High-fidelity quantum cloning of two nonorthogonal quantum states via weak measurements

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We propose a scheme to enhance the fidelity of a symmetric quantum cloning machine via weak measurements. By adjusting the intensity of the weak measurement parameter p, we obtain copies of initial states with different values of fidelity. Choosing a proper value of p, we can obtain perfect copies. In this paper, we focus on 1-2 quantum cloning for two nonorthogonal states. Sets containing more than two linear independent states are also briefly discussed. Due to the probabilistic nature of weak measurements, we obtain high-fidelity copies probabilistically. If the weak measurement is a success, we do subsequent operations to obtain high-fidelity copies; otherwise, the cloning process fails and we quit. From this perspective, the scheme we propose is economical for saving quantum resources and time. Our scheme may be very useful in quantum information processing.

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I. INTRODUCTION

In recent years, quantum information has been developed very quickly. The study of quantum information processing (QIP) has attracted much attention from various research communities. Various schemes for logic gates, such as controlled NOT (CNOT) and SWAP, as required for classical computers, were proposed theoretically and implemented experimentally in many systems including optical systems [1–5], trapped ions [6–8], cavity quantum electrodynamics [9–11], and liquid-state nuclear magnetic resonance [12–14]. There seems to be a great promising future in quantum computers. The peculiar principles such as linearity, unitarity, and inseparability have been utilized to realize quantum computers [15]. On one hand, these principles enhance the capacity of information processing, but on the other hand, they put up some obstacles [16]. A fundamental restriction in QIP is that an unknown quantum state cannot be copied perfectly [17], in contrast with replicating information ubiquitously in the classical world. This is a consequence of the linearity of quantum mechanics and makes a qubit so distinct from a classical bit. This limitation is known as the no-cloning theorem and has found its application in quite different fields of quantum information theory, such as quantum computation and quantum cryptography [18].

However, if we pay some price, then an approximate or even exact cloning is possible. It does not prohibit the possibility of approximate cloning of an arbitrary state of a quantum mechanical system. Bužek and Hillery first presented a scheme in which, given an unknown qubit, two identical output qubits as approximate as possible to the input qubits are produced [19]. After their seminal paper, quantum cloning has been extensively studied and lots of topmost achievements have been made, both theoretically and experimentally [20,21]. This Bužek-Hillery quantum cloning machine is state independent and known as the universal quantum cloning machine (UQCM). The optimal fidelity of a cloning bound achieves $5/6$ for UQCM. Shortly after, another quantum cloning machine was proposed [22–24], called the quantum phase-covariant cloning machine (QPCCM). The fidelity of the QPCCM reaches 0.854, which is higher than that of UQCM. The QPCCM is significantly important in quantum cryptography as it provides the optimal eavesdropping method for a large class of attacks on quantum cryptograph protocols [25–27]. Besides these deterministic quantum cloning machines, Duan and Guo [28,29] revealed that a quantum state secretly chosen from a linearly independent set of states can be probabilistically cloned with unit fidelity. They called this quantum cloning probabilistic quantum cloning (PQC). This kind of quantum cloning is different from the deterministic quantum cloning machine. It has nonzero probability that the cloning process fails. However, once it succeeds, perfect copies of the initial qubits are obtained. The scheme we proposed in this paper is different from the PQC. The more detailed differences between them are shown later.

It has been well realized that we can clone qubits in a better way. The QPCCM is one example. Recently, an optimal quantum cloning machine, which clones qubits of arbitrary symmetrical distribution around the Bloch vector, was investigated [30]. More generally, states in the block region, which is a simply connected region enclosed by a "longitude-latitude grid" on the Bloch sphere, was also investigated [31]. All those quantum cloning machines are based on the maximin principle by making full use of *a priori* information of amplitude and phase about the to-be-cloned qubits. As expected, the performance of these machines is better than the UQCM. In addition to the price of limiting the range of input states, other resources can also be sacrificed, such as the probability of successful cloning. PQC is one of those machines.

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Inspired by those previous works, we propose a scheme that combines the weak measurements and unitary transformations. Assuming that a given qubit is secretly chosen from a state set $\{|\psi_1\rangle, |\psi_2\rangle\}$, our task is to duplicate the given qubit. Many relevant studies are reported [29,32,33]. In their schemes, they all clone given qubits directly. Compared with their schemes, our scheme has peculiar advantages. Before complicated quantum transformations, we pretreat the given qubits. This pretreatment allows us to obtain high-fidelity copies. What is more, the fidelity can be controlled by adjusting the value of the parameter p. Whether we do further operations depends on the results of measurements. This economizes quantum resources and makes our scheme an economical one. The rest of this paper is organized as follows. In Sec. II, we make a brief review of weak measurements and optimal quantum cloning for two nonorthogonal states. In Sec. III, we show our scheme for high-fidelity copies in detail. In Sec. IV, comparisons with other schemes are made. We also clarify the generalization of our scheme. Finally, a concise summary is given in Sec. V.

II. THEORY

A. Weak measurements

The projection postulate is one of the basic postulates of standard quantum theory and it states that measurement of a variable of a quantum system irrevocably collapses the initial state to one of the eigenstates (corresponding to the measurement outcomes) of the measurement operator. Once the initial state collapses due to a projection measurement on a quantum system, it can never be recovered. However, the situation is different for the case that the measurement is not sharp, i.e., a nonprojective measurement [34]. For weak measurements, the information extracted from the quantum system is deliberately limited, thereby keeping the measured system's state from randomly collapsing towards an eigenstate. It is possible to reverse the measurement-induced state collapse and the unsharpness of a measurement has been shown to be related to the probabilistic nature of the reversing operation which can serve as a probabilistic quantum error correction [35].

Consider that the initial state of a qubit is a pure state $|\phi\rangle$ and the measurement operators $\hat{P}_1$ and $\hat{P}_2$ are orthogonal projectors whose sum $\hat{P}_1 + \hat{P}_2 = \hat{I}$ is identity. We introduce the operators

$$\hat{M}_{\text{yes}} = \sqrt{p}\hat{P}_1 + \hat{P}_2, \quad \hat{M}_{\text{no}} = \sqrt{1-p}\hat{P}_1, \quad p \in [0,1]. \tag{1}$$

It should be noted that $\hat{M}^{\dagger}_{\text{yes}}\hat{M}_{\text{yes}} + \hat{M}^{\dagger}_{\text{no}}\hat{M}_{\text{no}} = \hat{I}$ and therefore $\hat{M}_{\text{yes}}$ and $\hat{M}_{\text{no}}$ describe a measurement. Consider the effect of the operators $\hat{M}$ on a pure state $|\phi\rangle$. The state can be rewritten as $|\phi\rangle = \sqrt{p_1}\,|\phi_1\rangle + \sqrt{p_2}\,|\phi_2\rangle$, where $|\phi_{1,2}\rangle = \hat{P}_{1,2}\,|\phi\rangle / \sqrt{p_{1,2}}$ are the two possible outcomes of the projective measurement and $p_{1,2} = \langle\phi|\,\hat{P}_{1,2}\,|\phi\rangle$ are the corresponding probabilities. The operator $\hat{M}_{\text{yes}}$ decreases the ratio $\frac{p_1}{p_2}$, causing $\hat{M}_{\text{yes}}\,|\phi\rangle$ to move toward $|\phi_2\rangle$ while operator $\hat{M}_{\text{no}}$ collapses the $|\phi\rangle$ into $|\phi_1\rangle$.

B. Optimal quantum cloning for two nonorthogonal states

Suppose we are given with equal probability one quantum state from a set including two known nonorthogonal quantum states in the form

$$\begin{aligned} |\psi_1\rangle &= \cos(\xi)\,|0\rangle + \sin(\xi)\,|1\rangle\,, \\ |\psi_2\rangle &= \sin(\xi)\,|0\rangle + \cos(\xi)\,|1\rangle\,, \end{aligned} \tag{2}$$

where $\xi \in [0, \pi/4]$, with the scalar product

$$\langle\psi_1|\psi_2\rangle = \sin(2\xi). \tag{3}$$

The transformation of symmetric 1-2 state-dependent cloning takes the following form:

$$\begin{aligned} |00\rangle &\to a\,|00\rangle + b(|01\rangle + |10\rangle) + c\,|11\rangle\,, \\ |10\rangle &\to a\,|11\rangle + b(|10\rangle + |01\rangle) + c\,|00\rangle\,, \end{aligned} \tag{4}$$

where we assume the cloning coefficients a, b, and c are real numbers. Due to the unitarity of the transformation, the following formulas must be satisfied:

$$a^2 + 2b^2 + c^2 = 1, \quad ac + b^2 = 0. \tag{5}$$

Solving Eq. (5), we obtain the following equations:

$$a = \tfrac{1}{2}(\sqrt{1-4b^2} + 1), \quad c = \tfrac{1}{2}(\sqrt{1-4b^2} - 1). \tag{6}$$

In previous work, fidelity has often been used as a factor of merit, which is defined as $F = \langle\phi_{\text{in}}|\,\rho_{\text{out}}\,|\phi_{\text{in}}\rangle$, where ρ_{out} is a reduced density matrix of output state 1 or 2 and $|\psi_{\text{in}}\rangle$ is a to-be-cloned state. Due to the symmetry of transformation given by Eq. (4), we obtain the fidelity as

$$\begin{aligned} F(|\psi_1\rangle) = F(|\psi_2\rangle) = \tfrac{1}{4}(3a^2 &+ 4(a+b)(b+c)\sin 2\xi \\ &+ (a+c)(a-2b-c)\cos 4\xi \\ &+ 2ab + 4b^2 + 2bc + c^2). \end{aligned} \tag{7}$$

With some calculation and using the method of Lagrange multipliers, we can determine the cloning coefficient b as

$$b = \tfrac{1}{8}(1 - \csc 2\xi + \csc 2\xi\sqrt{9\sin^2 2\xi - 2\sin 2\xi + 1}). \tag{8}$$

Combining with Eq. (6), we obtain the detailed transformation and the maximum fidelity for optimal cloning.

III. SCHEME FOR HIGHER FIDELITY

In this section, we are going to investigate how to enhance the fidelity by using weak measurements. Unlike the schemes proposed by others, we do a weak measurement as a pretreatment before transformation. After the successful weak measurement, we obtain new intermediate qubits. Putting them into a quantum machine, we obtain the final copies with high fidelity at the output of the machine. Let us now present our scheme in detail. Suppose we are given a qubit selected randomly from Eq. (2). We first do a weak measurement described as in Eq. (1) on the given state. Let $\hat{P}_1 = |+\rangle\langle+|$ and $\hat{P}_2 = |-\rangle\langle-|$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Thus $|\psi_{1,2}\rangle$ can be rewritten as

$$|\psi_{1,2}\rangle = \frac{\sqrt{2}}{2}(\cos\xi + \sin\xi)\,|+\rangle \pm \frac{\sqrt{2}}{2}(\cos\xi - \sin\xi)\,|-\rangle\,.$$

After the weak measurement, if the outcome is "yes," we obtain the intermediate states

$$|\psi'_{1,2}\rangle = \frac{\sqrt{p}[\sin(\xi)+\cos(\xi)]}{\sqrt{(p-1)\sin(2\xi)+p+1}}\,|+\rangle \pm \frac{\cos(\xi)-\sin(\xi)}{\sqrt{(p-1)\sin(2\xi)+p+1}}\,|-\rangle\,, \tag{9}$$

with the success probability

$$p_{\text{yes}} = \tfrac{1}{2}[(p-1)\sin(2\xi)+p+1]. \tag{10}$$

Similarly, we obtain the scalar product of two possible intermediate states as

$$\langle\psi'_1|\psi'_2\rangle = \frac{(p+1)\sin(2\xi)+p-1}{(p-1)\sin(2\xi)+p+1}. \tag{11}$$

Next, we do a transformation on the intermediate qubit $|\psi'_{1,2}\rangle$ and ancillary qubit which is originally in a blank state $|0\rangle$. For the sake of convenience, we substitute $\frac{\sqrt{p}(\sin\xi+\cos\xi)}{\sqrt{(p-1)\sin 2\xi+p+1}}$ and $\frac{\cos\xi-\sin\xi}{\sqrt{(p-1)\sin 2\xi+p+1}}$ with $\frac{\sqrt{2}}{2}(\cos\xi'+\sin\xi')$, and $\frac{\sqrt{2}}{2}(\cos\xi'-\sin\xi')$ respectively, so the intermediate qubits can be rewritten as

$$\begin{aligned}|\psi'_1\rangle &= \frac{\sqrt{2}}{2}(\cos(\xi')+\sin(\xi'))\,|+\rangle + \frac{\sqrt{2}}{2}(\cos(\xi')-\sin(\xi'))\,|-\rangle \\ &= \cos(\xi')\,|0\rangle + \sin(\xi')\,|1\rangle\,, \\ |\psi'_2\rangle &= \frac{\sqrt{2}}{2}(\cos(\xi')+\sin(\xi'))\,|+\rangle - \frac{\sqrt{2}}{2}(\cos(\xi')-\sin(\xi'))\,|-\rangle \\ &= \sin(\xi')\,|0\rangle + \cos(\xi')\,|1\rangle\,. \end{aligned} \tag{12}$$

Combining them with Eq. (11), we obtain the important equation as

$$\sin(2\xi') = \frac{(p+1)\sin(2\xi)+p-1}{(p-1)\sin(2\xi)+p+1}. \tag{13}$$

By applying transformation (4), we obtain the copy fidelity as

$$F(|\psi_1\rangle) = F(|\psi_2\rangle) = \tfrac{1}{2}(1+(a+c)\cos 2\xi\cos 2\xi' + 2b(a+c)\sin 2\xi(\sin 2\xi'+1)). \tag{14}$$

We want to derive the optimal fidelity, with the constraints as in Eq. (5). After some algebra, we determine the cloning coefficients as

$$b = \frac{\csc 2\xi\sqrt{8(\sin 2\xi'+1)^2\sin^2 2\xi+\cos^2 2\xi'\cos^2 2\xi}}{8(\sin 2\xi'+1)} - \frac{\cos 2\xi'\cot 2\xi}{8(\sin 2\xi'+1)}. \tag{15}$$

So the optimal fidelity has the expression

$$\begin{aligned}F = \frac{1}{32}\Bigg\{16 &+ \frac{3\sqrt{2}\cos(2\xi')\cos(2\xi)}{(\sin(2\xi')+1)}[4\sin^2(2\xi')+8\sin(2\xi')-\cos^2(2\xi')\cot^2(2\xi) \\ &+ \cos(2\xi')\cot(2\xi)\sqrt{\{\cos^2(2\xi')\cot^2(2\xi)+8[1+\sin(2\xi')]^2\}}+4]^{1/2} \\ &+ \frac{\sqrt{2}\sin(2\xi)}{(\sin(2\xi')+1)^2}\sqrt{\{\cos^2(2\xi')\cot^2(2\xi)+8[1+\sin(2\xi')]^2\}}[4\sin^2(2\xi')+8\sin(2\xi')-\cos^2(2\xi')\cot^2(2\xi) \\ &+ \cos(2\xi')\cot(2\xi)\sqrt{\{\cos^2(2\xi')\cot^2(2\xi)+8[1+\sin(2\xi')]^2\}}+4]^{1/2}\Bigg\}.\end{aligned} \tag{16}$$

Generally, ξ is given in advance and ξ' is determined by the parameter p, which can be adjusted and controlled. By adjusting the value of the parameter p, we can obtain copies with different values of fidelity. As shown in Fig. 1, we show the dependence of the fidelity on the parameters ξ and ξ'. The color represents the fidelity of the copies. Given initial states with a fixed value of ξ, the fidelity changes with different ξ'. We can find that under the condition $\sin(2\xi') = \sin^2(2\xi)$, F always reaches 1, which means we obtain two perfect copies of the initial states. Combining with Eq. (13), we obtain a critical value $p_c = \frac{1+\sin^2 2\xi}{(1+\sin 2\xi)^2}$. For instance, $\xi = \pi/8$ and $\xi' = \pi/12$ satisfy the above condition. Then all parameters are determined and we obtain the initial qubits

$$\begin{aligned}|\psi_1\rangle &= \tfrac{1}{2}\sqrt{2+\sqrt{2}}\,|0\rangle + \tfrac{1}{2}\sqrt{2-\sqrt{2}}\,|1\rangle\,, \\ |\psi_2\rangle &= \tfrac{1}{2}\sqrt{2-\sqrt{2}}\,|0\rangle + \tfrac{1}{2}\sqrt{2+\sqrt{2}}\,|1\rangle\,,\end{aligned} \tag{17}$$

intermediate qubits

$$\begin{aligned}|\psi'_1\rangle &= \tfrac{1}{4}(\sqrt{2}+\sqrt{6})\,|0\rangle + \tfrac{1}{4}(\sqrt{6}-\sqrt{2})\,|1\rangle\,, \\ |\psi'_2\rangle &= \tfrac{1}{4}(\sqrt{6}-\sqrt{2})\,|0\rangle + \tfrac{1}{4}(\sqrt{2}+\sqrt{6})\,|1\rangle\,,\end{aligned} \tag{18}$$

and transformation coefficients

$$b = \frac{1}{2\sqrt{3}},\quad a = \frac{1}{2}\left(\sqrt{\frac{2}{3}}+1\right),\quad c = \frac{1}{2}\left(\sqrt{\frac{2}{3}}-1\right). \tag{19}$$

The final qubits would be

$$\begin{aligned}|\psi_1^f\rangle = &\left(\tfrac{1}{2}\sqrt{2+\sqrt{2}}\,|0\rangle + \tfrac{1}{2}\sqrt{2-\sqrt{2}}\,|1\rangle\right) \\ &\otimes\left(\tfrac{1}{2}\sqrt{2+\sqrt{2}}\,|0\rangle + \tfrac{1}{2}\sqrt{2-\sqrt{2}}\,|1\rangle\right),\end{aligned} \tag{20}$$

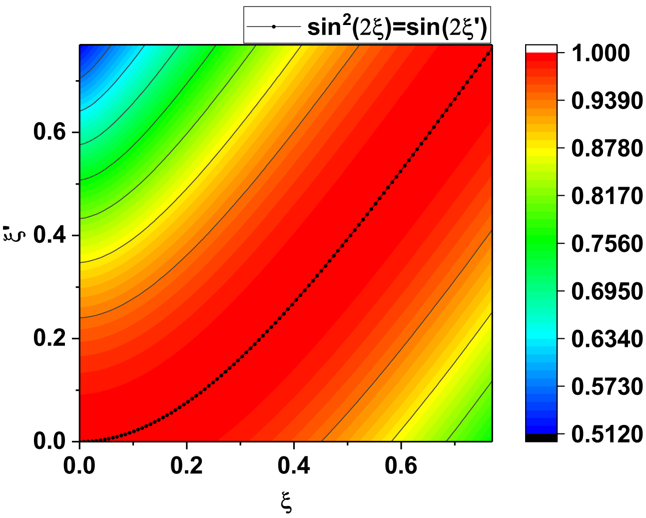

FIG. 1. The dependence of the fidelity on the parameters ξ and ξ'. The dotted line corresponds to $\sin(2\xi') = \sin^2(2\xi)$, where the fidelity reaches its maximum value of 1.

$$|\psi_2^f\rangle = \left(\tfrac{1}{2}\sqrt{2-\sqrt{2}}\,|0\rangle + \tfrac{1}{2}\sqrt{2+\sqrt{2}}\,|1\rangle\right) \otimes \left(\tfrac{1}{2}\sqrt{2-\sqrt{2}}\,|0\rangle + \tfrac{1}{2}\sqrt{2+\sqrt{2}}\,|1\rangle\right), \qquad (21)$$

which are exactly the perfect copies of the initial qubits.

Next, we discuss the success probability, which has the formula as in Eq. (10). It is a function of ξ and p, where ξ ranges from 0 to $\pi/4$ and p ranges from 0 to 1. As we mentioned before, ξ is already known and unchangeable. We choose a proper value of p so that we can keep a balance between success probability and fidelity. The dependence of the success probability and the fidelity on p is shown in Fig. 2 with a fixed value of ξ. The success probability is monotone increasing with p while the fidelity increases before p_c and then decreases after it. It can be proved that the fidelity reaches 1 at p_c. And the corresponding success probability is $p_{\text{yes}} = \frac{1}{1+\sin(2\xi)}$. This is just the Duan-Guo bound [29].

IV. COMPARISONS AND DISCUSSION

In this section, we make some comparisons with other schemes and give some discussions on our scheme. To clone states, a trivial cloning strategy is the measurement-based procedure: one measures initial states and produces two copies of the states, according to the measurement results. For a given state chosen randomly from two nonorthogonal states, the unambiguous discrimination provides a measurement scheme to identify the state unambiguously with nonzero probability. Once we identify the state, we produce two perfect copies (actually we can produce any number of copies). For the given states in the form of Eq. (2) with equal *a priori* probabilities, the minimum probability value of an inconclusive outcome [or Ivanovic-Dieks-Peres (IDP) limit] [36] is given by

$$p_? = |\langle\psi_1||\psi_2\rangle| = \sin 2\xi. \qquad (22)$$

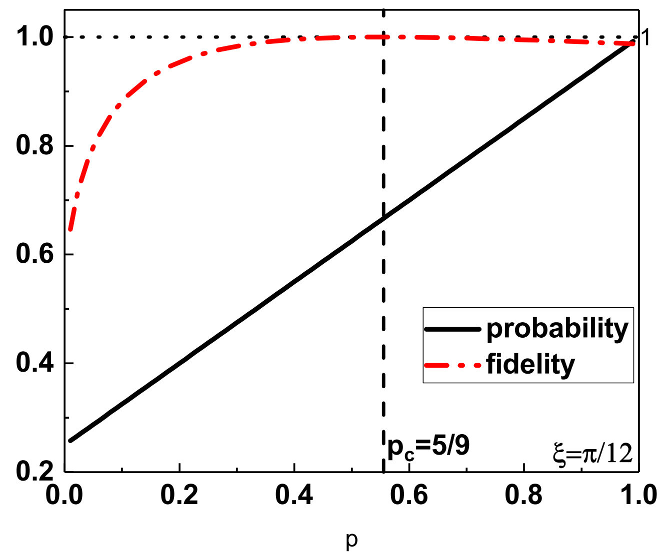

FIG. 2. The dependence of the success probability and the fidelity on the parameters p and ξ. Here, the figure is depicted with $\xi = \pi/12$ and similar curves can be obtained with other values of ξ. The success probability is monotone increasing with p while the fidelity increases before p_c and then decreases after it. For the case of $\xi = \pi/12$, $p_c = 5/9$.

In our scheme, if $p = p_c$, we would obtain two perfect copies. In this condition, the p_{no} is given by

$$p_{\text{no}} = \frac{\sin 2\xi}{1 + \sin 2\xi}. \qquad (23)$$

Actually, under these circumstances, our scheme is equivalent to the probabilistic quantum cloning, so it is optimal. As shown in Fig. 3, p_{no} is always smaller than $p_?$ in the region $(0, \pi/4)$, which shows the advantage of our scheme.

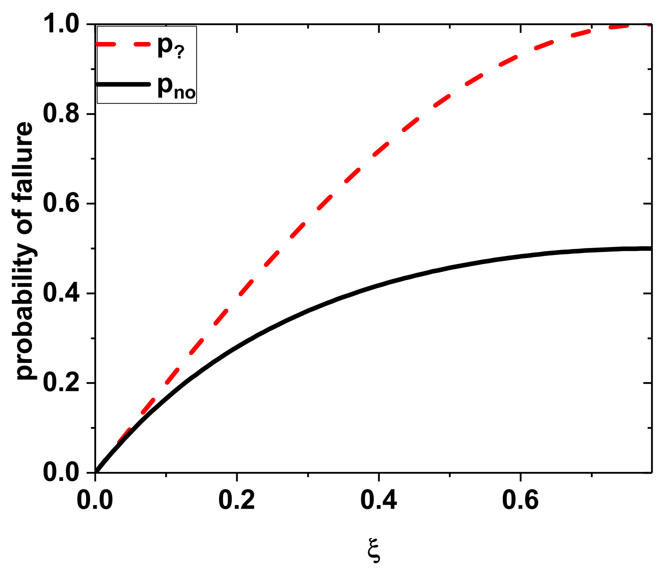

FIG. 3. The dependence of the probability of obtaining an inconclusive result on the parameter ξ. Here, we set $p = p_c$ so that we could obtain two perfect copies.

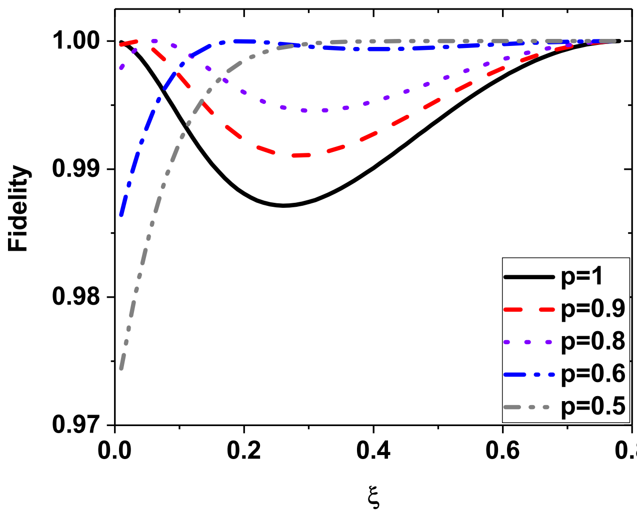

FIG. 4. The dependence of the fidelity on the parameter ξ as $p = 1, 0.9, 0.8, 0.6$, and 0.5. We can obtain higher fidelity by adjusting the value of the parameter p in the region $(p_c, 1)$.

There is another cloning scheme, called optimal state-dependent cloning [32]. This scheme produces approximate copies with optimal fidelity deterministically. Actually, setting $p = 1$, i.e., $\xi = \xi'$, we find that the fidelity we obtained in Eq. (16) is equivalent to that of optimal $1 \rightarrow 2$ state-dependent quantum cloning (SDQC) [32,33], which clones the states directly without pretreatment. Adjusting the value of p in the region $(p_c, 1)$, we obtain higher-fidelity copies compared with that of optimal $1 \rightarrow 2$ SDQC. The fidelity would reach 1 if $p = p_c$. However, it would decrease again if $p < p_c$. We show the dependence of the fidelity on the parameter ξ with $p = 1, 0.9, 0.8, 0.6$, and 0.5 in Fig. 4. For the case of ξ being close to zero and p being small enough, our scheme may behave badly as shown in Fig. 4. Actually, the value of p is adjustable and can be confined in the region $(p_c, 1)$. This ensures that we obtain copies with higher fidelity.

In short, we obtain copies with different values of the fidelity probabilistically by adjusting the value of p. The optimal state-dependent cloning, perfect-probabilistic cloning, and unambiguous discrimination can be regarded as particular cases. In the case of $p = 1$, our scheme is equivalent to the optimal state-dependent cloning. In the case of $p = p_c$, it is equivalent to the perfect-probabilistic cloning. In the case of $p_d = \frac{1-\sin 2\xi}{1+\sin 2\xi}$, states after successful weak measurements are orthogonal, and thus distinguishable, which is equivalent to states discrimination. Here, p_c is always greater than p_d, accounting for higher success probability. The trick is that, for proper p, the overlap of states has been adjusted after successful weak measurements, which is beneficial for the following cloning transformation.

The most obvious advantage of our scheme is that we present a method to balance the copy fidelity and the success probability via weak measurements. This makes our scheme different from previous ones: one can only obtain perfect copies with nonzero probability using PQC or approximate copies deterministically. However, using our scheme, we can adjust p to achieve a tradeoff between the success probability and the fidelity. This flexibility may be of great use in QIP. For instance, for some quantum key distributions, Eve can sacrifice the success probability in order to obtain information. Alice and Bob cannot detect whether they are eavesdropped upon or not, if they only use the disturbance criterion. Eve hides herself in qubit loss. In general, the best eavesdropping scheme for Eve is to choose a proper p to keep a balance between the success probability and the fidelity so that she could obtain information as much as possible and hide herself from discovery in the qubit loss or noise.

Next, we give some brief discussions of generalization to the cases that state sets contain more than two linear independent states. Consider a state chosen secretly from a linearly independent and nonorthogonal set $S = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_N\rangle\}$. The weak measurement is characterized by two operators $\hat{M}_{\rm yes}$ and $\hat{M}_{\rm no}$. They satisfy $\hat{M}_{\rm yes}^\dagger \hat{M}_{\rm yes} + \hat{M}_{\rm no}^\dagger \hat{M}_{\rm no} = I$. We may write $\hat{M}_{\rm yes}|\psi_i\rangle = \sqrt{p_i}|\phi_i\rangle$ $(i = 1, 2, \ldots, N)$, where $0 < p_i < 1$ and $|\phi_i\rangle$ are not orthogonal necessarily. If $|\phi_i\rangle$ are orthogonal, it turns out to be unambiguous discrimination [36–38]. Thus, we obtain the following equations:

$$\langle\psi_i|\hat{M}_{\rm yes}^\dagger \hat{M}_{\rm yes}|\psi_j\rangle = \sqrt{p_i p_j}\langle\phi_i||\phi_j\rangle \;\; (i, j = 1, 2, \ldots, N). \tag{24}$$

Since $|\psi_i\rangle$ are linearly independent and nonorthogonal, we can establish the form of $M_{\rm yes}$ that satisfies Eq. (24) as

$$\hat{M}_{\rm yes} = \sum_i \frac{\sqrt{p_i}}{\langle\psi_i^\perp||\psi_i\rangle}|\phi_i\rangle\langle\psi_i^\perp|, \tag{25}$$

where $|\psi_i^\perp\rangle$ is orthogonal to all $|\psi_j\rangle$ for $j \neq i$. Thus, we could pretreat the initial states by choosing a proper set of p_i and $|\phi_i\rangle$. The average success probability is

$$p_{\rm pro} = \frac{1}{N}\sum_i p_i. \tag{26}$$

We note that the $|\phi_i\rangle$ does not appear in Eq. (26), which means the choice of $|\phi_i\rangle$ does not affect the average success probability. However, $|\phi_i\rangle$ do play an important role in the subsequent transformation task. Finally, we obtain high-fidelity clones with a proper set of p_i and $|\phi_i\rangle$.

The general solution to the weak measurements and cloning transformations for an arbitrary state set $\{|\psi\rangle\}$ is complicated and presently unknown. Some other special cases, for example, symmetric sets of pure states [39], may be solved analytically. This is an open question and needs further investigation. The cloning of two nonorthogonal states is relatively simple. However, for typical distributions such as universal or phase-covariant distribution, states to be cloned are linearly dependent. Thus, there do not exist $\hat{M}_{\rm yes}$ that pretreat initial states arbitrarily. For example, $\hat{M}_{\rm yes}$ may decrease the overlap between $|\psi_1\rangle$ and $|\psi_2\rangle$. Nevertheless, it would increase the overlap between some other states, for instance, $|\psi_3\rangle$ and $|\psi_4\rangle$, inevitably. $|\psi_k\rangle$ $(k = 1, 2, 3, 4)$ are four states in the initial state set. For the uniform distribution, our scheme may not improve the performance of cloning. However, for nonuniform distributions, for instance, $p_1, p_2 \gg p_3, p_4$, p_k being the probability that state is $|\psi_k\rangle$, it may be possible to find proper weak measurements and unitary transformations that clone initial states with high fidelity.

V. SUMMARY

In this paper, we propose a scheme to clone qubits chosen randomly from a nonorthogonal state set using weak measurements. We demonstrate that weak measurements can be useful for high fidelity in quantum cloning processing. First, we do a weak measurement on the qubits to make them easier to clone. If the result of the measurement is "yes," we feed the qubits to the subsequent unitary transformation. After the cloning process is accomplished, we obtain copies with high fidelity dependent on the value of p. By choosing a proper p, we can even obtain perfect copies. It is easily implemented with current experimental techniques since many experimental quantum clonings have been reported [40–44]. It is also clarified that our scheme is valid for state sets that contains more than two linear independent states. Analogously, what we need to do is to find a set of proper measurement operators which pretreats the initial states before the subsequent unitary transformations. Unambiguous discrimination is one of the pretreatments [37]. Obviously, this is not optimal. Since the weak measurement is nonunitary, the whole process is probabilistic and the success probability is dependent on p. Sometimes, we may get nothing. But, once we succeed, we would obtain copies with high fidelity. What is more, the practical cloning transformations usually consist of a series of complicated quantum logic gates in the experiments. Only when the outcome is "yes" do we perform further cloning transformations; otherwise, we quit. This can save us lots of quantum resources. Thus, our scheme is economical from this perspective. Finally, the method we used here is not restricted to the quantum cloning process and could be found in some other applications in QIP. The core thought is that we pretreat the target qubits, for which some price may be paid. Nonetheless, we may obtain final qubits with high fidelity.

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