q}_z(\Gamma_{|z_0,\mathbf{r}}) = 2\ln[\cosh(R) + z_0 \sinh(R)], \tag{5}$$

where $R = dt\sum_n r_n/\tau$. When $z_0 = 0$ (i.e., when $|x_0\rangle$ lays on the equator of the Bloch sphere), one finds that $\mathcal{Q}_z(\Gamma_{|z_0,\mathbf{r}}) > 0$ for any $\mathbf{r}$, leading to a strictly positive average [44]. This special case is analogous to the example of free expansion of a single particle gas where the entropy production is always positive, subsequently violating the Jarzynski equality [25]. We revisit this example in Appendix D, and analytically verify the FT presented in this paper.

In the remainder of this paper, we show that our arrow of time measure satisfies a FT similar to the Integral Fluctuation Theorem for the entropy production, extensively studied in the case of a quantum system in contact with a thermal reservoir [8,11,38,42,71,72]. We will apply our general results to four different measurement schemes: the two examples already presented, and the detection of the fluorescence of the qubit via a Heterodyne setup (i.e., after a phase-preserving amplification of the field yielding information on both its quadratures I_n and Q_n, stored in the record $r_n = I_n - iQ_n \in \mathbb{C}$) and a Homodyne setup (after a phase-sensitive amplification of the field gathering information about one quadrature stored in $r_n \in \mathbb{R}$) [73]. The Kraus operators encoding the effect of such measurements during a small time step dt read:

$$M_{\rm He}(r_n) = \frac{e^{-|r_n|^2/2}}{\sqrt{\pi}}\begin{pmatrix}\sqrt{1-\epsilon} & 0\\ \sqrt{\epsilon}\, r_n^* & 1\end{pmatrix},$$
$$M_{\rm Ho}(r_n) = \frac{e^{-r_n^2/2}}{\pi^{1/4}}\begin{pmatrix}\sqrt{1-\epsilon/2} & 0\\ \sqrt{\epsilon}\, r_n & 1\end{pmatrix}, \qquad (6)$$

where $\epsilon = \gamma dt$, with γ the spontaneous emission rate of the qubit. The backward evolution operators and the arrow of time measure $\mathcal{Q}$ can be computed following the same protocol described in Eqs. (3) and (4). Their probability distributions are plotted in Fig. 1 for the three different continuous detection schemes, highlighting their strictly positive average value. Interestingly, the average value of the arrow of time measure depends on the measurement scheme, although the measurement rates are chosen to be identical $\gamma = 1/\tau$. The present approach therefore brings new tools to compare the irreversibility of different measurement channels. We also study the case of a qubit simultaneously driven and continuously monitored in the Appendix.

III. FLUCTUATION THEOREM

To obtain our FT, we compute the average value of $e^{-\mathcal{Q}(\Gamma)} = P_B^{\rm AC}(\tilde{\Gamma})/P_F(\Gamma)$ over the forward trajectories Γ, i.e., $\langle e^{-\mathcal{Q}(\Gamma)}\rangle = \int D\Gamma P_F(\Gamma)e^{-\mathcal{Q}(\Gamma)}$. Since we need to integrate over all possible realizations, the constraint that the measurement readout $\mathbf{r}$ and the quantum state dynamics $\mathbf{x}$ at each step correspond via the Bayesian update rule for each individual realizations is imposed by defining $\int D\Gamma$ appropriately as $\int D\Gamma = \int D\mathbf{x} \int D\mathbf{r}\delta[\mathbf{x} - \mathbf{x}(\mathbf{r})]$ (see Appendix B). We find the central result of this paper:

$$\langle e^{-\mathcal{Q}(\Gamma)}\rangle = 1 - \mu, \qquad (7)$$

where μ is a parameter equal to (see Appendix B)

$$\mu = 1 - \int D\Gamma P_B^{\rm AC}(\Gamma) = \int D\mathbf{r}\frac{|\langle\bar{x}_0|\mathcal{M}^\dagger(\mathbf{r})\mathcal{M}(\mathbf{r})|x_0\rangle|^2}{\langle x_0|\mathcal{M}^\dagger(\mathbf{r})\mathcal{M}(\mathbf{r})|x_0\rangle}, \qquad (8)$$

where $\mathcal{M}(\mathbf{r}) = \overleftarrow{\prod_n} M(r_n)$ is the global Kraus operator of the sequence of measurements and $|\bar{x}_0\rangle$ is the normalized state orthogonal to $|x_0\rangle$. From Eq. (8) it is clear that $\mu \geqslant 0$. The equality $\mu = 0$ can be reached solely if $|x_0\rangle$ is an eigenstate of the global effect operator $\mathcal{E}(\mathbf{r}) = \mathcal{M}(\mathbf{r})^\dagger\mathcal{M}(\mathbf{r})$ for any $\mathbf{r}$. Applying the Cauchy-Schwartz inequality for vectors $|\psi\rangle = \mathcal{M}(\mathbf{r})|x_0\rangle$ and $|\phi\rangle = \mathcal{M}(\mathbf{r})|\bar{x}_0\rangle$ yields $|\langle\phi|\psi\rangle|^2/\langle\psi|\psi\rangle \leqslant \langle\phi|\phi\rangle$, which demonstrates that $\int D\mathbf{r}\langle\bar{x}_0|\mathcal{M}^\dagger(\mathbf{r})\mathcal{M}(\mathbf{r})|\bar{x}_0\rangle = 1$ is an upper bound for μ.

Equation (7) constrains the fluctuations and average of the arrow of time measure. In particular, it readily imposes via Jensen's inequality a lower bound on the average arrow of time:

$$\langle\mathcal{Q}(\Gamma)\rangle \geqslant -\ln(1-\mu). \qquad (9)$$

IV. ABSOLUTE IRREVERSIBILITY

The right-hand side of the FT in Eq. (7) is strictly lower than 1 when the initial state is not an eigenstate of the effect matrix $\mathcal{E}(\mathbf{r})$, leading to a strictly positive value of $\langle\mathcal{Q}(\Gamma)\rangle$. In the literature related to FTs, this feature has been referred to as absolute irreversibility [25,27], and it reveals existence of time-reversed trajectories that are accounted for by probability law P_B, but which do not bring the system back to its initial state $|x_0\rangle$. For such trajectories, the forward probability is zero so that the ratio $P_B(\tilde{\mathbf{r}}|x_N)/P_F(\mathbf{r}|x_0)$ and the arrow of time diverges [27]. Taking the absolutely continuous part $P_B^{\rm AC}$ of P_B in the definition of $\mathcal{Q}(\Gamma)$ is required to restrict the average in Eq. (7) to allowed forward trajectories.

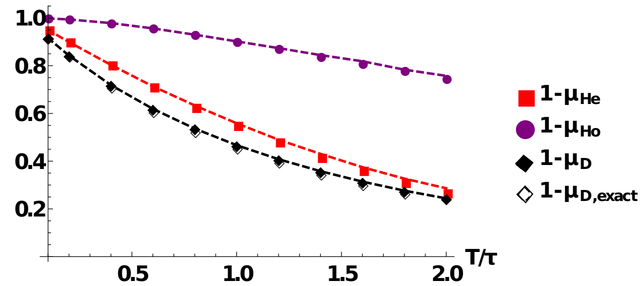

FIG. 2. Absolute irreversibility of the three studied continuous detection schemes: Left-hand side of the FT $\langle e^{-\mathcal{Q}(\Gamma)}\rangle$ (dashed) and parameter μ (dotted) computed from Eq. (8), as a function of the duration of the measurement T/τ, starting from $T/\tau = 0.1$. The qubit is initialized in the eigenstate of σ_x with eigenvalue 1. We have simulated 1×10^6 trajectories, setting $\tau^{-1} = \gamma$. The analytically obtained value $1-\mu_{\rm D,exact}$ for the dispersive measurement with no Rabi drive is also marked in the figure.

Physically, absolute irreversibility evaluated by Eq. (8) quantifies how much two initially orthogonal quantum states $|x_0\rangle$ and $|\bar{x}_0\rangle$ become indistinguishable when they evolve subject to the same measurement record $\mathbf{r}$. It reflects the fact that as information about the system is acquired, the measurement brings any state towards the same fixed point indicated by the measurement readout. As a consequence, μ increases with the measurement duration (see Fig. 2) and reaches unity when all the possible information is obtained (when the measurement becomes projective). Absolute irreversibility only disappears when different realizations have no effect on the qubit's state; i.e., when the qubit is already in an eigenstate of $\mathcal{E}(\mathbf{r})$ for any $\mathbf{r}$. Note that one may still obtain a nonzero $\langle\mathcal{Q}\rangle$ even though μ is zero, for example, when applying the measurement operators in Eq. (1) to an eigenstate of σ_z. Here the readout fluctuates and keep bringing (redundant) information about the state of the qubit. A perfectly reversible situation ($\langle\mathcal{Q}\rangle = 0$) requires in addition that the measurement gathers no information at all, i.e., when the coupling of the qubit with the measuring device approaches zero. This illustrates that irreversibility ($\langle\mathcal{Q}\rangle > 0$) and absolute irreversibility ($\mu \neq 0$) are two different properties defined for a set of forward trajectories which help characterize information acquisition during quantum measurements. We finally emphasize that one can generalize Eq. (7) to the case where the initial state of the system is drawn from an ensemble $\{|x_0\rangle\}$ with probability $p(x_0)$. This situation still leads to absolute irreversibility in general (see Appendix C).

V. ANALYSIS OF THE EXAMPLES

We first apply our results to the weak measurement characterized by $M_k(\pm 1)$ defined in Eq. (1). Here the parameter μ

can be computed analytically:

$$\mu_k = \left[(1-2k)^2\left(1-z_0^2\right)\right]/\left[1-(1-2k)^2 z_0^2\right], \quad (10)$$

which for $k \in [0, 1/2]$ indeed belongs to [0,1]. We retrieve in this example that $\mu_k = 0$ for $z_0 = \pm 1$ and μ_k is strictly positive otherwise. The limit $k \to 0$ (strong measurement) corresponds to $\mu_k \to 1^-$, such that the bound $-\ln(1-\mu_k)$ goes to $+\infty$, capturing that the arrow of time measure diverges for a strong measurement. Conversely, for $k \to 1/2$, μ_k goes to 0 for any value of z_0: the measurement in this limit does not gather any information and has no effect of the qubit, such that the process becomes absolutely reversible, and $\langle \mathcal{Q}_k \rangle \to 0$. Interestingly, for a fixed $z_0 \in [-1, 1]$, the parameter k allows to go from a perfectly strong measurement to a weak measurement, and even to no measurement at all. This transition is accompanied by $\langle \mathcal{Q}_k \rangle$ going from $+\infty$ to 0, and absolute irreversibility is present but its amount, quantified by μ_k decreases and finally reaches 0 when the measurement has no back-action anymore on the qubit's state.

For the dispersive σ_z, homodyne and heterodyne measurements on a qubit for a finite duration $T = Ndt$, we verify the FT by simulating qubit trajectories to compute $\langle e^{-\mathcal{Q}(\Gamma)} \rangle$ and numerically integrate Eq. (8). One can see on Fig. 2 the agreement between both sides of Eq. (7), which numerically validates our FT, and proves the presence of absolute irreversibility as μ is found greater than zero. We also compare our results to the analytic expression of μ for the dispersive measurement with no Rabi drive, discussed in the Appendix. Just as parameter k in the two-outcome measurement example, the measurement time allows to switch between an extremely weak measurement (for $T \ll \tau$) such that $\mu \ll 1$ and $\langle \mathcal{Q}(\Gamma) \rangle \geqslant 0$ to a strong measurement (for $T \gg \tau$) such that μ goes to 1 and the lower bound for the average arrow of time diverges. The agreement to our FT for single step measurements, and for continuously monitoring a qubit undergoing Rabi oscillations are also presented in the Appendix.

VI. CONCLUSION

We have proved a fluctuation theorem for the arrow of time in continuous quantum measurements, analogous to FTs for transformation of a quantum system in contact with a thermal reservoir. We have shown that this FT exhibits absolute irreversibility, leading to a strictly positive average arrow of time. This property is inherent to the wavefunction collapse induced by a continuous quantum measurement, and relates the irreversibility to information gained during the measurement process.

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APPENDIX A: AVERAGE VALUE OF $\mathcal{Q}_k$ FOR THE TWO OUTCOME SPIN MEASUREMENT

For the single step, two outcome spin measurement described by measurement operators,

$$M_k(1) = \begin{pmatrix} \sqrt{1-k} & 0 \\ 0 & \sqrt{k} \end{pmatrix},$$
$$M_k(-1) = \begin{pmatrix} \sqrt{k} & 0 \\ 0 & \sqrt{1-k} \end{pmatrix}. \quad (A1)$$

we compute the average value of $\mathcal{Q}_k(\Gamma)$ as $\langle \mathcal{Q}_k(\Gamma) \rangle = P_F(\Gamma_1)\mathcal{Q}_k(\Gamma_1) + P_F(\Gamma_{-1})\mathcal{Q}_k(\Gamma_{-1})$, where $\mathcal{Q}_k(\Gamma_r)$ is computed using the formula $\mathcal{Q}_k(\Gamma_r) = \ln\{[(r + z_0 - 2kz_0)^2]/[4k(1-k)]\}$, for $r \in \{-1, 1\}$. In Fig. 3, we plot the average $\langle \mathcal{Q}_k \rangle$ for the case $z_0 = 0$, that demonstrate the essential features discussed in the main text, its nonnegativity, and positive divergence as $k \to 0$.

APPENDIX B: DERIVATION OF THE FLUCTUATION THEOREM

Here we derive the identity $\langle e^{-\mathcal{Q}(\Gamma)} \rangle = 1 - \mu$, by considering discrete state update using Kraus operators and then taking the continuum limit. We first note that the probability distribution function of the forward state update for a sequence of N measurements—that imposes the constraint that a given pair $\Gamma = (\mathbf{x}, \mathbf{r})$ has a nonvanishing probability if and only if the sequence of states $\mathbf{x} = \{x_k\}_{k=0}^{N}$ and the measurement readouts $\mathbf{r} = \{r_k\}_{k=0}^{N-1}$ correspond via the Bayesian update rule: $\mathbf{x}(\mathbf{r}) = \{x_0,\ x_1(r_0|x_0),\ x_2(r_1|x_1) \ldots x_N(r_{(N-1)}|x_{(N-1)})\}$—can be written as follows [51,57]:

$$\mathcal{P}_F(\Gamma) = \delta(x_0 - x_{\rm in}) \prod_{k=0}^{N-1} P_F(x_{k+1}|x_k, r_k) P_F(r_k|x_k). \quad (B1)$$

Here the term $P_F(x_{k+1}|x_k, r_k)$ represents a deterministic state update given the dynamics, imposed as a three-dimensional δ function for each component of spin along the Bloch sphere coordinates,

$$P_F(x_{k+1}|x_k, r_k) = \prod_{i=1}^{3} \delta\left[x_{k+1}^i - {\rm Tr}\left(\hat{\sigma}^i \frac{U_k M(r_k)\rho_k M(r_k)^\dagger U_k^\dagger}{{\rm Tr}[M(r_k)\rho_k M(r_k)^\dagger]}\right)\right]$$
$$= \delta[x_{k+1} - (x_{k+1}|x_k, r_k)], \quad (B2)$$

and the probability of obtaining a readout r_k given x_k is given by the expression

$$P_F(r_k|x_k) = {\rm Tr}[M(r_k)\rho_k M(r_k)^\dagger]. \quad (B3)$$

Note that imposing a delta function boundary condition at each step as in Eq. (B1) ensures that the trajectories where r_k and x_k do not correspond to each other have probability zero. These trajectories—completely determined by the initial state x_0 and the measurement readout $\mathbf{r}$—are labeled by the notation $\Gamma_{|x_0,\mathbf{r}}$ in the main text, referring to individual realizations of the measurement process.

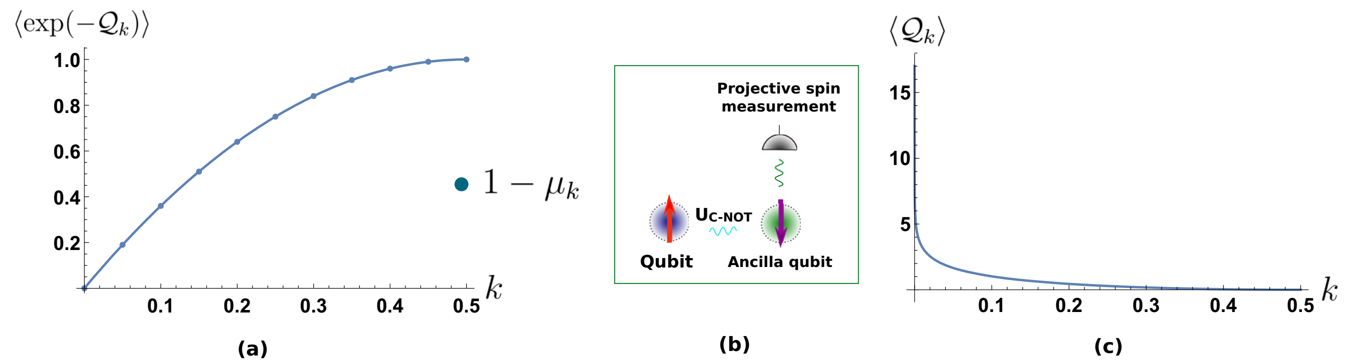

FIG. 3. Here we consider a single-step weak discrete measurement of qubit population, when the qubit initialized at $x = 1$. In Fig. 3(a), we show that the identity $\langle \exp(-\mathcal{Q}_k)\rangle$ [solid line] $= 1 - \mu_k$ [dotted] is satisfied for different values of the measurement strength $k \in [0, \frac{1}{2}]$. A possible experimental implementation of this measurement scheme is shown in Fig. 3(b), where the quantum system (qubit) and the measuring device (ancilla qubit) evolve via the controlled-NOT unitary. The measurement is completed by projecting the ancilla qubit onto the spin basis. (c) Here we plot the average value $\langle \mathcal{Q}_k\rangle$ for $k \in [0, 1/2]$ for a qubit initialized at $z = 0$, considering the two outcome z measurement discussed in the main text.

For any given final state x_f obtained at the end of the forward measurement, the backward probability distribution can be written similarly,

$$\mathcal{P}_B(\tilde{\Gamma}) = \delta(x_N - x_f)\prod_{k=N}^{1} P_B(x_{k-1}|x_k, r_{k-1})P_B(r_{k-1}|x_k), \quad \text{(B4)}$$

where we have

$$\begin{aligned} &P_B(x_{k-1}|x_k, r_{k-1}) \\ &= \prod_{i=1}^{3}\delta\Bigg[x^i_{k-1} \\ &\quad -\mathrm{Tr}\left(\hat{\sigma}^i\frac{\tilde{M}(r_{k-1})U^\dagger_{k-1}\rho_k U_{k-1}\tilde{M}(r_{k-1})^\dagger}{\mathrm{Tr}[\tilde{M}(r_{k-1})U^\dagger_{k-1}\rho_k U_{k-1}\tilde{M}(r_{k-1})^\dagger]}\right)\Bigg] \\ &= \delta[x_{k-1} - (x_{k-1}|x_k, r_{k-1})]. \end{aligned} \quad \text{(B5)}$$

The update operator $\tilde{M}(r_k) = \theta^{-1}M(r_k)^\dagger\theta$, where θ is the time reversal operator, and the backward probabilities,

$$P_B(r_{k-1}|x_k) = \mathrm{Tr}[\tilde{M}(r_{k-1})U^\dagger_{k-1}\rho_k U_{k-1}\tilde{M}(r_{k-1})^\dagger]. \quad \text{(B6)}$$

We now proceed to compute the quantity $\langle e^{-\mathcal{Q}(\Gamma)}\rangle$ as a statistical average over all possible forward trajectories in the ensemble being considered. The integration measure over all the possible trajectories Γ with nonvanishing forward probabilities can also be expressed in terms of the readouts $\mathbf{r}$ and the corresponding Bloch sphere coordinates $\mathbf{x}$ as

$$\int D\Gamma = \int D\mathbf{x}\int D\mathbf{r}\,\delta[\mathbf{x} - \mathbf{x}(\mathbf{r})], \quad \text{(B7)}$$

where we assume $\int D\mathbf{x} \equiv \int\prod_{k=1}^{N} Dx_k$. Note that the Bloch sphere coordinates x_k take continuum of values in the interval $[-1, 1]$, and the readout(s) $\mathbf{r}$ for the homodyne/heterodyne measurements are also continuous variables. The δ function imposes the constraints of the initial state and the Bayesian state update,

$$\delta[\mathbf{x} - \mathbf{x}(\mathbf{r})] = \delta(x_0 - x_{\mathrm{in}})\prod_{k=0}^{N-1}\delta[x_{k+1} - (x_{k+1}|x_k, r_k)]. \quad \text{(B8)}$$

The quantity $\langle e^{-\mathcal{Q}(\Gamma)}\rangle$ pertinent to our time-reversal scheme is defined as the following integral over paths:

$$\langle e^{-\mathcal{Q}(\Gamma)}\rangle = \int D\Gamma\, P_F[\Gamma]\,\frac{P_B^{\mathrm{AC}}[\Gamma]}{P_F[\Gamma]}. \quad \text{(B9)}$$

Here for a given trajectory Γ, we have defined $P_F[\Gamma] = \prod_{k=0}^{N-1} P_F(r_k|x_k)$. We have also defined $\mathcal{Q} = \ln\frac{P_F[\Gamma]}{P_B^{\mathrm{AC}}[\Gamma]}$, where $P_B^{\mathrm{AC}}[\Gamma]$ correspond to the probability of obtaining a backward trajectory which has a corresponding forward trajectory (having forward probability $P_F[\Gamma]$) in the ensemble of all forward trajectories (denoted by the superscript AC, implying absolute continuous part of the backward distribution, relative to the forward distribution, used in the context of Lebesgue's decomposition theorem [70]). This probability of obtaining a readout backward, given the intital state state x_0 and measurement record $\mathbf{r}$ can be written more concisely in terms of the effect matrix as

$$P_B^{\mathrm{AC}}[\Gamma] = \prod_{k=N}^{1} P_B(r_{k-1}|x_k) = \frac{\mathrm{Det}[\mathcal{E}(\mathbf{r})]}{\mathrm{Tr}[\rho_{x_0}\mathcal{E}(\mathbf{r})]}. \quad \text{(B10)}$$

Using Eq. (B1), we have

$$\begin{aligned} \langle e^{-\mathcal{Q}(\Gamma)}\rangle &= \int D\Gamma\, P_F[\Gamma]\,\frac{P_B^{\mathrm{AC}}[\Gamma]}{P_F[\Gamma]} \\ &= \int D\mathbf{x}\int D\mathbf{r}\,\mathcal{P}_F\,\frac{\prod_{k=N}^{1} P_B(r_{k-1}|x_k)}{\prod_{k=0}^{N-1} P_F(r_k|x_k)} \quad \text{(B11)} \\ &= \int D\mathbf{x}\int D\mathbf{r}\,\delta(x_0 - x_{\mathrm{in}})\prod_{k=0}^{N-1}\delta[x_{k+1} \\ &\quad - (x_{k+1}|x_k, r_k)]\prod_{k=N}^{1} P_B(r_{k-1}|x_k) \\ &= \int D\mathbf{x}\int D\mathbf{r}\,\delta[\mathbf{x} - \mathbf{x}(\mathbf{r})]\,\frac{\mathrm{Det}[\mathcal{E}(\mathbf{r})]}{\mathrm{Tr}[\rho_{x_0}\mathcal{E}(\mathbf{r})]} \\ &= \int D\mathbf{r}\,\frac{\mathrm{Det}[\mathcal{E}(\mathbf{r})]}{\mathrm{Tr}[\rho_{x_0}\mathcal{E}(\mathbf{r})]}. \end{aligned} \quad \text{(B12)}$$

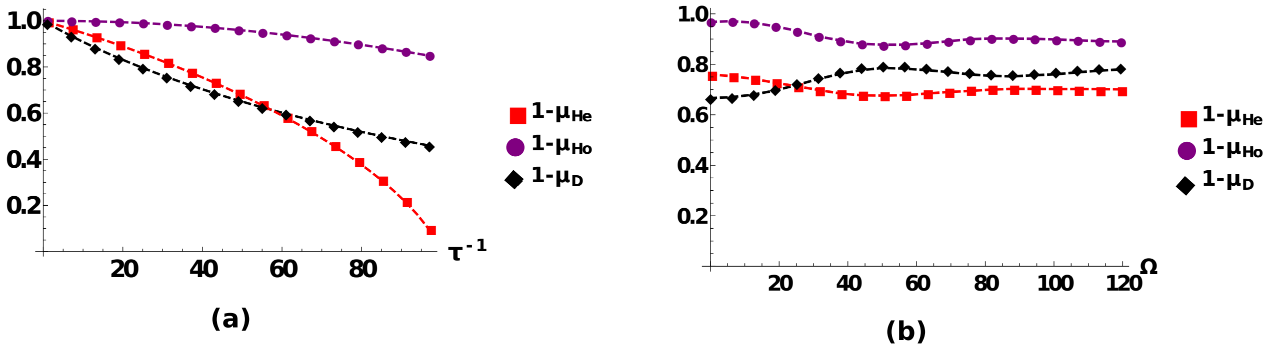

FIG. 4. (a) Absolute irreversibility of the three studied continuous detection schemes for a single step measurement: Left-hand side of the FT $\langle e^{-\mathcal{Q}(\Gamma)}\rangle$ (dashed) and parameter μ (dotted) computed from Eq. (B15), as a function of the duration of the measurement rate $\tau^{-1}=\gamma$. The qubit is initialized in the eigenstate of σ_x with eigenvalue 1. (b) Verifying the FT for the three studied continuous detection schemes for different Rabi drive frequency Ω: Left-hand side of the FT $\langle e^{-\mathcal{Q}(\Gamma)}\rangle$ (dashed) and parameter μ (dotted) computed from Eq. (B15) for $T=0.5\tau$. The qubit is initialized in the eigenstate of σ_x with eigenvalue 1. We have simulated 1×10^6 trajectories, setting $\tau^{-1}=\gamma$.

We performed the integration over $\mathbf{x}$ since the integrant depends only on $\mathbf{r}$ and x_0. We now write the effect matrix $\mathcal{E}(\mathbf{r})$ in the basis of $\{|x_0\rangle,\ |\bar{x}_0\rangle\}$, where $\rho_{x_0}=|x_0\rangle\langle x_0|$, and $\langle x_0|\bar{x}_0\rangle=0$ as

$$\mathcal{E}(\mathbf{r})=\begin{bmatrix} a(\mathbf{r}) & c(\mathbf{r})\\ c^*(\mathbf{r}) & b(\mathbf{r})\end{bmatrix}. \tag{B13}$$

For a given initial state, sum over all probabilities in the forward direction is equal to one implies that the effect matrix $\mathcal{E}(\mathbf{r})$ satisfies the following relation:

$$\int D\mathbf{r}\,\mathcal{E}(\mathbf{r})=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}. \tag{B14}$$

We therefore obtain

$$\begin{aligned}\langle e^{-\mathcal{Q}(\Gamma)}\rangle &= \int D\mathbf{r}\,\frac{\mathrm{Det}[\mathcal{E}(\mathbf{r})]}{\mathrm{Tr}[\rho_{x_0}\mathcal{E}(\mathbf{r})]}=\int D\mathbf{r}\,\frac{a(\mathbf{r})b(\mathbf{r})-|c(\mathbf{r})|^2}{a(\mathbf{r})}\\ &= \int D\mathbf{r}\,b(\mathbf{r})-\int D\mathbf{r}\frac{|c(\mathbf{r})|^2}{a(\mathbf{r})}=1-\mu,\end{aligned} \tag{B15}$$

where we have defined

$$\int D\mathbf{r}\,\frac{|c(\mathbf{r})|^2}{a(\mathbf{r})}\equiv\mu, \tag{B16}$$

leading to Eq. (7) of the main text. We verify this identity in Fig. 4, considering (a) single step measurement described by measurement operator M_X, and (b) continuously monitoring a qubit subject to Rabi drive, where the effective time evolution operator is $\mathcal{U}(r_n,dt)=M_X(r_n)\,e^{-\frac{i}{\hbar}Hdt}$ (for $H=\hbar\Omega\sigma_y/2$), with $X=z$, He, Ho, labeling continuous dispersive σ_z measurement, Heterodyne and Homodyne detection of qubit's fluorescence respectively. Equation (B15) can be analytically verified in certain special cases. An example of such a case is presented in Sec. D, where we look at the dispersive spin measurement with no Rabi drive, and obtain a probability distribution that estimates μ analytically.

APPENDIX C: FT IN THE CASE OF A RANDOM INITIAL QUBIT STATE

We now assume that the initial state of the system is drawn from a set $\{|x_0\rangle\}$ according to a probability law $p(x_0)$. As the consequence, the average over the trajectory involved in the fluctuation theorem Eq. (B15) now corresponds to $\langle\cdot\rangle=\int dx_0 p(x_0)\int D\Gamma_{|x_0}P_F[\Gamma](\cdot)$ instead of $\langle\cdot\rangle_{|x_0}=\int D\Gamma_{|x_0}P_F[\Gamma](\cdot)$ we used earlier, although we had suppressed the conditioning on x_0 for brevity in our earlier discussions [and in Eq. (7) of the main text], by absorbing it to the delta function constraint involved in the integration measure $\int D\Gamma$. However, the definition of the arrow of time measure $\mathcal{Q}(\Gamma_{|x_0,\mathbf{r}})$ associated with a given initial state x_0 and record $\mathbf{r}$ is unchanged. We emphasize that the sum over x_0 runs onto the qubit's Hilbert space, and the distribution $p(x_0)$ is allowed to be either discrete (e.g., when the preparation is due to the projective measurement of an observable) or continuous.

In this situation, the IFT becomes

$$\langle e^{-\mathcal{Q}(\Gamma)}\rangle=1-\langle\mu\rangle_{x_0}, \tag{C1}$$

where $\langle\mu\rangle_{x_0}$ is the average of μ over the distribution of initial state:

$$\begin{aligned}\langle\mu\rangle_{x_0} &= \int dx_0 p(x_0)\mu\\ &= \int dx_0 p(x_0)\int D\mathbf{r}\frac{|\langle\bar{x}_0|\mathcal{M}^\dagger(\mathbf{r})\mathcal{M}(\mathbf{r})|x_0\rangle|^2}{\langle x_0|\mathcal{M}^\dagger(\mathbf{r})\mathcal{M}(\mathbf{r})|x_0\rangle}.\end{aligned} \tag{C2}$$

In general, such average is not a sufficient condition to have $\langle\mu\rangle_{x_0}=0$, even when drawing the state from a set of states preserved by the measurement. A simple example is the case of the two-outcome spin measurement described by Eq. (A1), applied to a state drawn from the circle of the qubit states of zero y coordinate in the Bloch sphere. One gets

$$\langle\mu\rangle_{x_0}=\int_{-1}^{1}dz_0 p(z_0)\frac{(1-2k)^2(1-z_0^2)}{1-(1-2k)^2z_0^2}, \tag{C3}$$

which takes for instance the value $1-4k(1-k)\mathrm{ArcTanh}(1-2k)/(1-2k)\neq0$ for a flat probability distribution $p(z_0)=1/2$ of the initial z coordinate denoted z_0.

This contrast with usual FTs with absolute irreversibility is explained by our choice of (i) defining the arrow of time from the probabilities of the forward (respectively, backward) trajectory, conditioned to the initial (respectively, final) state, rather than from a joint probability $p(x_0)P_F[\Gamma_{|x_0,\mathbf{r}}]$ (respectively, $p(x_N)P_B^{\mathrm{AC}}[\tilde{\Gamma}_{|x_N,\tilde{\mathbf{r}}}]$) of picking the initial (respectively,

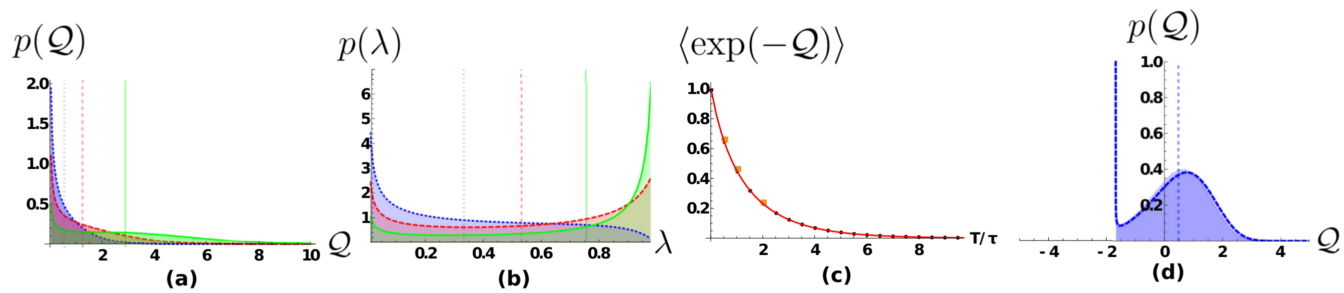

FIG. 5. Here we plot (a) the distribution of $\mathcal{Q}$ [44], indicating their strictly positive average value, and (b) the distribution of λ, indicating their mean value $\langle\lambda\rangle = \mu$, for different durations: $T/\tau = 0.5$ (dotted, blue), $T/\tau = 1$ (dashed, red), and $T/\tau = 2$ (joined, green), and compare with the numerical simulation of 10^6 trajectories in each case. (c) We verify the fluctuation theorem for dispersive qubit measurement with no Rabi drive starting at $z_0 = 0$ for different values of T/τ. Left-hand side of the FT $\langle e^{-\mathcal{Q}}\rangle$ (solid) and parameter $1-\mu$ (dotted) computed from Eq. (B15), using the analytical approach discussed in Sec. D. The data obtained using numerical simulations used in (a) and (b) are indicated using (overlapping) blue circle and orange square markers. (d) Here we compare the analytical solution obtained in Sec. E with the numerical simulation of 10^6 trajectories for $\epsilon' = T/\tau = 0.5$.

final state) and obtaining the record $\mathbf{r}$; and (ii) not performing a final projective measurement on the system. If one adds these two conditions, one finds another fluctuation theorem of the form

$$\langle e^{-\mathcal{Q}(\Gamma)-\Delta s[\Gamma]}\rangle = 1 - \mu'. \tag{C4}$$

Here, $\Delta s[\Gamma] = \ln[p(x_0)/p(x_N)]$ is a boundary contribution that corresponds to the difference of stochastic entropies of the initial and final set of qubit states. In this case, the absolute irreversibility parameter μ' vanishes provided $p(x_0)$ has a support spanning every final states of the reversed trajectories. The price to pay is that the fluctuation theorem does not involve only the arrow of time measure, but also $\Delta s[\Gamma]$. In addition, one can expect that the final projective measurement has a strong impact, possibly overcoming the contribution of the weak continuous measurement under study.

APPENDIX D: SPECIAL CASE: DISPERSIVE MEASUREMENT WITH NO RABI DRIVE

Here we look at the particular case of dispersive measurement with no Rabi drive, where the total integrated signal $R = \frac{1}{\tau}\int_0^T dt\, r(t)$ completely describes the measurement dynamics. The probability distribution of $\mathcal{Q}$ in this case can be obtained by methods described in Ref. [44] that allows us to compute $\langle\exp(-\mathcal{Q})\rangle$ analytically as the integral $\langle\exp(-\mathcal{Q})\rangle = \int d\mathcal{Q}\exp(-\mathcal{Q})\mathcal{P}(\mathcal{Q})$. Here we note that a similar analytical result can be obtained for μ as well, that permits us to analytically verify the identity $\langle\exp(-\mathcal{Q})\rangle = 1-\mu$. To achieve this, we define μ as the mean value of the probability distribution of a random variable λ

$$\lambda(R) = \frac{\mathrm{Tr}[\rho(0)\mathcal{E}(R)\tilde{\rho}(0)\mathcal{E}(R)]}{\mathrm{Tr}[\rho(0)\mathcal{E}(R)]^2}. \tag{D1}$$

Note that $\lambda = \frac{|c(R)|^2}{a(R)^2}$, by multiplying and dividing the integrand of the left-hand side of Eq. (B16) by the forward probability $a(R)$. Here $\rho(0)$ is the initial state, which is assumed to be pure, and $\tilde{\rho}(0)$ is the state orthogonal to that. The probability distribution $\mathcal{P}(\lambda)$ can be obtained from the probability distribution $\mathcal{P}(\mathcal{Q})$ by noting that

$$\mathcal{P}(\mathcal{Q})\, d\mathcal{Q} = \mathcal{P}(\lambda) d\lambda \tag{D2}$$

or

$$\mathcal{P}(\lambda) = \mathcal{P}(\mathcal{Q})\frac{d\mathcal{Q}}{d\lambda}\bigg|_{\mathcal{Q}=\mathcal{Q}(\lambda)}. \tag{D3}$$

We note that for the case when qubit is initialized at $z=0$, this result is rather simple. In this case, we obtain $\lambda(R) = (\tanh R)^2 = 1-\exp(-\mathcal{Q})$, where $\mathcal{Q} = 2\ln\cosh R$ for the initial state $z=0$, as obtained in Ref. [44]. We obtain

$$\frac{d\mathcal{Q}}{d\lambda} = \frac{1}{1-\lambda}. \tag{D4}$$

Using the relation $\mathcal{Q}(\lambda) = -\ln(1-\lambda)$, we obtain the following expression for $\mathcal{P}(\lambda)$ (for qubit initialized at $z=0$),

$$\mathcal{P}(\lambda) = \sqrt{\frac{\tau}{2\pi T}}\frac{1}{(1-\lambda)^2}\sqrt{\frac{1-\lambda}{\lambda}} \times \exp\left(-\frac{T}{2\tau} - \frac{\tau}{2T}\left[\mathrm{arccosh}\frac{1}{\sqrt{1-\lambda}}\right]^2\right), \quad \lambda\epsilon[0,1]. \tag{D5}$$

We note that $\mu = \langle\lambda\rangle = \int_0^1 d\lambda\,\lambda\,\mathcal{P}(\lambda)$, that satisfies $\langle\exp(-\mathcal{Q})\rangle = 1-\mu$. Please refer to Fig. 5 where we numerically verify this identity for different durations of the measurement T/τ.

APPENDIX E: HOMODYNE MEASUREMENT

From the Kraus operator M_{Ho} given in main text, we first compute the arrow of time measure corresponding to a single step homodyne measurement performed during dt. We use the identity

$$\mathcal{Q}(\mathbf{r}) = -\ln\left(\frac{|\mathrm{Det}[M(r)]|^2}{\mathrm{Tr}\{\rho_{x_0}M^\dagger(r)M(r)\}^2}\right). \tag{E1}$$

We find for x_0 being the eigenstate of σ_x of eigenvalue $+1$:

$$\mathcal{Q}_{\mathrm{Ho}}(r) = \ln\left(\frac{1-\epsilon/4+\sqrt{\epsilon}r+\epsilon r^2/2}{1-\epsilon/2}\right). \tag{E2}$$

This expression allows to check that, $\mathcal{Q}_{\mathrm{Ho}}(\mathbf{r})$ admits a minimum negative value $\mathcal{Q}_{\min} = 2\ln[\sqrt{1-\epsilon/2}/2]$, reached for $r_{\min} = -1/\sqrt{\epsilon}$. The probability $P_{\mathrm{Ho}}^{(dt)}(\mathcal{Q})$ for $\mathcal{Q}_{\mathrm{Ho}}$ to take the value $\mathcal{Q}$ is given by

$$P^{(dt)}_{\mathrm{Ho}}(\mathcal{Q}) = P(r|x_0)\left(\frac{d\mathcal{Q}_{\mathrm{Ho}}(r)}{dr}\right)^{-1}\Bigg|_{r=r(\mathcal{Q})}, \tag{E3}$$

with

$$P(r|x_0) = \frac{e^{-r^2}}{\sqrt{\pi}}\left(1 + \sqrt{\epsilon}r - \frac{\epsilon}{4} + \frac{r^2\epsilon}{2}\right) \tag{E4}$$

and $r(\mathcal{Q})$ is obtained inverting Eq. (E2):

$$r(\mathcal{Q}) = \frac{1}{\sqrt{\epsilon}}\left(\sqrt{e^{\mathcal{Q}/2}\sqrt{1-\frac{\epsilon}{2}} + \frac{\epsilon}{2} - 1} - 1\right). \tag{E5}$$

For finite durations of the measurement, the concatenated measurement operators can be written as a single effective measurement,

$$\mathcal{M}_{\mathrm{Ho}}(r) = \frac{e^{-\sum_{n=1}^{N} r_n^2/2}}{\pi^{N/4}}\begin{pmatrix}(1-\epsilon/2)^{N/2} & 0\\ \sqrt{N\epsilon}\, y(\mathbf{r}) & 1\end{pmatrix}$$
$$\simeq \frac{e^{-\sum_{n=1}^{N} r_n^2/2}}{\pi^{N/4}}\begin{pmatrix}\sqrt{1-\epsilon'/2} & 0\\ \sqrt{\epsilon'}\, y(\mathbf{r}) & 1\end{pmatrix}, \tag{E6}$$

with the effective readout $y(\mathbf{r}) = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} r_n(1-\epsilon/2)^{(n-1)/2}$, and $\epsilon' = N\epsilon$, and this approximation is valid when $\epsilon \ll 1$. We use this approximation to reproduce the shape of the distribution of $\mathcal{Q}$ for the Homodyne measurement with no Rabi drive (presented in Fig. 1 of the main text), in Fig. 5(d).

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