Symmetric versus bosonic extension for bipartite states

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A bipartite state ρ^{AB} has a k-symmetric extension if there exists a $(k+1)$-partite state $\rho^{AB_1B_2\ldots B_k}$ with marginals $\rho^{AB_i} = \rho^{AB}, \forall i$. The k-symmetric extension is called bosonic if $\rho^{AB_1B_2\ldots B_k}$ is supported on the symmetric subspace of $B_1B_2\ldots B_k$. Understanding the structure of symmetric and bosonic extension has various applications in the theory of quantum entanglement, quantum key distribution, and the quantum marginal problem. In particular, bosonic extension gives a tighter bound for the quantum marginal problem based on separability. In general, it is known that a ρ^{AB} admitting symmetric extension may not have bosonic extension. In this work, we show that, when the dimension of the subsystem B is 2 (i.e., a qubit), ρ^{AB} admits a k-symmetric extension if and only if it has a k-bosonic extension. Our result has an immediate application to the quantum marginal problem and indicates a special structure for qubit systems based on group representation theory.

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I. INTRODUCTION

Entanglement is one of the central mysteries of quantum mechanics—two or more parties can be correlated in the way that is much stronger than they can be in any classical way [1]. Due to its striking features, entanglement plays a key role in many quantum information processing tasks such as teleportation and quantum key distribution [2]. However, while entanglement has been investigated fairly extensively in the research literature, identifying entangled state remains a challenging task. Indeed, even for bipartite quantum systems, there is no generic procedure that can tell us whether a given bipartite state is entangled or not. Actually, the entanglement detection problem has long been known to be NP hard in general [3].

Consider a bipartite quantum system with Hilbert space $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$; here the subsystems are labeled A and B. A state ρ^{AB} is separable if it can be written as the convex combination $\sum_i p_i \rho^{A,i} \otimes \rho^{B,i}$ for a probability distribution p_i and states $\rho^{A,i}$ and $\rho^{B,i}$; otherwise, it is entangled [4]. In practice, one typically constructs detection criteria based on simple properties that are obeyed by all separable states; therefore, these are necessary but not sufficient conditions for separability. A most favored approach is the known partial transpose (PPT) criterion [5,6]. Another commonly used method is through the k-symmetric extension hierarchy.

A bipartite state ρ^{AB} is k-symmetric extendible if there is a global quantum state $\rho^{AB_1B_2\cdots B_k}$ whose marginals on A, B_i are equal to the given ρ_{AB} for $i = 1, 2, \ldots, k$. It was found that the set of all k-extendible states, denoted by Θ_k, is convex, with a hierarchy structure $\Theta_k \supset \Theta_{k+1}$, and besides, in the $k \to \infty$ limit, Θ_∞ converges exactly to the set of separable states which is also convex [7]. In other words, separable states are the only states that have k-copy symmetric extensions for all $k \geqslant 2$. This leads to a separability criteria which consists of a hierarchy of tests: one asks about whether or not a given state belongs to the k-extendible set Θ_k for increasing k.

A bipartite state ρ^{AB} is k-bosonic extendible if the global quantum state $\rho^{AB_1B_2\cdots B_k}$ with $\rho^{AB_i} = \rho^{AB}$ is supported on the symmetric subspace of $B_1B_2\ldots B_k$. Similarly, the set of all k-bosonic extendible states, denoted by $\bar{\Theta}_k$, is convex, with a hierarchy structure $\bar{\Theta}_k \supset \bar{\Theta}_{k+1}$, and in the $k \to \infty$ limit, $\bar{\Theta}_\infty = \Theta_\infty$ converges also to the set of separable states. Obviously $\bar{\Theta}_k \subseteq \Theta_k$; the k-bosonic extension is stronger than the k-symmetric extension. Based on k-symmetric and bosonic extension, effective numerical tests for separability have been developed [8–10].

It is natural to ask whether $\bar{\Theta}_k$ is strictly contained in Θ_k for any finite k. It turns out that the answer depends on k and the dimension of the system B. It is known that $\bar{\Theta}_2 = \Theta_2$ for $d_A = d_B = 2$, and $\bar{\Theta}_2 \subset \Theta_2$ for $d_A = d_B = 3$ [11]. An example of ρ^{AB} with two-symmetric extension that has no bosonic extension can be constructed from a pure three-qutrit state $\rho^{AB_1B_2}$ that is supported on the antisymmetric subspace of B_1B_2. This may indicate that $\bar{\Theta}_k \subset \Theta_k$ for $d_B > 2$. In this sense, the $d_B = 2$ case is of particular interest, given that $\bar{\Theta}_k = \Theta_k$ for $k = 2$ and ∞. One would naturally wonder whether it is also the case for any other k. Our main result of this work, as summarized below, shows it is indeed the case.

Main result. For $d_B = 2$, ρ^{AB} admits a k-symmetric extension if and only if it has a k-bosonic extension, for any k. That is, $\bar{\Theta}_k = \Theta_k$ for $d_B = 2$.

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This result finds an immediate application to the quantum marginal problem, also known as the consistency problem, which asks for the conditions under which there exist an N-particle density matrix ρ_N whose reduced density matrices (quantum marginals) on the subsets of particles $S_i \subset \{1, 2, \ldots, N\}$ are equal to the given density matrices ρ_{S_i} for all i [12]. The related problem in fermionic (bosonic) systems is the so-called N-representability problem, which inherits a long history in quantum chemistry [13,14].

In this sense, the k-symmetric extension problem is a special case of the quantum marginal problem, and the k-bosonic extension problem is intimately related to the N-representability problem [15]. Also it worth mentioning that the quantum marginal problem and the N-representability problem are in general very difficult. They were shown to be the complete problems of the complexity class QMA, even for the relatively simple case where the given marginals are two-particle states [16–18]. In other words, even with the help of a quantum computer, it is very unlikely that the quantum marginal problems can be solved efficiently in the worst case.

An interesting necessary condition of the k-symmetric and bosonic extension problem is derived in [19], based on the separability of ρ^{AB}. It shows that if ρ^{AB} has k-symmetric extension then the state $\tilde{\rho}_k^{AB} = (d_B\rho^A \otimes I_B + k\rho^{AB})/(d_B^2 + k)$ is separable. This condition can be strengthened if ρ^{AB} has k-bosonic extension, where the state $\tilde{\rho}_k^{AB} = (\rho^A \otimes I_B + k\rho^{AB})/(d_B + k)$ is separable.

Our main result hence has an immediate corollary as summarized below. And it has shown that this result leads to strong conditions for detecting the consistency of overlapping marginals [19].

Corollary. For $d_B = 2$, if ρ^{AB} has k-symmetric extension, then the state $\tilde{\rho}_k^{AB} = (\rho^A \otimes I_B + k\rho^{AB})/(k+2)$ is separable.

We organize our paper as follows: in Sec. II, we review some background of the known results for the relationship between $\bar{\Theta}_k$ and Θ_k; in Sec. III, we use the $d_B = 2, k = 3$ case as an example to demonstrate the proof idea of our main result; in Sec. IV, we discuss the proof idea for the general case; some further discussions are given in Sec. V; some technical details of the proof are discussed in the Appendices.

II. BACKGROUND AND PREVIOUS RESULTS

Consider the following notations:

$$\mathrm{Tr}_{B_2B_3\ldots B_k}[\rho^{AB_1\cdots B_k}] = \rho^{AB_1}, \tag{2.1a}$$

$$(\mathbb{1}^A \otimes P^{ij})\rho^{AB_1\cdots B_k}(\mathbb{1}^A \otimes P^{ij})^\dagger = \rho^{AB_1\cdots B_k}, \tag{2.1b}$$

where the operator $P^{ij} \in S_k$ is an element in permutation group S_k, which swaps the ith subsystem B_i and the jth subsystem B_j. The global state $\rho^{AB_1\cdots B_k}$ is called a k-symmetric extension of ρ^{AB_1}.

Equation (2.1b) requires the global state $\rho^{AB_1\cdots B_k}$ be invariant under any exchange of B_i and B_j, but it does not require that $\rho^{AB_1\cdots B_k}$ must support on a subspace with specific permutation symmetry; e.g., for a two-symmetric extendible state, its extension can be bosonic, which supports on the symmetric subspace only, or fermionic, whose support only resides on the antisymmetric subspace or, more generally, can be a mixture of both.

The following has already been known.

Fact 1. Given any two-symmetric extendible state ρ^{AB}, if $d_B = 2$, then a bosonic extension always exists.[1]

The original proof can be found in [20]. For consistency and readability, we include the proof here.

Proof. For $k = 2$, Eq. (2.1b) reduces to

$$(\mathbb{1}^A \otimes P^{B_1B_2})\rho^{AB_1B_2}(\mathbb{1}^A \otimes P^{B_1B_2})^\dagger = \rho^{AB_1B_2}, \tag{2.2}$$

which means that $\rho^{AB_1B_2}$ commutes with $(\mathbb{1}_A \otimes P_{B_1B_2})$. Therefore, they have common eigenvectors, say $\{|\phi_j\rangle\}$. Since $(\mathbb{1}^A \otimes P^{B_1B_2})^2 = \mathbb{1}$, we have

$$\mathbb{1}^A \otimes P^{B_1B_2}|\phi_j\rangle = \pm|\phi_j\rangle, \ \forall j. \tag{2.3}$$

Thus, generically, $\rho^{AB_1B_2}$ can be decomposed as

$$\rho^{AB_1B_2} = \sum_j \lambda_j^+|\phi_j^+\rangle\langle\phi_j^+| + \sum_l \lambda_l^-|\phi_l^-\rangle\langle\phi_l^-|, \tag{2.4}$$

where $\mathbb{1}^A \otimes P^{B_1B_2}|\phi_j^\pm\rangle = \pm|\phi_j^\pm\rangle$. Owning to the fact that B_1 and B_2 are two qubits, $|\phi_j^\pm\rangle$ can be further decomposed as

$$|\phi_j^+\rangle = \sum_k |\psi_{j,k}\rangle_A|\psi_k^+\rangle_{B_1B_2}, \tag{2.5a}$$

$$|\phi_j^-\rangle = |\xi_j\rangle_A|\psi^-\rangle_{B_1B_2}, \tag{2.5b}$$

where $|\psi_{j,k}\rangle_A$ and $|\xi_j\rangle_A$ are vectors of subsystem A, while $|\psi_k^+\rangle_{B_1B_2}$ and $|\psi^-\rangle_{B_1B_2}$ are the triplet and singlet states, respectively.[2] Replacing the singlet state $|\psi^-\rangle_{B_1B_2} \equiv \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ with $|\psi^+\rangle_{B_1B_2} \equiv \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, we get a new global state $\sigma^{AB_1B_2}$

$$\sigma^{AB_1B_2} = \sum_j \lambda_j^+ \sum_{k,k'} |\psi_{j,k}\rangle_A\langle\psi_{j',k'}|_A|\psi_k^+\rangle_{B_1B_2}\langle\psi_{k'}^+|_{B_1B_2} + \sum_l \lambda_l^-|\xi_l^-\rangle_A\langle\xi_l^-|_A|\psi^+\rangle_{B_1B_2}\langle\psi^+|_{B_1B_2}. \tag{2.6}$$

Obviously, $\sigma^{AB_1B_2}$ and $\rho^{AB_1B_2}$ have identical reduced density matrix ρ^{AB_1}, and $\sigma^{AB_1B_2}$ supports on the bosonic subspace. ■

The above proof can be roughly divided into two steps.

(1) Find the general form of global state after symmetric extension, which probably is a convex combination of bosonic extension and nonbosonic extension.

(2) Demonstrate that a bosonic extension, which preserves the reduced density matrix untouched, could be yielded by replacing the nonbosonic component with a bosonic one.

However, the above two-symmetric extension possesses properties that are not true for general k.

(a) When considering two-symmetric extension states, the permutation group contains only one nontrivial element $P^{B_1B_2}$; thus all permutations commute with global density matrix and have the common set of eigenvectors. While for general k, the permutation group itself is a non-Abelian group, thus Eq. (2.3) will not always hold.

[1]This claim does not always work when the subsystem has higher dimension.

[2]Here we do not require $|\psi_{j,k}\rangle_A$ and $|\xi_j\rangle_A$ to be normalized for simplicity in description.

(b) The dimension of nonbosonic subspace in two-qubit is one; hence we do not have to consider the off diagonal terms for a nonbosonic component. Again, it is no longer true for general k.

The above differences imply that the decomposition of general k-symmetric extendible states after extension will be more complicated than Eq. (2.6).

III. THREE-SYMMETRIC AND/OR BOSONIC EXTENSION

Before starting the proof for general k, we first take a look at the $k = 3$ case.

Consider the Hilbert space $\mathcal{T} \equiv V^{(1)} \otimes V^{(2)} \otimes V^{(3)}$ constituted by three-qubit B_1, B_2, and B_3, where each $V^{(i)}$ represents a qubit. $\mathcal{T}$ is spanned by eight vectors: $\{|000\rangle, |001\rangle, \dots, |111\rangle\}$. The bosonic subspace contains four linear independent vectors:

$$|000\rangle, \quad \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle),$$
$$\frac{1}{\sqrt{3}}(|011\rangle + |101\rangle + |110\rangle), \quad |111\rangle.$$

Since the cross term of bosonic subspace and nonbosonic subspace is forbidden, we shall only consider the density matrix that supports on nonbosonic subspace.

Notice that, permutation can only swap two or more subsystems, but keep the numbers of $|0\rangle$ and $|1\rangle$ constant. Thus the nonbosonic subspace could be further divided into two subspaces $V^{(a)}$ and $V^{(b)}$. $V^{(a)}$ has 2 $|0\rangle$ and 1 $|1\rangle$, while the other has 1 $|0\rangle$ and 2 $|1\rangle$.

First let us consider a density matrix $\tilde{\rho}^{AB_1B_2B_3}$ that purely supports on $\mathrm{End}(V_A) \otimes \mathrm{End}(V^{(a)})$. The dimension of this subspace is two and one can find a basis

$$|\psi_1^{(a)}\rangle \equiv \frac{1}{\sqrt{6}}(2|001\rangle - |010\rangle - |100\rangle), \tag{3.1a}$$

$$|\psi_2^{(a)}\rangle \equiv \frac{1}{\sqrt{2}}(|010\rangle - |100\rangle). \tag{3.1b}$$

It can be easily checked that, if Eq. (2.1b) was satisfied,

$$\begin{aligned}\tilde{\rho}^{AB_1B_2B_3} &= \tilde{\rho}^A \otimes \tilde{\rho}^{B_1B_2B_3}\\ &= \tilde{\rho}^A \otimes \left(\tfrac{1}{2}|\psi_1^{(a)}\rangle\langle\psi_1^{(a)}| + \tfrac{1}{2}|\psi_2^{(a)}\rangle\langle\psi_2^{(a)}|\right)\\ &\propto \tilde{\rho}^A \otimes \mathbb{1}^{\mathrm{End}(V^{(a)})},\end{aligned} \tag{3.2}$$

where $\mathbb{1}^{\mathrm{End}(V^{(a)})}$ is the identity operator in $\mathrm{End}(V^{(a)})$.

It is straightforward to check that $V^{(a)}$ is an invariant space under permutation group S_3; thus the representation on $V^{(a)}$ must be irreducible. Equation (2.1b) essentially requires that each group element of S_3 must commute with $\tilde{\rho}^{B_1B_2B_3}$. By Schur's Lemma, it must be proportional to $\mathbb{1}^{\mathrm{End}(V^{(a)})}$.

Likewise, one can write down a density matrix that purely supports on $\mathrm{End}(V_A) \otimes \mathrm{End}(V^{(b)})$

$$\begin{aligned}\bar{\rho}^{AB_1B_2B_3} &= \bar{\rho}^A \otimes \left(\tfrac{1}{2}|\psi_1^{(b)}\rangle\langle\psi_1^{(b)}| + \tfrac{1}{2}|\psi_2^{(b)}\rangle\langle\psi_2^{(b)}|\right)\\ &= \bar{\rho}^A \otimes \bar{\rho}^{B_1B_2B_3},\end{aligned} \tag{3.3}$$

where

$$|\psi_1^{(b)}\rangle \equiv \frac{1}{\sqrt{6}}(2|110\rangle + |101\rangle - |011\rangle), \tag{3.4a}$$

$$|\psi_2^{(b)}\rangle \equiv \frac{1}{\sqrt{2}}(|101\rangle - |011\rangle). \tag{3.4b}$$

Examining the density matrix supporting purely on the bosonic subspace, one could find that there might exist cross terms like $|\psi\rangle_A\langle\psi'|_A \otimes |000\rangle_{B_1B_2B_3}\langle 111|_{B_1B_2B_3}$. Thus it is reasonable to assume that cross terms mapping from $V^{(a)}$ to $V^{(b)}$ would also exist and vice versa.

After calculation, one could verify that a Hermitian cross term satisfying Eq. (2.1b) must be of the following form:

$$\hat{\rho}^{AB_1B_2B_3} = \hat{\rho}^A \otimes \hat{\rho}^{B_1B_2B_3} + \text{H.c.}, \tag{3.5}$$

where

$$\hat{\rho}^{B_1B_2B_3} = \tfrac{1}{2}|\psi_1^{(a)}\rangle\langle\psi_1^{(b)}| + \tfrac{1}{2}|\psi_2^{(a)}\rangle\langle\psi_2^{(b)}|. \tag{3.6}$$

A general density matrix $\rho^{AB_1B_2B_3}$ supporting on $\mathrm{End}(V_A) \otimes \mathrm{End}(V^{(a)} \oplus V^{(b)})$ should be a linear combination of $\tilde{\rho}^{AB_1B_2B_3}$, $\bar{\rho}^{AB_1B_2B_3}$, and $\hat{\rho}^{AB_1B_2B_3}$,[3]

$$\rho^{AB_1B_2B_3} = \alpha\tilde{\rho}^{AB_1B_2B_3} + \beta\bar{\rho}^{AB_1B_2B_3} + \gamma\hat{\rho}^{AB_1B_2B_3}. \tag{3.7}$$

Define

$$\tilde{\sigma}^{B_1B_2B_3} \equiv |\phi_1\rangle_{B_1B_2B_3}\langle\phi_1|_{B_1B_2B_3}, \tag{3.8a}$$

$$\bar{\sigma}^{B_1B_2B_3} \equiv |\phi_2\rangle_{B_1B_2B_3}\langle\phi_2|_{B_1B_2B_3}, \tag{3.8b}$$

$$\hat{\sigma}^{B_1B_2B_3} \equiv |\phi_1\rangle_{B_1B_2B_3}\langle\phi_2|_{B_1B_2B_3} + \text{H.c.}, \tag{3.8c}$$

where

$$|\phi_1\rangle_{B_1B_2B_3} \equiv \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle), \tag{3.9a}$$

$$|\phi_2\rangle_{B_1B_2B_3} \equiv \frac{1}{\sqrt{3}}(|110\rangle + |101\rangle + |011\rangle). \tag{3.9b}$$

It is easy to check that a bosonic global state $\sigma^{AB_1B_2B_3}$, which satisfies

$$\mathrm{Tr}_{(AB_1)^c}\,\sigma^{AB_1B_2B_3} = \mathrm{Tr}_{(AB_1)^c}\,\rho^{AB_1B_2B_3} = \rho^{AB_1}, \tag{3.10}$$

can be obtained by a simple replacement,

$$\tilde{\rho}^{B_1B_2B_3} \leftrightarrow \tilde{\sigma}^{B_1B_2B_3}, \tag{3.11a}$$

$$\bar{\rho}^{B_1B_2B_3} \leftrightarrow \bar{\sigma}^{B_1B_2B_3}, \tag{3.11b}$$

$$\hat{\rho}^{B_1B_2B_3} \leftrightarrow \hat{\sigma}^{B_1B_2B_3}, \tag{3.11c}$$

and a coefficient modification $\gamma' \leftrightarrow -\gamma$.[4] Clearly, as long as $\rho^{AB_1B_2B_3}$ is a density matrix, which means it has to be positive definite, normalized, and Hermitian, $\sigma^{AB_1B_2B_3}$ must also be a proper density matrix.

[3]Of course, such coefficients have to satisfy some constraints to ensure that $\rho^{AB_1B_2B_3}$ is a legal density matrix.

[4]By multiplying a global phase on basis in $V^{(b)}$, we could always absorb this minus sign or any other phase.

Though the idea of proof for $k = 3$ is quite similar to $k = 2$, discrepancies mentioned in Sec. II led to a more complicated version.

Our proof for general k will also be divided into three steps.

(1) At first we write down a general matrix (not necessarily a density matrix) after symmetric extension. This step could be further divided into two steps. First, we shall write down a general matrix in $\mathcal{H} \equiv \text{End}(V_A) \otimes \text{End}(\otimes_{i=1}^{k} V_{B_i})$. The key point of this step is to find an orthogonal and complete basis in $\mathcal{H}$. Such a basis should be able to conveniently describe the permutation symmetry. Secondly, we shall restrict such general matrix form according to Eq. (2.1b). As one could expect, not only does there exist the diagonal terms that represent mapping inside irreducible subspaces, cross terms that describe mapping between different irreducible subspaces also arise, as long as the representation on both irreducible subspaces are equivalent.

(2) Then, we shall verify that the former part will become the diagonal terms after partial trace, while the latter one contributes to the off-diagonal terms. Under our specific situation that B_1 is a qubit, only one independent off-diagonal term survives; thus cross terms can always be replaced with a bosonic version by properly modifying coefficients. On the other hand, the ratio between diagonal terms is always the same, regardless of whether they are obtained from a bosonic extension or not.

(3) The last piece is to demonstrate that the global matrix, obtained by replacing nonbosonic entries with bosonic ones, is positive semidefinite.

IV. CASE OF k EXTENSION

In this section we will prove the following main result.

Theorem 2. For any k-extendible state ρ^{AB}, if $d_B = 2$, then a k-bosonic extension always exists.

Consider a Hilbert space $\mathcal{T} = \bigotimes_{i=1}^{k} V^{(i)}$ constituted by k-qubit $B_1, B_2, \ldots, B_k$, whose computational basis is $\{\Phi_{i_1,i_2,\ldots,i_k} \equiv |i_1, i_2, \ldots, i_k\rangle\}$, where $i_1, i_2, \ldots, i_k = 0, 1$.

Each subsystem $V^{(i)}$ is invariant under SU(2) rotation, and transforms according to the two-dimensional irreducible representation $D^{(2)}$. Therefore, the Lie algebra su(2), which describes the infinite small rotation of SU(2), has the following matrix form on each $V^{(i)}$:

$$J_z^{(i)} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J_+^{(i)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$J_-^{(i)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

where we have set

$$|1\rangle^{(i)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |0\rangle^{(i)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$\mathcal{T}$ is also invariant under global SU(2) rotation, whose corresponding su(2) algebra is given by $\mathbf{J}_z \equiv \sum_i J_z^{(i)}$, $\mathbf{J}_\pm \equiv \sum_i J_\pm^{(i)}$. $\mathcal{T}$ transforms under representation $\otimes^k D^{(2)}$, which is not irreducible, but can be decomposed as a direct sum of a series of irreducible representations

$$\bigotimes^k D^{(2)} = \bigoplus_j m_j D^{(2j+1)}, \tag{4.1}$$

where m_j is the multiplicity of irreducible representation $D^{(2j+1)}$. This is equivalent to saying that $\mathcal{T}$ can be partitioned as a direct sum of orthogonal subspaces:

$$\mathcal{T} = \bigoplus_j m_j \mathcal{T}^{(2j+1)}. \tag{4.2}$$

In Appendix A, we manifest that such $\mathcal{T}^{(2j+1)}$ has particular permutation symmetry described by Young diagram $[\lambda]$.[5]

Since irreducible representation can be labeled by partition $[\lambda]$, we can rewrite Eq. (4.1) with new notation,

$$\bigotimes^k D^{[1]} = \bigoplus_{[\lambda]} C_k^{[\lambda]} D^{[\lambda]}, \tag{4.3}$$

where $C_k^{[\lambda]}$ is the multiplicity of SU(2) irreducible representation $D^{[\lambda]}$.

Two irreducible representation spaces $\mathcal{T}_\mu^{[\lambda]}$ and $\mathcal{T}_\nu^{[\lambda]}$ corresponding to the same Young diagram but different Young tableaus are orthogonal to each other. It is also known that there is no multiplicity in any weight subspace in a SU(2) representation, as long as it is irreducible. Thus one can safely use weight ω, the eigenvalue of $\mathbf{J}_z$, to label different states inside an irreducible subspace $\mathcal{T}_\mu^{[\lambda]}$. Therefore, $\{|[\lambda], \mu, \omega\rangle\}$ labels a complete basis of $\mathcal{T}$ one by one, where $[\lambda]$ tells inequivalent SU(2) representations, while μ differentiate equivalent ones. They together determine an orthogonal irreducible subspace and ω labels every different vector inside.

On the other hand, $\{|[\lambda], \mu, \omega\rangle\}$ can be interpreted in another way: ω describes the weight and $[\lambda]$ tells inequivalent S_k representations; thus these two parameter differentiate orthogonal invariant subspaces, while μ labels vectors inside.[6] From now on we shall use $|\omega_\mu^{[\lambda]}\rangle$ short for $|\omega, [\lambda], \mu\rangle$.

Any matrix $\rho^{AB_1B_2\cdots B_n} \in \text{End}(V_A) \otimes \text{End}(\mathcal{T})$ can be expressed as

$$\rho^{AB_1B_2\cdots B_k} = \sum_{[\lambda],[\lambda']} \sum_{\mu,\mu'} \sum_{\omega,\omega'} \sum_{\alpha,\alpha'} \left|\psi^\alpha_{\omega,[\lambda],\mu}\right\rangle\!\left\langle\psi^{\alpha'}_{\omega',[\lambda'],\mu'}\right| \times \otimes \left|\omega_\mu^{[\lambda]}\right\rangle\!\left\langle\omega'^{[\lambda']}_{\mu'}\right|, \tag{4.4}$$

where $|\psi^\alpha_{\omega,[\lambda],\mu}\rangle$ is a non-normalized state in V_A and α labels different states in V_A.

Inserting Eq. (4.4) into Eq. (2.1b), $\forall \pi \in S_k$ we get a series of constraints for $\rho^{AB_1B_2\cdots B_k}$:

$$\forall [\lambda], [\lambda']\omega, \omega' \text{ and } \mu, \mu', \sum_{\alpha,\alpha'} \left|\psi^\alpha_{\omega,[\lambda],\mu}\right\rangle\!\left\langle\psi^{\alpha'}_{\omega',[\lambda'],\mu'}\right| \times \sum_{\nu,\nu'} \mathcal{A}(\pi)^{[\lambda]}_{\mu,\nu} \mathcal{A}(\pi)^{[\lambda']*}_{\nu',\mu'} \left|\omega_\nu^{[\lambda]}\right\rangle\!\left\langle\omega'^{[\lambda']}_{\nu'}\right| = \sum_{\alpha,\alpha'} \left|\psi^\alpha_{\omega,[\lambda],\mu}\right\rangle\!\left\langle\psi^{\alpha'}_{\omega',[\lambda'],\mu'}\right| \left|\omega_\mu^{[\lambda]}\right\rangle\!\left\langle\omega'^{[\lambda']}_{\mu'}\right|, \tag{4.5}$$

where $\mathcal{A}^{[\lambda]}$ and $\mathcal{A}^{[\lambda']}$ are irreducible representations of permutation group S_k.

[5]Here $[\lambda] \equiv \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is a partition of integer k, where all λ_i are integers satisfying $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0$, $\sum_{i=1}^n \lambda_i = k$. Such partition describes an n-row Young diagram.

[6]The validity of this explanation is verified in Appendix B.

Define matrix

$$M(\omega,\omega',[\lambda],[\lambda']) \equiv \sum_{\mu,\mu'} M(\omega,\omega',[\lambda],[\lambda'])_{\mu\mu'} |\omega_\mu^{[\lambda]}\rangle\langle\omega_{\mu'}'^{[\lambda']}|, \quad (4.6)$$

where

$$M(\omega,\omega',[\lambda],[\lambda'])_{\mu\mu'} \equiv \sum_{\alpha,\alpha'} |\psi^{\alpha}_{\omega,[\lambda],\mu}\rangle\langle\psi^{\alpha'}_{\omega',[\lambda'],\mu'}|. \quad (4.7)$$

Thus $\forall \pi \in S_k$

$$\mathcal{A}^{[\lambda]}(\pi) M(\omega,\omega',[\lambda],[\lambda']) \mathcal{A}^{[\lambda']}(\pi)^\dagger = M(\omega,\omega',[\lambda],[\lambda']). \quad (4.8)$$

Schur's lemma guarantees that (a) when $[\lambda] \neq [\lambda']$, $M = 0$ and (b) when $[\lambda] = [\lambda']$, M is invertible.

Choose $|\omega_\mu^{[\lambda]}\rangle$ carefully such that the representations $\mathcal{A}^{[\lambda]}$ are identical, not just isomorphic, for different weights ω. Hence all $M(\omega,\omega',[\lambda],[\lambda])$ can be proportional to the corresponding identity matrix. Therefore, one could eliminate plenty of cross terms and restrict $\rho^{AB_1B_2\cdots B_k}$ to

$$\rho^{AB_1B_2\cdots B_k} = \sum_{[\lambda]}\sum_{\omega,\omega'}\left(\sum_{\alpha,\alpha'} |\psi^{\alpha}_{\omega,[\lambda]}\rangle\langle\psi^{\alpha'}_{\omega',[\lambda]}|\right) \times \otimes \frac{1}{d^{[\lambda]}} \sum_\mu |\omega_\mu^{[\lambda]}\rangle\langle\omega_\mu'^{[\lambda]}|, \quad (4.9)$$

where $d^{[\lambda]}$ is the dimension of S_k irreducible representation corresponding to Young diagram $[\lambda]$.

Determining RDM ρ^{AB_1} can be immediately reduced to calculating every possible combination of $[\lambda]$, ω, and ω'. For given $[\lambda]$, ω, and ω', one could temporally ignore system A and concentrate on group $\{B_1, B_2, \dots, B_k\}$. Then the task left is to calculate

$$\frac{1}{d^{[\lambda]}} \mathrm{Tr}_{(B_1)^c} \sum_\mu |\omega_\mu^{[\lambda]}\rangle\langle\omega_\mu'^{[\lambda]}|. \quad (4.10)$$

If $\omega = \omega'$, it is equivalent to consider a mixed state within a constant weight subspace. Hence

$$\frac{1}{d^{[\lambda]}} \mathrm{Tr}_{(B_1)^c} \sum_\mu |\omega_\mu^{[\lambda]}\rangle\langle\omega_\mu^{[\lambda]}| = t_0|0\rangle\langle 0| + t_1|1\rangle\langle 1|. \quad (4.11)$$

According to [21],[7]

$$\frac{t_0}{t_1} = \frac{k-2\omega}{k+2\omega}. \quad (4.12)$$

Since the ratio between diagonal terms is solely determined by the number of subsystems k and weight ω, any nonbosonic extensions can be directly replaced by a bosonic version in same weight subspace

$$\frac{1}{d^{[\lambda]}} \sum_\mu |\omega_\mu^{[\lambda]}\rangle\langle\omega_\mu^{[\lambda]}| \leftrightarrow |\omega^S\rangle\langle\omega^S|. \quad (4.13)$$

If $\omega - \omega' = \pm 1$, nonzero contribution of Eq. (4.10) would be proportional to $|1\rangle\langle 0|$ and $|0\rangle\langle 1|$, respectively. Different $[\lambda]$ only affect the proportion coefficients. Choosing proper coefficients for bosonic extension will exactly recover the result of nonbosonic ones,[8]

$$\frac{1}{d^{[\lambda]}} \sum_\mu |\omega_\mu^{[\lambda]}\rangle\langle\omega_\mu'^{[\lambda]}| \leftrightarrow \beta^{[\lambda]}_{\omega,\omega'} |\omega^S\rangle\langle\omega'^S|, \quad (4.14)$$

where

$$\beta^{[\lambda]}_{\omega,\omega'} = \sqrt{\frac{\left(\frac{\lambda_1-\lambda_2}{2}-\omega\right)\left(\frac{\lambda_1-\lambda_2}{2}+\omega+1\right)}{\left(\frac{k}{2}-\omega\right)\left(\frac{k}{2}+\omega+1\right)}}\,\delta_{\omega\pm 1,\omega'}.$$

Obviously, $0 < \beta^{[\lambda]}_{\omega,\omega'} \leqslant 1$.

If $|\omega - \omega'| \geqslant 2$, Eq. (4.10) would vanish. According to Eq. (4.9), $\rho^{AB_1B_2\cdots B_k}$ has a series of bosonic version $\sigma^{AB_1B_2\cdots B_k}$ (not all of them are proper density matrix)

$$\sigma^{AB_1B_2\cdots B_k} \equiv \sum_{[\lambda]}\sum_{\omega,\omega'}\left(\sum_{\alpha,\alpha'} |\psi^{\alpha}_{\omega,[\lambda]}\rangle\langle\psi^{\alpha'}_{\omega',[\lambda]}|\right) \times \otimes p^{[\lambda]}_{\omega,\omega'} |\omega^S\rangle\langle\omega'^S|, \quad (4.15)$$

where $p^{[\lambda]}_{\omega,\omega'}$ are coefficients

$$p^{[\lambda]}_{\omega,\omega'} = \begin{cases} 1, & \omega = \omega', \\ \sqrt{\frac{(\frac{\lambda_1-\lambda_2}{2}-\omega)(\frac{\lambda_1-\lambda_2}{2}+\omega+1)}{(\frac{k}{2}-\omega)(\frac{k}{2}+\omega+1)}}, & \omega \pm 1 = \omega', \\ \text{arbitraryvalue}, & \text{others}. \end{cases} \quad (4.16)$$

We can always find proper $p^{[\lambda]}_{\omega,\omega'}$ such that $\sigma^{AB_1B_2\cdots B_k}$ can be decomposed as a convex combination of a series of pure bosonic extensions.[9] Hence $\sigma^{AB_1B_2\cdots B_k}$ is positive definite.

Therefore, we have finished the proof of Theorem 2.

V. DISCUSSION

We have shown that if a bipartite state ρ^{AB} has a k-symmetric extension

$$(\mathbb{1}^A \otimes \pi)\rho^{AB_1B_2\cdots B_k}(\mathbb{1}^A \otimes \pi)^\dagger = \rho^{AB_1B_2\cdots B_k}, \quad (5.1)$$

with $\rho^{AB_i} = \rho^{AB}$, it must also have a bosonic extension $\sigma^{AB_1B_2\cdots B_k}$ satisfying

$$\sigma^{AB_1B_2\cdots B_k} = \sum_\alpha p_\alpha |\phi_\alpha\rangle^{AB_1B_2\cdots B_k}\langle\phi_\alpha|^{AB_1B_2\cdots B_k}, \quad (5.2a)$$

$$(\mathbb{1}^A \otimes \pi)|\phi_\alpha\rangle^{AB_1B_2\cdots B_k} = |\phi_\alpha\rangle^{AB_1B_2\cdots B_k}, \quad (5.2b)$$

where $\pi \in S_k$ is an arbitrary permutation operator and p_α is a probability distribution as long as B_1 is a qubit.

Notice that Eq. (4.9) was essentially saying that $\rho^{AB_1B_2\cdots B_k}$ could be further decomposed into two major parts. The first part contributed to the "diagonal terms," whose ratio is identical as long as global state laid in same weight subspaces. Hence this part did not contain information about permutation symmetry. The second type contributed to the "off-diagonal

[7] See details in Appendix C.

[8] See details in Appendix D.

[9] See Appendix E for details.

terms." They probably carried information about permutation symmetry.

Since we have discussed the case for k-qubit extension, it is natural to consider the k-qudit problem, i.e., $d_B > 2$. However, it turns out that the situation is much more complicated, since there is now more than one pair of off-diagonal terms after partial trace. In the qubit case, we can peel off all the off-diagonal terms at one time for each given $[\lambda]$. Due to the properties of $\beta^{[\lambda]}_{\omega,\omega'}$, the residual "diagonal" matrix is guaranteed to be positive definite. In the qudit case, we might have to peel off the off-diagonal terms in several steps. After peeling off all off-diagonal terms, the residual might not be always positive definite.

Example 3. Consider a tripartite pure state on $V_A \otimes V_{B_1} \otimes V_{B_2}$ of the form

$$|\psi\rangle = \alpha(|012\rangle - |021\rangle) + \beta(|120\rangle - |102\rangle) + \gamma(|201\rangle - |210\rangle), \quad (5.3)$$

where $\alpha, \beta, \gamma \neq 0$ are all different. Clearly, $|\psi\rangle$ is a fermionic extension of $\rho^{AB_1} \equiv \mathrm{Tr}_{B_2}[|\psi\rangle\langle\psi|]$. The diagonal terms of $\bar{\rho}^{AB_1}$ are

$$\bar{\rho}^{AB_1} = \alpha\alpha^*(|01\rangle\langle 01| + |02\rangle\langle 02|) + \beta\beta^*(|12\rangle\langle 12| + |10\rangle\langle 10|) + \gamma\gamma^*(|20\rangle\langle 20| + |21\rangle\langle 21|), \quad (5.4)$$

while the off-diagonal $\tilde{\rho}^{AB_1}$ read

$$\tilde{\rho}^{AB_1} = -\alpha\beta^*|01\rangle\langle 10| - \alpha\gamma^*|02\rangle\langle 20| - \beta\gamma^*|12\rangle\langle 21| + \mathrm{H.c.} \quad (5.5)$$

If ρ^{AB_1} had a bosonic extension $\sigma^{AB_1B_2}$, then $\mathrm{Tr}_{B_2}\sigma^{AB_1B_2}$ should produce exact off-diagonal terms as $\tilde{\rho}^{AB_1}$.

$-\alpha\beta^*|01\rangle\langle 10|$ can be obtained in three different ways:[10]

$$\mathrm{Tr}_{B_2}[\alpha\beta^*|0\rangle\langle 1| \otimes (|10\rangle + |01\rangle)\langle 00|], \quad (5.6a)$$

$$\mathrm{Tr}_{B_2}[\alpha\beta^*|0\rangle\langle 1| \otimes |11\rangle(\langle 10| + \langle 01|)], \quad (5.6b)$$

$$\mathrm{Tr}_{B_2}[\alpha\beta^*|0\rangle\langle 1| \otimes (|12\rangle + |21\rangle)(\langle 02| + \langle 20|)]. \quad (5.6c)$$

However, in order to keep $\sigma^{AB_1B_2}$ positive definite, the first two choices will have to introduce $|00\rangle\langle 00|$ and $|11\rangle\langle 11|$, respectively, which did not appear in diagonal terms, and hence destroy the positive definiteness. Therefore, we have to use Eq. (5.6c) to obtain $-\alpha\beta^*|01\rangle\langle 10|$ in $\mathrm{Tr}_{B_2}\sigma^{AB_1B_2}$.

After replacing all terms in $\tilde{\rho}^{AB_1}$ with corresponding bosonic extension, the off-diagonal terms in $\sigma^{AB_1B_2}$ are

$$\tilde{\sigma}^{AB_1B_2} = -\{\alpha\beta^*|0\rangle\langle 1| \otimes (|12\rangle + |21\rangle)(\langle 02| + \langle 20|) + \alpha\gamma^*|0\rangle\langle 2| \otimes (|12\rangle + |21\rangle)(\langle 01| + \langle 10|) + \beta\gamma^*|1\rangle\langle 2| \otimes (|20\rangle + |02\rangle)(\langle 01| + \langle 10|) + \mathrm{H.c.}\}. \quad (5.7)$$

Because of the global minus sign, it is impossible to peel off all three pairs of off-diagonal terms or any two of them at one time. In other words, we have to peel them off pair by pair. After matching the corresponding diagonal terms with the off-diagonal pair, the "residual" diagonal matrix in Tr_{B_3} will be

$$(\alpha\alpha^* - pp^* - ss^*)(|01\rangle\langle 01| + |02\rangle\langle 02|) + (\beta\beta^* - qq^* - uu^*)(|12\rangle\langle 12| + |10\rangle\langle 10|) + (\gamma\gamma^* - tt^* - vv^*)(|20\rangle\langle 20| + |21\rangle\langle 21|), \quad (5.8)$$

where

$$pq^* = \alpha\beta^*, \quad st^* = \alpha\gamma^*, \quad uv^* = \beta\gamma^*. \quad (5.9)$$

It is easy to check that

$$\alpha\alpha^* - pp^* - ss^* + \beta\beta^* - qq^* - uu^* + \gamma\gamma^* - tt^* - vv^* < 0. \quad (5.10)$$

Therefore, there does not exist a bosonic extension for ρ^{AB_1}.

However, there may be some coincident situation, under which the "residual" diagonal matrix is positive definite; hence the symmetric extendibility of ρ^{AB_1} is equivalent to its bosonic extendibility. We do not know whether such coincidences are purely accidental or there are some underlining profound reasons. We leave this for future research.

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APPENDIX A: PERMUTATION SYMMETRY OF IRREDUCIBLE SUBSPACE $\mathcal{T}^{(2j+1)}$

We define P^{mn}, an element in permutation group S_k, as

$$P^{mn}\Phi_{i_1,\dots,i_m,\dots,i_n,\dots,i_k} = \Phi_{i_1,\dots,i_n,\dots,i_m,\dots,i_k}. \quad (A1)$$

It is obvious that the permutation of the indices of subsystems commutes with the tensor product of individual SU(2) rotation, and hence the global SU(2) rotation. Therefore, any subspace

$$\mathcal{T}^{[\lambda]}_\mu \equiv \mathcal{Y}^{[\lambda]}_\mu \mathcal{T} \quad (A2)$$

projected by a standard Young tableau[11] $\mathcal{Y}^{[\lambda]}_\mu$ is also invariant under the global SU(2) rotation. Here $[\lambda]$ describes an n-row Young diagram and μ differentiates standard Young tableaus corresponding to the same $[\lambda]$.

Since $\{\Phi_{i_1,i_2,\dots,i_k}\}$ is a complete basis of $\mathcal{T}$, $\{\mathcal{Y}^{[\lambda]}_\mu\Phi_{i_1,i_2,\dots,i_k}\}$ must be a supercomplete basis of $\mathcal{T}^{[\lambda]}_\mu$. From now on, to describe the vector $\mathcal{Y}^{[\lambda]}_\mu\Phi_{i_1,i_2,\dots,i_k}$, we shall use a graphic way: insert i_j into the entry of Young diagram $\mathcal{Y}^{[\lambda]}$ that is filled by j in standard Young tableau $\mathcal{Y}^{[\lambda]}_\mu$.

[10] Of course it can be a mixture of these three.

[11] Standard Young tableau is obtained by filling $\{1, 2, \dots, k\}$ into all entries in such a manner that each row and each column keeps in increasing order.

Example 4:

$$\begin{aligned}\begin{array}{|c|c|}\hline i_1 & i_3 \\ \hline i_2 \\ \cline{1-1}\end{array} &\equiv \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\Theta_{i_1,i_2,i_3} \\ &= (E+(13))(E-(12))\Phi_{i_1,i_2,i_3} \\ &= \Phi_{i_1,i_2,i_3} + \Phi_{i_3,i_2,i_1} - \Phi_{i_2,i_1,i_3} - \Phi_{i_3,i_1,i_2}. \end{aligned} \quad \text{(A3)}$$

Because each $V^{(i)}$ is a two-dimensional subsystem, and for any given Young tableau, its corresponding Young operator antisymmetrize each column, a Young diagram with three or more rows would project into the empty space; thus from now on we only consider a one- or two-row Young diagram.

Moreover, if two states belonging to a same invariant subspace, say $\mathcal{T}_\mu^{[\lambda]}$, are identical except interchanging entries in a given column, their difference would only be a factor -1.

Example 5.

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 \\ \cline{1-1}\end{array} = -1 \times \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}.$$

Therefore, if a state fills two zeros or two ones in the same column, such state must vanish.

It can be easily verified that the global SU(2) representation on $\mathcal{T}_\mu^{[\lambda]}$ is irreducible, by the following observation.

Observation 6. There is a single highest weight state $|\omega_M\rangle$ in each $\mathcal{T}_\mu^{[\lambda]}$

$$|\omega_M\rangle = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & \ldots\ldots & 1 \\ \hline 0 & 0 & \ldots & 0 \\ \cline{1-4}\end{array}. \quad \text{(A4)}$$

Proof. It can be easily verified that the effect of $\mathbf{J}_+$ acting on any state in $\mathcal{T}$ is just simply lifting $|0\rangle$ to $|1\rangle$ or annihilating $|1\rangle$, then summing all modified states together. Therefore, $\mathbf{J}_+|\omega_M\rangle$ would vanish since each modified state will either be annihilated directly or have two $|1\rangle$'s filling in a column. Thus $|\omega_M\rangle$ is a highest weight state.

States with the same amount of $|1\rangle$ and $|0\rangle$ as $|\omega_M\rangle$ would be either zero or can be obtained by simply interchanging entries within one or more columns of $|\omega_M\rangle$, which would at most contribute an additional factor -1.

States with more $|1\rangle$ than $|\omega_M\rangle$ vanish directly.

States that replace several $|1\rangle$ to $|0\rangle$ in the first row will survive after the action of $\mathbf{J}_+$, and thus cannot be a highest weight state.

Therefore, $|\omega_M\rangle$ indeed is the single highest weight state in $\mathcal{T}_\mu^{[\lambda]}$.

APPENDIX B: CONSTRUCT IRREDUCIBLE SUBSPACE OF S_k

For any element $\pi \in S_k$, $\pi\mathcal{T}_\mu^{[\lambda]}$ will either leave $\mathcal{T}_\mu^{[\lambda]}$ untouched or map it integrally to another $\mathcal{T}_\nu^{[\lambda]}$

$$\pi\mathcal{T}_\mu^{[\lambda]} = \pi\mathcal{Y}_\mu^{[\lambda]}\mathcal{T} = \mathcal{Y}_\nu^{[\lambda]}\pi\mathcal{T} = \mathcal{T}_\nu^{[\lambda]}. \quad \text{(B1)}$$

Furthermore, any element $\pi \in S_k$ cannot change the state weight. Thus states with the same weight ω_0, irreducible label $[\tilde{\lambda}]$, but different μ span an invariant subspace $\mathcal{T}^{[\tilde{\lambda}]}(\omega_0)$ under permutation. The representation, say $\mathcal{A}$, on $\mathcal{T}^{[\tilde{\lambda}]}(\omega_0)$ can be decomposed as a direct sum of irreducible representations of S_k, and so does $\mathcal{T}^{[\tilde{\lambda}]}(\omega_0)$ itself.

Any state in $\mathcal{T}^{[\tilde{\lambda}]}(\omega_0)$ is obtained by a projection operator corresponding to Young diagram $[\tilde{\lambda}]$. Hence in $\mathcal{T}^{[\tilde{\lambda}]}(\omega_0)$ only appears an irreducible representation described by $\mathcal{A}^{[\tilde{\lambda}]}$. Furthermore, it can be quickly obtained that the multiplicity of $\mathcal{A}^{[\tilde{\lambda}]}$ must be one, as a quick corollary of the following statement.

Observation 7. Suppose $C_k^{[\lambda]}$ is the multiplicity of SU(2) irreducible representation $D^{[\lambda]}$ appearing in Eq. (4.3); then $C_k^{[\lambda]}$ equals $d^{[\lambda]}$, the dimension of irreducible representation $\mathcal{A}^{[\lambda]}$ of *permutation group* S_k.

Proof. When $k=1$, the statement is trivial.

Suppose the statement holds when $k=m$. For $k=m+1$,

$$\begin{aligned}\bigotimes^{m+1} D^{[1]} &= \left(\bigoplus_{\lambda_1+\lambda_2=m} C_m^{[\lambda]}D^{[\lambda]}\right)\bigotimes D^{[1]} \\ &= \bigoplus_{\lambda_1'+\lambda_2'=m+1} C_{m+1}^{[\lambda']}D^{[\lambda']}, \end{aligned} \quad \text{(B2)}$$

and, according to the Littlewood-Richardson rule,

$$C_{m+1}^{\{\lambda_1',\lambda_2'\}} = C_m^{\{\lambda_1'-1,\lambda_2\}} + C_m^{\{\lambda_1',\lambda_2'-1\}}, \quad \text{(B3)}$$

where $C_m^{\{\lambda_1,\lambda_2\}}$ equals $d^{[\lambda]}$, the dimension of S_m irreducible representation $\mathcal{A}^{\{\lambda_1,\lambda_2\}}$. $d^{[\lambda]}$ can be easily computed from its Young diagram by a result known as the hook-length formula

$$d^{\{\lambda_1,\lambda_2\}} = \frac{(\lambda_1+\lambda_2)!(\lambda_1-\lambda_2+1)}{\lambda_2!(\lambda_1+1)!}. \quad \text{(B4)}$$

By simple calculation, one can verify that

$$C_{m+1}^{\{\lambda_1',\lambda_2'\}} = \frac{(\lambda_1'+\lambda_2')!(\lambda_1'-\lambda_2'+1)}{\lambda_2'!(\lambda_1'+1)!} = d^{\{\lambda_1',\lambda_2'\}}, \quad \text{(B5)}$$

and thus the proof is accomplished. ■

This indicates that an irreducible subspace of permutation group S_k in $\mathcal{T}$ can be uniquely determined by weight ω and irreducible representation label $[\lambda]$.

APPENDIX C: RATIO BETWEEN DIAGONAL TERMS

The combinatorial properties of the constant weight condition impose strong constraints on the reduced density matrices [21]

$$\sum_i (k-2\omega-2)\rho_{01,01}^{B_1B_i} = \sum_i (k+2\omega)\rho_{00,00}^{B_1B_i}, \quad \text{(C1a)}$$

$$\sum_i (k+2\omega-2)\rho_{10,10}^{B_1B_i} = \sum_i (k-2\omega)\rho_{11,11}^{B_1B_i}. \quad \text{(C1b)}$$

Assume that

$$\sum_i \rho_{00,00}^{B_1B_i} = (k-2\omega-2)t_0, \quad \text{(C2a)}$$

$$\sum_i \rho_{10,10}^{B_1B_i} = (k-2\omega)t_1. \quad \text{(C2b)}$$

Then we can express the matrix elements using t_0 and t_1

$$\rho^{B_1}_{0,0} = \frac{1}{k-1}\sum_i \left(\rho^{B_1B_i}_{00,00} + \rho^{B_1B_i}_{01,01}\right) = 2t_0, \quad \text{(C3a)}$$

$$\rho^{B_1}_{1,1} = \frac{1}{k-1}\sum_i \left(\rho^{B_1B_i}_{10,10} + \rho^{B_1B_i}_{11,11}\right) = 2t_1, \quad \text{(C3b)}$$

$$\begin{aligned}\sum_i \rho^{B_i}_{0,0} &= \sum_i \rho^{B_1B_i}_{00,00} + \sum_i \rho^{B_1B_i}_{10,10}\\ &= (k-2\omega-2)t_0 + (k-2\omega)t_1, \quad \text{(C3c)}\end{aligned}$$

$$\begin{aligned}\sum_i \rho^{B_i}_{1,1} &= \sum_i \rho^{B_1B_i}_{00,00} + \sum_i \rho^{B_1B_i}_{01,01}\\ &= (k+2\omega)t_0 + (k+2\omega-2)t_1. \quad \text{(C3d)}\end{aligned}$$

Compatibility of the k-symmetric extendible state requires that

$$\rho^{B_1}_{0,0} : \rho^{B_1}_{1,1} = \rho^{B_i}_{0,0} : \rho^{B_i}_{1,1}, \quad \text{(C4)}$$

i.e.,

$$\frac{2t_0}{2t_1} = \frac{(k-2\omega-2)t_0 + (k-2\omega)t_1}{(k+2\omega)t_0 + (k+2\omega-2)t_1},$$

which leads to

$$\frac{t_0}{t_1} = \frac{k-2\omega}{k+2\omega}. \quad \text{(C5)}$$

Here we have dropped the solution $t_0 + t_1 = 0$.

APPENDIX D: ADJUSTING COEFFICIENT FROM NONBOSONIC EXTENSION TO BOSONIC EXTENSION

Suppose there are two k-qubit pure states $|\psi\rangle$ and $|\phi\rangle$, which lie in different constant weight subspaces V_ω and $V_{\omega+1}$,

$$|\psi\rangle = \sum_{i_1,\dots,i_k=0,1} a_{i_1,\dots,i_k}|i_1,\dots,i_k\rangle, \quad \text{(D1a)}$$

$$|\phi\rangle = \sum_{j_1,\dots,j_k=0,1} b_{j_1,\dots,j_k}|j_1,\dots,j_k\rangle. \quad \text{(D1b)}$$

Define $\rho \equiv |\psi\rangle\langle\phi|$ and consider its one-particle partial trace $\{\rho^{(m)} \equiv \mathrm{Tr}_{m^c}\,\rho\}$. It is easy to verify that $\rho^{(m)}$ has only one nonzero element

$$\begin{aligned}\rho^{(m)} = &\sum_{\substack{i_1,\dots,i_{m-1},i_{m+1}\dots,i_k\\ j_1,\dots,j_{m-1},j_{m+1}\dots,j_k}} a_{i_1,\dots,i_k} b_{j_1,\dots,j_k}\delta_{i_m0}\delta_{j_m1}\\ &\times\delta_{i_1j_1}\dots\delta_{i_{m-1}j_{m-1}}\delta_{i_{m+1}j_{m+1}}\dots\delta_{i_kj_k}|0\rangle\langle 1|. \quad \text{(D2)}\end{aligned}$$

$J_+^{(m)}$ acts on the mth particle, elevating $|0\rangle$ to $|1\rangle$ and annihilating the $|1\rangle$; hence

$$\begin{aligned}\mathrm{Tr}(J_+^{(m)}\rho) = &\sum_{\substack{i_1,\dots,i_{m-1},i_{m+1}\dots,i_k\\ j_1,\dots,j_{m-1},j_{m+1}\dots,j_k}} a_{i_1,\dots,i_k} b_{j_1,\dots,j_k}\delta_{i_m0}\delta_{j_m1}\\ &\times\delta_{i_1j_1}\cdots\delta_{i_{m-1}j_{m-1}}\delta_{i_{m+1}j_{m+1}}\cdots\delta_{i_kj_k}. \quad \text{(D3)}\end{aligned}$$

Therefore, $\rho^{(m)} = \mathrm{Tr}(J_+^{(m)}\rho)|0\rangle\langle 1|$. Moreover,

$$\sum_m \rho^{(m)} = \mathrm{Tr}(\mathbf{J}_+\rho)|0\rangle\langle 1|. \quad \text{(D4)}$$

Now recalling the task in Eq. (4.10), one immediately obtains

$$\begin{aligned}&\frac{1}{d^{[\lambda]}}\mathrm{Tr}_{(B_1)^c}\sum_\mu \left|\omega^{[\lambda]}_\mu\right\rangle\!\left\langle\omega'^{[\lambda]}_\mu\right|\\ &= \frac{1}{kd^{[\lambda]}}\sum_i \mathrm{Tr}_{(B_i)^c}\sum_\mu \left|\omega^{[\lambda]}_\mu\right\rangle\!\left\langle\omega'^{[\lambda]}_\mu\right|\\ &= \frac{1}{kd^{[\lambda]}}\mathrm{Tr}\left(\mathbf{J}_+\sum_\mu \left|\omega^{[\lambda]}_\mu\right\rangle\!\left\langle\omega'^{[\lambda]}_\mu\right|\right)|0\rangle\langle 1|. \quad \text{(D5)}\end{aligned}$$

Since for each given μ, all possible $\{|\omega^{[\lambda]}_\mu\rangle\}$ forms the same irreducible SU(2) representation $D^{[\lambda]}$, corresponding to irreducible representation $j = \frac{1}{2}(\lambda_1 - \lambda_2)$. Within such representation, $\mathbf{J}_+$ elevates $|\omega\rangle$ to $|\omega+1\rangle$, for every weight ω that is not the highest weight,

$$\frac{1}{d^{[\lambda]}}\mathrm{Tr}_{(B_1)^c}\sum_\mu \left|\omega^{[\lambda]}_\mu\right\rangle\!\left\langle\omega'^{[\lambda]}_\mu\right| = \alpha^{[\lambda]}_{\omega,\omega'}\,|0\rangle\langle 1|, \quad \text{(D6)}$$

where

$$\alpha^{[\lambda]}_{\omega,\omega'} = \frac{\delta_{\omega+1,\omega'}}{k}\sqrt{\left(\frac{\lambda_1-\lambda_2}{2}-\omega\right)\left(\frac{\lambda_1-\lambda_2}{2}+\omega+1\right)}.$$

APPENDIX E: DENSITY MATRIX GIVEN IN EQ. (4.15) COULD BE POSITIVE DEFINITE

Notice that $\sigma^{AB_1B_2\cdots B_k}$ in Eq. (4.15) is a convex combination of different $[\lambda]$ ingredients:

$$\sigma^{AB_1B_2\cdots B_k} = \sum_{[\lambda]}\sigma^{AB_1B_2\cdots B_k}_{[\lambda]}, \quad \text{(E1)}$$

where

$$\begin{aligned}\sigma^{AB_1B_2\cdots B_k}_{[\lambda]} \equiv &\sum_{\omega,\omega'}\left(\sum_{\alpha,\alpha'}\left|\psi^\alpha_{\omega,[\lambda]}\right\rangle\!\left\langle\psi^{\alpha'}_{\omega',[\lambda]}\right|\right)\\ &\otimes p^{[\lambda]}_{\omega,\omega'}|\omega^S\rangle\langle\omega'^S|. \quad \text{(E2)}\end{aligned}$$

Here $p^{[\lambda]}_{\omega,\omega'}$ is defined in Eq. (4.16). Therefore, it is sufficient to verify that $\sigma^{AB_1B_2\cdots B_k}_{[\lambda]}$ can be a positive-definite matrix.

We give a construction as below:

$$\begin{aligned}\sigma^{AB_1B_2\cdots B_k}_{[\lambda]} = &\left(\sum_\omega \xi^{[\lambda]}_\omega|\omega^S\rangle\right)\left(\sum_{\omega'}\xi^{*[\lambda]}_{\omega'}\langle\omega'^S|\right)\\ &+\sum_\omega(1-|\xi^{[\lambda]}_\omega|^2)|\omega^S\rangle\langle\omega^S|, \quad \text{(E3)}\end{aligned}$$

where

$$\xi^{[\lambda]}_\omega\xi^{*[\lambda]}_{\omega+1} = p^{[\lambda]}_{\omega,\omega+1}, \quad \text{(E4)}$$

$$\left|\xi^{[\lambda]}_\omega\right|^2 \leqslant 1. \quad \text{(E5)}$$

The first term of the right-hand side in Eq. (E3) contributes all the off-diagonal terms of $\sigma_{[\lambda]}^{AB_1B_2\cdots B_k}$ and the second term consists of purely diagonal terms. The remaining part is to give an explicit example of $\{\xi_\omega^{[\lambda]}\}$.

Set $\xi_{\frac{k}{2}+1}^{[\lambda]}=1$ and $\xi_{\frac{k+1}{2}}^{[\lambda]}=\xi_{\frac{k+1}{2}}^{[\lambda]}=\sqrt{p_{\frac{k-1}{2},\frac{k+1}{2}}^{[\lambda]}}$ for even and odd k; the remaining $\xi_\omega^{[\lambda]}$ can be obtained by iteration relation Eq. (E4).

Equation (E5) is satisfied due to the fact that (a) the maximum value of $p_{\omega,\omega+1}^{[\lambda]}$ is obtained at $\omega_m=\frac{k}{2}+1$ and $\omega_m=\frac{k\pm1}{2}$ when k is even and odd, respectively, and (b) $p_{\omega,\omega+1}^{[\lambda]}$ increases strictly when $\omega<\omega_m$ and decreases strictly when $\omega>\omega_m$.

It is straightforward to verify that $\sigma_{[\lambda]}^{AB_1B_2\cdots B_k}$ is positive definite as long as its counterpart in the nonbosonic extension is. Therefore, $\sigma^{AB_1B_2\cdots B_k}$ can always be positive definite.

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