Regularized maximal fidelity of the generalized Pauli channels

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We consider the asymptotic regularization of the maximal fidelity for the generalized Pauli channels, which is a problem similar to the classical channel capacity. In particular, we find the exact formulas for the extremal channel fidelities and the maximal output ∞-norm. For wide classes of channels, we show that these quantities are weakly multiplicative. Finally, we find the regularized maximal fidelity for the channels satisfying the time-local master equations.

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I. INTRODUCTION

The noisiness of a quantum channel is characterized by its capacity, which determines the maximal amount of quantum information that it can reliably transmit. Finding the channel capacity is still an open problem, even in the simplest case of the classical capacity

$$C(\Lambda)=\lim_{n\to\infty}\frac{1}{n}\chi(\Lambda^{\otimes n}), \tag{1}$$

given in terms of the Holevo capacity χ [1,2] of the channel Λ. This is the case because calculating the Holevo capacity requires finding the maximal value of the entropic expression

$$\chi(\Lambda)=\max_{\{p_k,\rho_k\}}\left[S\left(\sum_k p_k\Lambda[\rho_k]\right)-\sum_k p_kS(\Lambda[\rho_k])\right] \tag{2}$$

over ensembles of states ρ_k and their probabilities of occurrence p_k. The problem of finding the classical capacity simplifies significantly for unitarily covariant quantum channels, for which $C(\Lambda)=\chi(\Lambda)$ as long as the minimal output entropy $S_{\min}(\Lambda)$ is additive [3,4]. The additivity of $S_{\min}(\Lambda)$ was first showed by King for unital qubit channels [5] and depolarizing channels [6].

Recently, Ernst and Klesse [7] proposed a toy problem that is structurally similar to finding the classical capacity. Namely, their goal is to determine the asymptotic regularization

$$f_{\max}^{(\infty)}(\Lambda)=\lim_{n\to\infty}\sqrt[n]{f_{\max}(\Lambda^{\otimes n})} \tag{3}$$

of the maximal channel fidelity $f_{\max}$, which measures the distortion of states under the action of the channel. It turns out that there is a relation between $f_{\max}^{(\infty)}(\Lambda)$ and the maximal output ∞-norm $\nu_\infty(\Lambda)$, which is a measure of the optimal output purity. Moreover, if the maximal output ∞-norm is weakly multiplicative, then $f_{\max}^{(\infty)}(\Lambda)=\nu_\infty(\Lambda)$.

Our goal is to analyze the regularized maximal fidelity for the generalized Pauli channels. For these channels, we find the exact formulas for the extremal values of the channel fidelity on pure input states and the maximal output ∞-norm. We derive the formula for $\nu_\infty(\Lambda)$, which is reached on the projectors onto the mutually unbiased bases vectors. This fact was proven by Nathanson and Ruskai [8]. Next, we show the weak multiplicativity of $\nu_\infty(\Lambda)$ for certain families of channels. Finally, we find the regularized maximal fidelity for a wide class of the generalized Pauli channels.

II. GENERALIZED PAULI CHANNELS

The generalized Pauli channels were first considered by Nathanson and Ruskai [8] as the *Pauli diagonal channels constant on axes*. Ohno and Petz [9] analyzed even more general channels, of which the generalized Pauli channels are the special case with the commutative subalgebras $\{\mathbb{I},U_\alpha^k\mid k=1,\dots,d-1\}$. Their applications range between the quantum process tomography [10], optimal parameter estimation [11], and geometrical quantum mechanics [12]. In the theory of open quantum systems and non-Markovian dynamics, the channels were analyzed in both the time-local [13,14] and memory kernel approaches [15]. In the present paper, we focus on their other properties, like state distortion and purity.

When constructing the generalized Pauli channels, one considers the d-dimensional Hilbert space $\mathcal{H}$ with the maximal number $N(d)=d+1$ of mutually unbiased bases $\{\psi_0^{(\alpha)},\dots,\psi_{d-1}^{(\alpha)}\}$ [16,17]. Let us recall that the bases are mutually unbiased if their vectors satisfy the following conditions:

$$\langle\psi_k^{(\alpha)}|\psi_l^{(\alpha)}\rangle=\delta_{kl},\quad \left|\langle\psi_k^{(\alpha)}|\psi_l^{(\beta)}\rangle\right|^2=\frac{1}{d},\quad \alpha\neq\beta. \tag{4}$$

The generalized Pauli channels $\Lambda:\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{H})$ are defined by [8,13]

$$\Lambda=\frac{dp_0-1}{d-1}\mathbb{1}+\frac{d}{d-1}\sum_{\alpha=1}^{d+1}p_\alpha\Phi_\alpha, \tag{5}$$

where p_α denotes the probability distribution, $\mathbb{1}$ is the identity channel,

$$\Phi_\alpha[X]=\sum_{k=0}^{d-1}P_k^{(\alpha)}XP_k^{(\alpha)}, \tag{6}$$

and $P_k^{(\alpha)} := |\psi_k^{(\alpha)}\rangle\langle\psi_k^{(\alpha)}|$ is a rank-1 projector. For $d=2$, Eq. (5) reduces to the Pauli channel

$$\Lambda = \sum_{\alpha=0}^{3} p_\alpha \sigma_\alpha \rho \sigma_\alpha, \tag{7}$$

where $\sigma_0 = \mathbb{I}$, σ_1, σ_2, σ_3 are the Pauli matrices.

The generalized Pauli channels satisfy the eigenvalue equations

$$\Lambda\big[U_\alpha^k\big] = \lambda_\alpha U_\alpha^k, \quad k=1,\dots,d-1, \tag{8}$$

and $\Lambda[\mathbb{I}] = \mathbb{I}$. In the above equation, the eigenvectors are the unitary operators constructed from the projectors onto the mutually unbiased bases vectors,

$$U_\alpha^k = \sum_{l=0}^{d-1} \omega^{kl} P_l^{(\alpha)}, \tag{9}$$

with $\omega = e^{2\pi i/d}$. The eigenvalues λ_α are $(d-1)$-times degenerate, and they are related to the probability distribution via

$$\lambda_\alpha = \frac{1}{d-1}[d(p_0 + p_\alpha) - 1]. \tag{10}$$

The inverse relation reads

$$p_0 = \frac{1}{d^2}\left[1 + (d-1)\sum_{\alpha=1}^{d+1}\lambda_\alpha\right], \tag{11}$$

$$p_\alpha = \frac{d-1}{d^2}\left[1 + d\lambda_\alpha - \sum_{\beta=1}^{d+1}\lambda_\beta\right]. \tag{12}$$

The necessary and sufficient conditions for the generalized Pauli channel to be a completely positive and trace-preserving map are the generalized Fujiwara–Algoet conditions [8,18,19]

$$-\frac{1}{d-1} \leqslant \sum_{\beta=1}^{d+1}\lambda_\beta \leqslant 1 + d\min_\beta \lambda_\beta. \tag{13}$$

III. CHANNEL FIDELITY

The fidelity is a measure of distance between two quantum states [19–21]. It helps us to determine how distinguished the states are from one another. Uhlmann [22] defined the fidelity between the density operators ρ and σ by

$$F(\rho,\sigma) := \left(\mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)^2, \tag{14}$$

where $0 \leqslant F(\rho,\sigma) \leqslant 1$ and $F(\rho,\sigma) = 1$ if and only if $\rho = \sigma$. On the basis of this simple formula, many other types of fidelity were derived, like the entanglement fidelity [23], average fidelity [24], or regularized maximum pure state input-output fidelity [7]. One can also introduce the notion of channel fidelity $F(\rho, \Lambda[\rho])$, which measures the fidelity between the input ρ and output $\Lambda[\rho]$ states [25]. One defines the minimal and maximal channel fidelity on pure input states [26] by

$$\begin{aligned} f_{\min}(\Lambda) &= \min_P F(P,\Lambda[P]) = \min_P \mathrm{Tr}(P\Lambda[P]),\\ f_{\max}(\Lambda) &= \max_P F(P,\Lambda[P]) = \max_P \mathrm{Tr}(P\Lambda[P]), \end{aligned} \tag{15}$$

where P are rank-1 projectors. They measure how much a given quantum channel can distort an initial state. The more Λ resembles the identity channel $\mathbb{1}$, the less ρ changes under a single action of the channel. However, the maximal fidelity $f_{\max}(\Lambda) = 1$ does not imply $\Lambda = \mathbb{1}$, as the maximal value is also reached, e.g., for $\Lambda[\rho] = P_0 \mathrm{Tr}\rho$ with P_0 being a fixed rank-1 projector.

Because of its concavity, the minimal value of $F(\rho,\Lambda[\rho])$ for mixed inputs is reached at pure states, and therefore $f_{\min}(\Lambda)$ is the minimal channel fidelity for mixed states. There is no similar relation for the maximal channel fidelity, as $\max_\rho F(\rho,\Lambda[\rho]) = F(\rho_*,\Lambda[\rho_*]) = 1$, where $\rho_* := \mathbb{I}/d$ is the state invariant under Λ.

Theorem 1. For the generalized Pauli channel Λ defined by Eq. (5), the minimal and maximal channel fidelities on pure input states are equal to

$$f_{\min}(\Lambda) = \frac{1}{d}[1 + (d-1)\lambda_{\min}], \tag{16}$$

$$f_{\max}(\Lambda) = \frac{1}{d}[1 + (d-1)\lambda_{\max}], \tag{17}$$

where $\lambda_{\max} = \max_\alpha \lambda_\alpha$ and $\lambda_{\min} = \min_\alpha \lambda_\alpha$.

Proof. To calculate the channel fidelity, we need to know how Λ transforms pure initial states. Any rank-1 projector P can be written in the unitary basis $\{\mathbb{I}, U_\alpha^k\}$ introduced in Eq. (9). Namely,

$$P = \frac{1}{d}\left(\mathbb{I} + \sum_{\alpha=1}^{d+1}\sum_{k=1}^{d-1} x_{\alpha k} U_\alpha^k\right), \tag{18}$$

where $x_{\alpha k}$ are complex parameters. Now, we find that

$$\Lambda[P] = \frac{1}{d}\left[\mathbb{I} + \sum_{\alpha=1}^{d+1}\sum_{k=1}^{d-1} \lambda_\alpha x_{\alpha k} U_\alpha^k\right], \tag{19}$$

which allows us to obtain the channel fidelity for the generalized Pauli channels,

$$F(P,\Lambda[P]) = \mathrm{Tr}(P\Lambda[P]) = \frac{1}{d}\left(1 + \sum_{\alpha=1}^{d+1}\lambda_\alpha \sum_{k=1}^{d-1}|x_{\alpha k}|^2\right). \tag{20}$$

Recall that P is a rank-1 projector, and hence

$$\mathrm{Tr}P^2 = \frac{1}{d}\left(1 + \sum_{\alpha=1}^{d+1}\sum_{k=1}^{d-1}|x_{\alpha k}|^2\right) = 1, \tag{21}$$

where we used the fact that U_α^k are traceless and mutually orthogonal. The above condition is equivalent to

$$\sum_{\alpha=1}^{d+1}\sum_{k=1}^{d-1}|x_{\alpha k}|^2 = d-1. \tag{22}$$

Therefore, $F(P,\Lambda[P])$ reaches its minimal and maximal values if the only nonvanishing coefficients are $x_{\alpha_* k}$ and $x_{\alpha_\# k}$, respectively, where $\lambda_{\alpha_*} = \lambda_{\min}$ and $\lambda_{\alpha_\#} = \lambda_{\max}$. The minimal and maximal channel fidelities are reached at the projectors onto the mutually unbiased bases vectors,

$$f_{\min}(\Lambda) = F\big(P_k^{(\alpha_*)}, \Lambda\big[P_k^{(\alpha_*)}\big]\big), \tag{23}$$

$$f_{\max}(\Lambda) = F\big(P_k^{(\alpha_\#)}, \Lambda\big[P_k^{(\alpha_\#)}\big]\big), \tag{24}$$

where, from Eq. (9),

$$P_k^{(\alpha)} = \frac{1}{d}\left(\mathbb{I} + \sum_{l=1}^{d-1} \omega^{-kl} U_\alpha^l\right). \tag{25}$$

■

Remark 1. The minimal and maximal channel fidelities from Theorem 1 can be equivalently written in terms of the probability distribution p_α as

$$f_{\min}(\Lambda) = p_0 + \min_{\alpha>0} p_\alpha, \tag{26}$$

$$f_{\max}(\Lambda) = p_0 + \max_{\alpha>0} p_\alpha. \tag{27}$$

IV. MAXIMAL OUTPUT *p*-NORM

Quantum channels Λ transform input states ρ into output states $\Lambda[\rho]$. While it is relatively easy to control the input, the attainable output states depend on the channel's properties. In particular, it is not always possible to find such ρ for which $\Lambda[\rho]$ is pure. In these cases, one can ask how close the outputs are to pure states. This is measured by optimal output purity measures. One such measure is the maximal output p-norm defined as follows:

$$\nu_p(\Lambda) := \max_P \|\Lambda[P]\|_p, \tag{28}$$

where the Schlatten p-norm reads

$$\|\Lambda[P]\|_p := (\mathrm{Tr}\Lambda[P]^p)^{1/p}, \quad 1 \leqslant p < \infty, \tag{29}$$

$$\|\Lambda[P]\|_\infty := \max_Q \mathrm{Tr}(Q\Lambda[P]), \tag{30}$$

and Q is a rank-1 projector. For product channels, it is known that

$$\nu_p(\Lambda\otimes\Phi) \geqslant \nu_p(\Lambda)\nu_p(\Phi). \tag{31}$$

The maximal output p-norm is multiplicative if

$$\nu_p(\Lambda\otimes\Phi) = \nu_p(\Lambda)\nu_p(\Phi) \tag{32}$$

and weakly multiplicative if the above equality is satisfied only for $\Phi = \Lambda$. Fukuda [27] proved that, if Eq. (32) is satisfied for all $\Phi : \mathbb{I}_{d_1}/d_1 \longmapsto \mathbb{I}_{d_2}/d_2$, then it holds for any Φ.

Nathanson and Ruskai [8] derived the exact formula for the maximal output 2-norm of the generalized Pauli channel,

$$\nu_2(\Lambda) = \sqrt{\frac{1}{d}\big[1 + (d-1)\max_\alpha \lambda_\alpha^2\big]}. \tag{33}$$

They also conjectured that the maximal output p-norm is achieved on the projectors $P_k^{(\alpha)}$ onto the mutually unbiased bases vectors and proved it for two special cases: $p = 2$ and $p = \infty$. Indeed, the maximal value of $\|\Lambda[P]\|_2$ is reached at $P_k^{(\alpha_0)}$, where α_0 numbers the eigenvalue whose module is maximal. Finally, Nathanson and Ruskai [8] showed that $\nu_2(\Lambda\otimes\Phi)$ is multiplicative for the generalized Pauli channels Λ (with arbitrary Φ). From the multiplicativity of $\nu_2(\Lambda)$, it follows that the maximal channel fidelity can be weakly multiplicative.

Proposition 1. Assume that the generalized Pauli channel has nonnegative eigenvalues $\lambda_\alpha \geqslant 0$. Then, its maximal channel fidelity on pure input states is weakly multiplicative in the sense that

$$f_{\max}(\Lambda\otimes\Lambda) = f_{\max}^2(\Lambda). \tag{34}$$

Proof. Note that every generalized Pauli channel Λ with non-negative eigenvalues can be written as the composition

$$\Lambda = \Lambda'^\dagger\Lambda' \tag{35}$$

of the generalized Pauli channel Λ' and its adjoint $\Lambda'^\dagger$ defined by $\mathrm{Tr}(X\Lambda'[Y]) =: \mathrm{Tr}(\Lambda'^\dagger[X]Y)$. This is possible under the condition that the eigenvalues $\lambda'_\alpha = \sqrt{\lambda_\alpha}$. Next, observe that the maximal output 2-norm in Eq. (33) and the maximal channel fidelity in Theorem 1 are related via

$$\nu_2^2(\Lambda') = f_{\max}(\Lambda'^\dagger\Lambda'). \tag{36}$$

Finally, from the multiplicativity of the maximal output 2-norm, it follows that

$$f_{\max}(\Lambda) = \nu_2^2(\Lambda') = \nu_2(\Lambda'\otimes\Lambda') = \sqrt{f_{\max}(\Lambda\otimes\Lambda)}. \tag{37}$$

■

Remark 2. The maximal output 2-norm $\nu_2(\Lambda)$ and the maximal channel fidelity on pure input states $f_{\max}(\Lambda)$ are attained at the same state $P_k^{(\alpha_\#)}$ if and only if $\lambda_\alpha \geqslant 0$, where $\lambda_{\alpha_\#} = \max_\alpha \lambda_\alpha$.

Now, we derive the formula for the maximal output ∞-norm.

Proposition 2. For the generalized Pauli channel Λ, the maximal output ∞-norm is given by

$$\nu_\infty(\Lambda) = \max_{P,Q} \mathrm{Tr}(Q\Lambda[P]) = \frac{1}{d}\max\{1 + (d-1)\lambda_{\max}, 1 - \lambda_{\min}\}, \tag{38}$$

where $\lambda_{\max} = \max_\alpha \lambda_\alpha$ and $\lambda_{\min} = \min_\alpha \lambda_\alpha$.

Proof. To calculate $\mathrm{Tr}(Q\Lambda[P])$ for the generalized Pauli channel Λ, let us parametrize the projectors P, Q by Eq. (18) and

$$Q = \frac{1}{d}\left(\mathbb{I} + \sum_{\alpha=1}^{d+1}\sum_{k=1}^{d-1} y_{\alpha k} U_\alpha^k\right). \tag{39}$$

Note that the condition $0 \leqslant \mathrm{Tr}PQ \leqslant 1$ is equivalent to

$$-1 \leqslant \sum_{\alpha=1}^{d+1}\sum_{k=1}^{d-1} x_{\alpha k}\bar{y}_{\alpha k} \leqslant d-1. \tag{40}$$

Moreover, from Eq. (19), it follows that

$$\mathrm{Tr}(Q\Lambda[P]) = \frac{1}{d}\left[1 + \sum_{\alpha=1}^{d+1}\sum_{k=1}^{d-1} \lambda_\alpha x_{\alpha k}\bar{y}_{\alpha k}\right]. \tag{41}$$

The maximal value of the above quantity is reached when $x_{\alpha k} = y_{\alpha k} = 0$ for $\alpha \neq \alpha_*$.

(1) If $\lambda_{\alpha_*} \geqslant 0$, then the maximal value

$$\max_{P,Q} \mathrm{Tr}(Q\Lambda[P]) = \frac{1}{d}\left[1 + \lambda_{\alpha_*}\sum_{k=1}^{d-1} x_{\alpha_* k}\bar{y}_{\alpha_* k}\right] \tag{42}$$

of Eq. (41) is attained for the maximal bound of Eq. (40),

$$\sum_{k=1}^{d-1} x_{\alpha_* k}\bar{y}_{\alpha_* k} = d-1, \tag{43}$$

and $\lambda_{\alpha_*} = \lambda_{\max}$. In this case,

$$\max_{P,Q} \mathrm{Tr}(Q\Lambda[P]) = \frac{1}{d}[1+(d-1)\lambda_{\max}]. \tag{44}$$

From the results for 2-norms, we know that the above formula corresponds to the choice

$$Q = P = P_k^{(\alpha_*)} = \frac{1}{d}\left(\mathbb{I} + \sum_{l=1}^{d-1}\omega^{-kl}U_{\alpha_*}^l\right). \tag{45}$$

(2) If $\lambda_{\alpha_*} \leqslant 0$, then the maximal value

$$\max_{P,Q} \mathrm{Tr}(Q\Lambda[P]) = \frac{1}{d}\left[1 - |\lambda_{\alpha_*}|\sum_{k=1}^{d-1} x_{\alpha_* k}\bar{y}_{\alpha_* k}\right] \tag{46}$$

of Eq. (41) is attained for the minimal bound of (40),

$$\sum_{k=1}^{d-1} x_{\alpha_* k}\bar{y}_{\alpha_* k} = -1, \tag{47}$$

and $\lambda_{\alpha_*} = \lambda_{\min}$. In this case,

$$\max_{P,Q} \mathrm{Tr}(Q\Lambda[P]) = \tfrac{1}{d}[1-\lambda_{\min}]. \tag{48}$$

The projectors P and Q that maximize Eq. (41) are as follows:

$$P = P_k^{(\alpha_*)} = \frac{1}{d}\left(\mathbb{I} + \sum_{l=1}^{d-1}\omega^{-kl}U_{\alpha_*}^l\right), \tag{49}$$

$$Q = P_m^{(\alpha_*)} = \frac{1}{d}\left(\mathbb{I} + \sum_{l=1}^{d-1}\omega^{-ml}U_{\alpha_*}^l\right), \tag{50}$$

where $k \neq m$.

■

Let us compare our results with the analysis done for the Pauli channels in Ref. [7]. For $d=2$, Eq. (38) produces

$$\nu_\infty(\Lambda) = \max_{P,Q} \mathrm{Tr}(Q\Lambda[P]) = \tfrac{1}{2}\max\{1+\lambda_{\max}, 1-\lambda_{\min}\}. \tag{51}$$

In terms of the probability distribution,

$$\nu_\infty(\Lambda) = \tfrac{1}{2}(1+\lambda_{\max}) = p_0 + p_{\max}, \tag{52}$$

provided that $\lambda_{\max}+\lambda_{\min} \geqslant 0$. Now, $\mathrm{Tr}(Q\Lambda[P])$ reaches the maximal value for $P=Q=P_0^{(\alpha_*)}$ or $P=Q=P_1^{(\alpha_*)}$, where $\lambda_{\max} = \lambda_{\alpha_*}$. Note that

$$\lambda_{\max}+\lambda_{\min} = p_0 - p_{\mathrm{mid}} \geqslant 0, \tag{53}$$

where $p_{\min} \leqslant p_{\mathrm{mid}} \leqslant p_{\max}$ and $\{p_{\min}, p_{\mathrm{mid}}, p_{\max}\} = \{p_1, p_2, p_3\}$. On the other hand,

$$\nu_\infty(\Lambda) = \tfrac{1}{2}(1-\lambda_{\min}) = p_{\mathrm{mid}} + p_{\max} \tag{54}$$

only when $\lambda_{\max}+\lambda_{\min} \leqslant 0$, i.e., $p_0 - p_{\mathrm{mid}} \leqslant 0$. This time, the maximum of $\mathrm{Tr}(Q\Lambda[P])$ is reached for $\{P = P_0^{(\alpha_*)}, Q = P_1^{(\alpha_*)}\}$ or $\{P = P_1^{(\alpha_*)}, Q = P_0^{(\alpha_*)}\}$, where $\lambda_{\min} = \lambda_{\alpha_*}$. These results coincide with the maximal output ∞-norms for the Pauli channels found in Ref. [7].

V. ASYMPTOTIC REGULARIZATIONS OF THE CHANNEL FIDELITY

After Ernst and Klesse [7], we introduce the nth regularization of the maximal channel fidelity on pure input states and the maximal output p-norm,

$$f_{\max}^{(n)}(\Lambda) = \sqrt[n]{f_{\max}(\Lambda^{\otimes n})}, \tag{55}$$

$$\nu_p^{(n)}(\Lambda) = \sqrt[n]{\nu_p(\Lambda^{\otimes n})}. \tag{56}$$

In particular, one talks about the asymptotic regularization if $n=\infty$. The authors consider the asymptotic regularization of the maximal channel fidelity as a *toy model* in the channel capacity problem. They prove the following relation:

$$f_{\max}(\Lambda) \leqslant \nu_\infty(\Lambda) \leqslant \nu_\infty^{(\infty)}(\Lambda) = f_{\max}^{(\infty)}(\Lambda). \tag{57}$$

From this formula and Proposition 1, there follow the corollaries below:

Corollary 1. If $\lambda_\alpha \geqslant 0$, then $f_{\max}^{(n)}(\Lambda)$ is weakly multiplicative, and hence

$$f_{\max}^{(n)}(\Lambda) = f_{\max}(\Lambda), \quad n = 1, 2, \ldots, \infty. \tag{58}$$

In particular,

$$f_{\max}(\Lambda) = \nu_\infty(\Lambda) = \nu_\infty^{(\infty)}(\Lambda) = f_{\max}^{(\infty)}(\Lambda). \tag{59}$$

Corollary 2. For $\lambda_{\max} \geqslant -\frac{1}{d-1}\lambda_{\min}$, the maximal output ∞-norm and the maximal channel fidelity coincide, $\nu_\infty(\Lambda) = f_{\max}(\Lambda)$. If additionally $\lambda_{\min} \geqslant 0$, then $\nu_\infty(\Lambda)$ is weakly multiplicative.

In the theory of open quantum systems, the evolution is given by dynamical maps $\Lambda(t)$—that is, the families of quantum channels parametrized by time $t \geqslant 0$. Assume that $\Lambda(t)$ is the solution of the master equation

$$\dot{\Lambda}(t) = \mathcal{L}(t)\Lambda(t), \quad \Lambda(0) = \mathbb{1}, \tag{60}$$

with the time-local generator $\mathcal{L}(t)$. An important property of such a dynamical map is that its eigenvalues are non-negative, $\lambda_\alpha(t) \geqslant 0$. It is easy to check that $\lambda_\alpha(t)$ satisfy the inequalities in Corollaries 1 and 2. Therefore, the generalized Pauli channels being the solutions of Eq. (60) are good examples of the quantum channels for which the maximal channel fidelity on pure input states and the maximal output ∞-norm are weakly multiplicative.

VI. CONCLUSIONS

We analyzed the channel fidelity for the generalized Pauli channels, which is the measure of distortion between input and output states. We found general analytical formulas for the minimal and maximal channel fidelity on pure input states and showed that the latter satisfies the weak multiplicativity conjecture for some classes of the generalized Pauli channels. Next, we focused our attention on the matter of purity of the output states, which is measured by the maximal output p-norm. For $p=\infty$, we derived the exact formula for the maximal output norm, and we also showed that it can be weakly multiplicative. Finally, we analyzed the regularized maximal channel fidelity on pure inputs, which is technically simpler than the channel capacity problem. It turns out that our results lead to interesting implications in the theory of open quantum systems. Namely, if the generalized Pauli channel is generated by the time-local generator, then its maximal channel fidelity on pure states and maximal output ∞-norm are both weakly multiplicative.

Many questions still remain unanswered. It would be interesting to determine whether the extremal channel fidelities and the maximal output ∞-norm are weakly multiplicative for the whole spectrum of eigenvalues λ_α. Also, one might wonder whether the channel fidelity for mixed states reaches its maximal value on pure states, which is known to be the case for its minimal value. The behavior of the regularized channel fidelities for $n < \infty$ requires further studies. Other open questions include the characterizations of the *Holevo-like* quantity [28]

$$\widetilde{\chi}_{1/2}(\Lambda) := \min_{\sigma} \max_{\rho} \ln(F(\sigma, \Lambda[\rho])), \tag{61}$$

as well as the minimum entanglement fidelity [29,30]

$$f_{\min}^{\text{ent}}(\Lambda) := \min_{P \in \mathcal{B}(\mathcal{H}_2)} F(P, (\mathbb{1}_1 \otimes \Lambda)[P]), \tag{62}$$

where Λ is the generalized Pauli channel on $\mathcal{B}(\mathcal{H}_2)$, and $\mathbb{1}_1$ is the identity channel on $\mathcal{B}(\mathcal{H}_1)$.

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