Three-body pseudopotential for atoms confined in one dimension

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(Received 15 June 2018; published 24 January 2019)

Following a strong analogy with two-dimensional physics, the three-body pseudopotential in one dimension is derived. The Born approximation is then considered in the context of ultracold atoms in a linear harmonic waveguide. The equivalence between this model and the three-body wave equation obtained directly from the Schrödinger equation, including the confinement potential of the waveguide, is demonstrated in the vicinity of the dimer threshold.

DOI: 10.1103/PhysRevA.99.012711

I. INTRODUCTION

At sufficiently low temperature, ultracold atoms in very elongated harmonic traps can be considered as in a one-dimensional (1D) geometry. For bosons, the idealization of the trap in terms a 1D harmonic waveguide permits one to achieve a mapping with the Lieb-Liniger model [1,2]. In this context, few-body systems in 1D have attracted recent interest [3–6]. As a consequence of the Yang-Baxter criterion [7], the three-body problem occupies also a central place in studying the breakdown of integrability in these systems [8–11]. Interestingly, virtual excitations of pairs of atoms in the transverse modes of the waveguide led to the introduction of a 1D zero-range three-body potential [8,9]. This potential has important consequences in the prediction of a 1D dilute liquid state [12,13]. Moreover, it permits one to predict the existence of an excited trimer state in the vicinity of the dimer threshold [14,15]. This prediction coincides exactly with that derived directly from the dimensional reduction of the three-dimensional (3D) Schrödinger equation in the presence of a 1D waveguide [16].

In Ref. [15] a renormalization procedure is used to cure the divergencies coming from the bare zero-range three-body potential. In Ref. [14] a three-body contact condition is used instead to implement the zero-range model. These two-last studies lead to exactly the same reduced equation of the three-body problem, referred to hereafter as the 1D Skorniakov Ter-Martirosian (STM) equation. Nevertheless the 1D STM equation in Ref. [16] differs from the latter, even though the same prediction for the excited trimer state is obtained near the dimer threshold.

In this work, using a mapping with two-dimensional (2D) physics, a three-body zero-range pseudopotential leading to a mathematically well-behaved problem is derived both in configuration (Sec. II) and in momentum space (Sec. III). The pseudopotential has the same form as the two-body Λ potential in the 2D space [17]. Then the equivalence between the zero-range model and the renormalization procedure of Ref. [15] is obtained in Sec. IV. In Sec. V, the STM equation for three bosons interacting via the two- and three-body zero-range potentials is derived. This STM equation is identical to that obtained in Refs. [14,15]. The Λ parameter of the pseudopotential can be any wave number. However using the pseudopotential at first order of the perturbation theory breaks this invariance in a change of Λ. In Sec. VI, it is shown that a specific value of Λ permits one to justify the use of the pseudopotential at the first order of the perturbation and to recover the renormalized strength of the zero-range potential. Finally, in Sec. VII, the equivalence between the zero-range 1D model and the dimensional reduction method is obtained in the vicinity of the dimer threshold where the first-order perturbation theory is accurate.

II. REGULARIZED ZERO-RANGE THREE-BODY POTENTIAL

In this section the 1D three-body contact potential is introduced in complete analogy with the 2D two-body problem. Let us consider three particles of the same mass m labeled by the index $i \in (1, 2, 3)$. The positions of the three particles are given by the coordinates z_i. The center of mass of the system is $C = (z_1 + z_2 + z_3)/3$, and the two other Jacobi coordinates are

$$z_{ij} = z_i - z_j \quad ; \quad Z_{ij} = \frac{2}{\sqrt{3}}\left(z_k - \frac{z_i + z_j}{2}\right), \tag{1}$$

where all the indices i, j, k are distinct and are a cyclic permutation of the triplet $(1, 2, 3)$. The general form of the zero-range three-body potential can be written as

$$V_3^{\rm pp}\Psi(z_1, z_2, z_3) = \frac{\hbar^2}{m}\delta(z_{12})\delta(z_{23})\psi_3(C). \tag{2}$$

The function $\langle C|\psi_3\rangle$ is denoted hereafter as the three-body contact. It will be shown later that it characterizes the singular behavior of the wave function at the contact of the three particles. For a given pair (ij), one introduces the hyper-radius $R = \sqrt{Z_{ij}^2 + z_{ij}^2}$ and the hypercoordinates that allow for the 2D mapping,

$$\mathbf{R} = z_{ij}\hat{\mathbf{e}}_z + Z_{ij}\hat{\mathbf{e}}_Z, \tag{3}$$

where $(\hat{\mathbf{e}}_z, \hat{\mathbf{e}}_Z)$ is an orthonormal basis. In this system of coordinates, the potential in Eq. (2) can be expressed as

$$V_3^{\mathrm{pp}}\Psi(z_1, z_2, z_3) = \frac{2\hbar^2}{m\sqrt{3}}\delta^2(\mathbf{R})\psi_3(C). \tag{4}$$

Due to its s-wave character, the expression of the zero-range potential in Eq. (4) does not depend on the choice of a specific pair of particles in the definition of the hypercoordinate $\mathbf{R}$. For a bare three-body zero-range potential, the three-body contact $\psi_3(C)$ is just the value of the wave function at the contact of the three particles. However, similarly to the two-dimensional Green's function, for a given value of the three-body contact, the two-dimensional δ distribution in Eq. (4) gives rise to a logarithmic singularity of the wave function at $R = 0$: $\Psi(z_1, z_2, z_3) \sim \ln(R/l)\psi_3(C)/(\pi\sqrt{3}) + \cdots$, where the length l is needed for having a dimensionless argument of the logarithmic function. As the wave function is infinite at the contact $R = 0$, the bare zero-range potential thus leads to a mathematically not-well-behaved model. In practice, the bare potential can be used at the first order of perturbation, where the unperturbed wave function is regular at the contact [13] (see Sec. VI). When the bare potential is used nonperturbatively as in Ref. [15], a renormalizing procedure of the strength is necessary (see Sec. IV).

As a consequence of the 2D mapping in Eq. (3), a mathematically consistent zero-range model of the three-body interaction in 1D is given by considering a self-adjoint extension of the 2D Laplacian $\Delta_{\mathbf{R}}$ [18]. In this modeling, the domain of the Hamiltonian corresponds to wave functions that are defined everywhere except at the contact $\mathbf{R} = 0$ and which also verify the contact condition:

$$\Psi(z_1, z_2, z_3) \underset{R\to 0}{=} \frac{\psi_3(C)}{\pi\sqrt{3}} \ln\left(\frac{R}{a_3}\right) + \cdots. \tag{5}$$

In Eq. (5), a_3 is a fixed length that characterizes the underlying short-range physics during a three-body collision. It is denoted hereafter as the 1D three-body scattering length. Equation (5) shows that the three-body scattering length is analogous to a disk radius: it fixes the same node at a given hyper-radius for all the eigenstates of the Hamiltonian. The zero-range three-body potential is fully defined by the boundary condition in Eq. (5) and has been introduced in Ref. [14]. The three-body contact $\psi_3(C)$ in Eq. (5) depends on the state considered. For instance, in the free 1D space and in the absence of two-body potential, a bound state is necessarily given by the 2D Green's function at negative energy. Thus $\Psi(R) = \mathcal{N}K_0(qR)$ ($\mathcal{N}$ is the normalization factor and K_0 is a modified Bessel function of second kind). Considering the singularity of this state for a vanishing hyper-radius, one finds, by identification with the contact condition in Eq. (5), that the contact is $\psi_3 = -\mathcal{N}\pi/\sqrt{3}$ and the binding wave number is $q = 2e^{-\gamma}/a_3$. In this case, there is thus only one bound state of binding energy $E_3 = -\frac{4\hbar^2 e^{-2\gamma}}{m a_3^2}$.

Another equivalent way to implement the zero-range model of the three-body interaction is to introduce the operator that permits one to calculate the contact $\psi_3(C)$ from the wave function. For this purpose, one can notice again that Eqs. (4) and (5) are formally equivalent to the definition of a zero-range potential in the 2D two-body problem [17,19]. It is then straightforward to use, for the three-body zero-range potential, the known expression of the 2D Λ potential:

$$\begin{aligned} V_3^{\mathrm{pp}}&\Psi(z_1, z_2, z_3) \\ &= \frac{-\pi\sqrt{3}\hbar^2}{m\ln(e^\gamma\Lambda a_3/2)}\delta(z_{12})\delta(z_{13}) \\ &\quad\times \lim_{R\to 0}\left[1 - \ln\left(\frac{e^\gamma\Lambda R}{2}\right)R\frac{\partial}{\partial R}\right]\Psi(z_1, z_2, z_3), \end{aligned} \tag{6}$$

where the parameter Λ is any wave number. From Eq. (6), one can identify the Λ-dependent strength in the pseudopotential,

$$g_3(\Lambda) = \frac{-\pi\sqrt{3}\hbar^2}{m\ln(e^\gamma\Lambda a_3/2)}, \tag{7}$$

and the regularizing operator,

$$\langle C|R_\Lambda|\Psi\rangle = \lim_{R\to 0}\left[1 - \ln\left(\frac{e^\gamma\Lambda R}{2}\right)R\frac{\partial}{\partial R}\right]\Psi(z_1, z_2, z_3). \tag{8}$$

Using the identity $\Delta_{\mathbf{R}}\ln(R) = 2\pi\delta(\mathbf{R})$, one can verify that the δ terms in the Schrödinger equation are eliminated if and only if the wave function verifies the contact condition in Eq. (5). Consequently, in the pseudopotential approach, imposing that the eigenstates are solutions of the Schrödinger equation for all values of the coordinates, including at the point of contact $R = 0$, is a way to select the set of wave functions satisfying the correct contact condition. Using Eq. (5), one finds $g_3(\Lambda)\langle C|R_\Lambda|\Psi\rangle = \frac{\hbar^2}{m}\psi_3(C)$ for any value of the parameter Λ and thus one recovers Eq. (4) from the Λ potential in Eq. (6).

III. THREE-BODY PSEUDOPOTENTIAL IN THE MOMENTUM REPRESENTATION

In what follows, the three-body zero-range pseudopotential is derived in the momentum representation. The momentum of the particle i is denoted by k_i. To avoid any ambiguities with the equations in configuration space, the bra-ket notation will be used below. The analogy with the two-body problem in 2D can be pursued, and the derivation follows along the same lines as in Refs. [20,21]. For this purpose, one defines the Jacobi coordinates in the momentum space. The momentum of the center of mass is $k_C = k_1 + k_2 + k_3$. The two other Jacobi coordinates are defined by

$$k_{ij} = \frac{k_i - k_j}{2} \quad ; \quad K_{ij} = \frac{2k_k - (k_i + k_j)}{3}. \tag{9}$$

In what follows, the notations $k = k_{12}$ and $K = K_{12}$ are used. Similarly to what has been done in the previous section, one introduces the hypermomentum

$$\mathbf{Q} = k\hat{\mathbf{e}}_z + \frac{\sqrt{3}}{2}K\hat{\mathbf{e}}_Z. \tag{10}$$

The stationary Schrödinger equation at energy E, for a system with only one interaction term given by the three-body potential of Eq. (2), is in the momentum space

$$\left(Q^2 + k_C^2/6 - mE/\hbar^2\right)\langle\mathbf{Q}, k_C|\Psi\rangle = -\langle k_C|\psi_3\rangle. \tag{11}$$

Equation (11) gives the high-momentum behavior of the wave function for all energies and also in the possible presence of other nonsingular three-body potential. The three-body contact can be thus defined in the momentum representation by

$$\langle k_C|\psi_3\rangle = -\lim_{Q\to\infty} Q^2\langle \mathbf{Q}, k_C|\Psi\rangle. \tag{12}$$

Let us consider the Green's function at the negative energy $-\hbar^2\Lambda^2/m$ in the center-of-mass frame for a vanishing hyperradius ($R \to 0$):

$$\int \frac{d^2Q}{(2\pi)^2}\frac{\exp(i\mathbf{Q}\cdot\mathbf{R})}{Q^2+\Lambda^2} = -\frac{1}{2\pi}\ln\left(\frac{e^\gamma R\Lambda}{2}\right)+\cdots. \tag{13}$$

Equation (13) can be used for any positive value of the parameter Λ and permits one to express the three-body contact condition in Eq. (5) as

$$\frac{2}{\sqrt{3}}\int\frac{d^2Q}{(2\pi)^2}\left[\langle \mathbf{Q}, k_C|\Psi\rangle + \frac{\langle k_C|\psi_3\rangle}{Q^2+\Lambda^2}\right] = \frac{\hbar^2\langle k_C|\psi_3\rangle}{mg_3(\Lambda)}. \tag{14}$$

From Eqs. (2) and (14), one can deduce the three-body pseudopotential in the momentum representation,

$$\langle k_1, k_2, k_3|V_3^{\rm pp}|\Psi\rangle = g_3(\Lambda)\langle k_C|R_\Lambda|\Psi\rangle, \tag{15}$$

where

$$\langle k_C|R_\Lambda|\Psi\rangle = \frac{2}{\sqrt{3}}\int\frac{d^2Q}{(2\pi)^2}\left[\langle \mathbf{Q}, k_C|\Psi\rangle + \frac{\langle k_C|\psi_3\rangle}{Q^2+\Lambda^2}\right], \tag{16}$$

and $\langle k_C|\psi_3\rangle$ is defined by Eq. (12)

IV. LINK WITH THE RENORMALIZATION OF THE THREE-BODY POTENTIAL

Using the notation of the present paper, the renormalization method of the three-body interaction in Ref. [15] can be introduced by using a separable three-body potential $V(Q_c)$ with a strength g_3 that depends on the ultraviolet momentum cutoff Q_c [22]:

$$\langle k_1, k_2, k_3|V(Q_c)|k_1', k_2', k_3'\rangle = (2\pi)\delta(k_C - k_C')g_3(Q_c)\theta(Q_c - |Q|)\theta(Q_c - |Q'|), \tag{17}$$

where θ is the Heaviside function. Thus, for $Q < Q_c$,

$$\langle \mathbf{Q}, k_C|V(Q_c)|\Psi\rangle = \frac{2g_3(Q_c)}{\sqrt{3}}\int_{Q'<Q_c}\frac{d^2Q'}{(2\pi)^2}\langle \mathbf{Q}', k_C|\Psi\rangle. \tag{18}$$

Let us show in what follows that in the limit of an infinite cutoff, this potential leads to the contact condition associated with the pseudopotential. For that purpose, one introduces the reference state $|\Psi_\Lambda\rangle$ such that

$$\frac{\hbar^2}{m}(Q^2+\Lambda^2)\langle \mathbf{Q}, k_C|\Psi_\Lambda\rangle = -\langle \mathbf{Q}, k_C|V(Q_c)|\Psi\rangle. \tag{19}$$

From this last definition, the action of the cutoff-dependent potential on the difference between the wave function and the reference state can be written as

$$\langle \mathbf{Q}, k_C|V(Q_c)|\Psi - \Psi_\Lambda\rangle = \langle \mathbf{Q}, k_C|V(Q_c)|\Psi\rangle\left[1 + \frac{2mg_3(Q_c)}{\hbar^2\sqrt{3}} \times \int_{Q'<Q_c}\frac{d^2Q'}{(2\pi)^2}\frac{1}{Q'^2+\Lambda^2}\right]. \tag{20}$$

In the limit where the cutoff Q_c goes to infinity, one can identify from Eq. (11) the three-body contact as

$$\langle \mathbf{Q}, k_C|V(Q_c)|\Psi\rangle \underset{Q_c\to\infty}{=} \frac{\hbar^2}{m}\langle k_C|\psi_3\rangle. \tag{21}$$

In the limit of an arbitrarily large cutoff, one then finds, from Eq. (20),

$$\langle \mathbf{Q}, k_C|V(Q_c)|\Psi - \Psi_\Lambda\rangle = \frac{\hbar^2 g_3(Q_c)}{mg_3(\Lambda)}\langle k_C|\psi_3\rangle. \tag{22}$$

Now, the left-hand-side of Eq. (20) can be also directly expressed for $Q < Q_c$ as

$$\langle \mathbf{Q}, k_C|V(Q_c)|\Psi - \Psi_\Lambda\rangle = \frac{2g_3(Q_c)}{\sqrt{3}}\int_{Q'<Q_c}\frac{d^2Q'}{(2\pi)^2}\left[\langle \mathbf{Q}', k_C|\Psi\rangle + \frac{\langle k_C|\psi_3\rangle}{Q'^2+\Lambda^2}\right]. \tag{23}$$

By equalizing the right-hand side of Eqs. (22) and (23), one then finds the contact condition in Eq. (14).

V. STM EQUATION

In this section the bound states made of three identical bosons are considered. The three particles interact via the three-body pseudopotential of Eq. (15). Moreover, each pair of particles (ij) interacts via the zero-range potential of the Lieb-Liniger model,

$$V(z_{ij}) = -\frac{2\hbar^2}{ma_2}\delta(z_{ij}), \tag{24}$$

where a_2 is the 1D two-body scattering length. One introduces the two-body contact, which corresponds to the value of the wave function at the contact of two particles considered. For the contact of the pair (12), one has in the momentum representation,

$$\langle K, k_C|\psi_2\rangle = \int\frac{dk}{2\pi}\langle k, K, k_C|\Psi\rangle. \tag{25}$$

In the center-of-mass frame the wave function can be factorized as $\langle k, K, k_C|\Psi\rangle = (2\pi)\delta(k_C)\langle k, K|\phi\rangle$. The two- and three-body contacts are also factorized as $\langle K, k_C|\psi_2\rangle = (2\pi)\delta(k_C)\langle K|S_2\rangle$ and $\langle k_C|\psi_3\rangle = (2\pi)\delta(k_C)S_3$. The three-body Schrödinger equation for a bound state of energy $E = -\frac{\hbar^2q^2}{m}$ is thus

$$\left(k^2 + \frac{3}{4}K^2 + q^2\right)\langle k, K|\phi\rangle = -S_3 + \frac{2}{a_2}(\langle K|S_2\rangle + \langle -k - K/2|S_2\rangle + \langle k - K/2|S_2\rangle). \tag{26}$$

The STM equation follows from Eq. (26) after integration over the relative momentum k:

$$\left(a_2 - \frac{1}{\sqrt{q^2 + \frac{3K^2}{4}}}\right)\langle K|S_2\rangle + \frac{a_2 S_3}{\sqrt{4q^2+3K^2}} = 4\int \frac{dK'}{2\pi}\frac{\langle K'|S_2\rangle}{K^2 + K'^2 + KK' + q^2}. \tag{27}$$

Injection of Eq. (26) in the contact condition (14) gives

$$\frac{3}{a_2}\int \frac{dK}{2\pi}\frac{\langle K|S_2\rangle}{\sqrt{\frac{3}{4}K^2+q^2}} = -\frac{S_3\ln(qa_3e^\gamma/2)}{\pi\sqrt{3}}. \tag{28}$$

This last relation permits one to close Eq. (27) and to recover the set of equations for the two-body contact obtained in Refs. [14,15]. Remarkably, an exact implicit equation on the binding wave number q has been derived from Eqs. (27) and (28) in Ref. [13].

VI. Λ-DEPENDENT AND RENORMALIZED STRENGTHS

In the derivation of the trimer spectrum $E = -\hbar^2q^2/m$ from Eqs. (27) and (28), the renormalized strength $g_3(q)$ was used in Ref. [15] associated with a zero-range potential without the regularizing operator

$$V_3^{\text{ren}} = g_3(q)\delta(z_{12})\delta(z_{23}). \tag{29}$$

This choice of the momentum scale in the interaction strength is associated with the reference state for the wave number $\Lambda = q$ in Eq. (19) and thus

$$\langle \mathbf{Q}, k_C|\Psi_{\Lambda=q}\rangle = -(2\pi)\delta(k_C)\frac{S_3}{Q^2+q^2}. \tag{30}$$

Indeed, for this value of the parameter Λ, one can decompose the three-body state as

$$|\Psi\rangle = |\Psi_{\Lambda=q}\rangle + |\Psi_{\text{reg}}\rangle, \tag{31}$$

where for an infinite momentum cutoff Q_c, $\langle C, \mathbf{R}|\Psi_{\text{reg}}\rangle$ is, by construction, a regular wave function when the hyper-radius R tends to zero. Using again Eqs. (21), (22), and (23), one finds for this choice of reference state

$$\langle \mathbf{Q}, k_C|V(Q_c)|\Psi\rangle \underset{Q_c\to\infty}{=} \frac{2g_3(q)}{\sqrt{3}}\int \frac{d^2Q'}{(2\pi)^2}\langle \mathbf{Q}', k_C|\Psi_{\text{reg}}\rangle, \tag{32}$$

so that one can write

$$V(Q_c)|\Psi\rangle \underset{Q_c\to\infty}{=} V_3^{\text{ren}}|\Psi_{\text{reg}}\rangle, \tag{33}$$

an identity which provides the way to use the renormalized potential in Eq. (29).

Equation (33) can be also used for a perturbative treatment of the three-body potential. For this purpose, let us consider the regime where the three-body potential gives a small contribution with respect to the two-body interactions. One can then approximate, at the leading order, the regular state $|\Psi_{\text{reg}}\rangle$ by an eigenstate of the three-body problem without the three-body potential, denoted by $|\phi^{(0)}\rangle$ in the center-of-mass frame:

$$\langle \mathbf{Q}, k_C|\Psi_{\text{reg}}\rangle = (2\pi)\delta(k_C)\langle \mathbf{Q}|\phi^{(0)}\rangle. \tag{34}$$

The singular behavior at the contact of the three particles is solely included in the perturbation $\langle \mathbf{Q}|\delta\phi\rangle$, which is much smaller than $\langle \mathbf{Q}|\phi^{(0)}\rangle$, except in the high-momentum limit where $(2\pi)\delta(k_C)\langle \mathbf{Q}|\delta\phi\rangle \sim \langle \mathbf{Q}, k_C|\Psi_{\Lambda=q}\rangle$. Considering the renormalized potential in Eq. (29) as a perturbation, one then obtains, at first order, the correction in the energy $\delta E = \langle\phi^{(0)}|V_3^{\text{ren}}|\phi^{(0)}\rangle$. This method was used in Ref. [15] to derive the binding energies of the trimers in the perturbative regime of the three-body potential.

From the point of view of the pseudopotential, one finds that for $\Lambda = q$, the explicit dependence on the three-body contact $\langle k_C|\psi_3\rangle$ in Eq. (16) is exactly canceled:

$$\langle k_C|R_{\Lambda=q}|\Psi\rangle = \frac{2}{\sqrt{3}}\int \frac{d^2Q}{(2\pi)^2}\langle \mathbf{Q}, k_C|\Psi_{\text{reg}}\rangle. \tag{35}$$

Similarly to the renormalization method, this gives us the opportunity to use the zero-range pseudopotential perturbatively. Again, one uses Eq. (34) and finds

$$R_{\Lambda=q}|\phi\rangle \sim \frac{2}{\sqrt{3}}\int \frac{d^2Q}{(2\pi)^2}\langle \mathbf{Q}|\phi^{(0)}\rangle = \phi^{(0)}(R=0). \tag{36}$$

Thus at the first order of the perturbation, where $1/|\ln(qa_3)|$ is the small parameter, using the Λ potential for the specific case $\Lambda = q$ is equivalent to using the renormalized zero-range potential in Eq. (29) with the energy correction

$$\delta E = g_3(q)\langle\phi^{(0)}|R_{\Lambda=q}|\phi^{(0)}\rangle. \tag{37}$$

VII. THREE-BODY PROBLEM NEAR THE DIMER THRESHOLD IN A 1D WAVEGUIDE

In the case of a 1D harmonic atomic waveguide, the effect of virtual excitations of pairs of particles in the direction transverse to the free motion breaks the integrability in the many-body quasi-1D problem. For taking this effect into account, it is necessary to introduce a perturbation to the Lieb-Liniger model. It has been shown that this perturbation can be modeled at the first order of the Born approximation by a zero-range bare three-body potential with strength [8]

$$g_3^{\text{Born}} = -\frac{6\hbar^2a_\perp^2}{ma_2^2}\ln\left(\frac{4}{3}\right), \tag{38}$$

where $a_\perp = \sqrt{2\hbar/(m\omega)}$ is the characteristic length of the waveguide.

When the two-body scattering length a_2 is large and positive, i.e., at the threshold of the dimer of binding wave number $1/a_2$, the Lieb-Liniger model predicts a shallow trimer (the McGuire trimer) in the 1D waveguide. Virtual excitations of pairs of particles induce a perturbation of this trimer of binding wave number $q = 2/a_2$ [23]. From the discussion in Sec. VI, this last wave number gives the value of the parameter Λ when the pseudopotential is used at the first order of the perturbation theory [see Eq. (37)]. One can then identify the bare strength in Eq. (38) with $g_3(2/a_2)$. One finds $\ln(a_3/a_2) \sim \frac{\pi a_2^2}{2\sqrt{3}\ln(4/3)a_\perp^2}$ for $a_2 \to \infty$. This last equation remains valid in the regime of large two-body scattering lengths when one considers any three-body process with an energy of the order of $\hbar^2/ma_2^2$ in the center-of-mass frame. In this regime, in the presence of the three-body potential,

in addition to the perturbed McGuire state, another trimer is found near the atom-dimer continuum. Importantly, the asymptotic law near the threshold for the two trimers coincides exactly with those found from the dimensional reduction of the 3D Schrödinger equation in the presence of a harmonic waveguide [14–16,24]:

$$q_0 \sim \frac{2}{a_2} + \frac{4a_\perp^2}{a_2^3}\ln\left(\frac{4}{3}\right);\ q_1 \sim \frac{1}{a_2} + \frac{2a_\perp^4}{3a_2^5}\ln^2\left(\frac{4}{3}\right), \quad (39)$$

where q_0 and q_1 are the binding wave numbers.

In the dimensional reduction method of Ref. [16], there is no three-body potential in the Hamiltonian and the quasi-1D character of the system is revealed by a summation over the transverse modes in the STM equation:

$$\left(\frac{1}{\sqrt{\frac{3}{4}K^2+q^2}} - a_2\right)\langle K|S_2\rangle = -2\int\frac{dK'}{\pi}\sum_{n=0}^{\infty}\frac{1}{4^n}\frac{\langle K'|S_2\rangle}{\frac{4n}{a_\perp^2}+q^2+K^2+K'^2+KK'}. \quad (40)$$

The momentum $1/a_\perp$ plays the role of a cutoff in the integral term of Eq. (40). In the limit where the momentums K, K', and q are much smaller that $1/a_\perp$, one has

$$\sum_{n=1}^{\infty}\frac{1}{4^n}\frac{1}{\frac{4n}{a_\perp^2}+q^2+K^2+K'^2+KK'} \sim \ln\left(\frac{4}{3}\right)\frac{a_\perp^2}{4}. \quad (41)$$

The approximation in Eq. (41) was the one used in Ref. [16] for the derivation of the asymptotic law for the two trimers in Eq. (39).

In what follows it is shown that this approximation is equivalent to the Born approximation of Eq. (38), performed with the zero-range pseudopotential. For that purpose, one considers formally the effective three-body potential V_3, which gives Eq. (40). The Hamiltonian is

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{3}\frac{\partial^2}{\partial z_i^2} - \frac{2\hbar^2}{ma_2}\sum_{i<j\leqslant 3}\delta(z_{ij}) + V_3. \quad (42)$$

The absence of singular behavior of the eigenstates of Eq. (40) shows that necessarily the potential V_3 is nonlocal in the configuration space. One can remark that the regime where the two-body contact $\langle K|S_2\rangle$ is negligible for a momentum K of the order of $1/a_\perp$ corresponds to the regime where the zero-range three-body potential can be treated in the Born approximation. Then, using the approximation Eq. (41), one finds an energy-dependent potential valid in the Born approximation or, equivalently, at the first order of the perturbation theory. For a hypermomentum $Q \lesssim 1/a_\perp$, one can write, in the center-of-mass frame,

$$\langle k, K|V_3|\phi\rangle = -\frac{\hbar^2}{m}\pi\ln(4/3)\left(k^2+\frac{3}{4}K^2+q^2\right)\frac{a_\perp^3}{a_2} \times \int_<\frac{dk'dK'}{(2\pi)^2}\langle k', K'|\phi\rangle, \quad (43)$$

and $\langle k, K|V_3|\phi\rangle \sim 0$ for $Q \gtrsim 1/a_\perp$. In Eq. (43), $\int_<$ means an integration in the domain where the hypermomentum $Q' = \sqrt{k'^2+\frac{3}{4}K'^2}$ is smaller than $1/a_\perp$. Multiplying Eq. (43) by $\langle\phi|k, K\rangle$ and assuming that V_3 is a small perturbation, one can use the relation

$$\langle\phi|\mathcal{H}_0 - E|k, K\rangle \sim \frac{2\hbar^2}{ma_2}(\langle S_2|K\rangle + \langle S_2|-k-K/2\rangle + \langle S_2|k-K/2\rangle), \quad (44)$$

where $\mathcal{H}_0$ is the kinetic operator. The integration of the resulting equation on the hypermomentum in the domain $Q < 1/a_\perp$ gives the expectation value of the effective three-body potential in the center-of-mass frame:

$$\langle\phi|V_3|\phi\rangle \sim -\frac{6\hbar^2a_\perp^2}{ma_2^2}\ln\left(\frac{4}{3}\right)|\phi(R=0)|^2. \quad (45)$$

One thus recovers the same strength of the three-body potential as that of the zero-range model in the Born approximation in Eq. (38).

VIII. CONCLUSION

The three-body Λ potential introduced in this paper leads to a mathematically well-behaved Schrödinger equation, thus avoiding a renormalization procedure of a bare zero-range potential. This pseudopotential is used in the context of atoms in a 1D waveguide where the virtual excitations in the transverse modes give rise to an effective zero-range three-body potential, in addition to the usual two-body potential of the Lieb-Liniger model. For the three-body problem, in the limiting case of a large two-body scattering length, a three-body potential is obtained from the STM equation derived directly from the 3D Schrödinger equation in the presence of a 1D waveguide. This potential, used at the first order of the perturbation theory, explains the equivalence found near the dimer threshold between the zero-range potential approach of Refs. [14,15] and the direct method of Ref. [16].

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