Positive-partial-transpose square conjecture for $n = 3$

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We present the positive-partial-transpose (PPT) square conjecture introduced by M. Christandl *Banff International Research Station Workshop: Operator Structures in Quantum Information Theory* (Banff International Research Station, Alberta, 2012). We prove the conjecture in the case $n = 3$ as a consequence of the fact that two-qutrit PPT states have Schmidt number of at most 2. The PPT square conjecture in the case of $n \geqslant 4$ is still open. We present an example to support the conjecture for $n = 4$.

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I. INTRODUCTION

Quantum operations are implemented by quantum channels. The quantum channel is a completely positive trace preserving (CPTP) map between two matrix algebras [1]. The composition of quantum channels is a fundamental operation in quantum information, and we need quantum channels to create useful entangled states for quantum-information tasks. Such channels are called positive-partial-transpose (PPT) channels. These channels turn out to be useful in quantum key distribution and dense coding protocols [2]. It has been further conjectured by Christandl that the compositions of PPT channels is an entanglement-breaking channel [3]. That is, the Choi matrix of the composition is a separable state [4]. The conjecture is known as the PPT square conjecture. Other results of limitations on the entanglement of the output state of a quantum key repeater protocol supports the PPT square conjecture. It has been proved that the state of Alice and Bob conditioned on any measurement by Charlie is always separable if Alice and Charlie share a PPT state and Bob and Charlie also share a PPT state [2,5,6]. Proving it would provide a deeper understanding of the difference between PPT and separable states, as well as a method for attacking the separability problem.

The conjecture has received a lot of attention recently. First the conjecture holds asymptotically when the distance between the iterates of every unital or trace-preserving PPT channel and the set of entanglement breaking maps tends to zero [7]. Furthermore, every unital PPT channel becomes entanglement breaking after a finite number of iterations [8]. For the finite-dimensional cases, it has been proved recently that the conjecture is valid in dimension 3 and some examples like Gaussian quantum channels are proved to support the conjecture in all dimensions [9]. As the first main result of this paper, we independently prove the same result by proposing a way of deciding separable states in $M_3 \otimes M_3$. We have claimed the independence of our result from [9] by private communication with A. Müller Hermes. This is shown in Theorem 3. Besides, our method can be extended to decide the separability of some $n \times 3$ PPT states in Theorem 5. This is the second main result of this paper. As far as we know, this is the latest progress on this long-standing open problem. To further investigate the PPT square conjecture for higher dimensions, we construct a 4×4 PPT entangled state, extract its PPT map, and show that the Choi matrix of composition of the PPT map is a separable state. So it supports the PPT square conjecture. We further discuss the example, and construct a more general family of maps satisfying Conjecture 2 in Theorem 7.

The rest of this paper is organized as follows. In Sec. II, we construct the definitions we use in this paper. We present the main problem as Conjecture 2, and prove the special case on two-qutrit states. We further generalize our results to $n \times 3$ states in Corollary 4. In Sec. III, we construct an example of the 4×4 PPT state to support Conjecture 2, and further investigate the example in Sec. IV. Finally we conclude in Sec. V.

II. PROVING THE PPT SQUARE CONJECTURE FOR $n = 3$

We shall work with bipartite states on the space $\mathcal{H}_A \otimes \mathcal{H}_B$. Linear maps that are both completely positive and completely copositive are called PPT maps. So the Choi matrix of the PPT map ϕ is a (non-normalized) PPT entangled state, i.e., $(I \otimes \phi)(|\psi\rangle\langle\psi|)$ is PPT entangled and $|\psi\rangle$ is the bipartite maximally entangled state. Let us consider the composition $\phi_2 \circ \phi_1$ of two PPT maps ϕ_1 and ϕ_2 where $\phi_1, \phi_2 \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$. Let $M_{m,n}$ be the set of $m \times n$ complex matrices and $B(M_{m,n})$ be the set of all linear maps on $M_{m,n}(\mathbb{C})$. If $m = n$ then we denote $M_n := M_{n,m}$. We have the Kraus decomposition for a quantum channel Λ, that is, $\Lambda(*) = \sum_i A_i(*)A_i^\dagger$ and $\sum_i A_i^\dagger A_i = I$.

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Definition 1. Let $C_{\phi_1\circ\phi_2}$ be the Choi matrix of the composition of two channels ϕ_1 and ϕ_2. We shall call $C_{\phi_1\circ\phi_2}$ the composition of two Choi matrices C_{ϕ_1} and C_{ϕ_2}.

All PPT maps on $M_2(\mathbb{C})$ are separable due to the Peres-Horodecki separation criterion [10,11]. So the first nontrivial case lies in $B(M_3(\mathbb{C}))$ and we shall always confine ourselves to the $n \geqslant 3$ cases. Obviously, the composition of two PPT maps is still PPT. The following conjecture is referred to as the PPT square conjecture. It is known that the two conjectures in Conjecture 2 are equivalent. We refer readers to recent progress on the conjecture in [12].

Conjecture 2. (i) If ϕ is a PPT map on M_n, then $\phi\circ\phi$ is separable.

(ii) If ϕ_1 and ϕ_2 are PPT maps on M_n, then $\phi_1\circ\phi_2$ is separable.

The following result proves a special case of Conjecture 2 (i). This is the first main result of this paper.

Theorem 3. Conjecture 2 (i) holds for $n = 3$.

proof. Suppose ρ is an arbitrary quantum state in $M_3 \otimes M_3$. So $\sigma := (I_3 \otimes \phi)(\rho)$ is a PPT state in $M_3 \otimes M_3$. It is known that σ has Schmidt number of at most 2 [13]. Let $\sigma = \sum_j p_j |a_j\rangle\langle a_j|$ where each $|a_j\rangle$ has Schmidt rank of at most 2. That is, $|a_j\rangle \in \mathcal{K}_j \simeq \mathbb{C}^3 \otimes \mathbb{C}^2$. So each state $(I_3 \otimes \phi)(|a_j\rangle\langle a_j|)$ is a PPT state in $M_3 \otimes M_2$ up to local equivalence. The Peres-Horodecki criterion says that $(I_3 \otimes \phi)(|a_j\rangle\langle a_j|)$ is separable. Using the convex sum of σ we obtain that $(I_3 \otimes \phi)(\sigma) = [I_3 \otimes (\phi\circ\phi)](\rho)$ is separable. Choosing ρ as the maximally entangled state implies the assertion. ■

The following is a corollary of Theorem 3. It provides a method of deciding separable states in $M_n \otimes M_3$.

Corollary 4. If ϕ_1, ϕ_2 are two PPT maps on M_3 and $\rho \in M_n \otimes M_3$ then the state $[I_n \otimes (\phi_1\circ\phi_2)]\rho$ is separable.

proof. It suffices to prove the assertion when ρ_A is the maximally mixed state, namely $\rho_A = \frac{1}{n} I_n$. We assume $\rho = \sum_j |\psi_j\rangle\langle\psi_j|$, where $|\psi_j\rangle = (A_j \otimes I_3)|\Psi_3\rangle$, where the isometry $A_j : \mathbb{C}^3 \to \mathbb{C}^n$ and $|\Psi_3\rangle = \frac{1}{\sqrt{3}} \sum_{i=0}^{2} |ii\rangle$ is the two-qutrit maximally entangled state. Using Theorem 3 and the equivalence of the two statements in Conjecture 2, we obtain that the state $[I_n \otimes (\phi\circ\phi)](|\psi_j\rangle\langle\psi_j|)$ is separable for any j. So the state $[I_n \otimes (\phi_1\circ\phi_2)]\rho$ is separable. This completes the proof. ■

Note that the proofs of Theorem 3 and Corollary 4 both apply to the case when ϕ, ϕ_1, ϕ_2 are not trace preserving. Further, the proof of Theorem 3 does not rely on the fact that $(I_3 \otimes \phi)(\rho)$ is a PPT. In fact, it relies on the fact that $(I_3 \otimes \phi)(\rho)$ has Schmidt number of at most 2 [13]. So we have the following result. This is the second main result of this paper.

Theorem 5. Suppose ϕ_1 is a PPT map on M_3, ϕ_2 is a completely positive map on M_3, and the bipartite state $\rho \in M_n \otimes M_3$. If $(I_n \otimes \phi_2)\rho$ has Schmidt number of at most 2 then the state $[I_n \otimes (\phi_1\circ\phi_2)]\rho$ is separable.

Corollary 4 and Theorem 5 provide two channels ϕ_1, ϕ_2 such that their combination becomes both entanglement-breaking channels. If ϕ_2 is a PPT map then $(I_n \otimes \phi_2)\rho$ has Schmidt number of at most 2. So Theorem 5 is stronger than Corollary 4.

The next case for studying Conjecture 2 is $M_4 \otimes M_4$. We propose the following conjecture. It is not included in Theorem 5, because the latter discusses only states in $M_n \otimes M_3$.

Conjecture 6. If ϕ is a PPT map on M_4 and $\rho \in M_4 \otimes M_4$ has Schmidt rank 2, then the state $[I_4 \otimes (\phi\circ\phi)]\rho$ is separable.

In the next section we construct an example supporting Conjecture 6 and thus Conjecture 2.

III. AN EXAMPLE TO SUPPORT THE PPT SQUARE CONJECTURE FOR *n* = 4

It is unknown whether Conjecture 2 for $n \geqslant 4$ is true. Some examples satisfying the conjecture have been constructed in [12]. They respectively rely on randomness, graphs, and Gaussian channels. In this section we shall construct an example satisfying Conjecture 2 for $n = 4$, and it does not rely on the above properties. In particular, we construct a 4×4 PPT entangled state $(I_4 \otimes \phi)(|\psi\rangle\langle\psi|)$ in (10) with Kraus operators in (11), such that the state $[I_4 \otimes (\phi\circ\phi)](|\psi\rangle\langle\psi|)$ defined via (26) turns out to be separable. Here ϕ is the PPT map on $\mathcal{H}_B$ defined in (10) and $|\psi\rangle = |00\rangle + |11\rangle + |22\rangle + |33\rangle$ is the 4×4 maximally entangled state.

We construct the following state inspired by [[14], Sec. VII B].

$$\rho_2 = (|00\rangle + |11\rangle + |22\rangle)(\langle 00| + \langle 11| + \langle 22|) + |02\rangle\langle 02| + |20\rangle\langle 20| \tag{1}$$

$$+ (|01\rangle + |10\rangle + |33\rangle)(\langle 01| + \langle 10| + \langle 33|) + |03\rangle\langle 03| + |31\rangle\langle 31| \tag{2}$$

$$+ |12\rangle\langle 12| + |13\rangle\langle 13| + |30\rangle\langle 30| + |21\rangle\langle 21|. \tag{3}$$

The partial transpose of ρ_2 is

$$\rho_2^\Gamma = (|00\rangle\langle 00| + |11\rangle\langle 11| + |22\rangle\langle 22| + |01\rangle\langle 10| + |10\rangle\langle 01| + |02\rangle\langle 20| + |20\rangle\langle 02| + |12\rangle\langle 21| + |21\rangle\langle 12|) \tag{4}$$

$$+ (|01\rangle\langle 01| + |10\rangle\langle 10| + |33\rangle\langle 33| + |00\rangle\langle 11| + |11\rangle\langle 00| + |31\rangle\langle 03| + |03\rangle\langle 31| + |13\rangle\langle 30| + |30\rangle\langle 13|) \tag{5}$$

$$+ |02\rangle\langle 02| + |20\rangle\langle 20| + |03\rangle\langle 03| + |31\rangle\langle 31| + |12\rangle\langle 12| + |13\rangle\langle 13| + |30\rangle\langle 30| + |21\rangle\langle 21|. \tag{6}$$

Let $\rho_2 \to \sigma := \frac{1}{3}[\mathrm{diag}(a,b,c,1)\otimes I_4]\rho_2[\mathrm{diag}(a,b,c,1)\otimes I_4]$ with positive a, b, c. Then

$$\sigma = \tfrac{1}{3}((a|00\rangle + b|11\rangle + c|22\rangle)(a\langle 00| + b\langle 11| + c\langle 22|) \tag{7}$$

$$+ (a|01\rangle + b|10\rangle + |33\rangle)(a\langle 01| + b\langle 10| + \langle 33|) \tag{8}$$

$$+ a^2|02\rangle\langle 02| + a^2|03\rangle\langle 03| + b^2|12\rangle\langle 12| + b^2|13\rangle\langle 13| + c^2|20\rangle\langle 20| + |30\rangle\langle 30| + c^2|21\rangle\langle 21| + |31\rangle\langle 31|). \tag{9}$$

To make ϕ a quantum channel, we require $\sigma_A = \frac{1}{3}(4a^2|0\rangle\langle 0| + 4b^2|1\rangle\langle 1| + 3c^2|2\rangle\langle 2| + 3|3\rangle\langle 3|) = I_4$. So we have $a = b = \frac{\sqrt{3}}{2}$, and $c = 1$. We have that

$$\sigma = (I_4 \otimes \phi)(|\psi\rangle\langle\psi|) := \sum_{j=1}^{10} (I_4 \otimes P_j)|\psi\rangle\langle\psi|(I_4 \otimes P_j^\dagger), \tag{10}$$

where the Kraus operators of map ϕ are

$$\begin{aligned} P_1 &= \frac{1}{\sqrt{3}}\,\mathrm{diag}(a, b, c, 0) = P_1^\dagger, \quad P_2 = \frac{1}{\sqrt{3}}(a|1\rangle\langle 0| + b|0\rangle\langle 1| + |3\rangle\langle 3|) = P_2^\dagger, \\ P_3 &= \frac{a}{\sqrt{3}}|2\rangle\langle 0|, \quad P_4 = \frac{a}{\sqrt{3}}|3\rangle\langle 0|, \quad P_5 = \frac{b}{\sqrt{3}}|2\rangle\langle 1|, \quad P_6 = \frac{b}{\sqrt{3}}|3\rangle\langle 1|, \\ P_7 &= \frac{c}{\sqrt{3}}|0\rangle\langle 2|, \quad P_8 = \frac{c}{\sqrt{3}}|1\rangle\langle 2|, \quad P_9 = \frac{1}{\sqrt{3}}|0\rangle\langle 3|, \quad P_{10} = \frac{1}{\sqrt{3}}|1\rangle\langle 3|, \end{aligned} \tag{11}$$

satisfy $\sum_j P_j^\dagger P_j = I_4$. So we obtain the PPT map $\phi : \mathbb{C}^4 \to \mathbb{C}^4$. For our purpose we investigate the PPT map $\phi \circ \phi$ with Kraus operators $\{P_i P_j\}$. By computing one can show that the nonzero operators in $\{P_i P_j\}$ are the following 44 matrices in $M_4 \otimes M_4$:

$$P_1 P_1 = \frac{1}{3}\,\mathrm{diag}(a^2, b^2, c^2, 0), \quad P_1 P_2 = P_2 P_1 = \frac{ab}{3}(|1\rangle\langle 0| + |0\rangle\langle 1|), \tag{12}$$

$$P_1 P_3 = \frac{ac}{3}|2\rangle\langle 0|, \quad P_1 P_5 = \frac{bc}{3}|2\rangle\langle 1|, \tag{13}$$

$$P_1 P_7 = \frac{ac}{3}|0\rangle\langle 2|, \quad P_1 P_8 = \frac{bc}{3}|0\rangle\langle 3|, \quad P_1 P_9 = \frac{a}{3}|1\rangle\langle 2|, \quad P_1 P_{10} = \frac{b}{3}|1\rangle\langle 3|, \tag{14}$$

$$P_2 P_2 = \frac{1}{3}(ab|0\rangle\langle 0| + ab|1\rangle\langle 1| + |3\rangle\langle 3|), \tag{15}$$

$$P_2 P_4 = \frac{a}{3}|3\rangle\langle 0|, \quad P_2 P_6 = \frac{b}{3}|3\rangle\langle 1|, \tag{16}$$

$$P_2 P_7 = \frac{ac}{3}|1\rangle\langle 2|, \quad P_2 P_8 = \frac{bc}{3}|0\rangle\langle 2|, \quad P_2 P_9 = \frac{a}{3}|1\rangle\langle 3|, \quad P_2 P_{10} = \frac{b}{3}|0\rangle\langle 3|, \tag{17}$$

and

$$P_3 P_1 = \frac{a^2}{3}|2\rangle\langle 0|, \quad P_3 P_2 = \frac{ab}{3}|2\rangle\langle 1|, \quad P_3 P_7 = \frac{ac}{3}|2\rangle\langle 2|, \quad P_3 P_9 = \frac{a}{3}|2\rangle\langle 3|, \tag{18}$$

$$P_4 P_1 = \frac{a^2}{3}|3\rangle\langle 0|, \quad P_4 P_2 = \frac{ab}{3}|3\rangle\langle 1|, \quad P_4 P_7 = \frac{ac}{3}|3\rangle\langle 2|, \quad P_4 P_9 = \frac{a}{3}|3\rangle\langle 3|, \tag{19}$$

$$P_5 P_1 = \frac{b^2}{3}|2\rangle\langle 1|, \quad P_5 P_2 = \frac{ab}{3}|2\rangle\langle 0|, \quad P_5 P_8 = \frac{bc}{3}|2\rangle\langle 2|, \quad P_5 P_{10} = \frac{b}{3}|2\rangle\langle 3|, \tag{20}$$

$$P_6 P_1 = \frac{b^2}{3}|3\rangle\langle 1|, \quad P_6 P_2 = \frac{ab}{3}|3\rangle\langle 0|, \quad P_6 P_8 = \frac{bc}{3}|3\rangle\langle 2|, \quad P_6 P_{10} = \frac{b}{3}|3\rangle\langle 3|, \tag{21}$$

$$P_7 P_1 = \frac{c^2}{3}|0\rangle\langle 2|, \quad P_7 P_3 = \frac{ac}{3}|0\rangle\langle 0|, \quad P_7 P_5 = \frac{bc}{3}|0\rangle\langle 1|, \tag{22}$$

$$P_8 P_1 = \frac{c^2}{3}|1\rangle\langle 2|, \quad P_8 P_3 = \frac{ac}{3}|1\rangle\langle 0|, \quad P_8 P_5 = \frac{bc}{3}|1\rangle\langle 1|, \tag{23}$$

$$P_9 P_2 = \frac{1}{3}|0\rangle\langle 3|, \quad P_9 P_4 = \frac{a}{3}|0\rangle\langle 0|, \quad P_9 P_6 = \frac{b}{3}|0\rangle\langle 1|, \tag{24}$$

$$P_{10} P_2 = \frac{1}{3}|1\rangle\langle 3|, \quad P_{10} P_4 = \frac{a}{3}|1\rangle\langle 0|, \quad P_{10} P_6 = \frac{b}{3}|1\rangle\langle 1|. \tag{25}$$

For convenience, we define the invertible diagonal matrix $D = \mathrm{diag}(4, 4, 3, 3)$. We perform the map ϕ on the state σ in (10), and investigate the separability of the resulting state as follows:

$$\gamma := (I_4 \otimes D)([I_4 \otimes (\phi \circ \phi)](|\psi\rangle\langle\psi|))(I_4 \otimes D) \tag{26}$$

$$= (I_4 \otimes D)\left(\sum_{j,k=1}^{10} (I_4 \otimes P_j P_k)\sigma(I_4 \otimes P_k^\dagger P_j^\dagger)\right)(I_4 \otimes D) \tag{27}$$

$$= (|00\rangle + |11\rangle + |22\rangle)(\langle 00| + \langle 11| + \langle 22|) \tag{28}$$

$$+ 2(|01\rangle + |10\rangle)(\langle 01| + \langle 10|) + (|00\rangle + |11\rangle + |33\rangle)(\langle 00| + \langle 11| + \langle 33|) \tag{29}$$

$$+ \frac{8}{3}|00\rangle\langle 00| + \frac{8}{3}|10\rangle\langle 10| + \frac{40}{9}|20\rangle\langle 20| + \frac{40}{9}|30\rangle\langle 30| + \frac{11}{3}|01\rangle\langle 01| + \frac{11}{3}|11\rangle\langle 11| + \frac{52}{9}|21\rangle\langle 21| + \frac{52}{9}|31\rangle\langle 31| \tag{30}$$

$$+ \frac{21}{16}|02\rangle\langle 02| + \frac{21}{16}|12\rangle\langle 12| + \frac{3}{4}|22\rangle\langle 22| + \frac{3}{4}|32\rangle\langle 32| + \frac{15}{8}|03\rangle\langle 03| + \frac{15}{8}|13\rangle\langle 13| + \frac{3}{2}|23\rangle\langle 23| + \frac{3}{2}|33\rangle\langle 33|. \tag{31}$$

The partial transpose of γ is

$$\gamma^\Gamma = |00\rangle\langle 00| + |11\rangle\langle 11| + |22\rangle\langle 22| + |01\rangle\langle 10| + |10\rangle\langle 01| + |12\rangle\langle 21| + |21\rangle\langle 12| + |20\rangle\langle 02| + |02\rangle\langle 20| \tag{32}$$

$$+ 2|01\rangle\langle 01| + 2|10\rangle\langle 10| + 2|00\rangle\langle 11| + 2|11\rangle\langle 00| \tag{33}$$

$$+ |00\rangle\langle 00| + |11\rangle\langle 11| + |33\rangle\langle 33| + |01\rangle\langle 10| + |10\rangle\langle 01| + |03\rangle\langle 30| + |30\rangle\langle 03| + |13\rangle\langle 31| + |31\rangle\langle 13| \tag{34}$$

$$+ \tfrac{8}{3}|00\rangle\langle 00| + \tfrac{8}{3}|10\rangle\langle 10| + \tfrac{40}{9}|20\rangle\langle 20| + \tfrac{40}{9}|30\rangle\langle 30| + \tfrac{11}{3}|01\rangle\langle 01| + \tfrac{11}{3}|11\rangle\langle 11| + \tfrac{52}{9}|21\rangle\langle 21| + \tfrac{52}{9}|31\rangle\langle 31| \tag{35}$$

$$+ \tfrac{21}{16}|02\rangle\langle 02| + \tfrac{21}{16}|12\rangle\langle 12| + \tfrac{3}{4}|22\rangle\langle 22| + \tfrac{3}{4}|32\rangle\langle 32| + \tfrac{15}{8}|03\rangle\langle 03| + \tfrac{15}{8}|13\rangle\langle 13| + \tfrac{3}{2}|23\rangle\langle 23| + \tfrac{3}{2}|33\rangle\langle 33|. \tag{36}$$

By splitting γ^Γ into the sum of a few two-qubit states $\sum_{i,j} p_{ij}|ij\rangle\langle ji|$ with $p_{ij} = 0,\ 1$, or 2 for $i \neq j$, one can show that each of the two-qubit states has PPT. So they are separable by the Peres-Horodecki criterion. Summing up then implies that γ^Γ is separable. The definition of γ implies that the state $[I_4 \otimes (\phi \circ \phi)](|\psi\rangle\langle\psi|)$ is also separable.

One can show that ρ_2^Γ is positive semidefinite. Further, ρ_2 is locally equivalent to the PPT entangled state in [[14], Sec. VII B]. So ρ_2 is a PPT entangled state, and $\mathrm{SN}(\rho_2) = \mathrm{SN}(\rho_2^\Gamma) = 2$. In particular $\mathrm{SN}(\rho_2) = 2$ follows from the fact that the two states in (1) and (2) both have Schmidt number 2. So Conjecture 6 holds for our example by choosing $\rho_2 = (I_4 \otimes \phi)\rho$ in Conjecture 6.

IV. DISCUSSION ON THE EXAMPLE

In this section we investigate the example of last section, and present Theorem 7 to cover the example. The state ρ_2 in (1)–(3) can be written as $\rho_2 = \rho_3 + \rho_4$ where the two states

$$\rho_3 = (|00\rangle + |11\rangle + |22\rangle)(\langle 00| + \langle 11| + \langle 22|) + |02\rangle\langle 02| + |20\rangle\langle 20| + |12\rangle\langle 12| + |21\rangle\langle 21|, \tag{37}$$

$$\rho_4 = (|01\rangle + |10\rangle + |33\rangle)(\langle 01| + \langle 10| + \langle 33|) + |03\rangle\langle 03| + |31\rangle\langle 31| + |13\rangle\langle 13| + |30\rangle\langle 30|. \tag{38}$$

So ρ_3 and ρ_4 respectively act on the space $\mathbb{C}^3 \otimes \mathbb{C}^3$ and $\mathrm{span}\{|0\rangle, |1\rangle, |3\rangle\} \otimes \mathrm{span}\{|0\rangle, |1\rangle, |3\rangle\}$. That is they are both two-qutrit states. Further, they both have Schmidt rank 2 from the last section. For the state ρ_2 in (1) and σ in (10), we obtain that $(I_4 \otimes \phi)\sigma$ is equivalent to the following state under stochastic local operations and classical communications (SLOCC):

$$(I_4 \otimes \phi)\rho_2 = \sum_{j=1}^{10} (I_4 \otimes P_j)\rho_2(I_4 \otimes P_j^\dagger) = \sum_{j=1}^{10} (I_4 \otimes P_j)\rho_3(I_4 \otimes P_j^\dagger) + \sum_{j=1}^{10} (I_4 \otimes P_j)\rho_4(I_4 \otimes P_j^\dagger), \tag{39}$$

where the last two sums respectively stand for the direct sum of a two-qutrit state and a product state in terms of the Kraus operators P_j's. So they are both separable states by Theorem 5. From this argument we derive Theorem 7. One can show that the example of the last section is a special case of Theorem 7.

Theorem 7. Suppose $\phi = \sum_j \phi_j$ such that for any j we have that $\phi_j : \mathbb{C}^{3\times p} \to \mathbb{C}^{3\times 3}$ are all PPT maps. Suppose ϕ' is a PPT map such that $(I \otimes \phi')\rho$ has Schmidt number 2. Then $(I_n \otimes \phi \circ \phi')(\rho)$ is separable.

V. CONCLUSIONS

We have shown that the PPT square conjecture holds for $n = 3$ as a consequence of the fact that 3×3 PPT states have a Schmidt number of at most 2. Further, we have proposed a conjecture as a special case of the PPT conjecture when $n = 4$ and the input quantum state ρ is of Schmidt number 2. We also have provided a nontrivial concrete example to support the PPT square conjecture when $n = 4$. In this case a counterexample is widely believed to exist. The next step for attacking the PPT square conjecture is to investigate more 4×4 PPT entangled states by checking their Schmidt numbers and the relevant PPT maps.

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